

Circular Scale of Time as a Guide of the Schrödinger's Perturbation Theory

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Abstract

The paper is a kind of a review which considers an investigation of the scale of time suggested by an application of the Schrödinger perturbation method, especially when the perturbation of a non-degenerate quantum state is examined. In fact the method was applied in numerous cases—also by Schrödinger himself—without any use of the notion of time. Simultaneously, because of the development of computers, their use in solving the perturbation problems gradually decreased. However, the point of importance in the paper became the time. We demonstrate that collisions of a quantum system with the perturbation potential can be arranged along a circular scale of time whose properties provide us precisely with the energy terms obtained by the Schrödinger perturbation theory. This validity of results is checked till the perturbation order $N = 7$.

Keywords

Scale of Time, Schrödinger's Perturbation Theory, Non-Degenerate Quantum State

1. Introduction. Different Kinds of Approach to the Time Parameter in the Everyday Life and Science

Evidently the time was an important parameter in the human existence from its very beginning.

Duration and repetition of the days, nights, seasons, years became a well-known observation of everybody. This situation did not change in course of centuries. Simultaneously we had, in general, a strict qualitative distinction between the intervals of time and space.

The space intervals were easy to manipulate in their arrangement, both in imagination and practice: there was no difficulty to have or put any such interval

in an arbitrary position or direction chosen by the observer. A totally different property concerned the intervals of time: they had always a definite property of a future object, or a past object, or an object being actually present in our interest.

Nevertheless the science, especially mechanics, could be developed in spite of a difficulty concerning the actual “historical” position of an interval of time. In effect the time interval entering the mechanical process could be considered independently from its “history” for, in many occasions, the mechanics could be liberated from its historical background associated with time.

A special point which made the sense of mechanical laws questionable was connected with an examination of the physical laws concerning the whole mechanical systems. In this case the main result became that a mechanical system, having a constant velocity, should not obey several kinetic laws other than those obtained for a system at rest; see e.g. [1]. This is usually presented by a requirement that the Galilean transformation laws for the mechanical parameters have to be valid. But the development done in physics in the 19th century led to conclusion that the laws of the Maxwell electrodynamics should be equally valid in a moving system as well as they are satisfied for a system at rest.

A well-known consequence of that conclusion was the replacement of the Galilean transformation of the mechanical parameters by the Lorentz one. In fact, the Galilean transformation keeps its good accuracy solely when the speed of the moving mechanical system remains low in comparison with the speed of light c —the effect which holds in the most part of situations met in the everyday life.

2. Present Approach to the Problem of Time and Its Scientific Position

A competition between the Galilean and Lorentz transformations done by the Lorentz formula, presented a well-known subject of the special relativistic theory. This theory applies the joint metrics of the time interval

$$dt \tag{1}$$

and space intervals

$$dx, dy, dz \tag{2}$$

by combining them into the formula

$$(ds)^2 = c^2 (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{3}$$

representing the square of a small distant ds of the moving body. In the general theory of relativity the metrics (3) is replaced by a more complicated one [2]

$$-(ds)^2 = g_{ik} dx_i dx_k. \tag{4}$$

in which summation does apply over the parameters i and k . Usually the space coordinates are x_1 , x_2 and x_3 and the time coordinate is denoted by x_0 . In general the terms in (4) being

$$g_{ik} \tag{5}$$

are some functions of x_i and x_k . A scientific advantage of the metrics due to

(4) and (5) over the metrics (3) is that (4) and (5) can take into account several special physical effects, like the gravitational interaction between mass and light confirmed next by the observation.

An outline of the ideas and formulas given above concerns mainly the classical physics. They allow us, however, to present the role of time in a different problem, referred mainly to the quantum theory. This theory, began by the Planck's treatment of the oscillators entering the black-body ensemble, allowed him to discover the oscillator quanta of energy, as well as the roles of the oscillator frequency and the constant carrying the Planck's name.

The next large step towards quanta was connected with a partly quantum and partly classical approach to the hydrogen atom developed by Bohr; see [3], Vol. 1. Because of its very good agreement with the observed data, the model was considered as practically perfect in calculating the light frequencies connected with the electron transitions in the atom. But next the applications of the quantum theory occurred rather limited because of the difficulty connected with a treatment of the many-electron systems present in the non-hydrogen atoms. This difficulty was successfully defeated by Schrödinger—and his successors—in the wave-mechanical approach to the electron structure of the atoms; see e.g. [3], Vol. 2.

3. Schrödinger's Quantum Problems and Simplification of Their Solutions

The main idea of Schrödinger was to follow the de Broglie concept and consider the electron as a wave-like particle of matter.

Then a corresponding wave-like equation can be built up and next solved. One side of the equation is a sum of the kinetic energy operator of one or many electrons presented in a system, and the next term in the sum is the potential energy operator which takes different particle interactions necessary to be considered into account. Another side of the Schrödinger equation is given by a product of the energy constant E multiplied by the electron wave function ψ . In effect we obtain the eigenequation for E and ψ . In general its solution represents a complicated mathematical task—only for very simple physical systems the equation can be rather readily solved.

The Schrödinger equation—on its one side—is a sum of the kinetic and potential energy operators \hat{E}_{kin} and \hat{E}_{pot} , viz.

$$\hat{H} = \hat{E}_{\text{kin}} + \hat{E}_{\text{pot}} \quad (6)$$

called the Hamiltonian—or energy—operator. This operator is acting on the wave function ψ , so

$$\hat{H}\psi = E\psi, \quad (7)$$

is giving the Schrödinger equation. The right-hand side of the Equation in (7) represents a product of the eigenenergy E , considered as a constant number, and ψ . The effect of solution of (7) is usually a discrete set of values of E and dis-

crete set of functions ψ . The case when solutions provide us with only different E in the set is called a non-degenerate case of solution, the degenerate case occurs when some of the E in the set are equal, though these E are corresponding to different eigenfunctions ψ .

4. A Simplification of the Solution of Equation (7) Done by Schrödinger

His simplified solution was usually based on a separation of the Hamiltonian \hat{H} into two parts, namely

$$\hat{H} = \hat{H}_0 + \hat{H}^{\text{per}} \quad (8)$$

where the eigenequation

$$\hat{H}_0 \psi_0 = E_0 \psi_0 \quad (9)$$

is expected to be more simple to solve than that given in (7). Briefly a more simple Equation (9) is called the unperturbed equation with eigenvalues E_0 equal to the unperturbed energies and ψ_0 are called the unperturbed eigenfunctions. Because of (8) the perturbation potential entering the unperturbed Equation (9) is equal to:

$$\hat{H} - \hat{H}_0 = \hat{H}^{\text{per}} = V^{\text{per}}. \quad (10)$$

Usually it is assumed that

$$\hat{H}^{\text{per}} = V^{\text{per}}(\vec{r}), \quad (10a)$$

so (10a) is taken—for the sake of convenience—as equal to a term independent of the momentum operator, or operators. Having solutions of (9) we can calculate the matrix elements

$$\langle m | V^{\text{per}} | p \rangle = \int \psi_0^{(m)} V^{\text{per}} \psi_0^{(p)} dV. \quad (11)$$

The matrix elements (11) combined with the eigenvalues E_0 entering (9) can provide us—according to the Schrödinger perturbation formalism—with the approximate energy eigenvalues of the more complicated eigenproblem (7).

This calculation can be done gradually for different perturbation orders N , beginning successfully with the lowest order $N = 1$. Huby [4] and Tong [5] calculated the number S_N of kinds of the perturbation terms which should be built up from the matrix elements (11) for a given order N , on condition the perturbation concerns a non-degenerate quantum state. This number is equal to

$$S_N = \frac{(2N-2)!}{N!(N-1)!}. \quad (12)$$

But the derivation of a detailed shape of terms entering the number S_N can be a complicated task, especially for large N . One of the aims of the present publication is to demonstrate that the perturbation calculation proposed by Schrödinger, especially its part referred to the S_N terms, can be drastically simplified if the collision events of an originally unperturbed system with the per-

turbation potential

$$V^{\text{per}} = V^{\text{per}}(\vec{r}) \quad (13)$$

are arranged along a special scale of time. The scale has a circular-like shape and the number of collision kinds with the perturbation potential for each N occurs precisely equal to S_N . Moreover, any collision kind, or collisions ensemble, is represented by a specified diagram created on the time scale. Simultaneously, the shape of the diagram provides us with a rule for calculating the corresponding contribution to the perturbation energy.

A final result for the perturbation energy obtained in this way for a given N agrees with a corresponding energy obtained by the Schrödinger method. The details of calculations concerning the time scale and its applications are presented in the original author's papers; see [6]-[22].

5. Use of a Circular Scale of Time in the Schrödinger's Perturbation Problem

In fact only the perturbation of a non-degenerate Schrödinger quantum state n was thoroughly considered with the aid of the mentioned scale. From the beginning of its application the circular scale of time was developed systematically for subsequent perturbation orders N :

$$N = 1, 2, 3, 4, \dots \quad (14)$$

A physical meaning of N was to give a number of collisions of an unperturbed system with the perturbation potential. This potential was usually assumed to depend solely on the position coordinate \vec{r} of the particle:

$$\hat{H}^{\text{per}} = \hat{H}^{\text{per}}(\vec{r}) = V^{\text{per}}(\vec{r}) \quad (15)$$

Therefore the considered perturbation is independent of the time parameter t .

The number $N = 1$ refers to a single collision of the system with the perturbation (10), the number $N = 2$ refers to two collisions with $V^{\text{per}}(\vec{r})$, etc. Any scale labelled by N is assumed to be composed of the beginning-end (b.e.) point of time, in effect the scale represented by $N = 1$ has solely a single time point (b.e.) necessary for consideration.

The scale of $N = 2$ —giving the perturbation order 2—has two points of importance: beyond of a single beginning-end point it has the second point which refers to any non-perturbed state p different than the considered unperturbed state n :

$$p \neq n. \quad (16)$$

The energy correction of state n due to the perturbation of order $N = 1$ is represented by a single term

$$\Delta E_1 = \langle n | V^{\text{per}} | n \rangle. \quad (17)$$

On the other hand, the perturbation energy belonging to the order $N = 2$ is given by a sum

$$\Delta E_2 = \sum_p \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | n \rangle}{E_n^{(0)} - E_p^{(0)}} \quad (18)$$

where evidently the relation (16) does hold.

According to the formulae given by Tong and Huby [4] [5], the number of kinds of the perturbation terms entering order N [see (12)] becomes:

$$S_1 = S_2 = 1 \quad (19)$$

which are in agreement with the number of S_N given by the formula (12), see **Figure 1** and **Figure 2**.

But in general we have

$$S_N > 1 \quad (20)$$

and our dominant interest is to calculate these S_N terms.

In the case of $N = 1$ the diagram has only a single point—the beginning-end point—presented by **Figure 1**; for $N = 2$ the diagram is represented by **Figure 2** having two isolated points. In the next step let us consider $N = 3$. In this case

$$S_3 = 2. \quad (21)$$

The circular scale for $N = 3$ has three points on it; see **Figure 3**. One point is the beginning-end (b.e.) point, the other time points are labelled on **Figure 3** by the numbers 1 and 2.

The first of the S_3 terms in (9) is represented by the formula ($p, q \neq n$)

$$\Delta E_3^{(\text{part 1})} = \sum_p \sum_q \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | q \rangle \langle q | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})(E_n^{(0)} - E_q^{(0)})}. \quad (22)$$



Figure 1. The diagram is corresponding to the perturbation order $N = 1$.

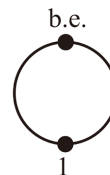


Figure 2. The diagram is corresponding to the perturbation order $N = 2$.

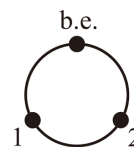


Figure 3. The diagram is corresponding to the perturbation order $N = 3$, part 1.

The second energy term dictated by (21) originates from contraction of the time points 1 and 2 which are present in **Figure 3**. This contraction 1:2 gives a diagram presented in **Figure 4**.

In fact the diagram in **Figure 4** can be considered as representing the product of two terms: the first term is a time loop identical with the diagram characteristic for $N = 1$ (see **Figure 1**), the other term is similar to the diagram for $N = 2$ (see **Figure 2**). A difference from the term given in (22) is represented by the formula

$$\Delta E_3^{(\text{part 2})} = - \sum_p \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})^2} \Delta E_1. \tag{23}$$

In effect the full perturbation energy term for $N = 3$ is

$$\Delta E^{(3)} = \Delta E_3^{(\text{part 1})} + \Delta E_3^{(\text{part 2})}. \tag{24}$$

The problem of sign attributed to $\Delta E_3^{(\text{part 1})}$ and $\Delta E_3^{(\text{part 2})}$ will be discussed below; see Section 6.

6. Abbreviated Formulae Applied in Calculating the Energy Perturbation Terms

The abbreviated formulae for ΔE_1 , ΔE_2 , $\Delta E_3^{(\text{first part})}$ and $\Delta E_3^{(\text{second part})}$ can be expressed as follows:

$$\Delta E_1 = \langle V \rangle, \tag{25}$$

$$\Delta E_2 = \langle VPV \rangle, \tag{26}$$

$$\Delta E_3^{(\text{first part})} = \langle VPVPV \rangle, \tag{27}$$

$$\Delta E_3^{(\text{second part})} = - \langle V \rangle \langle VP^2V \rangle, \tag{28}$$

where ΔE_1 is given by (17), ΔE_2 —by (18), $\Delta E_3^{(\text{part 1})}$ —by (22) and $\Delta E_3^{(\text{part 2})}$ —by (23).

A characteristic feature is that the case of $N = 1$ has no P terms entering ΔE_1 . The symbol P in (26) refers to the ratio

$$\frac{1}{E_n^{(0)} - E_p^{(0)}} \tag{29}$$

entering only once for any state p considered in the summation process in ΔE_2 . On the other hand ΔE_3 has two kinds of P terms, *viz.*

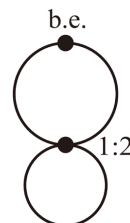


Figure 4. The diagram is corresponding to the perturbation order $N = 3$, part 2.

$$\frac{1}{E_n^{(0)} - E_p^{(0)}}, \frac{1}{E_n^{(0)} - E_q^{(0)}} \quad (30)$$

and product P^2 in (28) represents the term

$$\frac{1}{\left(E_n^{(0)} - E_p^{(0)}\right)^2} \quad (31)$$

entering the expression $\Delta E_3^{(\text{part } 2)}$ in (23).

In general, a sum of powers P entering any energy term ΔE_N should be equal to $N-1$.

The sign of ΔE_N is dependent on the number of terms entering the product representing a given ΔE_N : an odd number of terms in the product implies a positive sign before it, so it is in the case of ΔE_1 , ΔE_2 and $\Delta E_3^{(\text{part } 1)}$; an even number of terms entering the product implies a negative sign [see $\Delta E_3^{(\text{part } 2)}$ in (23)].

7. Contractions of the Time Points on the Scale of Time and the Number S_N

The number represented by S_N —and the formulae for the S_N terms—can be obtained by considering the allowed contractions of the time points on the scale. An example of such contractions is given by the time points 1 and 2 represented by the symbol

$$1:2 \quad (32)$$

entering the time scale for $N=3$. No other contractions than (32) can be admitted for $N=3$ and its time scale.

But let us consider the time scale for $N=5$. In this case—beyond of the beginning-end time point (b.e.)—we have the time points

$$1, 2, 3, \text{ and } 4 \quad (33)$$

on the scale; see **Figure 5**.

Since the point b.e. is excluded from contractions with the other time points, the allowed contractions to which the time points in **Figure 5** can be submitted are:

$$\begin{aligned} &1:2, 1:3, 1:4, 1:2:3, 1:2:4, 1:3:4, \\ &2:3, 2:4, 2:3:4, \\ &3:4, 1:2:3:4. \end{aligned} \quad (34)$$

We see that the time points entering any contraction (34) should increase from smaller ones to larger ones.

But this property does not complete the limits of contractions choice. There are allowed also two combined contractions:

$$1:2 \text{ and } 3:4, \quad (35)$$

$$1:4 \text{ and } 2:3 \quad (36)$$

but there are not allowed contractions like

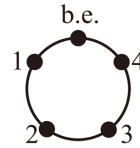


Figure 5. The time point b.e. and the free time points 1, 2, 3, and 4 on the scale.

$$1 : 3 \text{ and } 2 : 4. \tag{37}$$

The geometrical property which has to be satisfied for any set of contractions is that the lines (loops) associated with them should not cross.

In effect, together with the case of a single set of points given in (33), which are free from any contraction on the scale (see **Figure 5**), we have

$$1 + 6 + 5 + 2 = 14 = S_N = S_5 \tag{38}$$

diagrams concerning points 1, 2, 3, and 4 when the points are submitted to contractions. Expression (38) is equal precisely to the result of the formula (12):

$$S_5 = 14. \tag{39}$$

The above procedure can be extended to an arbitrary perturbation order N . The order $N = 7$ having $S_7 = 132$ terms was examined in [21] [22].

The diagrams representing the perturbation energies corresponding to contractions (34) are as in **Figure 6**.

The diagrams giving contractions (35) and (36) are as in **Figure 7**.

The diagram presented in **Figure 5** (having no contractions of the time points) gives the perturbation energy

$$\begin{aligned} \Delta E_5^{(1)} &= \langle VPVPVPVPV \rangle \\ &= \sum_p \sum_q \sum_r \sum_s \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | q \rangle \langle q | V^{\text{per}} | r \rangle \langle r | V^{\text{per}} | s \rangle \langle s | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})(E_n^{(0)} - E_q^{(0)})(E_n^{(0)} - E_r^{(0)})(E_n^{(0)} - E_s^{(0)})} \end{aligned} \tag{40}$$

where

$$p, q, r, s \neq n. \tag{40a}$$

The energy terms represented by **Figure 6** are:

$$\begin{aligned} 1 : 2 \text{ or (a)} &\rightarrow \Delta E_5^{(2)} = -\langle VP^2VPVPV \rangle \langle V \rangle \\ &= -\sum_p \sum_q \sum_r \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | q \rangle \langle q | V^{\text{per}} | r \rangle \langle r | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})^2 (E_n^{(0)} - E_q^{(0)})(E_n^{(0)} - E_r^{(0)})} \Delta E_1 \end{aligned} \tag{41}$$

$$\begin{aligned} 2 : 3 \text{ or (g)} &\rightarrow \Delta E_5^{(3)} = -\langle VPVP^2VPV \rangle \langle V \rangle \\ &= -\sum_p \sum_q \sum_r \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | q \rangle \langle q | V^{\text{per}} | r \rangle \langle r | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})(E_n^{(0)} - E_q^{(0)})^2 (E_n^{(0)} - E_r^{(0)})} \Delta E_1 \end{aligned} \tag{42}$$

$$\begin{aligned} 3 : 4 \text{ or (j)} &\rightarrow \Delta E_5^{(4)} = -\langle VPVPVP^2V \rangle \langle V \rangle \\ &= -\sum_p \sum_q \sum_r \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | q \rangle \langle q | V^{\text{per}} | r \rangle \langle r | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})(E_n^{(0)} - E_q^{(0)})(E_n^{(0)} - E_r^{(0)})^2} \Delta E_1 \end{aligned} \tag{43}$$

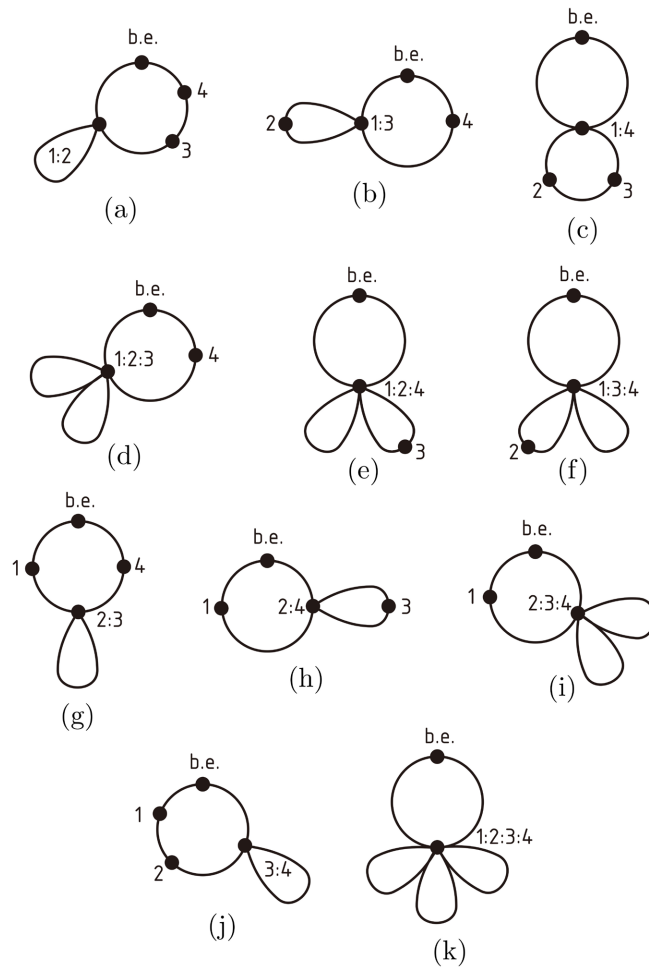


Figure 6. The diagrams representing the b.e. time points and contractions (34) on the time scale.

$$\begin{aligned}
 1:3 \text{ or (b)} &\rightarrow \Delta E_5^{(5)} = -\langle VP^2VPV \rangle \langle VPV \rangle \\
 &= -\sum_p \sum_q \frac{\langle n|V^{\text{per}}|p \rangle \langle p|V^{\text{per}}|q \rangle \langle q|V^{\text{per}}|n \rangle}{(E_n^{(0)} - E_p^{(0)})^2 (E_n^{(0)} - E_q^{(0)})} \Delta E_2
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 1:4 \text{ or (c)} &\rightarrow \Delta E_5^{(6)} = -\langle VP^2V \rangle \langle VPVPV \rangle \\
 &= -\sum_p \frac{\langle n|V^{\text{per}}|p \rangle \langle p|V^{\text{per}}|n \rangle}{(E_n^{(0)} - E_p^{(0)})^2} \Delta E_3^{(\text{part 1})}.
 \end{aligned} \tag{45}$$

In the next step

$$1:4 \cap 2:3 \rightarrow \Delta E_5^{(6)} = \langle VP^2V \rangle \langle VP^2V \rangle \langle V \rangle \tag{45a}$$

[see diagram (b') in **Figure 7** gives together with $\Delta E_5^{(6)}$ the result:

$$\begin{aligned}
 \Delta E_5^{(6)} + \Delta E_5^{(6')} &= -\langle VP^2V \rangle \left[\langle VPVPV \rangle - \langle VP^2V \rangle \langle V \rangle \right] \\
 &= -\langle VP^2V \rangle \left[\Delta E_3^{(\text{part 1})} + \Delta E_3^{(\text{part 2})} \right] \\
 &= -\langle VP^2V \rangle \Delta E_3;
 \end{aligned} \tag{46}$$

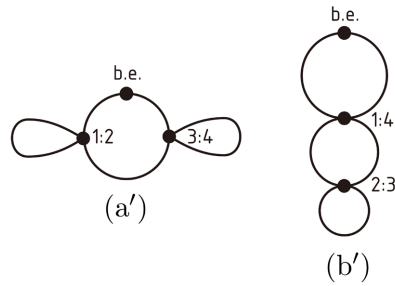


Figure 7. The diagrams representing the b.e. time points and contractions (35) and (36) on the time scale.

see (22) and (23). This implies that (46) provides us with two Schrödinger perturbation terms for energy.

The remaining contractions of the time points give:

$$\begin{aligned}
 1:2:3 \text{ or (d)} &\rightarrow \Delta E_5^{(7)} = \langle VP^3VPV \rangle (\langle V \rangle)^2 \\
 &= \sum_p \sum_q \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | q \rangle \langle q | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})^3 (E_n^{(0)} - E_q^{(0)})} (\Delta E_1)^2
 \end{aligned} \quad (47)$$

$$\begin{aligned}
 1:2:4 \text{ or (e)} &\rightarrow \Delta E_5^{(8)} = \langle VP^3VPV \rangle \langle V \rangle \langle VPV \rangle \\
 &= \sum_p \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})^3} \Delta E_1 \Delta E_2
 \end{aligned} \quad (48)$$

$$\begin{aligned}
 1:3:4 \text{ or (f)} &\rightarrow \Delta E_5^{(9)} = \langle VP^3VPV \rangle \langle VPV \rangle \langle V \rangle \\
 &= \sum_p \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})^3} \Delta E_2 \Delta E_1
 \end{aligned} \quad (49)$$

$$\begin{aligned}
 2:3:4 \text{ or (i)} &\rightarrow \Delta E_5^{(11)} = \langle VPVP^3V \rangle (\langle V \rangle)^2 \\
 &= \sum_p \sum_q \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | q \rangle \langle q | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)}) (E_n^{(0)} - E_q^{(0)})^3} (\Delta E_1)^2
 \end{aligned} \quad (50)$$

$$\begin{aligned}
 1:2:3:4 \text{ or (k)} &\rightarrow \Delta E_5^{(12)} = -\langle VP^3VPV \rangle (\langle V \rangle)^2 \\
 &= -\sum_p \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})^4} (\Delta E_1)^3.
 \end{aligned} \quad (51)$$

The energy term represented by (a') in **Figure 7** is

$$\begin{aligned}
 \Delta E_5^{(13)} &= \langle VP^2VP^2V \rangle (\langle V \rangle)^2 \\
 &= \sum_p \sum_q \frac{\langle n | V^{\text{per}} | p \rangle \langle p | V^{\text{per}} | q \rangle \langle q | V^{\text{per}} | n \rangle}{(E_n^{(0)} - E_p^{(0)})^2 (E_n^{(0)} - E_q^{(0)})^2} (\Delta E_1)^2.
 \end{aligned} \quad (52)$$

Equation (40a) should be satisfied in all summations.

8. Summary of Results

The main idea of the paper presented by the author—and in former his papers

quoted here—was to demonstrate that a rather tedious approach to the Schrödinger perturbation energy can be much simplified when a circular scale of time in classifying the perturbations events is applied. As a special case a perturbation of a non-degenerate quantum state is examined.

First the number of terms entering the calculation is in a perfect agreement with the number of the formulae expected by Huby and Tong. The next point is that an arrangement of the time points on the scale gives a ready mathematical access to the formulae entering the Schrödinger perturbation theory for a given perturbation order N . This is a convenient situation because—for example in applying the Feynmann diagrams—the number of the perturbation terms and their character are much larger and more complicated than those necessary to perform the proper Schrödinger calculations for a chosen $N \gg 1$.

In references [6]-[22] are given the diagrams of the time scale and their applications corresponding to the Schrödinger perturbation terms whose orders do not exceed $N = 7$. Some philosophical repercussions concerning the shape of the applied time scale and its use are also presented. In [9] a rather thorough comparison is done between the Feynman approach based on an infinite (straight-linear) time scale and a circular-time approach to the Schrödinger perturbation theory. Another comparison done in [22] does refer to the present perturbation calculation to the Leibniz theory and that outlined in the Ziman book [23].

The arrangement of the perturbation events on a special, *viz.* circular, scale of time, allowed us to obtain the perturbation energies of a non-degenerate quantum system without solving the corresponding perturbation equations.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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