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Geometric Phase in General Relativity

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Abstract

We study the transport of a small wave packet in the embedding of the Stueckelberg-Horwitz-Piron relativistic quantum theory into the manifold of general relativity around the Schwarzschild solution using a semiclassical approximation. We find that the parallel transport of the momentum leads to a geometrical (Berry type) phase.

Keywords

Geometrical Phase, Quantum Theory in General Relativity, Schwarzschild Solution

In this paper, we study the idea called geometric (Berry) phase found by Berry in 1984 [1] in the context of electromagnetism, in application to the effect of a change in direction of the momentum vector parallel transported around a closed path on a manifold with curvature, such as in gravitational field [2] (see also [3] [4] [5]). This change, as we show, leads to a phase change on its wave function [6] in the quantum theory [7] (see also [8] and [9]).

Stone *et al.* [10] have discussed the geometrical and Berry phase associated with Dirac and Weyl particles; our work deals with the phase generated by parallel transport for a quantum theory on the manifold of general relativity. Ghosh and Mukhopadhyay [11] have discussed the geometric (Berry) phase for a Dirac Hamiltonian theory in a gravitational field, pointing out the similarities between the effects of gravitation and magnetism. We study here the geometrical (Berry) phase on a wave function associated directly with a rigorous quantum theory on the manifold. In this paper, we follow a narrow wave packet transported on a geodesic around a black hole [12] and show that a semicassical argument leads to such a geometrical phase.

For any conserved vector S_μ infinitesimal parallel transport along a geodesic is given by

$$\mathrm{d}S_\mu = -\Gamma^\lambda_{\mu\nu}\mathrm{d}x^\nu S_\lambda \tag{1}$$

where $\mathrm{d}x^\mu$ is along the curve. For the Schwarzchild coordinates t,r,θ,φ, we assume a geodesic circle [12] at constant t,θ and r and carry out the integration over φ with measure $\mathrm{d}\varphi$. The change in the vector with φ is

$$\mathrm{d}S_\mu = -\Gamma^\lambda_{\mu\varphi}\mathrm{d}\varphi S_\lambda \tag{2}$$

The only non-vanishing components that enter (*e.g.* [2]) are:

$$\begin{aligned}\Gamma^\varphi_{r\varphi} &= \frac{1}{r}\\ \Gamma^\varphi_{\theta\varphi} &= \cot\theta\\ \Gamma^\theta_{\varphi\varphi} &= -\sin\theta\cos\theta\end{aligned} \tag{3}$$

The parallel transport equations can then be written

$$\begin{aligned}\frac{\mathrm{d}S_r}{\mathrm{d}\varphi} &= -\frac{1}{r}S_\varphi\\ \frac{\mathrm{d}S_\theta}{\mathrm{d}\varphi} &= -\cot\theta S_\varphi\\ \frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi} &= \sin\theta\cos\theta S_\theta\end{aligned} \tag{4}$$

Due to the non-diagonal structure, we see that this system is, in fact, second order. At fixed θ,r, differentiating the second and third equations of (4), with respect to φ, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2S_\theta}{\mathrm{d}\varphi^2} &= -k^2S_\theta\\ \frac{\mathrm{d}^2S_\varphi}{\mathrm{d}\varphi^2} &= -k^2S_\varphi\end{aligned} \tag{5}$$

where $k = |\cos\theta|$. The solutions of these oscillator type equations are given by

$$\begin{aligned}S_\theta &= A(\theta,r)\cos(k\varphi) + B(\theta,r)\sin(k\varphi)\\ S_\varphi &= C(\theta,r)\cos(k\varphi) + D(\theta,r)\sin(k\varphi)\end{aligned} \tag{6}$$

These solutions determine the equation for S_r:

$$\frac{\mathrm{d}S_r}{\mathrm{d}\varphi} = -\frac{1}{r}\left[C(\theta,r)\cos(k\varphi) + D(\theta,r)\sin(k\varphi)\right] \tag{7}$$

so that (up to a arbitrary function of θ,r,t which we set to zero).

$$S_r = -\frac{1}{kr}\left[C(\theta,r)\sin(k\varphi) - D(\theta,r)\cos(k\varphi)\right] \tag{8}$$

We must now set initial conditions at $\varphi = 0$. From Equation (6) we have

$$\begin{aligned}A(\theta,r) &= (S_\theta)_0\\ C(\theta,r) &= (S_\varphi)_0\end{aligned} \tag{9}$$

and from (8),

$$D(\theta, r) = kr(S_r)_0 \tag{10}$$

Since our equations are second order there must be just two independent constants of integration. One can eliminate, say, B, D, with initial conditions.

Using our solutions (6) and (8) and the original Equations (4), we see that

$$\begin{aligned} \left.\frac{\mathrm{d}S_r}{\mathrm{d}\varphi}\right|_0 &= -\frac{1}{r}C \\ \left.\frac{\mathrm{d}S_\theta}{\mathrm{d}\varphi}\right|_0 &= -\cot\theta = Bk \\ \left.\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi}\right|_0 &= \sin\theta\cos\theta A = Dk \end{aligned} \tag{11}$$

so that, substituting for B and D, we have the solutions

$$\begin{aligned} S_\theta &= A\cos(k\varphi) - C\frac{\cot\theta}{k}\sin(k\varphi) \\ S_\varphi &= C\cos(k\varphi) + A\frac{\sin\theta\cos\theta}{k}\sin(k\varphi) \\ S_r &= -\frac{1}{kr}\left(C\sin(k\varphi) - A\frac{\sin\theta\cos\theta}{k}\cos(k\varphi)\right) \end{aligned} \tag{12}$$

with A and C given by the initial conditions (9) one finds the vector for any φ; and in particular for $\varphi = 2\pi$ at any given θ for $k = |\cos\theta|$. Note that there is no singularity at $\theta = \pi/2$ ($k = 0$). Substituting (9) into (12), and replacing S_μ by p_μ we obtain:

$$\begin{aligned} \Delta p_\varphi &= p_\varphi(\varphi = 2\pi) - p_\varphi(\varphi = 0) = \cos(2\pi k) - 1 + A(\theta, r)\frac{\sin\theta\cos\theta}{k}\sin(2\pi k) \\ \Delta p_\theta &= p_\theta(\varphi = 2\pi) - p_\theta(\varphi = 0) = A(\theta, r)(\cos(2\pi k) - 1) - \frac{\cot\theta}{k}\sin(2\pi k) \\ \Delta p_r &= p_r(\varphi = 2\pi) - p_r(\varphi = 0) \\ &= -\frac{1}{kr}\left[\sin(2\pi k) + A(\theta, r)\sin\theta\cos\theta\cos(2\pi k)\left(1 + \frac{1}{k}\right)\right] \end{aligned} \tag{13}$$

Therefore, after transport in φ from 0 to 2π,

$$\begin{aligned} p'_\varphi &= p_\varphi(\varphi = 2\pi) = p_\varphi(\varphi = 0) + \Delta p_\varphi = p_\varphi + \Delta p_\varphi \\ p'_\theta &= p_\theta(\varphi = 2\pi) = p_\theta(\varphi = 0) + \Delta p_\theta = p_\theta + \Delta p_\theta \\ p'_r &= p_r(\varphi = 2\pi) = p_r(\varphi = 0) + \Delta p_r = p_r + \Delta p_r \end{aligned} \tag{14}$$

This means that the momentum after parallel transport is equal to the original momentum plus the change Δp_μ found explicitly in Equation (13). Now, for a wavepacket narrow in both energy-momentum and spacetime, we assume that the classical computations are a good semiclassical approximation. For a wave packet of the form [6]:

$$\psi(x^\mu) = \frac{1}{\sqrt{4\pi\hbar}}\int \mathrm{d}^4 p_\mu \Phi(p_\mu)\mathrm{e}^{ip_\mu x^\mu/\hbar} \tag{15}$$

We have then

$$\Phi\left(p_\mu\right)=\frac{1}{\sqrt{4\pi\hbar}}\int \mathrm{d}^4x_\mu\psi\left(x_\mu\right)\mathrm{e}^{-\mathrm{i}p_\mu x^\mu/\hbar} \tag{16}$$

where $x^\mu=\left(t,r,\theta,\varphi\right)$

After the change in the momentum in transport of the function from 0 to 2π in φ, we obtain

$$\begin{aligned}\Phi\left(p'_\mu\right)&=\frac{1}{\sqrt{4\pi\hbar}}\int \mathrm{d}^4x_\mu\psi\left(x_\mu\right)\mathrm{e}^{-\mathrm{i}p'_\mu x^\mu/\hbar}\\&=\frac{1}{\sqrt{4\pi\hbar}}\int \mathrm{d}^4x_\mu\psi\left(x_\mu\right)\mathrm{e}^{-\mathrm{i}\left(p_\mu+\Delta p_\mu\right)x^\mu/\hbar}\\&=\frac{1}{\sqrt{4\pi\hbar}}\int \mathrm{d}^4x_\mu\psi\left(x_\mu\right)\mathrm{e}^{-\mathrm{i}p_\mu x^\mu/\hbar}\mathrm{e}^{-\mathrm{i}\Delta p_\mu x^\mu/\hbar}\\&=\Phi\left(p_\mu\right)\mathrm{e}^{-\mathrm{i}\Delta p_\mu x^\mu/\hbar}\end{aligned} \tag{17}$$

where $p_\mu=\left(E,p_r,p_\theta,p_\varphi\right)$.

We remark that a convolution is not necessary since the support of the wave packet is very narrow.

Now substitute this result into the expression for $\psi\left(x^\mu\right)$ in (15) at the initial point x^μ at $\varphi=2\pi$ to obtain the additional phase factor $\mathrm{e}^{-\mathrm{i}\Delta p_\mu x^\mu/\hbar}$.

$$\psi'\left(x^\mu\right)=\psi\left(x^\mu\right)\mathrm{e}^{-\mathrm{i}\Delta p_\mu x^\mu/\hbar}$$

We therefore find, in our semiclassical calculation, that a wave packet transported on a closed geodesic curve around a black hole acquires a geometrical (Berry type) phase. Quantum scattering [6] on a black hole should display, as for the Aharonov-Bohm [13] experiment, a corresponding interference effect.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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