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Two Concepts in Optics of Anisotropic Dispersive Media and Polariton Case in Coordinate-Invariant Way

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Abstract

Two concepts of phenomenological optics of homogeneous, anisotropic and dispersive media are compared, the younger and more general concept of media with spatial dispersion and the older concept of (bi)-anisotropic media with material tensors for electric and magnetic induction which only depend on the frequency. The general algebraic form of the polarization vectors for the electric field and their one-dimensional projection operators is discussed without the degenerate cases of optic axis for which they become two-dimensional projection operators. Group velocity and diffraction coefficients in an approximate equation for the slowly varying amplitudes of beam solutions are calculated. As special case a polariton permittivity for isotropic media with frequency dispersion but without losses is discussed for the usual passive case and for the active case (occupation inversion of two energy levels that goes in direction of laser theory) and the group velocity is calculated. For this active case, regions of frequency and wave vector with group velocities greater than that of light in vacuum were found. This is not fully understood and due to large diffraction is likely only to realize in guided resonator form. The notion of “negative refraction” is shortly discussed but we did not find agreement with its assessment in the original paper.

Keywords

Spatial and Frequency Dispersion, Bi-Anisotropic Media, Uniaxial Media, Passive and Active Media, Negative Refraction, Operator Invariants, Complementary Operator, Group Velocity

To the Notations

Three-dimensional vectors: bold letters, e.g., $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\cdots$,

$\boldsymbol{ab}$ scalar products, $[\boldsymbol{a},\boldsymbol{b}]$ vector products, $[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$ volume products, $\boldsymbol{a}\cdot\boldsymbol{b}$ dyadic products,

AB operator products, $\mathsf{A}\boldsymbol{a},\tilde{\boldsymbol{a}}\mathsf{A}$ products of operators with vectors, $\tilde{\boldsymbol{a}}\mathsf{A}\boldsymbol{a}$ bilinear (and quadratic) forms.

In Euclidean spaces with a symmetric metric tensor g_{ij}, the dual tensor $b_{ik} \equiv \epsilon_{ijk} b_j$ to a vector b_j (ϵ_{ijk} Levi-Cicita symbol) can be also seen as anti-symmetric operator and in coordinate-invariant form we can write this anti-symmetric operator as $[\boldsymbol{b}]$ with the advantage that vector and also volume products can be written only by displacement of the squared brackets, e.g., $\boldsymbol{a}[\boldsymbol{b}] \equiv [\boldsymbol{a},\boldsymbol{b}]$, $[\boldsymbol{b}]\boldsymbol{c} \equiv [\boldsymbol{b},\boldsymbol{c}]$, $\boldsymbol{a}[\boldsymbol{b}]\boldsymbol{c} \equiv [\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c} \equiv \boldsymbol{a}[\boldsymbol{b},\boldsymbol{c}] \equiv [\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$.

In mathematical texts I write three-dimensional operators by serif-less Capital letters. In physical texts this makes sometimes difficulties because one cannot reasonably write all operators with physical meaning by Capital letters and all vectors with physical meaning by small letters. Furthermore, in case of Greek letters, "Latex" (and also printing) does not provide serif-less letters. In these cases I write operators as a compromise by bold letters, e.g., $\boldsymbol{\varepsilon},\boldsymbol{\mu}$ such as vectors to distinguish them, in particular, from scalars. This means that in present physical text one must know which kind of quantities one has: scalars, vectors or operators.

1. Introduction

There are two concepts of phenomenological macroscopic optics of the linear constitutive equations for anisotropic dispersive media. The first and younger concept is spatial dispersion in the first time mainly developed by Russian physicist in the fiftieths, in particular, Ginzburg and Agranovich [1], Ginzburg [2], Silin and Rukhadze [3] and in shorter form considered in a new chapter in the new edition of vol. 8 of the course of Landau and Lifshits [4].

The second and much older concept is to use two material equations for the electric and magnetic induction in dependence on the electric and magnetic field using tensors of second and sometimes in addition of third rank (optic activity) depending on frequency only (dispersion) and now often called "bi-anisotropic media" (including also electrical anisotropy only). Three representative monographs of the many possible possible ones are that of Tamm [5], that from Sommerfeld [6] and that of Born and Wolf [7] and in addition the comprehensive encyclopedic article from Szivessy [8]. We cite here also the most basic works of Fyodorov, the initiator of coordinate-invariant methods, and his followers from Minsk [9] [10] [11] who in addition to this concept use last methods and where the monograph [11] contains beside theory also experimental material to different media and crystals and by impression was mainly written by Filippov. Coordinate-invariant methods do not only write the starting equations in vector or tensor form but work from beginning up to the results only with vectors, operators and tensors which have a relation to the problem but not with arbitrary coordinate representations and which are mostly of advantage compared with often voluminous coordinate representations but with more sophisticated algebra. We also apply in this article widely coordinate-invariant methods where it is possible and used them also long ago in the past, e.g., [12] [13] [14] [15].

As a special case we discuss in detail a permittivity which we call polariton permittivity and which is related to phenomenological theory of excitons, e.g., in addition to [1] [2] [3] [4] by Knox, Agranovich, Davydov, Galanin and Pekar [16] [17] [18] [19] [20]. It admits to consider two essentially different special cases called the passive and the active case. The active case is connected with occupation inversion of at least two energy levels in the medium and is described in certain parts of frequency or connected wave vectors by amplification and leads into the neighborhood to laser theory. It contains also a very interesting phenomenon of propagation of excitations with velocities faster than light that is not fully understood.

In connection with my article [21] the notion of "negative refraction" of Pendry [22] came into the focus of my considerations[1]. I never have used it and likely never would use the notion "negative refraction" in connection with my own results in this field. In Section 11 and in Appendix D I try to represent my imaginations to the content of this notion which seems to me as incorrect ones.

Sections 2-8 are devoted to general characterization and comparison of both concepts including calculation of group velocities with and without taking into consideration the dispersion and Section 9 and Section 10 to the most simple model of a polariton permittivity. Section 11 was made to prepare short remarks to the notion of "negative refraction" in Appendix D.

2. The Concept of Spatial Dispersion

In this more general concept compared with the bi-anisotropic concept considered in next Section we write the equations of macroscopic electrodynamics in the form

$$\begin{aligned}&\left[\nabla,\boldsymbol{E}(\boldsymbol{r},t)\right]+\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{0},\quad \nabla\boldsymbol{B}(\boldsymbol{r},t)=0,\\&\left[\nabla,\boldsymbol{B}(\boldsymbol{r},t)\right]-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{D}(\boldsymbol{r},t)=\boldsymbol{0},\quad \nabla\boldsymbol{D}(\boldsymbol{r},t)=0,\end{aligned}\tag{2.1}$$

where $\boldsymbol{E}(\boldsymbol{r},t)$ is the macroscopic electric and $\boldsymbol{B}(\boldsymbol{r},t)$ the macroscopic magnetic field[2] The linear constitutive equation for spatially and temporally homogeneous but, in general, anisotropic dispersive media are written in the form

$$D_i(\boldsymbol{r},t)=\int \mathrm{d}^3r'\wedge\mathrm{d}t'\,\hat{\varepsilon}_{ij}(\boldsymbol{r}-\boldsymbol{r}',t-t')E_j(\boldsymbol{r}',t').\tag{2.2}$$

We now make a Fourier transformation for $\boldsymbol{E},\boldsymbol{B}$ and $\boldsymbol{D}$ according to the

[1]Long ago I visited a lecture of U. Leonhardt about this which I did not fully understand but also did not trace the original paper to this time. However, it seemed to me that U.L. assessed this paper very positively and as correct.

[2]Sometimes called magnetic induction but in this concept $\boldsymbol{B}$ and $\boldsymbol{H}$ are identical (see next Section) and $\boldsymbol{D}(\boldsymbol{r},t)$ is the electric induction.

scheme

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{r},t) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^3k\wedge\mathrm{d}\omega\,\boldsymbol{E}(\boldsymbol{k},\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{kr}-\omega t)},\\
\boldsymbol{E}(\boldsymbol{k},\omega) &= \int \mathrm{d}^3r\wedge\mathrm{d}t\,\boldsymbol{E}(\boldsymbol{r},t)\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{kr}-\omega t)},
\end{aligned}\tag{2.3}$$

we find the transformed constitutive relation in the form

$$D_i(\boldsymbol{k},\omega) = \varepsilon_{ij}(\boldsymbol{k},\omega)\,E_j(\boldsymbol{k},\omega),\tag{2.4}$$

with the definition of the general permittivity tensor $\varepsilon_{ij}(\boldsymbol{k},\omega)$

$$\varepsilon_{ij}(\boldsymbol{k},\omega) \equiv \int \mathrm{d}^3\rho\wedge\mathrm{d}\tau\,\hat{\varepsilon}_{ij}(\boldsymbol{\rho},\tau)\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k\rho}-\omega\tau)}.\tag{2.5}$$

In general, this tensor is non-symmetric.

After Fourier transformation of (2.1) these equations take on the form

$$\begin{aligned}
\left[\boldsymbol{k},\boldsymbol{E}(\boldsymbol{k},\omega)\right] - \frac{\omega}{c}\boldsymbol{B}(\boldsymbol{k},\omega) &= \boldsymbol{0}, \quad \boldsymbol{kB}(\boldsymbol{k},\omega) = 0,\\
\left[\boldsymbol{k},\boldsymbol{B}(\boldsymbol{k},\omega)\right] + \frac{\omega}{c}\boldsymbol{D}(\boldsymbol{k},\omega) &= \boldsymbol{0}, \quad \boldsymbol{kD}(\boldsymbol{k},\omega) = 0,
\end{aligned}\tag{2.6}$$

By elimination of $\boldsymbol{B}$ from these equations and using the constitutive Equation (2.4) we find the following operator equation for the Fourier components of the electric field (in case of $\omega \neq 0$)

$$\boldsymbol{0} = \left\{\frac{c^2}{\omega^2}\left(\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{k}^2\mathsf{I}\right) + \boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\right\}\boldsymbol{E}(\boldsymbol{k},\omega) \equiv \mathsf{L}(\boldsymbol{k},\omega)\,\boldsymbol{E}(\boldsymbol{k},\omega).\tag{2.7}$$

From this equation follows equivalently to the vanishing of the divergence of $\boldsymbol{D}(\boldsymbol{r},t)$

$$0 = \boldsymbol{k}\mathsf{L}(\boldsymbol{k},\omega)\,\boldsymbol{E}(\boldsymbol{k},\omega) = \boldsymbol{k}\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\,\boldsymbol{E}(\boldsymbol{k},\omega) = \boldsymbol{kD}(\boldsymbol{k},\omega).\tag{2.8}$$

Equation (2.7) is an operator equation for the Fourier transforms of the electric field to the eigenvalue "zero" which in the original form can be written

$$\boldsymbol{0} = \left\{\frac{c^2}{\frac{\partial^2}{\partial t^2}}\left(\nabla\cdot\nabla - \nabla^2\mathsf{I}\right) + \boldsymbol{\varepsilon}\left(-\mathrm{i}\nabla,\mathrm{i}\frac{\partial}{\partial t}\right)\right\}\boldsymbol{E}(\boldsymbol{r},t) \equiv \mathsf{L}\left(-\mathrm{i}\nabla,\mathrm{i}\frac{\partial}{\partial t}\right)\boldsymbol{E}(\boldsymbol{r},t),\tag{2.9}$$

with the differential operator (or integral operator in case of $\boldsymbol{\varepsilon}$)

$$\mathsf{L}\left(-\mathrm{i}\nabla,\mathrm{i}\frac{\partial}{\partial t}\right) \equiv \frac{c^2}{\frac{\partial^2}{\partial t^2}}\left(\nabla\cdot\nabla - \nabla^2\mathsf{I}\right) + \boldsymbol{\varepsilon}\left(-\mathrm{i}\nabla,\mathrm{i}\frac{\partial}{\partial t}\right).\tag{2.10}$$

For the solution of these operator equations it is favorable to consider some algebra of the operators before this.

The operator $\mathsf{L}(\boldsymbol{k},\omega)$ in the wave Equation (2.7) is defined by

$$\mathsf{L}(\boldsymbol{k},\omega) \equiv \frac{c^2}{\omega^2}\left(\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{k}^2\mathsf{I}\right) + \boldsymbol{\varepsilon}(\boldsymbol{k},\omega).\tag{2.11}$$

The invariants of the operator $\mathsf{L}(\boldsymbol{k},\omega)$ are ($\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$)

$$
\begin{aligned}
&\left|\mathsf{L}(\boldsymbol{k},\omega)\right| = \frac{c^4}{\omega^4}\boldsymbol{k}^2(\boldsymbol{k\varepsilon k}) - \frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k\varepsilon k} - \boldsymbol{k}\boldsymbol{\varepsilon}^2\boldsymbol{k}\right) + |\boldsymbol{\varepsilon}|,\\
&\left[\mathsf{L}(\boldsymbol{k},\omega)\right] = \frac{c^4}{\omega^4}\left(\boldsymbol{k}^2\right)^2 - \frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}^2 + \boldsymbol{k\varepsilon k}\right) + [\boldsymbol{\varepsilon}],\\
&\left\langle\mathsf{L}(\boldsymbol{k},\omega)\right\rangle = -2\frac{c^2}{\omega^2}\boldsymbol{k}^2 + \langle\boldsymbol{\varepsilon}\rangle,
\end{aligned} \tag{2.12}
$$

which are involved in the Cayley-Hamilton identity $\mathsf{L}^3 - \langle\mathsf{L}\rangle\mathsf{L}^2 + [\mathsf{L}]\mathsf{L} - |\mathsf{L}|\mathsf{I} = 0$ for the operator $\mathsf{L} \equiv \mathsf{L}(\boldsymbol{k},\omega)$ (see (A.1) in Appendix A).

The vanishing of the determinant of $\mathsf{L}(\boldsymbol{k},\omega)$

$$
\left|\mathsf{L}(\boldsymbol{k},\omega)\right| = 0, \tag{2.13}
$$

is the dispersion equation and describes a three-dimensional (hyper)-surface in the four-dimensional space of variables $(\boldsymbol{k},\omega)$. In the specialization to only frequency dispersion ($\boldsymbol{\varepsilon}(\boldsymbol{k},\omega) = \boldsymbol{\varepsilon}(\omega)$) it is identical in content but not in form with the Fresnel Equation (e.g., [6] [7]).

For the complementary operator $\overline{\mathsf{L}}(\boldsymbol{k},\omega)$ to $\mathsf{L}(\boldsymbol{k},\omega)$ we find (see Appendix A)

$$
\begin{aligned}
\overline{\mathsf{L}}(\boldsymbol{k},\omega) &\equiv \mathsf{L}^2 - \langle\mathsf{L}\rangle\mathsf{L} + [\mathsf{L}]\mathsf{I}\\
&= \frac{c^4}{\omega^4}\left(\boldsymbol{k}^2\right)\boldsymbol{k}\cdot\boldsymbol{k} - \frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{\varepsilon}\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{\varepsilon} + (\boldsymbol{k\varepsilon k})\mathsf{I}\right) + \overline{\boldsymbol{\varepsilon}},
\end{aligned}
$$

$$
\left\langle\overline{\mathsf{L}}(\boldsymbol{k},\omega)\right\rangle = \left[\mathsf{L}(\boldsymbol{k},\omega)\right],\ \langle\overline{\boldsymbol{\varepsilon}}\rangle = [\boldsymbol{\varepsilon}],\ \boldsymbol{\varepsilon}^{-1} = \frac{\overline{\boldsymbol{\varepsilon}}}{|\boldsymbol{\varepsilon}|},\ \boldsymbol{\varepsilon} = |\boldsymbol{\varepsilon}|\left(\overline{\boldsymbol{\varepsilon}}\right)^{-1},\ \overline{\overline{\boldsymbol{\varepsilon}}} = |\boldsymbol{\varepsilon}|\boldsymbol{\varepsilon}. \tag{2.14}
$$

The complementary operator $\overline{\mathsf{L}}(\boldsymbol{k},\omega)$ to $\mathsf{L}(\boldsymbol{k},\omega)$ plays an important role in optics of anisotropic media. If the determinant of L is vanishing, *i.e.*, $[\mathsf{L}] = 0$ then the squared complementary operator $\overline{\mathsf{L}}^2$ is proportional to $\overline{\mathsf{L}}$, more precisely[3]

$$
|\mathsf{L}| = 0: \quad \Rightarrow \left(\overline{\mathsf{L}}\right)^2 = \left\langle\overline{\mathsf{L}}\right\rangle\overline{\mathsf{L}},\quad \left\langle\overline{\mathsf{L}}\right\rangle = [\mathsf{L}], \tag{2.15}
$$

and Π according to the following definition

$$
\Pi \equiv \frac{\overline{\mathsf{L}}}{\left\langle\overline{\mathsf{L}}\right\rangle} = \frac{\overline{\mathsf{L}}}{[\mathsf{L}]},\quad \Rightarrow\quad \Pi^2 = \Pi,\quad \langle\Pi\rangle = 1, \tag{2.16}
$$

is projection operator to the eigenvalue $\lambda = 0$ of L. If $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$ are arbitrary vectors then non-vanishing vectors $\overline{\mathsf{L}}\boldsymbol{a}$ are right-hand eigenvector and non-vanishing vectors $\tilde{\boldsymbol{a}}\overline{\mathsf{L}}$ left-hand eigenvector of L to the eigenvalue $\lambda = 0$, *i.e.*

$$
|\mathsf{L}| = 0,\quad \overline{\mathsf{L}}\boldsymbol{a} \neq \boldsymbol{0},\quad \tilde{\boldsymbol{a}}\overline{\mathsf{L}} \neq 0: \quad\Rightarrow\quad \mathsf{L}\overline{\mathsf{L}}\boldsymbol{a} = |\mathsf{L}|\boldsymbol{a} = 0,\quad \tilde{\boldsymbol{a}}\overline{\mathsf{L}}\mathsf{L} = \tilde{\boldsymbol{a}}|\mathsf{L}| = 0. \tag{2.17}
$$

This follows from the Cayley-Hamilton identity. Arbitrary right-hand eigenvalues $\overline{\mathsf{L}}\boldsymbol{a}$ are proportional to possible solutions for the Fourier transform of the electric field according to the Equation (2.7). One may introduce mutually

[3]In the general case of three-dimensional operators if $|\mathsf{L}| \neq 0$ we have according to (A.5) the relation $\overline{\mathsf{L}}^2 = [\mathsf{L}]\overline{\mathsf{L}} + |\mathsf{L}|\left(\mathsf{L} - \langle\mathsf{L}\rangle\mathsf{I}\right)$ as can be straightforwardly calculated from the definitions using the Cayley-Hamilton identity.

normalized "polarization" vectors $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ to the electric field by the condition

$$\Pi = \boldsymbol{e}\cdot\tilde{\boldsymbol{e}}, \quad \langle \Pi \rangle = \tilde{\boldsymbol{e}}\boldsymbol{e} = 1, \quad \Pi^2 = \boldsymbol{e}\cdot\tilde{\boldsymbol{e}} = \Pi,$$

$$\boldsymbol{k\varepsilon}\Pi = (\boldsymbol{k\varepsilon e})\tilde{\boldsymbol{e}}, \quad \Pi\boldsymbol{\varepsilon k} = \boldsymbol{e}(\tilde{\boldsymbol{e}}\boldsymbol{\varepsilon k}), \quad \Rightarrow \quad \boldsymbol{k\varepsilon e} = 0, \quad \tilde{\boldsymbol{e}}\boldsymbol{\varepsilon k} = 0, \tag{2.18}$$

where in dependence on the symmetries of the operator $\mathsf{L}(\boldsymbol{k},\omega)$ the "co-vectors" $\tilde{\boldsymbol{e}}$ can be often specialized, for example, to $\tilde{\boldsymbol{e}} = \boldsymbol{e}^*$ for operators $\mathsf{L}(\boldsymbol{k},\omega) = \left(\mathsf{L}(\boldsymbol{k}^*,\omega^*)\right)^* \equiv \mathsf{L}^*(\boldsymbol{k},\omega)$.

Thus the explicit for of the projection operators for the determination of polarization vectors of the electric field are ($\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$)

$$\begin{aligned}\Pi(\boldsymbol{k},\omega) &= \frac{\overline{\mathsf{L}}(\boldsymbol{k},\omega)}{[\mathsf{L}(\boldsymbol{k},\omega)]} \\ &= \frac{\frac{c^4}{\omega^4}(\boldsymbol{k}^2)\boldsymbol{k}\cdot\boldsymbol{k} - \frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{\varepsilon k}\cdot\boldsymbol{k} - \boldsymbol{k}\cdot\boldsymbol{k\varepsilon} + (\boldsymbol{k\varepsilon k})\mathsf{I}\right) + \overline{\boldsymbol{\varepsilon}}}{\frac{c^4}{\omega^4}(\boldsymbol{k}^2)^2 - \frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}^2 + \boldsymbol{k\varepsilon k}\right) + [\boldsymbol{\varepsilon}]}, \\ \langle \Pi(\boldsymbol{k},\omega)\rangle &= 1, \quad (\Pi(\boldsymbol{k},\omega))^2 = \Pi(\boldsymbol{k},\omega).\end{aligned} \tag{2.19}$$

The degenerate case $\overline{\mathsf{L}}(\boldsymbol{k},\omega) = 0 \Rightarrow \langle\overline{\mathsf{L}}(\boldsymbol{k},\omega)\rangle = [\mathsf{L}(\boldsymbol{k},\omega)] = 0$ (but not true in inverse order) is the case of optic axes which we do not consider in present article in detail. However, isotropic media where all axes are "optic" axes also belong to this case.

In coordinate-invariant calculations of polarization vectors by means of the projection operator (2.19) as vectors $\boldsymbol{a}$ should be taken only vectors which possess a physical meaning of the considered system. According to (2.17) we have a great selection of possible choice of vectors $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$ for determination of such polarization vectors but not all are advantageous. According to $\boldsymbol{k\varepsilon e} = 0$ the right-hand polarization vectors $\boldsymbol{e}$ are perpendicularly to the vector $\boldsymbol{k\varepsilon}$ and one should not choose vectors which form a very small angle with this vector $\boldsymbol{k\varepsilon}$ as, for example, the vector $\boldsymbol{k}$ since then in limiting cases $\boldsymbol{k} \to \boldsymbol{k\varepsilon}$ it becomes undetermined. It seems to be favorable to choose for this purpose vector products of the vectors $\boldsymbol{k\varepsilon}$ or of $\boldsymbol{k}$ with other vectors where the last choice is favorable since in this case the most terms in the numerator of the projection operator (2.19) are canceled. We choose first the vector product $[\boldsymbol{k},\boldsymbol{k\varepsilon}]$ for which we find as (non-normalized) right-hand eigenvectors of the operator $\mathsf{L}(\boldsymbol{k},\omega)$ to the eigenvalue $\lambda = 0$

$$\begin{aligned}\Pi(\boldsymbol{k},\omega)[\boldsymbol{k},\boldsymbol{k\varepsilon}] &= \frac{\left(\overline{\boldsymbol{\varepsilon}} - \frac{c^2}{\omega^2}(\boldsymbol{k\varepsilon k})\mathsf{I}\right)[\boldsymbol{k},\boldsymbol{k\varepsilon}]}{\frac{c^4}{\omega^4}(\boldsymbol{k}^2)^2 - \frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}^2 + \boldsymbol{k\varepsilon k}\right) + [\boldsymbol{\varepsilon}]} \\ &= \frac{[\boldsymbol{k\varepsilon},\boldsymbol{k\varepsilon}^2] + \frac{c^2}{\omega^2}(\boldsymbol{k\varepsilon k})[\boldsymbol{k\varepsilon},\boldsymbol{k}]}{\frac{c^4}{\omega^4}(\boldsymbol{k}^2)^2 - \frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}^2 + \boldsymbol{k\varepsilon k}\right) + [\boldsymbol{\varepsilon}]},\end{aligned} \tag{2.20}$$

where the identity (B.6) was applied. This choice becomes inappropriate in limiting or other cases when $[\boldsymbol{k},\boldsymbol{k\varepsilon}]=\boldsymbol{0}$ that means if they become parallel.

If we directly choose as vector $\boldsymbol{a}$ one of the vectors vectors $\boldsymbol{k},\boldsymbol{\varepsilon k},\boldsymbol{\varepsilon}^2\boldsymbol{k}$ then we find as (non-normalized) polarization vectors of the electric field

$$\Pi(\boldsymbol{k},\omega)\boldsymbol{k}=\frac{\frac{c^2}{\omega^2}\boldsymbol{k}^2\left(\left(\frac{c^2}{\omega^2}\boldsymbol{k}^2-\langle\boldsymbol{\varepsilon}\rangle\right)\boldsymbol{k}+\boldsymbol{\varepsilon k}\right)+\overline{\boldsymbol{\varepsilon}}\boldsymbol{k}}{\frac{c^4}{\omega^4}\left(\boldsymbol{k}^2\right)^2-\frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}^2+\boldsymbol{k\varepsilon k}\right)+[\boldsymbol{\varepsilon}]},$$
$$\Pi(\boldsymbol{k},\omega)\boldsymbol{\varepsilon k}=\frac{\left(\frac{c^4}{\omega^4}\boldsymbol{k}^2(\boldsymbol{k\varepsilon k})-\frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k\varepsilon k}-\boldsymbol{k\varepsilon}^2\boldsymbol{k}\right)+|\boldsymbol{\varepsilon}|\right)\boldsymbol{k}}{\frac{c^4}{\omega^4}\left(\boldsymbol{k}^2\right)^2-\frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}^2+\boldsymbol{k\varepsilon k}\right)+[\boldsymbol{\varepsilon}]}=0, \tag{2.21}$$
$$\Pi(\boldsymbol{k},\omega)\boldsymbol{\varepsilon}^2\boldsymbol{k}=\frac{\frac{c^2}{\omega^2}\left(\frac{c^2}{\omega^2}\boldsymbol{k}^2\left(\boldsymbol{k\varepsilon}^2\boldsymbol{k}\right)-[\boldsymbol{\varepsilon}]\boldsymbol{k\varepsilon k}+|\boldsymbol{\varepsilon}|\boldsymbol{k}^2\right)\boldsymbol{k}+\left(\frac{c^2}{\omega^2}\boldsymbol{k\varepsilon}^2\boldsymbol{k}+|\boldsymbol{\varepsilon}|\right)\boldsymbol{\varepsilon k}-\frac{c^2}{\omega^2}(\boldsymbol{k\varepsilon k})\boldsymbol{\varepsilon}^2\boldsymbol{k}}{\frac{c^4}{\omega^4}\left(\boldsymbol{k}^2\right)^2-\frac{c^2}{\omega^2}\left(\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{k}^2+\boldsymbol{k\varepsilon k}\right)+[\boldsymbol{\varepsilon}]}.$$

where $\boldsymbol{\varepsilon k}$ is inappropriate since it provides the zero vector and expresses that polarization vectors of the electric field are perpendicularly to $\boldsymbol{k\varepsilon}$. One may check that $\boldsymbol{k\varepsilon}\Pi\boldsymbol{a}=0$ (2.18) in all cases.

Favorable representations of polarization vectors one may often find if we use in addition the vectors to optic axes in the representation of the permittivity tensor in principal axes form, most generally

$$\boldsymbol{\varepsilon}=\varepsilon_1\boldsymbol{c}_1\cdot\tilde{\boldsymbol{c}}_1+\varepsilon_2\boldsymbol{c}_2\cdot\tilde{\boldsymbol{c}}_2+\varepsilon_3\boldsymbol{c}_3\cdot\tilde{\boldsymbol{c}}_3,\quad \tilde{\boldsymbol{c}}_i\boldsymbol{c}_j=\delta_{ij}, \tag{2.22}$$

where all involved vectors and scalars may or may not depend on wave vector and frequency depending on the symmetry of the medium. In lossless case we have the simplification $\tilde{\boldsymbol{c}}_i=\boldsymbol{c}_i^*$ and under additional symmetry of the permittivity tensor $\boldsymbol{\varepsilon}$ for homogeneous waves (real wave vector and frequency) $\boldsymbol{c}_i^*=\boldsymbol{c}_i$. One may choose as vectors $\boldsymbol{a}$ for the determination of polarization vectors the vectors of the optic axes $\boldsymbol{c}_i,(i=1,2,3)$ themselves or the vector products $[\boldsymbol{k},\tilde{\boldsymbol{c}}_i]$.

Another possible approach is via the vector field of the electric induction. From (2.7) using the representation (A.3) for the inverse operator $\boldsymbol{\varepsilon}$ one may derive the following wave equation for the electric induction $\boldsymbol{D}(\boldsymbol{k},\omega)$

$$\boldsymbol{0}=\left\{\frac{c^2}{\omega^2|\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)|}\left(\boldsymbol{k}\cdot\boldsymbol{k}-\left(\boldsymbol{k}^2\right)\mathsf{I}\right)\overline{\boldsymbol{\varepsilon}}(\boldsymbol{k},\omega)+\mathsf{I}\right\}\boldsymbol{D}(\boldsymbol{k},\omega)\equiv\mathsf{L}^{\boldsymbol{D}}(\boldsymbol{k},\omega)\boldsymbol{D}(\boldsymbol{k},\omega). \tag{2.23}$$

where we define the operator $\mathsf{L}^{\boldsymbol{D}}(\boldsymbol{k},\omega)$ by

$$\mathsf{L}^{\boldsymbol{D}}(\boldsymbol{k},\omega)\equiv\frac{c^2}{\omega^2|\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)|}\left(\boldsymbol{k}\cdot\boldsymbol{k}-\left(\boldsymbol{k}^2\right)\mathsf{I}\right)\overline{\boldsymbol{\varepsilon}}(\boldsymbol{k},\omega)+\mathsf{I}-\frac{\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\cdot\boldsymbol{k}}{\boldsymbol{k\varepsilon}(\boldsymbol{k},\omega)\boldsymbol{k}},\quad \boldsymbol{k}\mathsf{L}^{\boldsymbol{D}}(\boldsymbol{k},\omega)=0. \tag{2.24}$$

We have substituted taking into account $\boldsymbol{kD}=0$ the three-dimensional unit operator I by a two-dimensional unit operator (or projection operator) I'

$$\mathsf{I}' \equiv \mathsf{I} - \frac{\boldsymbol{\varepsilon k} \cdot \boldsymbol{k}}{\boldsymbol{k \varepsilon k}}, \quad \mathsf{I}'^2 = \mathsf{I}', \quad \langle \mathsf{I}' \rangle = 2, \tag{2.25}$$

in such a way the operator $\mathsf{L}^{\boldsymbol{D}}$ possesses the properties

$$\boldsymbol{k}\mathsf{L}^{\boldsymbol{D}}(\boldsymbol{k},\omega) = \boldsymbol{0}, \quad \mathsf{L}^{\boldsymbol{D}}(\boldsymbol{k},\omega)\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)\boldsymbol{k} = \boldsymbol{0}. \tag{2.26}$$

It is now a two-dimensional operator by multiplication from the left in the plane perpendicular to $\boldsymbol{k}$ and from right in the plane perpendicular to $\boldsymbol{\varepsilon k}$. Landau and Lifshits [4] prefer for the treatment of some problems more directly the electric induction $\boldsymbol{D}$ but, clearly, without formalizing this with introduction of an operator $\mathsf{L}^{\boldsymbol{D}}$. The use of $\boldsymbol{D}$ instead of $\boldsymbol{E}$ possesses advantages (orthogonality to $\boldsymbol{k}$) but also disadvantages and we do not consider this.

3. The Concept of Bi-Anisotropic Constitutive Equations

The concept of bi-anisotropic media with the special case of bi-isotropic media is more specially than the concept of spatial dispersion discussed in last Section. The basic equations of macroscopic optics are written in this concept in the following way for the Fourier transforms

$$\begin{aligned} &\left[\boldsymbol{k}, \boldsymbol{E}(\boldsymbol{k},\omega)\right] - \frac{\omega}{c}\boldsymbol{B}(\boldsymbol{k},\omega) = \boldsymbol{0}, \quad \boldsymbol{kB}(\boldsymbol{k},\omega) = 0, \\ &\left[\boldsymbol{k}, \boldsymbol{H}(\boldsymbol{k},\omega)\right] + \frac{\omega}{c}\boldsymbol{D}'(\boldsymbol{k},\omega) = \boldsymbol{0}, \quad \boldsymbol{kD}'(\boldsymbol{k},\omega) = 0, \end{aligned} \tag{3.1}$$

where by definition

$$\boldsymbol{D}'(\boldsymbol{k},\omega) \equiv \boldsymbol{E}(\boldsymbol{k},\omega) + 4\pi\boldsymbol{P}'(\boldsymbol{k},\omega), \quad \boldsymbol{H}(\boldsymbol{k},\omega) \equiv \boldsymbol{B}(\boldsymbol{k},\omega) - 4\pi\boldsymbol{M}(\boldsymbol{k},\omega), \tag{3.2}$$

and where $\boldsymbol{P}'$ is the polarization in a narrow sense and $\boldsymbol{M}$ the magnetization and with constitutive equations of the following form for the Fourier transforms

$$\boldsymbol{D}'(\boldsymbol{k},\omega) = \boldsymbol{\varepsilon}(\omega)\boldsymbol{E}(\boldsymbol{k},\omega), \quad \boldsymbol{B}(\boldsymbol{k},\omega) = \boldsymbol{\mu}(\omega)\boldsymbol{H}(\boldsymbol{k},\omega), \tag{3.3}$$

where we do not assume that $\boldsymbol{\varepsilon}(\omega)$ and $\boldsymbol{\mu}(\omega)$ are symmetric tensors (e.g., magneto-optic effects). Usually, $\boldsymbol{H}$ is called the magnetic field and $\boldsymbol{B}$ the magnetic induction also $\boldsymbol{B}$ is the averaged microscopic magnetic field [4]. These notions are made for the symmetries between $\boldsymbol{E} \rightleftharpoons \boldsymbol{H}$ and $\boldsymbol{D} \rightleftharpoons \boldsymbol{B}$ in the field equations but this may be confusing.

In considered case it is favorable to introduce the notion of refraction vectors $\boldsymbol{n}$ by the definition ($\omega \neq 0$)

$$\boldsymbol{n} \equiv \frac{c}{\omega}\boldsymbol{k}. \tag{3.4}$$

The basic equations for the Fourier transforms of fields (3.1) simplify the slightly and are

$$\begin{aligned} &\left[\boldsymbol{n}, \boldsymbol{E}\right] - \boldsymbol{B} = \boldsymbol{0}, \quad \boldsymbol{nB} = 0, \\ &\left[\boldsymbol{n}, \boldsymbol{H}\right] + \boldsymbol{D}' = \boldsymbol{0}, \quad \boldsymbol{nD}' = 0, \end{aligned} \tag{3.5}$$

together with the constitutive equations

$$\boldsymbol{D}' = \boldsymbol{\varepsilon}\boldsymbol{E}, \quad \boldsymbol{B} = \boldsymbol{\mu}\boldsymbol{H}, \tag{3.6}$$

where we omitted to write the arguments in the fields and in the material tensors, e.g., $\boldsymbol{E}(\boldsymbol{k},\omega) \equiv \boldsymbol{E}$, $\boldsymbol{\varepsilon}(\omega) \equiv \boldsymbol{\varepsilon}$.

From (3.5) follow for a given refraction vectors $\boldsymbol{n}$ the orthogonalities

$$\boldsymbol{B}\boldsymbol{E} = [\boldsymbol{n},\boldsymbol{E}]\boldsymbol{E} = 0, \quad \boldsymbol{D}'\boldsymbol{H} = -[\boldsymbol{n},\boldsymbol{H}]\boldsymbol{H} = 0. \tag{3.7}$$

It is also important to mention here that the Equations (3.5) together with (3.6) remain unchanged under the simultaneous permutations

$$\boldsymbol{E} \leftrightarrow \boldsymbol{H}, \quad \boldsymbol{D}' \leftrightarrow \boldsymbol{B}, \quad \boldsymbol{\varepsilon} \leftrightarrow \boldsymbol{\mu}, \quad \boldsymbol{n} \leftrightarrow -\boldsymbol{n}, \tag{3.8}$$

but, clearly, all this is well known.

First, we derive a wave equation for the Fourier components $\boldsymbol{E}$ of the electric field. For this purpose we use the formula (A.3) for the inverse operator and the mathematical identity (B.6) and, furthermore, the transposition of the application of an operator to a vector $\mathsf{A}\boldsymbol{x} = \boldsymbol{x}\mathsf{A}^{\mathrm{T}}$ where A^{T} is the transposed operator to A and find

$$\begin{aligned}\boldsymbol{0} &= [\boldsymbol{n},\boldsymbol{H}] + \boldsymbol{D}' = \left[\boldsymbol{n},\boldsymbol{\mu}^{-1}\boldsymbol{B}\right] + \boldsymbol{D}' = \frac{[\boldsymbol{n},\overline{\boldsymbol{\mu}}[\boldsymbol{n},\boldsymbol{E}]]}{|\boldsymbol{\mu}|} + \boldsymbol{D}' = \frac{[\boldsymbol{n},[\boldsymbol{n}\boldsymbol{\mu},\boldsymbol{E}\boldsymbol{\mu}]]}{|\boldsymbol{\mu}|} + \boldsymbol{D}' \\ &= \frac{\left[\boldsymbol{n},\left[\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{n},\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{E}\right]\right]}{|\boldsymbol{\mu}|} + \boldsymbol{D}' = \left\{\frac{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{n}\cdot\boldsymbol{n}\boldsymbol{\mu}^{\mathrm{T}} - \left(\boldsymbol{n}\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{n}\right)\boldsymbol{\mu}^{\mathrm{T}}}{|\boldsymbol{\mu}|} + \boldsymbol{\varepsilon}\right\}\boldsymbol{E}.\end{aligned} \tag{3.9}$$

We write this Equation ($\boldsymbol{E} \equiv \boldsymbol{E}\left(\frac{\omega}{c}\boldsymbol{n},\omega\right)$)

$$\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n})\boldsymbol{E} = \boldsymbol{0}, \quad \Rightarrow \quad \boldsymbol{n}\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n})\boldsymbol{E} = \boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{E} = 0, \tag{3.10}$$

with an operator $\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n})$ defined by

$$\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n}) \equiv \frac{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{n}\cdot\boldsymbol{n}\boldsymbol{\mu}^{\mathrm{T}} - \left(\boldsymbol{n}\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{n}\right)\boldsymbol{\mu}^{\mathrm{T}}}{|\boldsymbol{\mu}|} + \boldsymbol{\varepsilon}. \tag{3.11}$$

Due to symmetry (3.8) in the starting Equations (3.5) and (3.6) one may immediately write down an analogous equation for $\boldsymbol{H}$

$$\mathsf{L}^{\boldsymbol{H}}(\boldsymbol{n})\boldsymbol{H} = \boldsymbol{0}, \quad \Rightarrow \quad \boldsymbol{n}\mathsf{L}^{\boldsymbol{H}}(\boldsymbol{n})\boldsymbol{H} = \boldsymbol{n}\boldsymbol{\mu}\boldsymbol{H} = 0, \tag{3.12}$$

with an operator $\mathsf{L}^{\boldsymbol{H}}(\boldsymbol{n})$ defined by

$$\mathsf{L}^{\boldsymbol{H}}(\boldsymbol{n}) \equiv \frac{\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{n}\cdot\boldsymbol{n}\boldsymbol{\varepsilon}^{\mathrm{T}} - \left(\boldsymbol{n}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{n}\right)\boldsymbol{\varepsilon}^{\mathrm{T}}}{|\boldsymbol{\varepsilon}|} + \boldsymbol{\mu}. \tag{3.13}$$

Both operators $\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n})$ and $\mathsf{L}^{\boldsymbol{H}}(\boldsymbol{n})$ possess the form of the operator L discussed in Appendix C with the following substitutions in case of $\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n})$ which we consider now

$$\mathsf{A} \to \boldsymbol{\mu}^{\mathrm{T}}, \quad \mathsf{B} \to \boldsymbol{\varepsilon}, \quad \boldsymbol{x} = \tilde{\boldsymbol{x}} \to \boldsymbol{n}. \tag{3.14}$$

According to (C.7) and (C.10) its invariants are $\boldsymbol{\mu} \equiv \boldsymbol{\mu}(\omega), \boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}(\omega)$

$$\begin{aligned}
\left\langle \mathsf{L}^{E}(\boldsymbol{n})\right\rangle &= \frac{\boldsymbol{n}\boldsymbol{\mu}^{2}\boldsymbol{n}-\langle\boldsymbol{\mu}\rangle\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n}+|\boldsymbol{\mu}|\langle\boldsymbol{\varepsilon}\rangle}{|\boldsymbol{\mu}|},\\
\left[\mathsf{L}^{E}(\boldsymbol{n})\right] &= \frac{(\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n})\boldsymbol{n}^{2}-\left(\left(\langle\boldsymbol{\mu}\rangle\langle\boldsymbol{\varepsilon}\rangle-\langle\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\varepsilon}\rangle\right)\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n}-\langle\boldsymbol{\varepsilon}\rangle\boldsymbol{n}\boldsymbol{\mu}^{2}\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\varepsilon}\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{n}\right)+|\boldsymbol{\mu}|[\boldsymbol{\varepsilon}]}{|\boldsymbol{\mu}|},\\
\left|\mathsf{L}^{E}(\boldsymbol{n})\right| &= \frac{(\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n})(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n})-\left(\langle\boldsymbol{\mu}^{\mathrm{T}}\overline{\boldsymbol{\varepsilon}}\rangle\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\mu}^{\mathrm{T}}\overline{\boldsymbol{\varepsilon}}\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{n}\right)+|\boldsymbol{\mu}||\boldsymbol{\varepsilon}|}{|\boldsymbol{\mu}|}.
\end{aligned} \tag{3.15}$$

We wrote here the general case $\boldsymbol{\mu}\neq\boldsymbol{\mu}^{\mathrm{T}}$ (and $\boldsymbol{\varepsilon}\neq\boldsymbol{\varepsilon}^{\mathrm{T}}$) but it was not necessary to write the sign "T" at $\boldsymbol{\mu}$ for transposition in all cases because, e.g., $\boldsymbol{n}\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{n}=\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n}$ and for all invariants holds, e.g., $\langle\boldsymbol{\mu}^{\mathrm{T}}\rangle=\langle\boldsymbol{\mu}\rangle$ (however, e.g., $\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\varepsilon}=\left(\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\mu}\right)^{\mathrm{T}}$)[4]. The complementary operator $\overline{\mathsf{L}^{E}}(\boldsymbol{n})$ obtainable from (C.11) is fairly complicated and we do not write it down.

The dispersion equation for a bi-anisotropic medium is

$$\left|\mathsf{L}^{E}(\boldsymbol{n})\right|=0, \tag{3.16}$$

or equivalently the analogous equation for the operator $\left|\mathsf{L}^{H}(\boldsymbol{n})\right|$. Polarization vectors $\boldsymbol{e}$ together with left-hand eigenvectors $\tilde{\boldsymbol{e}}$ to the operator $\mathsf{L}^{E}(\boldsymbol{n})$ for the electric field can be obtained by the projection operators

$$\Pi^{E}(\boldsymbol{n})=\frac{\overline{\mathsf{L}^{E}}(\boldsymbol{n})}{\left\langle\overline{\mathsf{L}^{E}}(\boldsymbol{n})\right\rangle}=\frac{\overline{\mathsf{L}^{E}}(\boldsymbol{n})}{\left[\mathsf{L}^{E}(\boldsymbol{n})\right]}=\boldsymbol{e}\cdot\tilde{\boldsymbol{e}},\quad \left\langle\Pi^{E}(\boldsymbol{n})\right\rangle=\tilde{\boldsymbol{e}}\boldsymbol{e}=1, \tag{3.17}$$

but this is complicated and we will it only do for the special case of bi-anisotropic uniaxial media in next Section.

4. Bi-Anisotropic Media as Special Case of Spatial Dispersion

The approach to the linear optics of media by spatial dispersion is much more general than the approach by bi-anisotropic media which is mainly interesting for its symmetries between electric and magnetic quantities and I am not a fan of the last for reason which will become clear in the following. Spatial dispersion is often discussed with expansion of the tensor $\varepsilon_{ij}(\boldsymbol{k},\omega)$ into powers of the wave vector as follows [1] [2] [3] [4]

$$\begin{aligned}
\varepsilon_{ij}(\boldsymbol{k},\omega) &= \varepsilon_{ij}(\omega)+\mathrm{i}\frac{c}{\omega}\gamma_{ijk}(\omega)k_{k}+\frac{c^{2}}{\omega^{2}}\alpha_{ijkl}(\omega)k_{k}k_{l}+\cdots,\\
\varepsilon_{ij}^{-1}(\boldsymbol{k},\omega) &= \varepsilon_{ij}^{-1}(\omega)+\mathrm{i}\frac{c}{\omega}\delta_{ijk}(\omega)k_{k}+\frac{c^{2}}{\omega^{2}}\beta_{ijkl}(\omega)k_{k}k_{l}+\cdots\\
&= \varepsilon_{ij}^{-1}(\omega)-\mathrm{i}\frac{c}{\omega}\varepsilon_{im}^{-1}(\omega)\gamma_{mnk}(\omega)\varepsilon_{nj}^{-1}(\omega)k_{k}\\
&\quad -\frac{c^{2}}{\omega^{2}}\varepsilon_{im}^{-1}(\omega)\left(\alpha_{mnkl}(\omega)+\gamma_{mpk}(\omega)\varepsilon_{pq}^{-1}(\omega)\gamma_{qnl}(\omega)\right)\varepsilon_{nj}^{-1}(\omega)k_{k}k_{l}-\cdots.
\end{aligned} \tag{4.1}$$

[4]All these problems could be removed if one defines $\boldsymbol{B}=\boldsymbol{H}\boldsymbol{\mu}$ instead of (3.6) but this possesses the danger to cause some confusion.

The concept of bi-anisotropic media with the constitutive Equations (3.3) can be expressed as a special case of the concept of spatial dispersion with the following dependence of tensor $\varepsilon_{ij}(\boldsymbol{k},\omega)$ on the wave vector $\boldsymbol{k}$

$$\begin{aligned}\varepsilon_{ij}(\boldsymbol{k},\omega) &= \varepsilon_{ij}(\omega)+\frac{c^2}{\omega^2}\alpha_{ijkl}(\omega)k_k k_l \\ &= \varepsilon_{ij}(\omega)+\frac{c^2}{\omega^2}\epsilon_{ikm}\epsilon_{jln}\left(\delta_{mn}-\mu_{mn}^{-1}(\omega)\right)k_k k_l.\end{aligned} \tag{4.2}$$

This can be expressed in a representation without indices in the form of Equation (2.7) with the general permittivity $\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$ by (A^{T} means transposition of A; definition $\boldsymbol{k}\equiv\frac{\omega}{c}\boldsymbol{n}$ would shorten the following representation)

$$\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)=\boldsymbol{\varepsilon}(\omega)+\frac{c^2}{\omega^2}\left(\frac{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{\mu}^{\mathrm{T}}-(\boldsymbol{k}\boldsymbol{\mu}\boldsymbol{k})\boldsymbol{\mu}^{\mathrm{T}}}{|\boldsymbol{\mu}|}-\left(\boldsymbol{k}\cdot\boldsymbol{k}-\boldsymbol{k}^2\mathsf{I}\right)\right). \tag{4.3}$$

In the general concept of spatial dispersion a bi-anisotropic medium appears as second-order effect in the expansion of the general tensor $\varepsilon_{ij}(\boldsymbol{k},\omega)$ in powers of the wave vector $\boldsymbol{k}$. It does, however, not possess the general form of a tensor of forth rank $\alpha_{ijkl}(\omega)$ with only symmetry in the last both indices (k,l) and, furthermore, sum terms which are linear in the wave vector $\boldsymbol{k}$ are completely absent (e.g., optic gyrotropy). The reason that the tensor $\alpha_{ijkl}(\omega)$ does not possess in the concept of bi-anisotropy the general form of such a tensor comes from the neglect of electric quadrupole terms and also of higher electric and magnetic multipole terms in the expansion of the general polarization $\boldsymbol{P}(\boldsymbol{k},\omega)$ in powers of the wave vector $\boldsymbol{k}$. Apart from the first term $\boldsymbol{P}(\omega)$ which mostly provides the greatest contribution and is uniquely defined the higher contributions are difficult to separate from each other since in multipole expansions only the first multipole moment which is non-vanishing is uniquely defined whereas the others depend on the chosen origin of the multipole expansion. The magnitude of the different terms from the multipole effects is difficult to assess but one has to assume that within terms of the same order they should be comparable.

The concept of bi-isotropy with $\varepsilon(\omega)$ and $\mu(\omega)$ as scalars in the constitutive equations is old and goes back under other names to the development of macroscopic electrodynamics by the Maxwell equations and its generalization to bi-anisotropy by transition to second-rank tensors $\varepsilon_{ij}(\omega)$ and $\mu_{ij}(\omega)$ is natural. However, the last leads usually to very complicated formulae if one calculates propagation and reflection and refraction problems (amplitudes included), moreover, if this is made by coordinate methods. The most comprehensive and unrivaled representation was given by Szivessy [8] and long ago I thought that it remains the last which works mainly with coordinate methods. However, more than in most other sources in this respect is made in the book of Fyodorov [9] with coordinate-invariant methods which he initiated and developed. In the book [10] of the same author the concept of bi-anisotropy (Fyodorov calls it "crystals with electric and magnetic anisotropy") is extended to inclusion of optic gyrotropy that even by coordinate-invariant treatment leads as a rule to very complicated formulae. The last chapter in this book contains linear algebra in three-dimensional Euclidean space in a form which is very useful for the application of coordinate-invariant calculations in three-dimensional spaces (chap. IV, pp. 362-450)[5].

An extended concept of bi-anisotropy in the basic equations is maintained, in particular, in the very versatile monograph of de Groot [23] and in nonlinear optics by Bloembergen [24] (called the "Netherland school" in [14] with inclusion of some other authors). Furthermore, there are articles to the calculation of the dyadic Green functions to the Huygens principle for bi-anisotropic media ([15] [25] and, e.g., Weiglhofer [26] with many citations published in: [27]).

It is necessary to report here also about an unprecedented scientific plagiarism in form of a book from 1983 by Hollis C. Chen, a professor of the Ohio University in U.S.A, about which I was informed by Fyodor Ivanovich Fyodorov in the middle of the eighties. This book and papers of Chen cannot be cited in normal way under references and I make some remarks to this case in the following footnote[6].

[5]For inclusion of higher than second-rank tensors the concept of coordinate-invariant treatment without using tensor indices fails but in three-dimensional case also third-rank tensors which are anti-symmetric in two indices can be included since they may be mapped onto second-rank pseudo-tensors.

[6]In the beginning of the eighties I sent my papers with application of coordinate-invariant methods (about 10) to Fyodorov who as it proved did not know them. Since this time we were in loose correspondence up to the end of the eighties and I used a big Optics Conference in Minsk in the eighties especially to meet him there personally and this took place in the main building of Academy of Sciences of Belarus on the main boulevard in Minsk. Once in the eighties I received a letter from Fyodorov about an unprecedented plagiarism in a book of H. C. Chen "Theory of Electromagnetic Waves, A Coordinate-Free Approach" from McGraw-Hill (1983). Fyodorov is not cited there and all is made to camouflage the real authorship of these methods. When I tried to see this book I could not find it and, as usual in such cases, a search programme in the libraries of GDR was started. It was not found in GDR and then in such cases it could be searched in West-Germany. In West-Berlin (about 15 km airline from me but unreachable that time) it was found in the Library of the Technical University and I got it for one month. Altogether, it lasted almost three quarters of a year to get it. All what Fyodorov said turned out to be true. The main part of Chen's book is almost a free translation of main chapters of the book of Fyodorov [9]. However, what Fyodorov obviously did not know was that two chapters of Chen's book are a plagiarism of my paper [15] to Huygens principle. Obviously also that Chen did not know my paper [25]. This journal was hardly known in the world and ceased to exist after the turn in GDR and he also did not use my papers to amplitude relations for reflection and refraction at anisotropic media in "Ann. d. Physik", likely because my notations were too different from that of Fyodorov and were more adapted to Landau and Lifshits [4]. When I tried to inform a branch office of McGraw-Hill in Hamburg (FRG) by a letter about this and to get an exemplar without payment (I could not pay to this time West-German currency) my chief at this time Witlof Brunner demanded to write not about the "plagiarism" that was not acceptable for me and I renounced to send this letter. Later, after the turn in GDR, I found this same exemplar of the book of Chen which I earlier have had for a month in the reading room of the Library of the Technical University in West-Berlin and made a copy. Obviously, to this time it was already taken from market (I tried earlier to get it through a colleague, H. Haake, from Essen in West Germany whom I met at a Workshop in Poland but he wrote me then that it was impossible to order). Fyodorov and his scientific colleagues reached that in scientific media in the West and in Russian newspapers was written about the plagiarism. Weiglhofer [26] (see [27]) and also some others did not know all this and cite the fraud of Chen instead of the genuine authors (but in [26] a book of Chen from 1992 is cited which obviously later could appear in U.S.A.). My chief to this time, W. Brunner, was indifferent and uninterested in all this and was not ready to support me.

5. Optic Uniaxial Bi-Anisotropic Media

We consider now the special case of optic uniaxial bi-anisotropic media which is determined by the following tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$

$$\begin{aligned}&\boldsymbol{\varepsilon}(\omega)=\varepsilon^{e}(\omega)\boldsymbol{c}\cdot\boldsymbol{c}+\varepsilon^{o}(\omega)(\mathsf{I}-\boldsymbol{c}\cdot\boldsymbol{c}),\\&\boldsymbol{\mu}(\omega)=\mu^{e}(\omega)\boldsymbol{c}\cdot\boldsymbol{c}+\mu^{o}(\omega)(\mathsf{I}-\boldsymbol{c}\cdot\boldsymbol{c}),\quad\left(\boldsymbol{c}^{2}=1\right).\end{aligned}\tag{5.1}$$

where the tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ considered as operators are symmetric and commute (as consequence of axial symmetry with the same axes for the electric and magnetic properties)

$$\boldsymbol{\mu}(\omega)=\boldsymbol{\mu}^{\mathrm{T}}(\omega),\quad\boldsymbol{\varepsilon}(\omega)=\boldsymbol{\varepsilon}^{\mathrm{T}}(\omega),\quad\boldsymbol{\mu}(\omega)\boldsymbol{\varepsilon}(\omega)=\boldsymbol{\varepsilon}(\omega)\boldsymbol{\mu}(\omega).\tag{5.2}$$

With $\boldsymbol{c}$ we have denoted a unit vector in direction of the common optic axis of the permittivity tensor $\boldsymbol{\varepsilon}$ and the permeability tensor $\boldsymbol{\mu}$ of the uniaxial bi-anisotropic medium (notations of Fyodorov [9]) and $\varepsilon^{e},\varepsilon^{o}$ and μ^{e},μ^{o} (upper indices "e" and "o" stand for "extraordinary" and "ordinary") are frequency depend material scalars. The complementary operators and the invariants, for example, for $\boldsymbol{\varepsilon}\equiv\boldsymbol{\varepsilon}(\omega)$ are (similarly, $\boldsymbol{\mu}(\omega)$)

$$\begin{aligned}&\overline{\boldsymbol{\varepsilon}}=\varepsilon^{o}\left(\varepsilon^{o}\boldsymbol{c}\cdot\boldsymbol{c}+\varepsilon^{e}\left(\mathsf{I}-\boldsymbol{c}\cdot\boldsymbol{c}\right)\right),\\&\langle\boldsymbol{\varepsilon}\rangle=\varepsilon^{e}+2\varepsilon^{o},\quad[\boldsymbol{\varepsilon}]=\varepsilon^{o}\left(2\varepsilon^{e}+\varepsilon^{o}\right)=\langle\overline{\boldsymbol{\varepsilon}}\rangle,\quad|\boldsymbol{\varepsilon}|=\varepsilon^{e}\left(\varepsilon^{o}\right)^{2}.\end{aligned}\tag{5.3}$$

Using this together with $\left|\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n})\right|$ in (3.15) and $\mathsf{L}^{\boldsymbol{H}}(\boldsymbol{n})$ in (3.13) one may specialize this determinant to

$$\mu^{e}\left(\mu^{o}\right)^{2}\left|\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n})\right|=\left(\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n}-\varepsilon^{o}\mu^{e}\mu^{o}\right)\left(\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}-\mu^{o}\varepsilon^{e}\varepsilon^{o}\right)=\varepsilon^{e}\left(\varepsilon^{o}\right)^{2}\left|\mathsf{L}^{\boldsymbol{H}}(\boldsymbol{n})\right|.\tag{5.4}$$

The dispersion equation that means the vanishing of the determinants $\left|\mathsf{L}^{\boldsymbol{E}}(\boldsymbol{n})\right|$ or $\left|\mathsf{L}^{\boldsymbol{H}}(\boldsymbol{n})\right|$ decomposes into a product of two separate equations as follows

$$\begin{aligned}&0=\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n}-\varepsilon^{o}\mu^{e}\mu^{o}\quad\leftrightarrow\quad\frac{(\boldsymbol{n}\boldsymbol{c})^{2}}{\varepsilon^{o}\mu^{o}}+\frac{[\boldsymbol{n},\boldsymbol{c}]^{2}}{\varepsilon^{o}\mu^{e}}=1,\\&0=\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{n}-\mu^{o}\varepsilon^{e}\varepsilon^{o}\quad\leftrightarrow\quad\frac{(\boldsymbol{n}\boldsymbol{c})^{2}}{\mu^{o}\varepsilon^{o}}+\frac{[\boldsymbol{n},\boldsymbol{c}]^{2}}{\mu^{o}\varepsilon^{e}}=1,\end{aligned}\tag{5.5}$$

which for real positive parameters $\varepsilon^{o},\varepsilon^{e},\mu^{o},\mu^{e}$ represent two rotation ellipsoids with axes lengths which are the square roots of the denominators in (5.5) and with equal axis length in direction of the optic axis. This means that the two rotation ellipsoids touches in axis direction. The determination of polarization vectors via projection operators as described in Section 2 seems to be too tedious and we choose a more special approach. Due to $\boldsymbol{n}\boldsymbol{\varepsilon}\boldsymbol{E}=0$ polarization vectors of the electric field have to be perpendicular to the vector $\boldsymbol{n}\boldsymbol{\varepsilon}$ and are therefore representable by the vector product of $\boldsymbol{n}\boldsymbol{\varepsilon}$ with a vector which possesses a component in the plane perpendicular to $\boldsymbol{n}\boldsymbol{\varepsilon}$.

We consider this for the first dispersion equation $\boldsymbol{n}\boldsymbol{\mu}\boldsymbol{n}=\varepsilon^{o}\mu^{e}\mu^{o}$ in (5.5). Using this equation a (non-normalized) polarization vector $\boldsymbol{e}'$ for the electric field with the proposition $\boldsymbol{e}' = [\boldsymbol{n\mu}, \boldsymbol{c}]$ according to (3.5) and (3.6) has to satisfy the equation

$$\begin{aligned} \mathbf{0} &= \left\{ \frac{\boldsymbol{\mu n} \cdot \boldsymbol{n\mu} - (\boldsymbol{n\mu n})\boldsymbol{\mu}}{|\boldsymbol{\mu}|} + \boldsymbol{\varepsilon} \right\} \boldsymbol{e}' = \frac{(\boldsymbol{\mu n} \cdot \boldsymbol{n\mu} - (\boldsymbol{n\mu n})\boldsymbol{\mu})[\boldsymbol{n\mu}, \boldsymbol{c}]}{|\boldsymbol{\mu}|} + \boldsymbol{\varepsilon e}' \\ &= -\frac{\varepsilon^o \mu^e \mu^o}{\mu^e (\mu^o)^2} \boldsymbol{\mu} [\boldsymbol{n\mu}, \boldsymbol{c}] + \boldsymbol{\varepsilon e}' = -\frac{\varepsilon^o}{\mu^o} \left(\mu^e \boldsymbol{c} \cdot \boldsymbol{c} + \mu^o (\mathsf{I} - \boldsymbol{c} \cdot \boldsymbol{c}) \right) \mu^o [\boldsymbol{n}, \boldsymbol{c}] + \boldsymbol{\varepsilon e} \\ &= -\varepsilon^o \mu^o [\boldsymbol{n}, \boldsymbol{c}] + \boldsymbol{\varepsilon e}' \end{aligned} \tag{5.6}$$

from which follows

$$\begin{aligned} \boldsymbol{e}' &= \varepsilon^o \mu^o \boldsymbol{\varepsilon}^{-1} [\boldsymbol{n}, \boldsymbol{c}] = \varepsilon^o \mu^o \left(\frac{1}{\varepsilon^e} \boldsymbol{c} \cdot \boldsymbol{c} + \frac{1}{\varepsilon^o} (\mathsf{I} - \boldsymbol{c} \cdot \boldsymbol{c}) \right) [\boldsymbol{n}, \boldsymbol{c}] \\ &= \mu^o [\boldsymbol{n}, \boldsymbol{c}] = \frac{\mu^o}{\varepsilon^o} [\boldsymbol{n\varepsilon}, \boldsymbol{c}]. \end{aligned} \tag{5.7}$$

For a (non-normalized) polarization vector $\boldsymbol{h}'$ of the magnetic field follows then from (3.5) using (5.7) and in addition the first of the Equations (5.5)

$$\begin{aligned} \boldsymbol{h}' &= \boldsymbol{\mu}^{-1} \boldsymbol{b}' = \boldsymbol{\mu}^{-1} [\boldsymbol{n}, \boldsymbol{e}'] = \mu^o \left(\frac{1}{\mu^e} \boldsymbol{c} \cdot \boldsymbol{c} + \frac{1}{\mu^o} (\mathsf{I} - \boldsymbol{c} \cdot \boldsymbol{c}) \right) [\boldsymbol{n}, [\boldsymbol{n}, \boldsymbol{c}]] \\ &= -\frac{\mu^o}{\mu^e} [\boldsymbol{n}, \boldsymbol{c}]^2 \boldsymbol{c} + (\boldsymbol{nc}) [\boldsymbol{c}, [\boldsymbol{n}, \boldsymbol{c}]] = (\boldsymbol{nc}) \boldsymbol{n} - \varepsilon^o \mu^o \boldsymbol{c} = \frac{1}{\mu^e} [\boldsymbol{n\mu}, [\boldsymbol{n}, \boldsymbol{c}]]. \end{aligned} \tag{5.8}$$

To get the analogous relations for the second dispersion equation $\boldsymbol{n\varepsilon n} = \mu^o \varepsilon^e \varepsilon^o$ in (5.5) one has only to apply the symmetry relations (3.8). Thus we find in this case a (non-normalized) polarization vector $\boldsymbol{h}$

$$\boldsymbol{h}' = \varepsilon^o [\boldsymbol{n}, \boldsymbol{c}] = \frac{\varepsilon^o}{\mu^o} [\boldsymbol{n\mu}, \boldsymbol{c}], \tag{5.9}$$

and a (non-normalized) polarization vector $\boldsymbol{e}$

$$\boldsymbol{e}' = (\boldsymbol{nc}) \boldsymbol{n} - \mu^o \varepsilon^o \boldsymbol{c} = \frac{1}{\mu^e} [\boldsymbol{n\varepsilon}, [\boldsymbol{n}, \boldsymbol{c}]]. \tag{5.10}$$

For non-normalized polarization vectors scalar factors are unimportant and can be omitted.

We now give preference to (non-normalized) polarization vectors of the electric field and omit there the unfavorable factors. Then we have for the first dispersion equations

$$\begin{aligned} &\boldsymbol{n\mu n} = \varepsilon^o \mu^e \mu^o : \\ &\boldsymbol{e} = [\boldsymbol{n\varepsilon}, \boldsymbol{c}] = \varepsilon^o [\boldsymbol{n}, \boldsymbol{c}], \quad \boldsymbol{h} = \frac{\varepsilon^o}{\mu^e \mu^o} [\boldsymbol{n\mu}, [\boldsymbol{n}, \boldsymbol{c}]] = \frac{\varepsilon^o}{\mu^o} ((\boldsymbol{nc}) \boldsymbol{n} - \varepsilon^o \mu^o \boldsymbol{c}), \end{aligned} \tag{5.11}$$

and for the second dispersion equation

$$\begin{aligned} &\boldsymbol{n\varepsilon n} = \mu^o \varepsilon^e \varepsilon^o : \\ &\boldsymbol{e} = [\boldsymbol{n\varepsilon}, [\boldsymbol{n}, \boldsymbol{c}]] = \varepsilon^e ((\boldsymbol{nc}) \boldsymbol{n} - \mu^o \varepsilon^o \boldsymbol{c}), \quad \boldsymbol{h} = \frac{\varepsilon^e \varepsilon^o}{\mu^o} [\boldsymbol{n\mu}, \boldsymbol{c}] = \varepsilon^e \varepsilon^o [\boldsymbol{n}, \boldsymbol{c}]. \end{aligned} \tag{5.12}$$

It is not difficult to make normalization of the polarization vectors. We left the factors in such a way that to each dispersion Equations (5.11) and (5.12) separately the vector $\boldsymbol{h}$ follows from $\boldsymbol{e}$ without changing a factor. It is also easy to make a transition to normalized polarization vectors that we do not write down.

The transition to the special case of "only electrically" uniaxial media can be made by definition in (5.13) by the substitution

$$\boldsymbol{\varepsilon}(\omega) = \varepsilon^{e}(\omega)\boldsymbol{c}\cdot\boldsymbol{c} + \varepsilon^{o}(\omega)(\mathsf{I} - \boldsymbol{c}\cdot\boldsymbol{c}), \quad \boldsymbol{\mu}(\omega) = \mathsf{I}, \quad \rightarrow \quad \mu^{e}(\omega) = \mu^{o}(\omega) = 1. \tag{5.13}$$

The two dispersion equations become asymmetric to each other and are the first for ordinary and the second for extraordinary waves

$$\begin{aligned} 0 &= \boldsymbol{n}^2 - \varepsilon^{o} \quad \leftrightarrow \quad \frac{(\boldsymbol{nc})^2 + [\boldsymbol{n},\boldsymbol{c}]^2}{\varepsilon^{o}} = \frac{\boldsymbol{n}^2}{\varepsilon^{o}} = 1, \\ 0 &= \boldsymbol{n\varepsilon n} - \varepsilon^{e}\varepsilon^{o} \quad \leftrightarrow \quad \frac{(\boldsymbol{nc})^2}{\varepsilon^{o}} + \frac{[\boldsymbol{n},\boldsymbol{c}]^2}{\varepsilon^{e}} = 1. \end{aligned} \tag{5.14}$$

The (non-normalized) polarization vectors for the electric and magnetic field (5.11) and (5.12) become for ordinary waves

$$\begin{aligned} &\boldsymbol{n}^2 = \varepsilon^{o}: \\ &\boldsymbol{e} = [\boldsymbol{n\varepsilon},\boldsymbol{c}] = \varepsilon^{o}[\boldsymbol{n},\boldsymbol{c}], \quad \boldsymbol{h} = \varepsilon^{o}[\boldsymbol{n},[\boldsymbol{n},\boldsymbol{c}]] = \varepsilon^{o}((\boldsymbol{nc})\boldsymbol{n} - \varepsilon^{o}\boldsymbol{c}), \end{aligned} \tag{5.15}$$

and for extraordinary waves

$$\begin{aligned} &\boldsymbol{n\varepsilon n} = \varepsilon^{e}\varepsilon^{o}: \\ &\boldsymbol{e} = [\boldsymbol{n\varepsilon},[\boldsymbol{n},\boldsymbol{c}]] = \varepsilon^{e}((\boldsymbol{nc})\boldsymbol{n} - \varepsilon^{o}\boldsymbol{c}), \quad \boldsymbol{h} = \varepsilon^{e}\varepsilon^{o}[\boldsymbol{n},\boldsymbol{c}]. \end{aligned} \tag{5.16}$$

Thus (electrically) ordinary waves are polarized perpendicular to the axis plane spanned by the axis vector $\boldsymbol{c}$ and the refraction vector $\boldsymbol{n}$ and extraordinary waves within this plane. Amplitude relations for reflection and refraction at the boundary between an isotropic and a uniaxial medium can be found, e.g., in [9], in [11] (a little too complicated) and in [13].

6. Group Velocity and Diffraction Coefficients

In a preliminary summary about the two discussed concepts one can say that spatial dispersion is the more general concept but the concept of bi-anisotropy leads to interesting symmetries between the electric and magnetic properties of media and is up to now often the only concept represented in excellent monographs, e.g., [7]. However, in the last concept it is difficult to include some phenomena such as, for example, natural optical activity although this is tried to make in the book [10] of Fyodorov. The concept of bi-anisotropy is used in the whole work of the Minsk Group [9] [10] [11]. Practically, in all older works about classical optics the concept of bi-anisotropy is used but not under this name and this development is comprehensively represented in the encyclopedic article of Szivessy [8]. One cannot be sure that all terms of a same level of spatial dispersion in a certain order in the wave vectors are included in this more symmetric bi-anisotropic concept or are included in doubled way. In every case one has to calculate anew such quantities as the group velocity in comparison to the classical optics with only one frequency-dependent permittivity tensor $\boldsymbol{\varepsilon}(\omega)$ with and without taking into account frequency dispersion and this is mostly not easy.

We now consider the concept of spatial dispersion with the permittivity tensor $\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)$ in the wave Equation (2.7) with the operator $\mathsf{L}(\boldsymbol{k},\omega)$ given in (2.11) together with their invariants in (2.12). The dispersion equation that is the vanishing of the determinant of $\mathsf{L}(\boldsymbol{k},\omega)$ can be resolved in a function $\omega=\omega(\boldsymbol{k})$ with different possible branches and if we insert this function into the dispersion equation one obtains identities of the form

$$0=\left|\mathsf{L}(\boldsymbol{k},\omega)\right|, \quad \Rightarrow \quad \omega=\omega(\boldsymbol{k}), \tag{6.1}$$

where $\left|\mathsf{L}(\boldsymbol{k},\omega(\boldsymbol{k}))\right|$ depends only on the wave vector $\boldsymbol{k}$. We introduce two important notions and prepare its calculation for specialized cases. If we differentiate the identity (6.1) one and two times with respect to $\boldsymbol{k}$ according to (we abbreviate $\mathsf{L}\equiv\mathsf{L}(\boldsymbol{k},\omega(\boldsymbol{k}))$)

$$\begin{aligned} 0&=\frac{\partial|\mathsf{L}|}{\partial k_i}+\frac{\partial|\mathsf{L}|}{\partial\omega}\frac{\partial\omega}{\partial k_i},\\ 0&=\frac{\partial^2|\mathsf{L}|}{\partial k_i\partial k_j}+\frac{\partial^2|\mathsf{L}|}{\partial k_i\partial\omega}\frac{\partial\omega}{\partial k_j}+\frac{\partial^2|\mathsf{L}|}{\partial k_j\partial\omega}\frac{\partial\omega}{\partial k_i}+\frac{\partial^2|\mathsf{L}|}{\partial\omega^2}\frac{\partial^2\omega}{\partial k_i\partial k_j}. \end{aligned} \tag{6.2}$$

We define the group velocity

$$v_i\equiv\frac{\partial\omega}{\partial k_i}=-\frac{\dfrac{\partial|\mathsf{L}|}{\partial k_i}}{\dfrac{\partial|\mathsf{L}|}{\partial\omega}}, \tag{6.3}$$

and in addition the symmetric diffraction coefficients

$$\begin{aligned} W_{ij}&\equiv\frac{\partial^2\omega}{\partial k_i\partial k_j}=-\frac{\dfrac{\partial^2|\mathsf{L}|}{\partial k_i\partial k_j}+\dfrac{\partial^2|\mathsf{L}|}{\partial k_i\partial\omega}\dfrac{\partial\omega}{\partial k_j}+\dfrac{\partial^2|\mathsf{L}|}{\partial k_j\partial\omega}\dfrac{\partial\omega}{\partial k_i}}{\dfrac{\partial^2|\mathsf{L}|}{\partial\omega^2}}\\ &=-\frac{1}{\dfrac{\partial^2|\mathsf{L}|}{\partial\omega^2}}\left\{\frac{\partial^2|\mathsf{L}|}{\partial k_i\partial k_j}-\frac{\dfrac{\partial^2|\mathsf{L}|}{\partial k_i\partial\omega}\dfrac{\partial|\mathsf{L}|}{\partial k_j}+\dfrac{\partial^2|\mathsf{L}|}{\partial k_j\partial\omega}\dfrac{\partial|\mathsf{L}|}{\partial k_i}}{\dfrac{\partial|\mathsf{L}|}{\partial\omega}}\right\}=W_{ji}, \end{aligned} \tag{6.4}$$

where $W_{ij}=W_{ji}$ is a symmetric bilinear form. Both become important for the beam propagation in second-order approximation.

The formula for the group velocity in (10.1) using (2.18) can be written

$$v_i=-\frac{\left\langle\overline{\mathsf{L}}\dfrac{\partial\mathsf{L}}{\partial k_i}\right\rangle}{\left\langle\overline{\mathsf{L}}\dfrac{\partial\mathsf{L}}{\partial\omega}\right\rangle}=-\frac{\boldsymbol{e}^*\dfrac{\partial\mathsf{L}}{\partial k_i}\boldsymbol{e}}{\boldsymbol{e}^*\dfrac{\partial\mathsf{L}}{\partial\omega}\boldsymbol{e}}, \quad \overline{\mathsf{L}}\propto\boldsymbol{e}\cdot\boldsymbol{e}^*, \tag{6.5}$$

where is taken into account that $\overline{\mathsf{L}}$ is proportional to the dyadic product of (in general, non-normalized) polarization vectors $\boldsymbol{e}\cdot\boldsymbol{e}^*$ of the electric field (see (2.18); $\tilde{\boldsymbol{e}}=\boldsymbol{e}^*$ since we consider the lossless case). We find

$$\begin{aligned}\frac{\partial L_{kl}}{\partial k_i}&=\frac{c^2}{\omega^2}\left(k_k\delta_{il}+\delta_{ik}k_l-2k_i\delta_{kl}\right)+\frac{\partial \varepsilon_{kl}}{\partial k_i},\\ \frac{\partial L_{kl}}{\partial \omega}&=-2\frac{c^2}{\omega^3}\left(k_k k_l-\boldsymbol{k}^2\delta_{kl}\right)+\frac{\partial \varepsilon_{kl}}{\partial \omega},\end{aligned}\tag{6.6}$$

from which follows

$$\begin{aligned}\left\langle\overline{\mathsf{L}}\frac{\partial \mathsf{L}}{\partial k_i}\right\rangle&=\frac{c^2}{\omega^2}\left(\left(\overline{\mathsf{L}}\boldsymbol{k}\right)_i+\left(\boldsymbol{k}\overline{\mathsf{L}}\right)_i-2\left\langle\overline{\mathsf{L}}\right\rangle k_i\right)+\left\langle\overline{\mathsf{L}}\frac{\partial \boldsymbol{\varepsilon}}{\partial k_i}\right\rangle,\\ \left\langle\overline{\mathsf{L}}\frac{\partial \mathsf{L}}{\partial \omega}\right\rangle&=-2\frac{c^2}{\omega^3}\left(\boldsymbol{k}\overline{\mathsf{L}}\boldsymbol{k}-\left\langle\overline{\mathsf{L}}\right\rangle\boldsymbol{k}^2\right)+\left\langle\overline{\mathsf{L}}\frac{\partial \boldsymbol{\varepsilon}}{\partial \omega}\right\rangle,\quad \left\langle\overline{\mathsf{L}}\right\rangle=\left[\mathsf{L}\right]\end{aligned}\tag{6.7}$$

Neglect of spatial dispersion means that we do not take into account the term $\left\langle\overline{\mathsf{L}}\frac{\partial \boldsymbol{\varepsilon}}{\partial k_i}\right\rangle$ in the numerator and neglect of frequency dispersion the term $\left\langle\overline{\mathsf{L}}\frac{\partial \boldsymbol{\varepsilon}}{\partial \omega}\right\rangle$ in the denominator of the formula for the group velocity (6.5).

Under neglect of dispersion one finds from $\left|\mathsf{L}\left(\boldsymbol{k},\omega\right)\right|$ explicitly given in (2.12)

$$v_i=\omega\frac{\frac{c^2}{\omega^2}\left(\boldsymbol{k}^2\left(\left(\boldsymbol{\varepsilon}\boldsymbol{k}\right)_i+\left(\boldsymbol{k}\boldsymbol{\varepsilon}\right)_i\right)+2\left(\boldsymbol{k}\boldsymbol{\varepsilon}\boldsymbol{k}\right)k_i\right)-\left\langle\boldsymbol{\varepsilon}\right\rangle\left(\left(\boldsymbol{\varepsilon}\boldsymbol{k}\right)_i+\left(\boldsymbol{k}\boldsymbol{\varepsilon}\right)_i\right)+\left(\left(\boldsymbol{\varepsilon}^2\boldsymbol{k}\right)_i+\left(\boldsymbol{k}\boldsymbol{\varepsilon}^2\right)_i\right)}{2\left(2\frac{c^2}{\omega^2}\left(\boldsymbol{k}^2\left(\boldsymbol{k}\boldsymbol{\varepsilon}\boldsymbol{k}\right)-\left\langle\boldsymbol{\varepsilon}\right\rangle\boldsymbol{k}\boldsymbol{\varepsilon}\boldsymbol{k}+\boldsymbol{k}\boldsymbol{\varepsilon}^2\boldsymbol{k}\right)\right)}.\tag{6.8}$$

From this follows after scalar multiplication with k_i follows, e.g., [4] [9]

$$\boldsymbol{k}\boldsymbol{v}=\omega,\quad\Rightarrow\quad\frac{\boldsymbol{k}}{\omega}\frac{\partial\omega}{\partial\boldsymbol{k}}\equiv\boldsymbol{n}\boldsymbol{s}=1,\tag{6.9}$$

with definition of the refraction vector $\boldsymbol{n}$ and of the ray vector $\boldsymbol{s}$ by (notations as in [4])

$$\boldsymbol{n}\equiv\frac{c\boldsymbol{k}}{\omega},\quad\boldsymbol{s}\equiv\frac{\boldsymbol{v}}{c}.\tag{6.10}$$

One should not forget that the relation (6.9) is derived under neglect of the dispersion of the permittivity tensor $\boldsymbol{\varepsilon}\left(\boldsymbol{k},\omega\right)$ and the differences between ray vector and group velocity in regions near to resonance frequencies or zeros of the permittivity tensor can become very important and even the direction of the group velocity can be changed by this additional terms.

7. Electrically and Magnetically Isotropic Media and Group Velocity

The constitutive equations for bi-isotropic or electrically and magnetically isotropic media are

$$\boldsymbol{D}(\boldsymbol{k},\omega)=\varepsilon(\omega)\boldsymbol{E}(\boldsymbol{k},\omega),\quad \boldsymbol{B}(\boldsymbol{k},\omega)=\mu(\omega)\boldsymbol{H}(\boldsymbol{k},\omega), \tag{7.1}$$

with scalar functions $\varepsilon(\omega)$ and $\mu(\omega)$. The equation for the electric field (3.10) in the concept of bi-anisotropy with the specialized operator (3.11) after multiplication with $\mu(\omega)$ becomes

$$\boldsymbol{0}=\mu(\omega)\mathsf{L}^{E}(\boldsymbol{n})\boldsymbol{E}\left(\frac{\omega}{c}\boldsymbol{n},\omega\right)=\left\{\boldsymbol{n}\cdot\boldsymbol{n}-\boldsymbol{n}^2\mathsf{I}+\varepsilon(\omega)\mu(\omega)\mathsf{I}\right\}\boldsymbol{E}\left(\frac{\omega}{c}\boldsymbol{n},\omega\right) \tag{7.2}$$

or equivalently by transition to the more general concept of spatial dispersion with $\boldsymbol{\varepsilon}(\boldsymbol{k},\omega)=\varepsilon(\omega)\mu(\omega)\mathsf{I}$

$$\boldsymbol{0}=\mathsf{L}(\boldsymbol{k},\omega)\boldsymbol{E}(\boldsymbol{k},\omega)=\left\{\frac{c^2}{\omega^2}\left(\boldsymbol{k}\cdot\boldsymbol{k}-\boldsymbol{k}^2\mathsf{I}\right)+\varepsilon(\omega)\mu(\omega)\mathsf{I}\right\}\boldsymbol{E}(\boldsymbol{k},\omega). \tag{7.3}$$

The dispersion equation for transversal waves polarized for both $\boldsymbol{E}$ and $\boldsymbol{B}$ in the plane perpendicular to wave vector $\boldsymbol{k}$

$$\frac{c^2}{\omega^2}\boldsymbol{k}^2\equiv\boldsymbol{n}^2=\varepsilon(\omega)\mu(\omega),\quad \boldsymbol{k}\boldsymbol{E}(\boldsymbol{k},\omega)=0, \tag{7.4}$$

and for longitudinal waves

$$\varepsilon(\omega)\mu(\omega)=0,\quad \boldsymbol{E}(\omega)\neq\boldsymbol{0},\quad \boldsymbol{B}(\omega)=0. \tag{7.5}$$

The longitudinal waves correspond in present approximation to pure temporal oscillations of the electric field with arbitrary possible direction of polarization (since $\boldsymbol{k}=\boldsymbol{0}$). We are interested here merely in the transversal waves.

The dispersion Equation (7.4) for transversal waves can be resolved in the form $\omega=\omega(\boldsymbol{k})$ (6.1) with different branches for $\omega(\boldsymbol{k})$. In Sections 9 and 10 we will consider in detail an example where the dispersion Equation (7.4) can be explicitly resolved in the form (6.1) with different branches, the permittivity for polaritons. By differentiation of the dispersion equation in the form (6.1) with respect to the wave vector $\boldsymbol{k}$ one may derive a general formula for the group velocity $\boldsymbol{v}\equiv\frac{\partial\omega}{\partial\boldsymbol{k}}$ for bi-isotropic media and also for higher coefficients $\frac{\partial^2\omega}{\partial k_i\partial k_j}$ and so on which play a role in higher approximations of propagation of beam-like waves in such media (diffraction). For the group velocity $\boldsymbol{v}$ one finds the general formula ($\boldsymbol{k}^2=|\boldsymbol{k}|^2$)

$$\boldsymbol{v}\equiv\frac{\partial\omega}{\partial\boldsymbol{k}}=\frac{2c^2\boldsymbol{k}}{\frac{\partial}{\partial\omega}\left(\omega^2\varepsilon(\omega)\mu(\omega)\right)}=c\frac{c|\boldsymbol{k}|}{\omega\left(\varepsilon(\omega)\mu(\omega)+\frac{\omega}{2}\frac{\partial}{\partial\omega}\left(\varepsilon(\omega)\mu(\omega)\right)\right)}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}. \tag{7.6}$$

Without taking into account the frequency dispersion of the permittivities we find

$$\boldsymbol{v}'\equiv\frac{\partial\omega}{\partial\boldsymbol{k}}=\frac{2c^2\boldsymbol{k}}{\varepsilon(\omega)\mu(\omega)\frac{\partial}{\partial\omega}\left(\omega^2\right)}=c\frac{c|\boldsymbol{k}|}{\omega\varepsilon(\omega)\mu(\omega)}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}=\frac{c}{\sqrt{\varepsilon(\omega)\mu(\omega)}}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}. \tag{7.7}$$

One may introduce a dispersion factor $\alpha_{\text{disp}}(\omega)$ by

$$\alpha_{\mathrm{disp}}(\omega) \equiv \frac{\varepsilon(\omega)\mu(\omega)\frac{\partial}{\partial\omega}(\omega^2)}{\frac{\partial}{\partial\omega}(\omega^2\varepsilon(\omega)\mu(\omega))} = \frac{\varepsilon(\omega)\mu(\omega)}{\varepsilon(\omega)\mu(\omega)+\frac{\omega}{2}\frac{\partial}{\partial\omega}(\varepsilon(\omega)\mu(\omega))}. \quad (7.8)$$

From (7.7) follows using the dispersion Equation (7.4)

$$\frac{\boldsymbol{k}}{\omega}\frac{\partial\omega}{\partial\boldsymbol{k}} = \frac{1}{1+\frac{\omega}{2\varepsilon(\omega)\mu(\omega)}\frac{\partial}{\partial\omega}(\varepsilon(\omega)\mu(\omega))} = \frac{1}{\frac{\omega}{2}\frac{\partial}{\partial\omega}\left(\log\left(\omega^2\varepsilon(\omega)\mu(\omega)\right)\right)} \equiv \alpha_{\mathrm{disp}}(\omega), \quad (7.9)$$

where we have given also a representation of the dispersion coefficient $\alpha_{\mathrm{disp}}(\omega)$ by a logarithmic derivative (useful or not?). Under neglect of dispersion that means if we do not take into account the first derivative of $\varepsilon(\omega)\mu(\omega)$ with respect to frequency ω we have ($\boldsymbol{s}$ is called ray vector, e.g., [4], §97, or [9] $(\boldsymbol{n},\boldsymbol{s})\to(\boldsymbol{m},\boldsymbol{p})$)

$$\frac{\boldsymbol{k}}{\omega}\frac{\partial\omega}{\partial\boldsymbol{k}} \equiv \boldsymbol{n}\underbrace{\frac{\boldsymbol{v}}{c}}_{\equiv\boldsymbol{s}} = 1, \quad \Rightarrow \quad \alpha_{\mathrm{disp}}(\omega) = 1. \quad (7.10)$$

Therefore, the dispersion coefficient $\alpha_{\mathrm{disp}}(\omega)$ says by which factor one has to modify the group velocity in comparison to neglect of dispersion if one take it into account. It goes also into some other formulae as, for example, the energy of the wave solution. On this very general level of treatment we cannot say whether or not $\alpha_{\mathrm{disp}}(\omega)$ is in every case positive for possible real-valued functions $\varepsilon(\omega)\mu(\omega)$. In last case of negative $\alpha_{\mathrm{disp}}(\omega)$ the genuine group velocity and the ray vector would have even opposite directions. The introduction of ray vectors $\boldsymbol{s}$ in addition to the refraction vectors $\boldsymbol{n}$ is appropriate to formulate duality (or symmetry) relations between electric and magnetic quantities which leave invariant the basic equations of macroscopic optics [4], (§ 97) such as (for $\mu(\omega)=1$)

$$\boldsymbol{E}\leftrightarrow\boldsymbol{D}, \quad \varepsilon_{ij}\leftrightarrow\varepsilon_{ij}^{-1}, \quad \boldsymbol{n}\leftrightarrow\boldsymbol{s}, \quad (7.11)$$

but one should not forget that these are only approximate relations and are only true under neglect of dispersion of the permittivities.

A second coefficient which we will consider is the relation of the modulus $|\boldsymbol{v}|$ of the group velocity $\boldsymbol{v}$ to the light velocity c for which one derives from (7.7) the relation

$$\frac{\boldsymbol{v}}{c} = \frac{\sqrt{\varepsilon(\omega)\mu(\omega)}}{\varepsilon(\omega)\mu(\omega)+\frac{\omega}{2}\frac{\partial}{\partial\omega}(\varepsilon(\omega)\mu(\omega))}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} = \beta(\omega)\frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \quad (\omega>0), \quad (7.12)$$

with definition

$$\beta(\omega) \equiv \frac{\boldsymbol{k}\boldsymbol{v}}{|\boldsymbol{k}|c} = \frac{\sqrt{\varepsilon(\omega)\mu(\omega)}}{\varepsilon(\omega)\mu(\omega)+\frac{\omega}{2}\frac{\partial}{\partial\omega}\left(\varepsilon(\omega)\mu(\omega)\right)} = \frac{\alpha_{\mathrm{disp}}(\omega)}{\sqrt{\varepsilon(\omega)\mu(\omega)}}. \tag{7.13}$$

For negative or complex $\varepsilon(\omega)\mu(\omega)$ it becomes imaginary or complex and is then not to interpret in easy way.

Let us write down at this opportunity the general form of the second-order coefficients $\frac{\partial^2\omega}{\partial k_i \partial k_j}$ for bi-isotropic media which are

$$\begin{aligned}\frac{\partial^2\omega}{\partial k_i \partial k_j} &= \frac{2c^2\delta_{ij} - \frac{\partial^2}{\partial\omega^2}\left(\omega^2\varepsilon(\omega)\mu(\omega)\right)v_i v_j}{\frac{\partial}{\partial\omega}\left(\omega^2\varepsilon(\omega)\mu(\omega)\right)} \\ &= \frac{2c^2}{\frac{\partial}{\partial\omega}\left(\omega^2\varepsilon(\omega)\mu(\omega)\right)} \\ &\quad \cdot\left\{\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2} + \left(1 - \frac{2\omega^2\varepsilon(\omega)\mu(\omega)\frac{\partial^2}{\partial\omega^2}\left(\omega^2\varepsilon(\omega)\mu(\omega)\right)}{\left(\frac{\partial}{\partial\omega}\left(\omega^2\varepsilon(\omega)\mu(\omega)\right)\right)^2}\right)\frac{k_i k_j}{|\boldsymbol{k}|^2}\right\}.\end{aligned} \tag{7.14}$$

As was to expect they are a linear combination of $\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}$ and $\frac{k_i k_j}{|\boldsymbol{k}|^2}$ the only second-rank symmetric tensors which can be built from vectors $\boldsymbol{k}$ alone and which are covariant under transformations of the rotation group $SO(3)$. The group velocity and the diffraction coefficients are involved in the expansion of the equation for the slowly varying amplitudes of beams with respect to spatial and temporal derivatives. We will shortly consider the corresponding equations in next Section.

8. Approximate Beam Equations for Homogeneous Isotropic Media

We consider an isotropic medium with permittivity $\varepsilon(\omega)$ and for simplicity with $\mu(\omega)=1$. The wave equation for the electric field with the involved operator $\mathsf{L}(\boldsymbol{k},\omega)$ in such a medium is

$$\mathbf{0} = \mathsf{L}\left(-\mathrm{i}\nabla, \mathrm{i}\frac{\partial}{\partial t}\right)\boldsymbol{E}(\boldsymbol{r},t), \quad \mathsf{L}(\boldsymbol{k},\omega) \equiv \frac{c^2}{\omega^2}\left(\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{k}^2\mathsf{I}\right) + \varepsilon(\omega)\mathsf{I}. \tag{8.1}$$

The necessary equation for solutions is the vanishing of the determinant

$$\mathbf{0} = \left|\mathsf{L}\left(-\mathrm{i}\nabla, \mathrm{i}\frac{\partial}{\partial t}\right)\right|\boldsymbol{E}(\boldsymbol{r},t). \tag{8.2}$$

We make now the proposition of slowly varying amplitudes $\boldsymbol{E}_0(\boldsymbol{r},t)$

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0(\boldsymbol{r},t)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t)} + \boldsymbol{E}_0^*(\boldsymbol{r},t)\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t)}. \tag{8.3}$$

Inserting this into (8.2) we find first

$$\begin{aligned}\mathbf{0} &= \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t)}\mathsf{L}\left(\boldsymbol{k}_0 - \mathrm{i}\nabla, \omega_0 + \mathrm{i}\frac{\partial}{\partial t}\right)\boldsymbol{E}_0(\boldsymbol{r},t) \\ &\quad + \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t)}\mathsf{L}\left(-\boldsymbol{k}_0 - \mathrm{i}\nabla, \omega_0 - \mathrm{i}\frac{\partial}{\partial t}\right)\boldsymbol{E}_0^*(\boldsymbol{r},t),\end{aligned} \tag{8.4}$$

We suppose that both sum parts are separated in a way that we can set them equal to zero independently. For example, we may think that we include into one part all frequencies $0 < \omega < +\infty$ and into the other part all frequencies $-\infty < \omega < 0$. With this assumption we have the equation

$$\begin{aligned}\mathbf{0} &= \mathsf{L}\left(\boldsymbol{k}_0 - \mathrm{i}\nabla, \omega_0 + \mathrm{i}\frac{\partial}{\partial t}\right)\boldsymbol{E}_0(\boldsymbol{r},t) \\ &= \left\{\mathsf{L}_0 - \mathrm{i}\left(\left(\frac{\partial \mathsf{L}}{\partial \boldsymbol{k}}\right)_0 \boldsymbol{\nabla} - \left(\frac{\partial \mathsf{L}}{\partial \omega}\right)_0 \frac{\partial}{\partial t}\right) - \frac{1}{2}\left(\left(\frac{\partial^2 \mathsf{L}}{\partial k_i \partial k_j}\right)_0 \nabla_i \nabla_j \right.\right. \\ &\quad \left.\left. - 2\left(\frac{\partial^2 \mathsf{L}}{\partial k_i \partial \omega}\right)_0 \nabla_i \frac{\partial}{\partial t} + \left(\frac{\partial^2 \mathsf{L}}{\partial \omega^2}\right)_0 \frac{\partial^2}{\partial t^2}\right) + \cdots\right\}\boldsymbol{E}_0(\boldsymbol{r},t),\end{aligned} \tag{8.5}$$

where we wrote the first terms in an expansion in powers of the differential operators. From this equation follows as necessary condition for all components of the solutions $\boldsymbol{E}_0(\boldsymbol{r},t)$ the vanishing of the determinant and with the analogous expansion as in (8.5)

$$\begin{aligned}\mathbf{0} &= \left|\mathsf{L}\left(\boldsymbol{k}_0 - \mathrm{i}\nabla, \omega_0 + \mathrm{i}\frac{\partial}{\partial t}\right)\right|\boldsymbol{E}_0(\boldsymbol{r},t) \\ &= \left\{|\mathsf{L}_0| - \mathrm{i}\left(\left(\frac{\partial |\mathsf{L}|}{\partial k_i}\right)_0 \nabla_i - \left(\frac{\partial |\mathsf{L}|}{\partial \omega}\right)_0 \frac{\partial}{\partial t}\right) - \frac{1}{2}\left(\left(\frac{\partial^2 |\mathsf{L}|}{\partial k_i \partial k_j}\right)_0 \nabla_i \nabla_j \right.\right. \\ &\quad \left.\left. - 2\left(\frac{\partial^2 |\mathsf{L}|}{\partial k_i \partial \omega}\right)_0 \nabla_i \frac{\partial}{\partial t} + \left(\frac{\partial^2 |\mathsf{L}|}{\partial \omega^2}\right)_0 \frac{\partial^2}{\partial t^2}\right) + \cdots\right\}\boldsymbol{E}_0(\boldsymbol{r},t),\end{aligned} \tag{8.6}$$

plus the corresponding complex conjugate equation. With the general formula for the differentiation of the determinant $\frac{\partial}{\partial \lambda}|\mathsf{A}| = \left\langle \overline{\mathsf{A}} \frac{\partial}{\partial \lambda} \mathsf{A} \right\rangle$ of an operator A with respect to a parameter λ this equation may be also written (we do not insert a more complicated formula for the second derivative of a determinant with respect to two parameters which also exists)

$$\begin{aligned}\mathbf{0} &= \left\{-\mathrm{i}\left(\left\langle \overline{\mathsf{L}}_0 \left(\frac{\partial \mathsf{L}}{\partial k_i}\right)_0 \right\rangle \nabla_i - \left\langle \overline{\mathsf{L}}_0 \left(\frac{\partial \mathsf{L}}{\partial \omega}\right)_0 \right\rangle \frac{\partial}{\partial t}\right) - \frac{1}{2}\left(\left(\frac{\partial^2 |\mathsf{L}|}{\partial k_i \partial k_j}\right)_0 \nabla_i \nabla_j \right.\right. \\ &\quad \left.\left. - 2\left(\frac{\partial^2 |\mathsf{L}|}{\partial k_i \partial \omega}\right)_0 \nabla_i \frac{\partial}{\partial t} + \left(\frac{\partial^2 |\mathsf{L}|}{\partial \omega^2}\right)_0 \frac{\partial^2}{\partial t^2}\right) + \cdots\right\}\boldsymbol{E}_0(\boldsymbol{r},t).\end{aligned} \tag{8.7}$$

with

$$|\mathsf{L}_0| \equiv |\mathsf{L}(\boldsymbol{k}_0, \omega_0)| = 0, \tag{8.8}$$

and where index 0 means that the derivatives are to take at $(\boldsymbol{k},\omega) = (\boldsymbol{k}_0,\omega_0)$. By division of this equation with $\left\langle\left(\bar{\mathsf{L}}_0 \frac{\partial \mathsf{L}}{\partial \omega}\right)_0\right\rangle$ one finds

$$
\begin{aligned}
0 = &\left\{ \mathrm{i}\left(\frac{\partial}{\partial t} - \frac{\left\langle \bar{\mathsf{L}}_0 \left(\frac{\partial \mathsf{L}}{\partial k_i} \right)_0 \right\rangle}{\left\langle \bar{\mathsf{L}}_0 \left(\frac{\partial \mathsf{L}}{\partial \omega} \right)_0 \right\rangle} \nabla_i \right) \right. \\
&\left. - \frac{1}{2} \frac{\left(\frac{\partial^2 |\mathsf{L}|}{\partial k_i \partial k_j} \right)_0 \nabla_i \nabla_j - 2 \left(\frac{\partial^2 |\mathsf{L}|}{\partial k_i \partial \omega} \right)_0 \nabla_i \frac{\partial}{\partial t} + \left(\frac{\partial^2 |\mathsf{L}|}{\partial \omega^2} \right)_0 \frac{\partial^2}{\partial t^2}}{\left\langle \left(\bar{\mathsf{L}}_0 \frac{\partial \mathsf{L}}{\partial \omega} \right)_0 \right\rangle} + \cdots \right\} \boldsymbol{E}_0(\boldsymbol{r}, t).
\end{aligned} \tag{8.9}
$$

The dispersion equation $|\mathsf{L}(\boldsymbol{k}, \omega)| = 0$ can be resolved in the form $\omega = \omega(\boldsymbol{k})$ for the different branches of the solution. In application to the slowly varying amplitudes with average wave vector $\boldsymbol{k}_0$ and frequency $\omega = \omega_0$ this means the resolution

$$
\begin{aligned}
\mathbf{0} &= \left\{ \omega_0 + \mathrm{i} \frac{\partial}{\partial t} - \omega(\boldsymbol{k}_0 - \mathrm{i}\nabla) \right\} \boldsymbol{E}_0(\boldsymbol{r}, t) \\
&= \left\{ \mathrm{i}\left(\frac{\partial}{\partial t} + \boldsymbol{v}_0 \nabla \right) + \frac{1}{2} \nabla \mathsf{W}_0 \nabla + \cdots \right\} \boldsymbol{E}_0(\boldsymbol{r}, t), \quad (\omega_0 = \omega_0(\boldsymbol{k}_0)),
\end{aligned} \tag{8.10}
$$

with the group velocity $\boldsymbol{v}_0$ and the quadratic form W_0 defined by

$$
\boldsymbol{v}_0 \equiv \left(\frac{\partial \omega}{\partial \boldsymbol{k}} \right)_0, \quad \mathsf{W}_0 \equiv \left(\frac{\partial^2 \omega}{\partial \boldsymbol{k} \cdot \partial \boldsymbol{k}} \right)_0, \quad \left(\text{or} \quad W_{0,ij} \equiv \left(\frac{\partial^2 \omega}{\partial k_i \partial k_j} \right)_0 \right). \tag{8.11}
$$

Obviously (8.9) and (8.10) are identical and one may find the correspondences.

We define the polarization vectors $\boldsymbol{e}_0$ and $\boldsymbol{e}_0^*$ which are right-hand and left-hand eigen-vectors to the operator L_0 to eigenvalue 0 according to

$$
0 = \mathsf{L}_0 \boldsymbol{e}_0, \quad 0 = \boldsymbol{e}_0^* \mathsf{L}_0, \quad \boldsymbol{k}_0 \boldsymbol{e}_0 = 0, \quad \boldsymbol{e}_0^* \boldsymbol{k}_0 = 0. \tag{8.12}
$$

By comparison of (8.9) with (8.10) we find for the group velocity v_0

$$
v_{0,i} \equiv \left(\frac{\partial \omega}{\partial k_i} \right)_0 = -\frac{\left(\frac{\partial |\mathsf{L}|}{\partial k_i} \right)_0}{\left(\frac{\partial |\mathsf{L}|}{\partial \omega} \right)_0} = -\frac{\left\langle \bar{\mathsf{L}}_0 \left(\frac{\partial \mathsf{L}}{\partial k_i} \right)_0 \right\rangle}{\left\langle \bar{\mathsf{L}}_0 \left(\frac{\partial \mathsf{L}}{\partial \omega} \right)_0 \right\rangle} = -\frac{\boldsymbol{e}_0^* \left(\frac{\partial \mathsf{L}}{\partial k_i} \right)_0 \boldsymbol{e}_0}{\boldsymbol{e}_0^* \left(\frac{\partial \mathsf{L}}{\partial \omega} \right)_0 \boldsymbol{e}_0}, \tag{8.13}
$$

where we applied $\mathsf{L}_0 \propto \boldsymbol{e}_0 \cdot \boldsymbol{e}_0^*$ meaning that L_0 is proportional to the dyadic product of the polarization vectors $\boldsymbol{e}_0$ and $\boldsymbol{e}_0^*$ (see also formulae (2.18) and (2.19)).

The beam solutions in their Fourier decomposition contain components to wave vectors and frequencies around the average wave vectors and frequency $(\boldsymbol{k}_0, \omega_0)$ and therefore the solution cannot possess solution which are exactly proportional to polarization vector $\boldsymbol{e}_0$. Therefore we make now the following proposition for solutions of the beam Equation (8.5)

$$\boldsymbol{E}_0(\boldsymbol{r},t)=\boldsymbol{e}_0A_0(\boldsymbol{r},t)+\left[\boldsymbol{e}_0,\boldsymbol{A}'(\boldsymbol{r},t)\right], \tag{8.14}$$

where $\boldsymbol{e}_0A_0(\boldsymbol{r},t)$ is the main part with polarization $\boldsymbol{e}_0$ and $\left[\boldsymbol{e}_0,\boldsymbol{A}'(\boldsymbol{r},t)\right]$ a small additional part with polarization perpendicular to $\boldsymbol{e}_0$. Both parts have to satisfy Equation (8.5) that means for the main part the following approximate scalar equation up to second-order derivatives of the slowly varying amplitude $A_0(\boldsymbol{r},t)$

$$0=\left\{\mathrm{i}\left(\frac{\partial}{\partial t}+\boldsymbol{v}_0\nabla\right)+\frac{1}{2}\nabla\mathsf{W}_0\nabla\right\}A_0(\boldsymbol{r},t),\quad\left(\nabla\mathsf{W}_0\nabla\equiv\nabla_iW_{0,ij}\nabla_j\right). \tag{8.15}$$

The additional part $\left[\boldsymbol{e}_0,\boldsymbol{A}'(\boldsymbol{r},t)\right]$ of the beam solution is not independent of the main part $\boldsymbol{e}_0A_0(\boldsymbol{r},t)$. Inserting both parts into Equation (8.5) we get approximately using $\mathsf{L}_0\boldsymbol{e}_0=\boldsymbol{0}$

$$\begin{aligned}\boldsymbol{0}&=\left\{\mathsf{L}_0-\mathrm{i}\left(\left(\frac{\partial\mathsf{L}}{\partial\boldsymbol{k}}\right)_0\nabla-\left(\frac{\partial\mathsf{L}}{\partial\omega}\right)_0\frac{\partial}{\partial t}\right)+\cdots\right\}\left(\boldsymbol{e}_0A_0(\boldsymbol{r},t)+\left[\boldsymbol{e}_0,\boldsymbol{A}_0'(\boldsymbol{r},t)\right]\right)\\&=\left\{\mathsf{L}_0\left[\boldsymbol{e}_0,\boldsymbol{A}_0'(\boldsymbol{r},t)\right]-\mathrm{i}\left(\left(\frac{\partial\mathsf{L}}{\partial\boldsymbol{k}}\right)_0\nabla-\left(\frac{\partial\mathsf{L}}{\partial\omega}\right)_0\frac{\partial}{\partial t}\right)\boldsymbol{e}_0A_0(\boldsymbol{r},t)+\cdots\right\},\end{aligned} \tag{8.16}$$

that has to be resolved to $\left[\boldsymbol{e}_0,\boldsymbol{A}_0'(\boldsymbol{r},t)\right]$. As approximation we use only the two explicitly written sum terms in the second line. First we find using the dispersion equation $\frac{c^2}{\omega_0^2}\boldsymbol{k}_0^2=\varepsilon_0$

$$\begin{aligned}\mathsf{L}_0\left[\boldsymbol{e}_0,\boldsymbol{A}_0'(\boldsymbol{r},t)\right]&=\left(\frac{c^2}{\omega_0^2}\left(\boldsymbol{k}_0\cdot\boldsymbol{k}_0-\boldsymbol{k}_0^2\mathsf{I}\right)+\varepsilon_0\mathsf{I}\right)\left[\boldsymbol{e}_0,\boldsymbol{A}_0'(\boldsymbol{r},t)\right]\\&=\frac{c^2}{\omega_0^2}\left[\boldsymbol{k}_0,\boldsymbol{e}_0,\boldsymbol{A}_0'(\boldsymbol{r},t)\right]\cdot\boldsymbol{k}_0=\varepsilon_0\frac{\left[\boldsymbol{k}_0,\boldsymbol{e}_0,\boldsymbol{A}_0'(\boldsymbol{r},t)\right]\cdot\boldsymbol{k}_0}{\boldsymbol{k}_0^2},\end{aligned} \tag{8.17}$$

that is proportional to the average wave vector $\boldsymbol{k}_0$. Furthermore follows for the operator part of the second sum term in (8.16) which acts onto $A_0(\boldsymbol{r},t)$ and here written with indices

$$\begin{aligned}&\left\{\left(\frac{\partial L_{ij}}{\partial k_k}\right)_0\nabla_k-\left(\frac{\partial L_{ij}}{\partial\omega}\right)_0\frac{\partial}{\partial t}\right\}e_{0,j}\\&=\left\{\frac{c^2}{\omega_0^2}\left(\delta_{ik}k_{0,j}+k_{0,i}\delta_{jk}-2k_{0,k}\delta_{ij}\right)\nabla_k+2\frac{c^3}{\omega_0^3}\left(k_{0,i}k_{0,j}-\boldsymbol{k}_0^2\delta_{ij}\right)\frac{\partial}{\partial t}\right\}e_{0,j}\\&=\frac{c^2}{\omega_0^2}\left\{k_{0,i}\boldsymbol{e}_0\nabla-2e_{0,i}\left(\boldsymbol{k}_0\nabla+\frac{c}{\omega_0}\frac{\partial}{\partial t}\right)\right\}.\end{aligned} \tag{8.18}$$

This possesses two sum terms proportional to the vector $\boldsymbol{k}_0$ and to the main polarization $\boldsymbol{e}_0$ and shows a typical difficulty consisting in the correct neglect of terms deriving equation for additional components with no contradictions. The second sum term is proportional to the polarization $\boldsymbol{e}_0$ of the main component and has to be neglected. If we do so we find from (8.17) and (8.18) the following formula for the additional component in direction of $\boldsymbol{k}_0$

$$\left[\boldsymbol{k}_0,\boldsymbol{e}_0,\boldsymbol{A}_0'(\boldsymbol{r},t)\right]=\mathrm{i}\left(\boldsymbol{e}_0\nabla\right)A_0(\boldsymbol{r},t). \tag{8.19}$$

In special case of vacuum we have $\boldsymbol{v}_0 = c\frac{\boldsymbol{k}_0}{|\boldsymbol{k}_0|}$ and $\mathrm{W}_0 = \frac{c}{|\boldsymbol{k}_0|}\left(\mathrm{I} - \frac{\boldsymbol{k}_0 \cdot \boldsymbol{k}_0}{|\boldsymbol{k}_0|^2}\right)$ and Equation (8.15) for the main component becomes

$$0 = \left\{ \mathrm{i}\left(\frac{\partial}{\partial t} + c\frac{\boldsymbol{k}_0}{|\boldsymbol{k}_0|}\nabla\right) + \frac{c}{2|\boldsymbol{k}_0|}\nabla\left(\mathrm{I} - \frac{\boldsymbol{k}_0 \cdot \boldsymbol{k}_0}{|\boldsymbol{k}_0|^2}\right)\nabla \right\} \boldsymbol{e}_0 A_0(\boldsymbol{r}, t). \tag{8.20}$$

In Section 9 we derive a more complicated case for media with the polariton permittivity.

Thus the approximate equations for beam solutions taking into account diffraction in first order consists of the Equation (8.15) for the main part of the slowly varying amplitude plus the Equation (8.19) for a "small" additional part in direction of $\boldsymbol{k}_0$ which can be determined alone from the main part by differentiations. We wanted to show how the group velocity and the diffraction coefficients are involved in approximate beam equations but a detailed consideration of these equations and of the solution of (8.15) requires much place and is here not intended.

9. Permittivity to Polariton Dispersion in Isotropic Media

We consider in this Section the following special permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$ of an isotropic medium with two real parameters ω_l and ω_t (or λ and ω_t) called polariton permittivity, e.g., [6] (§17, 18)

$$\begin{aligned} \varepsilon(\omega) &= 1 - \frac{\lambda}{\omega^2 - \omega_t^2} = 1 - \frac{\lambda}{2\omega_t}\left(\frac{1}{\omega - \omega_t} - \frac{1}{\omega + \omega_t}\right) = 1 - \frac{\omega_l^2 - \omega_t^2}{\omega^2 - \omega_t^2} \\ &= \frac{\omega^2 - \omega_l^2}{\omega^2 - \omega_t^2}, \quad (\mu(\omega) = 1), \quad \lambda \equiv \omega_l^2 - \omega_t^2, \end{aligned} \tag{9.1}$$

where for $\omega \approx (\geq)\omega_t$ the second sum term in round brackets can be neglected. In **Figure 1** this permittivity is illustrated for the two principal cases with different properties which we call the passive case $\omega_l \geq \omega_t$ and the active case $\omega_t > \omega_l$ (occupation inversion) and which are also characterized by (for $\omega > 0$)

$$\begin{aligned} \lambda > 0, \quad \omega_l > \omega_t : \quad \frac{\partial \varepsilon}{\partial \omega}(\omega) > 0, \quad (\text{passive case}), \\ \lambda < 0, \quad \omega_l < \omega_t : \quad \frac{\partial \varepsilon}{\partial \omega}(\omega) < 0, \quad (\text{active case}). \end{aligned} \tag{9.2}$$

The indices "l" and "t" in (9.1) mean "longitudinal" and "transversal". Polaritons (or real excitons) are a mixing of excitons in a medium and of photons in the vacuum (e.g., [1] [2] [17]) and correspond to the possible real light excitations in a medium. In this simple model ω_t is the frequency to a transition between two energy levels in the medium or a lattice oscillation and $\lambda = \omega_l^2 - \omega_t^2$ is beside other parameters proportional to the difference of the occupation of the two involved levels. For models of the medium in thermal equilibrium with temperature $T \neq 0$ the permittivity $\varepsilon(\omega)$ has to be generalized (see general form of the permittivity of an isotropic medium, e.g., in [4] chap XII, [3]). With

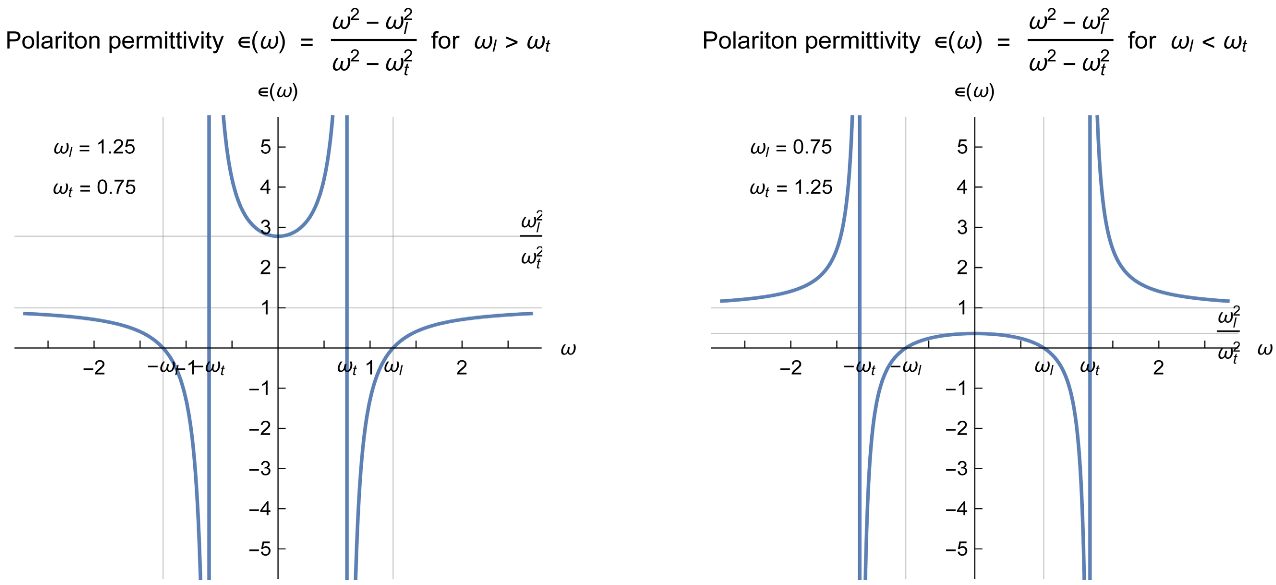

Figure 1. Polariton permittivity $\varepsilon(\omega)$ for $\omega_l > \omega_t$ (passive case) and for $\omega_l < \omega_t$ (active case). Apart from the jump from plus infinity to minus infinity (or vice versa) all derivatives of $\varepsilon(\omega)$ with respect to frequency are positive $\frac{\partial \varepsilon}{\partial \omega}(\omega) \ge 0$ in the passive case and negative in the active case $\frac{\partial \varepsilon}{\partial \omega}(\omega) \le 0$ (for $\omega > 0$ right half-plane.

respect to the propagation of light beams in such a medium it is equivalent to a medium which possesses the right-hand form for the product $\varepsilon(\omega)\mu(\omega)$ instead for $\varepsilon(\omega)$ alone and can be included into the last case.

One may think that the distinction in passive and active case in (9.2) corresponds in certain way to the usual distinction in normal and anomalous dispersion but beside analogies there are also essential differences. Normal dispersion is usually discussed for the passive case alone and appears for real frequencies if one adds in the denominators for $\varepsilon(\omega)$ in (9.1) an imaginary part to take into account losses in the medium and if we consider then the real part of arising permittivity and is present in (small) parts between the (main) regions of normal dispersion. In the model (9.1) these regions are reduced to the points $\omega^2 = \omega_t^2$ and normal dispersion is present in the whole region with exclusion of these points. In contrast, in the active case we have in the whole region "anomalous" dispersion also with exclusion of the points $\omega^2 = \omega_t^2$ only and imaginary parts in the denominator do not play a role. This picture can change in some way for thermal equilibrium but the distinction in (9.2) is meant without losses. The occupation inversion in active case corresponds in some sense to a negative absolute temperature (notion occasionally used in second half of last century) but a thermal equilibrium in this case is only possible for a finite number of energy levels.

For $\omega_l \equiv \omega_p$ and $\omega_t = 0$ we have the permittivity of a cold isotropic plasma

$$\omega_l \equiv \omega_p, \quad \omega_t = 0: \quad \Rightarrow \quad \varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}, \quad \omega_p \equiv \sqrt{\frac{4\pi n_e e^2}{m_e}}, \tag{9.3}$$

with ω_p the plasma frequency given here for an electron plasma (indices "e"; e, n_e, m_e mean electron charge, electron density, electron mass and electron charge). This is again idealized for temperature $T = 0$.

Longitudinal waves are in the idealized form (9.1) of the permittivity only possible for the frequency ω_l

$$\omega = \omega_l, \tag{9.4}$$

and are pure oscillations with no dispersion. Our main interest concern transversal waves which we now consider. The dispersion relation for transversal waves (7.4) specialized for the polariton case (9.1) becomes

$$\boldsymbol{k}^2 = \frac{\omega^2}{c^2}\varepsilon(\omega)\mu(\omega) = \frac{\omega^2}{c^2}\frac{\omega^2 - \omega_l^2}{\omega^2 - \omega_t^2}, \tag{9.5}$$

and depends only on the product $\varepsilon(\omega)\mu(\omega)$. For such waves the relation between the energy flow density $\boldsymbol{S}$ and the energy density w in the lossless case for quasi-plane and quasi-monochromatic waves and in the transition to the limiting case of plane monochromatic waves with real wave vector $\boldsymbol{k}$ and real frequency ω (homogeneous waves)

$$\boldsymbol{S} = \boldsymbol{v}w, \tag{9.6}$$

remains in every case the same and depends only from the dispersion relation ($\boldsymbol{v}$ is group velocity; see next Section). However, the splitting of the energy density w in (9.6) into a part from the electric field and into a part from the magnetic field depends on $\varepsilon(\omega)$ and $\mu(\omega)$ separately and therefore also the calculation of the corresponding energy flow density $\boldsymbol{S}$ which can be made from the energy density w by (9.6). As illustrated in **Figure 1** it is interesting to extend the permittivity (9.1) to $\lambda = \omega_l^2 - \omega_t^2 < 0$ which corresponds to a model medium with inverse occupation density of the two involved levels. The condition $\omega_l > \omega_t$ in the form of the permittivity (9.1) which belongs to the passive case is satisfied for taking into account only one transition with frequency ω_t between two energy levels and for sufficiently low temperatures. It may be converted into $\omega_l < \omega_t$ for pumping to a higher energy level of a laser medium to get inversion of occupation but to keep their difference $\omega_t^2 - \omega_l^2$ constant can be only a very rough approximation for the laser action near the threshold. Such a permittivity falls under the active case (not to be confused with notion (natural) optical activity!). By far, not all consequences for the active case are clear and are well understood.

Let us begin with a general consideration to dispersion equations in a homogeneous isotropic and infinitely extended medium. The general case is that both wave vector $\boldsymbol{k}$ and frequency ω are complex quantities. For the vacuum with the dispersion equation $c^2\boldsymbol{k}^2 = \omega^2$ with the splitting of wave vector and frequency in real and imaginary parts

$$\boldsymbol{k} = \boldsymbol{k}' + \mathrm{i}\boldsymbol{k}'', \quad \omega = \omega' + \mathrm{i}\omega'', \tag{9.7}$$

this leads to a complex equation with the following separation into a real and imaginary part

$$
\begin{aligned}
&c^2\left(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}''\right)^2=\left(\omega'+\mathrm{i}\omega''\right)^2, \quad \Rightarrow \\
&c^2\left(\boldsymbol{k}'^2-\boldsymbol{k}''^2\right)=\omega'^2-\omega''^2, \quad c^2\left(\boldsymbol{k}'\boldsymbol{k}''\right)=\omega'\omega''.
\end{aligned} \tag{9.8}
$$

These are 2 scalar equations for 5 real variables (for example, $|\boldsymbol{k}'|,|\boldsymbol{k}''|,\cos(\varphi)\equiv\frac{\boldsymbol{k}'\boldsymbol{k}''}{|\boldsymbol{k}'||\boldsymbol{k}''|},\omega',\omega''$) that restricts the number of free variables to 3 real variables. It is impossible to represent this in a single graphical representation and one has to make compromises. For example, for real frequency ($\omega'=\omega,\omega''=0$) we find from (9.8) the orthogonality $\boldsymbol{k}'\boldsymbol{k}''=0$ of real to imaginary part of the wave vector $\boldsymbol{k}$. It is known that such waves are generated in the vacuum under total reflection within an isotropic medium with $\boldsymbol{k}'$ parallel to the boundary plane and with $\boldsymbol{k}''$ in direction of the normal vector to the boundary plane corresponding to exponential decrease. Such waves are called inhomogeneous waves (not to confuse with inhomogeneous media!). All this is well known and understood. Which of the components $\left(\boldsymbol{k}',\boldsymbol{k}'',\omega',\omega''\right)$ are involved into a process can be only determined if one knows the boundary together with the boundary conditions. If we apply this to the polariton permittivity (9.1) with the complex dispersion Equation (9.5) we find the following equation

$$
c^2\left(\boldsymbol{k}'^2-\boldsymbol{k}''^2+\mathrm{i}2\boldsymbol{k}'\boldsymbol{k}''\right)=\left(\omega'^2-\omega''^2+\mathrm{i}2\omega'\omega''\right)\frac{\omega'^2-\omega''^2-\omega_l^2+\mathrm{i}2\omega'\omega''}{\omega'^2-\omega''^2-\omega_t^2+\mathrm{i}2\omega'\omega''}. \tag{9.9}
$$

Separated into real and imaginary part this leads to the two equations

$$
\begin{aligned}
0&=\left(\omega'^2-\omega''^2-\omega_t^2\right)\left(\omega'^2-\omega''^2-c^2\left(\boldsymbol{k}'^2-\boldsymbol{k}''^2\right)\right)\\
&\quad-4\omega'\omega''\left(\omega'\omega''-c^2\left(\boldsymbol{k}'\boldsymbol{k}''\right)\right)-\left(\omega_l^2-\omega_t^2\right)\left(\omega'^2-\omega''^2\right),\\
0&=\left(\omega'^2-\omega''^2-\omega_t^2\right)\left(\omega'\omega''-c^2\left(\boldsymbol{k}'\boldsymbol{k}''\right)\right)\\
&\quad+\omega'\omega''\left(\omega'^2-\omega''^2-c^2\left(\boldsymbol{k}'^2-\boldsymbol{k}''^2\right)\right)-\left(\omega_l^2-\omega_t^2\right)\omega'\omega'',
\end{aligned} \tag{9.10}
$$

which are of forth degree with respect to the real components of wave vector and frequency. Both equations have to be satisfied at the same time. This means that we have different possibilities of two-dimensional graphical representations for inhomogeneous waves and this is relatively complicated in such generality.

If we choose real wave vectors as free variable then the dispersion Equation (9.5) leads to a bi-quadratic equation for the frequency $\omega=\omega(\boldsymbol{k})$ in dependence on the modulus $|\boldsymbol{k}|$ of the wave vector as follows ($\boldsymbol{k}^2\equiv|\boldsymbol{k}|^2$)

$$
0=\omega^4-\left(\omega_l^2+c^2\boldsymbol{k}^2\right)\omega^2+\omega_t^2c^2\boldsymbol{k}^2, \tag{9.11}
$$

which resolved provides two branches of squared solutions

$$
\begin{aligned}
\omega_\pm^2(\boldsymbol{k})&=\frac{1}{2}\left\{\omega_l^2+c^2\boldsymbol{k}^2\pm\sqrt{\left(\omega_l^2+c^2\boldsymbol{k}^2\right)^2-4\omega_t^2c^2\boldsymbol{k}^2}\right\}\\
&=\frac{1}{2}\left\{\omega_l^2+c^2\boldsymbol{k}^2\pm\sqrt{\omega_l^4+2\left(\omega_l^2-2\omega_t^2\right)c^2\boldsymbol{k}^2+\left(c^2\boldsymbol{k}^2\right)^2}\right\},
\end{aligned} \tag{9.12}
$$

or of frequency solutions[7]

$$\begin{aligned}\omega_{\pm}^{(\pm)}(\boldsymbol{k}) &= (\pm)\frac{1}{2}\left\{\sqrt{\omega_l^2+2\omega_t c|\boldsymbol{k}|+c^2|\boldsymbol{k}|^2}\pm\sqrt{\omega_l^2-2\omega_t c|\boldsymbol{k}|+c^2|\boldsymbol{k}|^2}\right\}\\ &= (\pm)\frac{1}{2}\left\{\sqrt{\left(\omega_t+c|\boldsymbol{k}|\right)^2+\omega_l^2-\omega_t^2}\pm\sqrt{\left(\omega_t-c|\boldsymbol{k}|\right)^2+\omega_l^2-\omega_t^2}\right\}.\end{aligned} \tag{9.13}$$

This means that we have to given $|\boldsymbol{k}|$ two different solutions for $\omega_{(\pm)}^2$ signified by the upper indices "(±)" where one has to pay attention mainly to the two different lower signs "±" to the two sum terms with square roots. In the hatched region the solution (9.13) for the frequency in dependence on the modulus of the wave vector becomes complex and can be better represented in the form

$$\omega_{\pm}^{(\pm)}(\boldsymbol{k}) = (\pm)\frac{1}{2}\left\{\sqrt{\left(\omega_t+c|\boldsymbol{k}|\right)^2-\left(\omega_t^2-\omega_l^2\right)}\pm i\sqrt{\omega_t^2-\omega_l^2-\left(\omega_t-c|\boldsymbol{k}|\right)^2}\right\}. \tag{9.14}$$

Figure 2 represents the different branches of solutions $\omega=\omega(\boldsymbol{k})$ in dependence on the modulus $|\boldsymbol{k}|$ of the wave vector for the both principal cases $\omega_l>\omega_t$ and $\omega_l<\omega_t$. The left-hand picture is well known mostly in form of the right upper quadrant (see, e.g., [17], chap. III, Figure 8 or [2], chap. 11, Figure 11.4).

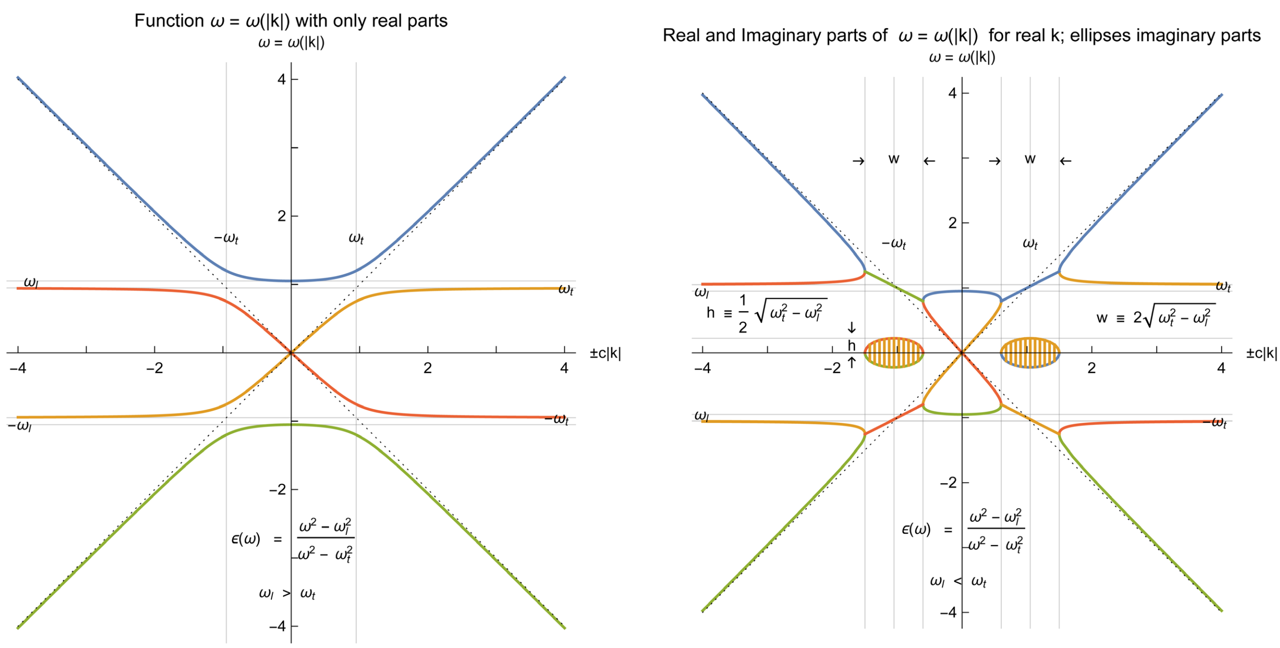

Figure 2. Frequencies in dependence on real wave vector for $\varepsilon(\omega)=\dfrac{\omega^2-\omega_l^2}{\omega^2-\omega_t^2}$, $\mu(\omega)=1$ for $\omega_l>\omega_t$ and $\omega_l<\omega_t$. The contours of the hatched parts in the right-hand picture are imaginary and describe amplification. For the figures we have chosen the values $\omega_l=1.05,\omega_t=0.95$ in passive case (to the left) and $\omega_l=0.95,\omega_t=1.05$ in active case (to the right).

[7] A bi-quadratic equation of the form $z^4-2pz^2+q^2=0$, $\left(p=p^*,q=q^*,\Rightarrow q^2\geq 0\right)$, with real p and q and therefore non-negative q^2 possesses 4 solutions which may be written in the following form $z_{\pm}^{(\pm)}=(\pm)\frac{1}{\sqrt{2}}\left(\sqrt{p+q}\pm\sqrt{p-q}\right)=(\pm)\frac{\sqrt{2}q}{\sqrt{p+q}\mp\sqrt{p-q}}$, with only real or imaginary sum terms $\sqrt{p\pm q}$. Clearly, these solutions are also true in general case but usually then do not provide separation into real and imaginary parts.

Asymptotically, for large $|\boldsymbol{k}|$ we have in both cases a branch $\omega = \pm c|\boldsymbol{k}|$ the same as for light beams in vacuum and a branch with constant $\omega = \pm\omega_t$ where ω_t is the resonance frequency to the energy difference of the considered two levels. The case $\omega_l - \omega_t > 0$ corresponds to lower occupation of higher level in comparison the lower level and the case $\omega_t - \omega_l > 0$ to inverse occupation. The last case can be achieved by pumping this level and as was to expect it possesses properties of amplification in a certain region of wave vectors. This is exactly separated by the two sum terms with square roots in the solutions in the form (9.13) or (9.14). We have hatched the imaginary parts inside their contours. Their real parts are in the figure to find over, respectively, under the hatched contours and look like straight lines but are such only in the limiting case $\omega_t - \omega_l \to 0$ as follows from the first sum term in (9.14).

In **Figure 3** the case $\omega_t > \omega_l$ is presented enlarged for the right upper quadrant of **Figure 2** in two numerical cases which show the dependence on the parameters ω_t and ω_l and, in particular, on the difference $\sqrt{\omega_t^2 - \omega_l^2}$. Microscopic models for the permittivity (9.1) show that the difference $\omega_t^2 - \omega_l^2$ is among other parameters as factors of the active transition levels proportional to the density occupation inversion $\sigma \propto \omega_t^2 - \omega_l^2 \equiv -\lambda$ of these levels. This means that in our idealized model the product of height h with width w of the amplifier contour (see **Figure 2**) is proportional to the square root of the density of inverse occupation of the considered active levels whereas h and w themselves are proportional to its square root. In laser theory to our knowledge the height h is

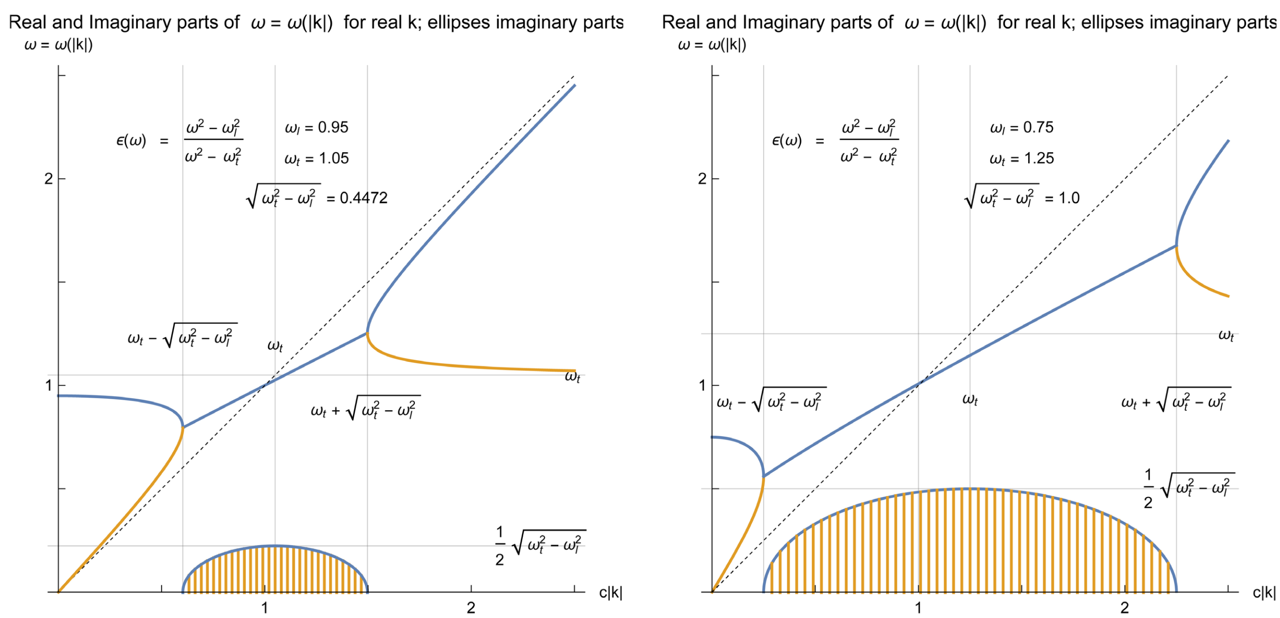

Figure 3. Function $\omega = \omega(\boldsymbol{k})$ for $\varepsilon(\omega) = \frac{\omega^2 - \omega_l^2}{\omega^2 - \omega_t^2}$, $\mu(\omega) = 1$ with $\omega_l < \omega_t$ (amplification) in two numerical cases. The left-hand figure is practically the right upper quadrant of **Figure 2** but amplified for better visibility and for making a comparison with a similar figure with other parameters. For the figures we have chosen the values $\omega_t = 1.05, \omega_l = 0.95$ and $\omega_t = 1.25, \omega_l = 0.75$.

usually assumed or calculated as direct proportional to the density of inverse occupation whereas for its width are made complicated considerations about natural line width and its enlargement.

For application to laser theory it is necessary to add feedback by a resonator. The unspecific losses of the involved resonator modes in the concerning region cut off an upper part of the amplification contour and thus lower it whereas their frequencies are determined mainly from the real part (first sum term in (9.14))

$$\begin{aligned}\omega_{\mathrm{Re}}\left(|k|\right) &= \frac{1}{2}\sqrt{\left(\omega_t + c|\boldsymbol{k}|\right)^2 - \left(\omega_t^2 - \omega_l^2\right)} \\ &= \frac{\sqrt{3\omega_t^2+\omega_l^2}}{2} + \frac{\omega_t}{\sqrt{3\omega_t^2+\omega_l^2}}\left(c|\boldsymbol{k}| - \omega_t\right)\left(1 - \frac{\omega_t^2-\omega_l^2}{4\left(3\omega_t^2+\omega_l^2\right)}\left(c|\boldsymbol{k}|-\omega_t\right) + \cdots\right), \\ &\left(\omega_t - \sqrt{\omega_t^2-\omega_l^2} \le c|\boldsymbol{k}| \le \omega_t + \sqrt{\omega_t^2-\omega_l^2}\right).\end{aligned} \tag{9.15}$$

For a long resonator in comparison to the transverse dimensions only the longitudinal modes without reflection at the side wands play a role. For such a resonator with ideal mirrors at the end the field at the mirrors has to be vanishing and the possible resonator modes have to possess a multiple m of the half the wavelengths λ, ($|\boldsymbol{k}| \equiv \frac{2\pi}{\lambda}$), which fit into the resonator length L. Thus for the possible wave vectors and frequencies we find in this idealized case ($m_0 \equiv m_{\min}$)

$$\begin{aligned}&|\boldsymbol{k}| = m\frac{\pi}{L}, \quad \omega_{\mathrm{Re}}\left(|k|\right) = \frac{1}{2}\sqrt{\left(\omega_t + m\pi\frac{c}{L}\right)^2 - \left(\omega_t^2-\omega_l^2\right)}, \\ &\omega_t - \sqrt{\omega_t^2-\omega_l^2} \le m\pi\frac{c}{L} \le \omega_t + \sqrt{\omega_t^2-\omega_l^2}, \quad \left(m = m_0, m_0+1, \cdots, m_{\max}\right).\end{aligned} \tag{9.16}$$

The resonator losses as said lower the amplification contour and therefore narrow the possible values for m and change also a little the relation (9.15) for the possible wave vectors and frequencies. With losses both the wave vectors and frequencies may become slightly complex. The imaginary part of the frequency determines also a line width by only classical considerations. For $\frac{\omega_t^2-\omega_l^2}{\omega_t^2} \le 1$ one finds in approximation from (9.15)

$$\frac{\omega_t^2-\omega_l^2}{\omega_t^2} \le 1: \quad \omega_{\mathrm{Re}}\left(\boldsymbol{k}\right) \approx \omega_t + \frac{1}{2}\left(c|\boldsymbol{k}| - \omega_t\right), \tag{9.17}$$

and the density of possible frequencies in the corresponding frequency interval is doubled in comparison to the density within a resonator with vacuum. If we assume that there is a process (pumping) which keeps constant the density of the inverse occupation then it may be considered as a very simple and idealized classical model of laser action, at least, near the threshold. However, this model cannot provide information at which level of occupation inversion the equilibrium between pumping and radiation is reached. In this sense it is similar to thermal equilibrium where without additional information it cannot be said how it was reached. Clearly, quantum-mechanical generalization makes further modifications also to the line widths.[8]

Generalizations of the permittivity (9.1) in different directions are possible and interesting, for example, by an additional constant sum term on the right-hand side taking into account summarily the contribution of all other resonances of two levels or taking into account losses by imaginary terms in the denominator. We consider now shortly the case of taking into account two resonance frequencies $\omega_{t,1}$ and $\omega_{t,2}$ leading to a permittivity of the form [6] (§18, Equation (6))

$$\varepsilon(\omega)=1-\frac{\lambda_1}{\omega^2-\omega_{t,1}^2}-\frac{\lambda_2}{\omega^2-\omega_{t,2}^2}, \tag{9.18}$$

with two further parameters λ_2 and $\omega_{t,2}$ which determine the strength of a second resonance. This can be also written in the following form

$$\varepsilon(\omega)=\frac{\left(\omega^2-\omega_{l,-}^2\right)\left(\omega^2-\omega_{l,+}^2\right)}{\left(\omega^2-\omega_{t,1}^2\right)\left(\omega^2-\omega_{t,2}^2\right)}, \tag{9.19}$$

with the definitions

$$\omega_{l,\pm}^2 \equiv \frac{1}{2}\left\{\omega_{t,1}^2+\lambda_1+\omega_{t,2}^2+\lambda_2 \pm \sqrt{\left(\omega_{t,1}^2+\lambda_1-\omega_{t,2}^2-\lambda_2\right)^2+4\lambda_1\lambda_2}\right\}, \tag{9.20}$$

where $\omega_{l,\mp}^2$ is real-valued or may become even complex-to complex-conjugate-valued and where $\omega_{l,-}$ and $\omega_{l,+}$ cannot be properly assigned to $\omega_{t,1}$ and $\omega_{t,2}$. The dispersion Equation (7.4) resolved to an equation for ω^2 in dependence on the squared wave vector $|\boldsymbol{k}|^2$ becomes a bi-cubic equation in $|\boldsymbol{k}|$ which is already difficult to solve for ω and to discuss. In dependence on the 4 parameters $\omega_{t,1},\omega_{t,2}$ and λ_1 and λ_2 one would have to distinguish many principal cases.

10. Group Velocity to the Polariton Permittivity in Passive Case and Group Velocities Faster Than Light Velocity in Active Case

We now consider the group velocity of transversal waves for the polariton permittivity (9.1). By differentiation of the dispersion Equation (9.5) with respect to $\boldsymbol{k}$ we calculate for the group velocity in direction $\frac{\boldsymbol{k}}{|\boldsymbol{k}|}$ of the wave vector and in dependence of its modulus on the frequency ω only

[8]Sometime in the eighties I asked H. Paul from our Institute to discuss with me the polariton model and he agreed. My first aim was to see did he know whether or not this model or something similar was already discussed in literature since I did not find neither theoretical considerations nor experimental hints for some results. He also did not know something and likely nothing existed. Thank! My second aim was to show him that the height as well as the width of the amplification contours in form of ellipses are proportional to the square root of the inverse occupation density and I made a drawing of this contour but nothing of the kind of the more general and complicated figures made by computer and presented here. I could not expect that we may discuss the formulae in detail (more than the permittivity he did not want to see) and soon H. Paul finished the discussion saying approximately: I believe you only if you have also the nonlinear terms in the equations. This, however, was not my intention, my possibility and my official task at that time.

$$\begin{aligned}\boldsymbol{v} &\equiv \frac{\partial\omega}{\partial\boldsymbol{k}} = \frac{2c^2\boldsymbol{k}}{\frac{\partial}{\partial\omega}\left(\omega^2\varepsilon(\omega)\right)} = \frac{c^2}{\omega}\frac{\left(\omega^2-\omega_t^2\right)^2}{\omega^4-2\omega_t^2\omega^2+\omega_l^2\omega_t^2}\boldsymbol{k} \\ &= c\underbrace{\sqrt{\frac{\omega^2-\omega_l^2}{\omega^2-\omega_t^2}}}_{\frac{c}{\omega}|\boldsymbol{k}|=\sqrt{\varepsilon(\omega)}}\frac{\left(\omega^2-\omega_t^2\right)^2}{\left(\omega^2-\omega_t^2\right)^2+\left(\omega_l^2-\omega_t^2\right)\omega_t^2}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \equiv \varphi(\omega)\frac{\boldsymbol{k}}{|\boldsymbol{k}|}.\end{aligned} \tag{10.1}$$

It possesses in this model in every case the direction of the wave vector also if it is complex-valued with different directions of real and imaginary part but the frequency-dependent coefficients $\varphi(\omega)$ can also become complex-valued (for real values of ω) due to presence of the square root. Without taking into account the dispersion of $\varepsilon(\omega)$ at the considered frequency ω (setting $\frac{\partial\varepsilon(\omega)}{\partial\omega}\to 0$) it is

$$\boldsymbol{v}' = \frac{2c^2\boldsymbol{k}}{\varepsilon(\omega)\frac{\partial}{\partial\omega}\left(\omega^2\right)} = c\frac{c\boldsymbol{k}}{\omega\varepsilon(\omega)} = c\frac{c|\boldsymbol{k}|}{\omega\varepsilon(\omega)}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} = \frac{c}{\sqrt{\varepsilon(\omega)}}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} = c\sqrt{\frac{\omega^2-\omega_t^2}{\omega^2-\omega_l^2}}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}, \tag{10.2}$$

but the dispersion cannot be switched off. For such points where $\varepsilon(\omega)$ possesses a minimum (or maximum) the derivative vanishes (*i.e.*, $\frac{\partial\varepsilon(\omega)}{\partial\omega}(\omega)=0$) and the group velocity with and without taking into account the dispersion are equal. The dispersion factor $\alpha_{\text{disp}}(\omega)\equiv\alpha(\omega)$ is

$$\begin{aligned}\alpha(\omega) &= \frac{\varepsilon(\omega)\frac{\partial}{\partial\omega}\left(\omega^2\right)}{\frac{\partial}{\partial\omega}\left(\omega^2\varepsilon(\omega)\right)} = \frac{\left(\omega^2-\omega_l^2\right)\left(\omega^2-\omega_t^2\right)}{\left(\omega^2-\omega_t^2\right)^2+\left(\omega_l^2-\omega_t^2\right)\omega_t^2} \equiv \frac{\left(\omega^2-\omega_l^2\right)\left(\omega^2-\omega_t^2\right)}{\left(\omega^2-\omega_-^2\right)\left(\omega^2-\omega_+^2\right)}, \\ \omega_\mp^2 &\equiv \omega_t\left(\omega_t\mp\sqrt{\omega_t^2-\omega_l^2}\right), \quad \omega_\mp \equiv \frac{\sqrt{\omega_t\left(\omega_t+\omega_l\right)}\mp\sqrt{\omega_t\left(\omega_t-\omega_l\right)}}{\sqrt{2}}.\end{aligned} \tag{10.3}$$

Due to an extremum of $\varepsilon(\omega)$ for $\omega=0$ its derivative with respect to frequency vanishes there and the dispersion factor becomes $\alpha(\omega=0)=1$.

One may distinguish 2 different cases and a limiting case between them represented by

$$|\boldsymbol{v}| = \varphi(\omega) = c\frac{\sqrt{1-\frac{\omega_l^2-\omega_t^2}{\omega^2-\omega_t^2}}}{1+\frac{\omega_l^2-\omega_t^2}{\omega^2-\omega_t^2}\frac{\omega_t^2}{\omega^2-\omega_t^2}} \to \begin{cases} <c, & \left(\omega_l>\omega_t\right) \\ =c, & \left(\omega_l=\omega_t\right), \\ >c, & \left(\omega_l<\omega_t\right) \end{cases} \tag{10.4}$$

which in relation to the light velocity obviously are determined by different properties. In last case of $\omega_l<\omega_t$ the group velocity can even be opposite to the direction of the wave vector in certain regions of the frequency.

If one wants to have the full dependence $\boldsymbol{v}\equiv\boldsymbol{v}(\boldsymbol{k})$ of the group velocity on the wave vector one has to distinguish the 4 branches in (9.13) or in (9.14) and finds by differentiation with respect to $\boldsymbol{k}$

$$\begin{aligned}\boldsymbol{v}_{\pm}^{(\pm)}(\boldsymbol{k}) &= (\pm)\frac{c}{2}\left\{\frac{\omega_t + c|\boldsymbol{k}|}{\sqrt{(\omega_t + c|\boldsymbol{k}|)^2 + \omega_l^2 - \omega_t^2}} \mp \frac{\omega_t - c|\boldsymbol{k}|}{\sqrt{(\omega_t - c|\boldsymbol{k}|)^2 + \omega_l^2 - \omega_t^2}}\right\}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \\ &= (\pm)\frac{c}{2}\left\{\frac{\omega_t + c|\boldsymbol{k}|}{\sqrt{(\omega_t + c|\boldsymbol{k}|)^2 - (\omega_t^2 - \omega_l^2)}} \pm i\frac{\omega_t - c|\boldsymbol{k}|}{\sqrt{(\omega_t^2 - \omega_l^2) - (\omega_t - c|\boldsymbol{k}|)^2}}\right\}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} \\ &\equiv c\beta(\omega)\frac{\boldsymbol{k}}{|\boldsymbol{k}|}.\end{aligned} \tag{10.5}$$

which we have written in two favorable representations for the passive case $\omega_l > \omega_t$ and the active case $\omega_t > \omega_l$.

The calculation of the second-order coefficients W_{ij} in the Equation (8.15) for beam propagation in isotropic media can be calculated, for example, from the principal structure of the group velocity

$$v_i \equiv \frac{\partial \omega}{\partial k_i} = \varphi(\omega)\frac{k_i}{|\boldsymbol{k}|}, \quad \varphi(\omega) \equiv c\frac{(\omega^2 - \omega_l^2)^{\frac{1}{2}}(\omega^2 - \omega_t^2)^{\frac{3}{2}}}{(\omega^2 - \omega_t^2)^2 + (\omega_l^2 - \omega_t^2)\omega_t^2}, \tag{10.6}$$

by further differentiation with respect to variables k_j. Using

$$\frac{\partial|\boldsymbol{k}|}{\partial k_i} = \frac{k_i}{|\boldsymbol{k}|}, \quad \Rightarrow \quad \frac{\partial^2|\boldsymbol{k}|}{\partial k_i \partial k_j} = \frac{\partial}{\partial k_j}\left(\frac{k_i}{|\boldsymbol{k}|}\right) = \frac{1}{|\boldsymbol{k}|}\left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right), \tag{10.7}$$

and due to $\omega \equiv \omega(\boldsymbol{k})$ and (10.7)

$$\frac{\partial \varphi}{\partial k_j}(\omega) = \frac{\partial \varphi}{\partial \omega}(\omega)\frac{\partial \omega}{\partial k_j} = \frac{\partial \varphi}{\partial \omega}(\omega)\varphi(\omega)\frac{k_j}{|\boldsymbol{k}|} = \frac{1}{2}\frac{\partial}{\partial \omega}\left((\varphi(\omega))^2\right)\frac{k_j}{|\boldsymbol{k}|}, \tag{10.8}$$

we find the general structure

$$W_{ij} \equiv \frac{\partial^2 \omega}{\partial k_i \partial k_j} = \varphi(\omega)\frac{1}{|k|}\left(\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right) + \frac{1}{2}\frac{\partial}{\partial \omega}\left((\varphi(\omega))^2\right)\frac{k_i k_j}{|\boldsymbol{k}|^2} \tag{10.9}$$

and, finally, with the special function $\varphi(\omega)$ in (10.6)

$$\begin{aligned}W_{ij} &\equiv \frac{\partial^2 \omega}{\partial k_i \partial k_j} = \frac{c^2}{\omega}\frac{(\omega^2 - \omega_t^2)^2}{(\omega^2 - \omega_t^2)^2 + (\omega_l^2 - \omega_t^2)\omega_t^2} \\ &\quad \cdot\left\{\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2} + (\omega_l^2 - \omega_t^2)\frac{\omega^2(\omega^4 + 2\omega_t^2\omega^2 - 3\omega_l^2\omega_t^2)}{\left((\omega^2 - \omega_t^2)^2 + (\omega_l^2 - \omega_t^2)\omega_t^2\right)^2}\frac{k_i k_j}{|\boldsymbol{k}|^2}\right\}.\end{aligned} \tag{10.10}$$

This result was checked by a slightly modified calculation which I do not present here. It possesses two sum terms proportional to the tensors $\delta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}$ and $\frac{k_i k_j}{|\boldsymbol{k}|^2}$ describing transversal and longitudinal diffraction (or diffusion) of the light beam during propagation.

In the special case of an isotropic cold plasma with the permittivity $\varepsilon(\omega)=1-\frac{\omega_p^2}{\omega^2}$ setting $\omega_l=\omega_p, \omega_t=0$ the formulae (10.1) for the group velocity and (10.10) for the diffraction coefficients simplify to

$$v_i=c\frac{c|\boldsymbol{k}|}{\omega}\frac{k_i}{|\boldsymbol{k}|}=c\sqrt{1-\frac{\omega_p^2}{\omega^2}}\frac{k_i}{|\boldsymbol{k}|}, \quad \Rightarrow \quad \boldsymbol{kv}=\omega\left(1-\frac{\omega_p^2}{\omega^2}\right), \tag{10.11}$$

that due to taking into account the frequency dispersion does not agree with (6.9). For the diffraction coefficients one finds

$$W_{ij}=\frac{c^2}{\omega}\left\{\delta_{ij}-\frac{k_ik_j}{|\boldsymbol{k}|^2}+\frac{\omega_p^2}{\omega^2}\frac{k_ik_j}{|\boldsymbol{k}|^2}\right\}, \tag{10.12}$$

and it contains also a term proportional to $\frac{k_ik_j}{|\boldsymbol{k}|^2}$ leading to a diffusion in longitudinal direction of the beam.

Without taking into account the dispersion we would obtain from (10.2)

$$W'_{ij}=\frac{c^2}{\sqrt{\varepsilon(\omega)}}\left(\delta_{ij}-\frac{k_ik_j}{|\boldsymbol{k}|^2}\right), \tag{10.13}$$

that is without a "longitudinal" contribution proportional to $\frac{k_ik_j}{|\boldsymbol{k}|^2}$.

In **Figure 4** we represent the permittivity $\varepsilon(\omega)$ (blue curves) together with the group velocity in relation to the light velocity without taking into account dispersion of the permittivity $\alpha(\omega)(\equiv\alpha_{\text{disp}}(\omega))$ (yellow curves) and with taking it into account as $\beta(\omega)\equiv\frac{|\boldsymbol{v}|}{c}$ (red curves) for the passive ($\omega_l>\omega_t$, to the left) and the active case ($\omega_t>\omega_l$ to the right). In the passive case we see that the group velocity remains smaller than the light velocity in every case and that it is real-valued in regions where $\varepsilon(\omega)$ is positive and that taking into account the dispersion the deviations in comparison to neglect of dispersion are important, in particular, in the neighborhood of the resonance frequency ω_t and in the neighborhood of the "(longitudinal)" frequency ω_l. In addition, the group velocity possesses in every case the same direction as the wave vector. In the region $\omega_t\le\omega\le\omega_l$ no relation of group velocity to light velocity (red) is drawn because the group velocity is there imaginary. All is so as expected. However, the same picture for the active case (to the right) contains an unexpected surprise.

In the active case $\omega_l\le\omega_t$ (right-hand picture in **Figure 4**) the group velocity is here in certain regions of frequency larger than the light velocity or it is in opposite direction to the wave vector. The regions of ω where it is larger than the light velocity but in direction of the wave vector $\boldsymbol{k}$ are

$$0<\omega<\omega_-, \quad \omega_+<\omega<\infty. \tag{10.14}$$

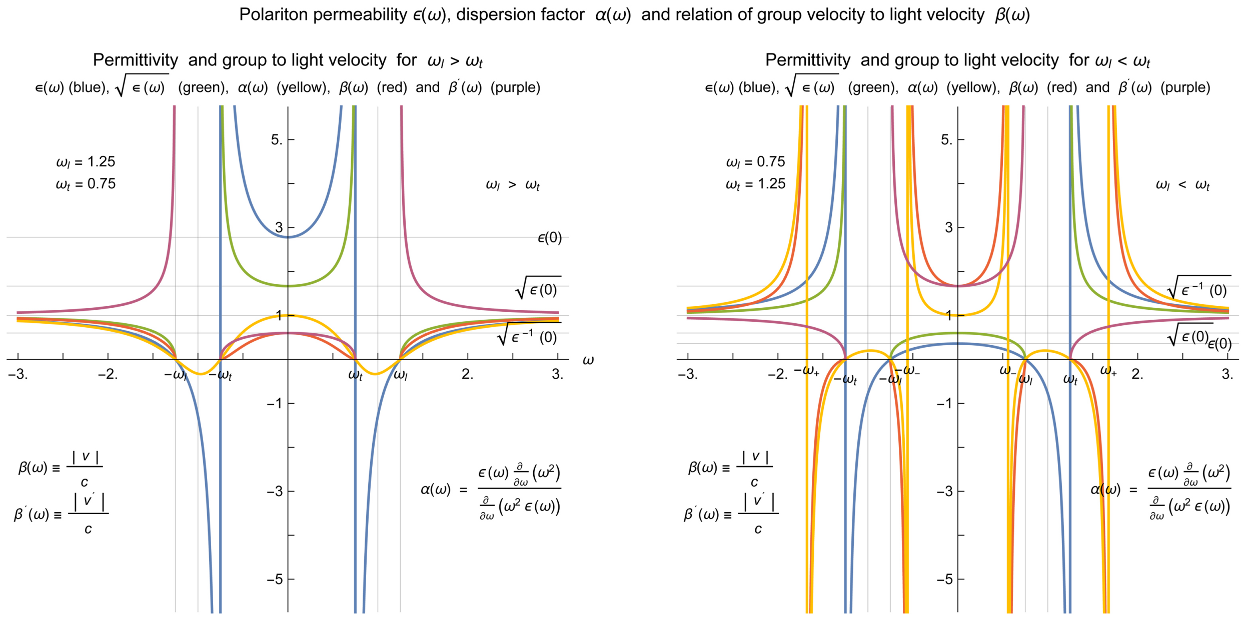

Figure 4. Polariton permittivity $\varepsilon(\omega)=\frac{\omega^2-\omega_l^2}{\omega^2-\omega_t^2},(\mu(\omega)=1)$, $\alpha(\omega)$ and $\beta(\omega)=\frac{|\mathbf{v}|}{c}$ for $\omega_l>\omega_t$ and for $\omega_l<\omega_t$. In regions $|\omega_t|<|\omega|<|\omega_l|$ for $\omega_l>\omega_t$ and $|\omega_l|<|\omega|<|\omega_t|$ for $\omega_l<\omega_t$ the relation of the group velocity to the light velocity (red curves with and purple without dispersion) becomes imaginary that is not drawn. In passive case $\omega_l>\omega_t$ wave vector and group velocity possess in every case the same direction and the last is smaller than the light velocity. In active case $\omega_l<\omega_t$ the group velocity is greater than the light velocity in the regions $0<\omega<\omega_-$ and $\omega_+<\omega$ but is in the same direction as the wave vector. In the region $\omega_t<\omega<\omega_+$ it is even in the opposite direction to the corresponding light velocity but only with taking into account the dispersion. The difference between $\beta(\omega)$ (red) and $\beta'(\omega)$ (purple is that last is calculated under neglect of the frequency dispersion of the permittivity. We have chosen for the pictures $\omega_l=1.25,\omega_t=0.75$ in passive case and $\omega_l=0.75,\omega_t=1.25$ in active case, the same as in **Figure 1**.

We will not come here with a quick physical explanation of this phenomenon although we considered the polariton permittivity with frequency $\omega_l<\omega<\omega_t$ already much earlier mainly with respect to its description of amplification in a certain region of wave vectors and frequencies. This phenomenon of group velocity greater than light velocity may likely come from a correlation within all parts of the medium by preparation of the occupation inversion of a level which is made already before it gets this property. The diffraction coefficients W_{ij} according to (10.10) possess the same zeros ($\omega_\pm$ in (10.3) and in **Figure 4**)) in the denominators as the group velocity v_i in (10.1) and can become very large. It is possible that the expansion (8.15) in the equation for the slowly varying amplitude (8.15) does not converge and that this equation is not applicable but a more general treatment should not change this basically. A practical use is likely very difficult to make and far from now. The most chances has its use for guided waves in long resonators and, in principle, such use due to the amplification properties of these similar media is already realized by lasers but not due to group velocity larger than light velocity as subsidiary effect. On the other side we cannot fully exclude that this phenomenon is already known and somewhere discussed in literature. Hypothetical particles in free space which move with a velocity greater than that of light are called tachyons and were occasionally considered from the sixtieths on, in detail, e.g., by Terletski [28] (Chapters V. and VI.) and by others but in our case they may be only quasi-particles within an active medium with occupation inversion if we associate them with the group velocity greater than that of light. It is known that the existence of tachyons with imaginary mass is not excluded by the Relativity theory but experimentally such are not found. However, in our case they cannot exist as such which usually form beams due to very large diffraction and in this sense they are not the same as were discussed as tachyons.

There is a second fully unexpected phenomenon. In the regions $\omega_{-} < \omega < \omega_{l}$ and $\omega_{t} < \omega_{+}$ the group velocity $\boldsymbol{v}$ is in opposite direction to the direction of the wave vector $\boldsymbol{k}$ but only if we take into account the dispersion of the permittivity that let us believe at first in an error of signs but all was calculated with the same reliable formulae as in the passive case and the formula (10.3) for the dispersion factor does not involve square roots where the sign is unclear. It seems that this opposite group velocity to refraction vector violets the causality but we cannot exclude an explanation by the established correlation between different parts of the medium. So the described phenomenon requires further attention.

In search for references to the unexpected phenomena I came via the interesting book of Vaas [29] (part II, Section 7, e.g., p. 145, 146) to the reference of authors Nimtz and Haibel [30]. In [30] it is claimed that G. Nimtz (together with H. Aichmann, p. 111) transmitted in 1994 a symphony using the tunneling effect through a sub-dimensioned wave guide (full length about 12 cm) with approximately the five-fold velocity of the light velocity in vacuum[9]. In the transmission of information (e.g. music) the frequency modulation of a signal is used and has to be detected from it. Despite the large spatial distortion of a beam via the propagation the frequency is more stable and not very distorted over longer length of a beam that may explain some results. All this is a challenging theme and one has to wait for further clarification.

11. Remark to Reflection and Refraction of Beams at Isotropic and Bi-Isotropic Media

The following considerations concern only passive cases which are not very problematic with respect to basic discussions. In reflection and refraction problems of beams at a boundary between isotropic and (or) bi-isotropic media maximum 4 different beams with the same average frequency ω can be related to each other since they must possess the same tangential component of the

[9]Long ago I heard or saw in popular sources something about experiments of G. Nimtz but did not take the super-luminal velocities seriously and did not try to find the original papers. Now, when I find as it seems to me by reliable mathematics such possibilities of group velocities greater than light velocity in vacuum I changed my opinion and look for physical explanations. However, I have to emphasize that I cannot support by own calculation the effects concerning tunneling and the conditions for them since I did not make such.

average wave vectors $\boldsymbol{k}_\nu^{i,r}$ or refraction vector $\boldsymbol{n}_\nu^{i,r} \equiv \frac{c}{\omega}\boldsymbol{k}_\nu^{i,r}$ (lower indices "1" and "2" stand for the both media and upper indices "*i*" for incident and "*r*" for reflected or refracted wave. The group velocities of the corresponding beams are calculated in (7.6). This is represented in **Figure 5** on the left-hand picture and the corresponding beam propagation on the right-hand picture. If $\boldsymbol{N}$ is a normal unit vector to the boundary at the considered point of beam reflection and refraction then the tangential component of all refraction vectors is

$$\overline{\boldsymbol{n}} \equiv \left[\boldsymbol{N},\left[\boldsymbol{n},\boldsymbol{N}\right]\right] = \boldsymbol{n} - \boldsymbol{n}\boldsymbol{N}\cdot\boldsymbol{N}, \quad \left(\boldsymbol{N}^2 = 1\right), \tag{11.1}$$

where for $\boldsymbol{n}$ an arbitrary of the involved refraction vectors can be inserted. Nothing changes in this picture if the refraction vectors in one or both media satisfy a dispersion equation $\boldsymbol{n}^2 = \varepsilon(\omega)\mu(\omega)$ in comparison to $\mu(\omega) = 1$. Also in the active case of isotropic media the group velocities $\boldsymbol{v}$ are in every case parallel to the corresponding refraction vectors $\boldsymbol{n}$ although we found that in this case the possibility exists that they are in opposite direction to the refraction vectors and may possess super-luminal velocities that is not yet fully understood and affirmed. Mostly one has only an incident beam from one side of the two media and only three waves (incident, reflected and refracted) are coupled at the boundary. All this is necessary to take into account if one discusses the work of Pendry [22] (see Appendix D).

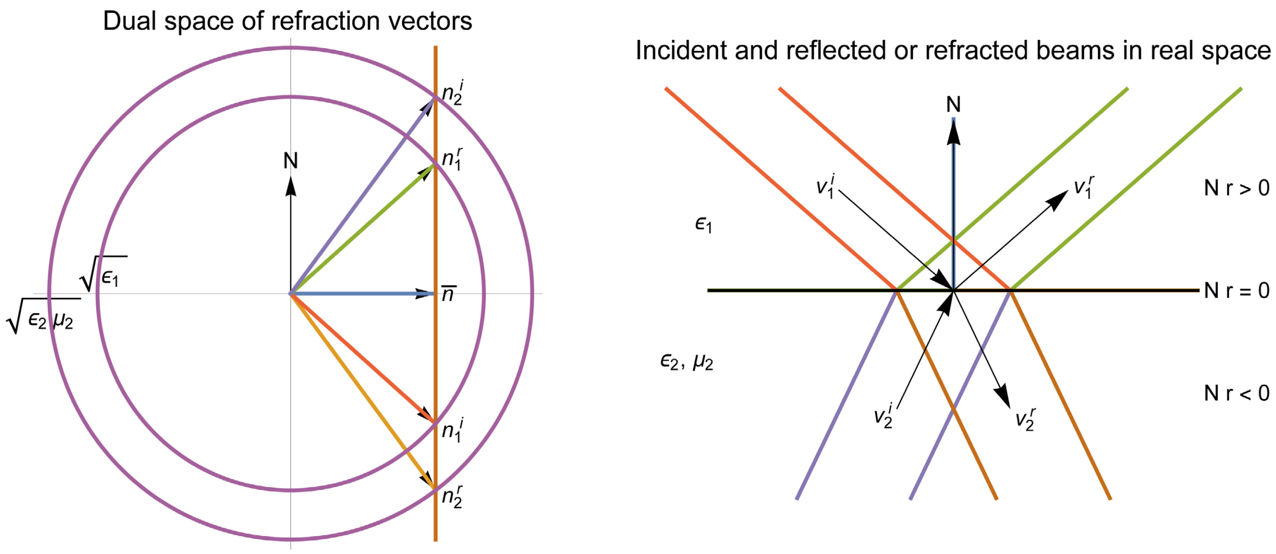

Figure 5. Refraction vectors $\boldsymbol{n}$ with equal tangential components and group velocities $\boldsymbol{v}$ at a boundary. The two media possess the lower indices 1 and 2 where medium 2 may possess an electric permittivity and a magnetic permeability. The upper indices "*i*" and "*r*" mean "incident" and "reflected or refracted" waves seen from the different sides of the boundary. It does not play a role whether a positive product $\varepsilon_2\mu_2$ is obtained from both positive or both negative values of ε_2 and μ_2. All refraction vectors $\boldsymbol{n}$ possess the same tangential component $\overline{\boldsymbol{n}}$ to the boundary $\boldsymbol{N}\boldsymbol{r} = 0$ and only their normal components are different in general case. For isotropic media the directions of the refraction vectors $\boldsymbol{n}$ and the corresponding group velocities $\boldsymbol{v}$ are the same. With respect to polarization of the 4 waves one may distinguish the two cases of polarization perpendicular and within the incidence plane spanned by the normal unit vector $\boldsymbol{N}$ to the boundary plane and an arbitrary of the refraction vectors $\boldsymbol{n}$.

The normal components $(\boldsymbol{nN})\cdot\boldsymbol{N}$ of all involved refraction vectors $\boldsymbol{n}$ can be obtained from the dispersion equations $\boldsymbol{n}_1^2=\varepsilon_1(\omega)$ and $\boldsymbol{n}_2^2=\varepsilon_2(\omega)\mu_2(\omega)$, respectively

$$\boldsymbol{n}_1^{i,r}\boldsymbol{N}=\pm\sqrt{\varepsilon_1(\omega)-\overline{\boldsymbol{n}}^2},\quad \boldsymbol{n}_2^{i,r}\boldsymbol{N}=\pm\sqrt{\varepsilon_2(\omega)\mu_2(\omega)-\overline{\boldsymbol{n}}^2}. \tag{11.2}$$

Furthermore, in **Figure 5** two case of polarizations are possible, with the electric field polarized in the incidence plane spanned by vectors $\boldsymbol{N}$ and $\overline{\boldsymbol{n}}$ and perpendicular to it. The discussion of the amplitude relations for the involved wave is not necessary here for the intended purpose.

12. Conclusion

We compared two concepts of representing the constitutive equations in classical macroscopic optics of homogeneous anisotropic media, first the more general concept of spatial dispersion and then the more special concept of bi-anisotropic media with two constitutive equations for the electric and the magnetic induction and made this in coordinate-invariant way. Then this was specialized to uniaxial and, finally, to isotropic media where a possible equation for quasi-plane and quasi-monochromatic beam propagation in approximation with the first two terms of an expansion of the slowly varying beam amplitudes with respect to derivatives in space and time was derived taking into account diffraction. This was then applied for the isotropic case to the polariton permittivity (9.1) with a detailed discussion of the passive and the active case. The active case is obtained from the passive case by changing a sign in the susceptibility and describes amplification in certain regions of the frequency that somehow goes connected with feedback in resonators in direction of (a prestep of) laser action in the stationary regime. Losses in form of imaginary parts in the permittivity, we did not include for more simplicity and calculability of the formulae.

When we calculated the group velocity for beams in the active case we found as a surprise for some regions of frequency the possibility of velocities faster than the light velocity in vacuum and as a yet greater surprise the possibility of an opposite direction to the direction of the wave vectors (but different signs with and without taking into account dispersion). A certain physical explanation is possibly by the preparation of occupation inversion in establishing the active case and therefore a correlation between all parts of the medium which is not contained in the equations. We found after this that it is not fully unknown from literature. We have to think furthermore about these phenomena and have to find access to more literature about this.

In the process of working with the topics into our viewpoint came also the notion of "negative refraction" but we could not find some positive aspects and the possibility of its realization in isotropic media and make a few remarks to this in the text and in Appendix D.

Primarily, I intended to include also such peculiar cases as inhomogeneous waves (for example, total reflection) and, in particular, the cases of optic axes with calculation of the two-dimensional projection operators for the electric field and the conical approximation of the dispersion surface in the neighborhood of optic axes in coordinate-invariant treatment. However, for extent and necessary time hoping to realize it we move this to a possible later time.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A. Some Important Relations for Three-Dimensional Operators

The most important relation for general three-dimensional operators A is the Cayley-Hamilton identity

$$\mathsf{A}^3 - \langle \mathsf{A} \rangle \mathsf{A}^2 + [\mathsf{A}]\mathsf{A} - |\mathsf{A}|\mathsf{I} = 0, \tag{A.1}$$

with the invariants with respect to similarity transformations which are the trace $\langle \mathsf{A} \rangle \equiv A_j^j$, the second invariant $[\mathsf{A}]$ and the determinant $|\mathsf{A}|$ according to (for spaces with metric tensor $g_{ij} = g_{ji}$ we may set $A_{ik} \equiv g_{ij} A_k^j$)

$$\langle \mathsf{A} \rangle \equiv A_{ii}, \quad [\mathsf{A}] \equiv \frac{1}{2}\left(\langle \mathsf{A} \rangle^2 - \langle \mathsf{A}^2 \rangle\right), \quad |\mathsf{A}| \equiv \frac{1}{6}\left(\langle \mathsf{A} \rangle^3 - 3\langle \mathsf{A} \rangle \langle \mathsf{A}^2 \rangle + 2\langle \mathsf{A}^3 \rangle\right). \tag{A.2}$$

As consequence of the Cayley-Hamilton identity the inverse operator to an arbitrary operator A can be represented by

$$\mathsf{A}^{-1} \equiv \frac{\bar{\mathsf{A}}}{|\mathsf{A}|}, \quad \bar{\mathsf{A}} \equiv \mathsf{A}^2 - \langle \mathsf{A} \rangle \mathsf{A} + [\mathsf{A}]\mathsf{I}, \quad \Rightarrow \quad \bar{\mathsf{A}}\mathsf{A} = \mathsf{A}\bar{\mathsf{A}} = |\mathsf{A}|\mathsf{I}, \tag{A.3}$$

where $\bar{\mathsf{A}}$ is the so-called complementary (or associated) operator to A. The invariants of this operator are

$$\langle \bar{\mathsf{A}} \rangle = [\mathsf{A}], \quad [\bar{\mathsf{A}}] = |\mathsf{A}|\langle \mathsf{A} \rangle, \quad |\bar{\mathsf{A}}| = |\mathsf{A}|^2, \tag{A.4}$$

and it possesses the properties

$$\bar{\mathsf{A}}^2 = [\mathsf{A}]\bar{\mathsf{A}} + |\mathsf{A}|\left(\mathsf{A} - \langle \mathsf{A} \rangle \mathsf{I}\right), \quad \bar{\bar{\mathsf{A}}} \equiv \overline{(\bar{\mathsf{A}})} = \frac{\mathsf{A}}{|\mathsf{A}|}. \tag{A.5}$$

All relations in (A.3), (A.4) and (A.5) are general relations for arbitrary three-dimensional operators A.

For the invariants of the sum of two operators A and B one derives the following identities (the complementary operators $\bar{\mathsf{A}}$ to operator A we define later)

$$\begin{aligned} &\langle \mathsf{A}+\mathsf{B} \rangle = \langle \mathsf{A} \rangle + \langle \mathsf{B} \rangle, \\ &[\mathsf{A}+\mathsf{B}] = [\mathsf{A}] + \langle \mathsf{A} \rangle \langle \mathsf{B} \rangle - \langle \mathsf{AB} \rangle + [\mathsf{B}], \\ &|\mathsf{A}+\mathsf{B}| = |\mathsf{A}| + \langle \bar{\mathsf{A}}\mathsf{B} \rangle + \langle \mathsf{A}\bar{\mathsf{B}} \rangle + |\mathsf{B}|. \end{aligned} \tag{A.6}$$

Special cases are $\mathsf{B} = \beta \mathsf{I}$

$$\begin{aligned} &\langle \mathsf{A}+\beta\mathsf{I} \rangle = \langle \mathsf{A} \rangle + 3\beta, \\ &[\mathsf{A}+\beta\mathsf{I}] = [\mathsf{A}] + 2\beta\langle \mathsf{A} \rangle + 3\beta^2, \\ &|\mathsf{A}+\beta\mathsf{I}| = |\mathsf{A}| + \beta[\mathsf{A}] + \beta^2\langle \mathsf{A} \rangle + \beta^3. \end{aligned} \tag{A.7}$$

The complementary operator of the sum of two operators is

$$\overline{\mathsf{A}+\mathsf{B}} \equiv \bar{\mathsf{A}} + \mathsf{AB} + \mathsf{BA} + \bar{\mathsf{B}} - \left(\langle \mathsf{B} \rangle \mathsf{A} + \langle \mathsf{A} \rangle \mathsf{B}\right) + \left(\langle \mathsf{A} \rangle \langle \mathsf{B} \rangle - \langle \mathsf{AB} \rangle\right)\mathsf{I}. \tag{A.8}$$

The projection operator Π for determination of eigenvectors of an operator A to non-degenerate eigenvalue α is

$$\Pi = \frac{\overline{\mathsf{A}-\alpha\mathsf{I}}}{\langle \overline{\mathsf{A}-\alpha\mathsf{I}} \rangle} = \frac{\mathsf{A}^2 - \left(\langle \mathsf{A} \rangle - \alpha\right)\mathsf{A} + \left([\mathsf{A}] - \alpha\langle \mathsf{A} \rangle + \alpha^2\right)\mathsf{I}}{[\mathsf{A}] - 2\alpha\langle \mathsf{A} \rangle + 3\alpha^2}, \quad \Pi^2 = \Pi. \tag{A.9}$$

For the differentiation of a determinant $|\mathsf{A}|$ of an operator A with respect to a parameter λ we find from (A.2) and (A.3) the identity

$$\frac{\partial|\mathsf{A}|}{\partial\lambda}=\left\langle\overline{\mathsf{A}}\frac{\partial\mathsf{A}}{\partial\lambda}\right\rangle=\frac{1}{2}\left\langle\frac{\partial\overline{\mathsf{A}}}{\partial\lambda}\mathsf{A}\right\rangle, \tag{A.10}$$

where the complementary operator $\overline{\mathsf{A}}$ to operator A is defined in (A.3).

Appendix B. Identities for Vector and Volume Products in Connection with Operators

We derive here mathematical identities for volume and vector products in connection with operators which are almost unknown or less known in case that they are already somewhere published. A part of them is used in the main text of our considerations.

We consider the volume product $[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$ of three vectors $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}$ and apply now to each vector the same operator A that means we consider the volume product $[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y},\mathsf{A}\boldsymbol{z}]$. Due to complete antisymmetry of the volume product with respect to permutations of neighbored vectors the volume product $[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y},\mathsf{A}\boldsymbol{z}]$ is proportional to the volume product $[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]$ with a proportionality factor which we denote by $|\mathsf{A}|$ and call the determinant of A that means

$$[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y},\mathsf{A}\boldsymbol{z}]=|\mathsf{A}|[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]. \tag{B.1}$$

One may convince oneself that this is really a good possibility to define the determinant of a three-dimensional operator. Now we write the chain of identities for a general volume product

$$[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y}]\mathsf{A}\boldsymbol{z}=[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y},\mathsf{A}\boldsymbol{z}]=[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]|\mathsf{A}|=[\boldsymbol{x},\boldsymbol{y}]|\mathsf{A}|\boldsymbol{z}=[\boldsymbol{x},\boldsymbol{y}]\overline{\mathsf{A}}\mathsf{A}\boldsymbol{z}, \tag{B.2}$$

where we substituted the determinant $|\mathsf{A}|$ according to $|\mathsf{A}|\to|\mathsf{A}|\mathsf{I}=\overline{\mathsf{A}}\mathsf{A}$ with $\overline{\mathsf{A}}$ the complementary operator to A (see also (A.3)). Since $\boldsymbol{z}$ is an arbitrary vector we may omit $\mathsf{A}\boldsymbol{z}$ in the identity (B.2) and obtain the identity for vector products

$$[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y}]=[\boldsymbol{x},\boldsymbol{y}]\overline{\mathsf{A}},\quad\Leftrightarrow\quad[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y}]\mathsf{A}=|\mathsf{A}|[\boldsymbol{x},\boldsymbol{y}]. \tag{B.3}$$

From this also follows almost immediately

$$\left[\overline{\mathsf{A}}\boldsymbol{x},\overline{\mathsf{A}}\boldsymbol{y}\right]=[\boldsymbol{x},\boldsymbol{y}]\overline{\left(\overline{\mathsf{A}}\right)}=|\mathsf{A}|[\boldsymbol{x},\boldsymbol{y}]\mathsf{A},\quad\left(\overline{\left(\overline{\mathsf{A}}\right)}\equiv\overline{\overline{\mathsf{A}}}=|\mathsf{A}|\mathsf{A}\right). \tag{B.4}$$

If we let act the operator A onto vectors $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{z}}$ to the left then from

$$[\tilde{\boldsymbol{x}}\mathsf{A},\tilde{\boldsymbol{y}}\mathsf{A},\tilde{\boldsymbol{z}}\mathsf{A}]=|\mathsf{A}|[\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}},\tilde{\boldsymbol{z}}], \tag{B.5}$$

follows in analogous way

$$[\tilde{\boldsymbol{x}}\mathsf{A},\tilde{\boldsymbol{y}}\mathsf{A}]=\overline{\mathsf{A}}[\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}],\quad\Leftrightarrow\quad\mathsf{A}[\tilde{\boldsymbol{x}}\mathsf{A},\tilde{\boldsymbol{y}}\mathsf{A}]=|\mathsf{A}|[\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}]. \tag{B.6}$$

If we substitute in (B.1) according to $\mathsf{A}\to\mathsf{A}-\alpha\mathsf{I}$ with arbitrary scalar α and use for determinants

$$|\mathsf{A}-\alpha\mathsf{I}|=|\mathsf{A}|-[\mathsf{A}]\alpha+\langle\mathsf{A}\rangle\alpha^2-\alpha^3, \tag{B.7}$$

then by collecting on both sides of the obtained identity the terms to equal powers of α we find in addition to (B.1) two further identities of the form

$$\begin{aligned}&[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y},\boldsymbol{z}]+[\mathsf{A}\boldsymbol{x},\boldsymbol{y},\mathsf{A}\boldsymbol{z}]+[\boldsymbol{x},\mathsf{A}\boldsymbol{y},\mathsf{A}\boldsymbol{z}]=[\mathsf{A}][\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}],\\&[\mathsf{A}\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]+[\boldsymbol{x},\mathsf{A}\boldsymbol{y},\boldsymbol{z}]+[\boldsymbol{x},\mathsf{A}\boldsymbol{y},\mathsf{A}\boldsymbol{z}]=\langle\mathsf{A}\rangle[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}].\end{aligned}\tag{B.8}$$

If we remove from (B.8) the vector $\boldsymbol{z}$ or make the substitution $\mathsf{A}\to\mathsf{A}-\alpha\mathsf{I}$ in the identity (B.3) using

$$\overline{\mathsf{A}-\alpha\mathsf{I}}=\overline{\mathsf{A}}-\left(\langle\mathsf{A}\rangle\mathsf{I}-\mathsf{A}\right)\alpha+\mathsf{I}\alpha^{2},\tag{B.9}$$

and collect all terms on both sides to equal powers of α we find in addition to (B.3) the identities

$$\begin{aligned}&[\mathsf{A}\boldsymbol{x},\mathsf{A}\boldsymbol{y}]+\left([\mathsf{A}\boldsymbol{x},\boldsymbol{y}]+[\boldsymbol{x},\mathsf{A}\boldsymbol{y}]\right)\mathsf{A}=[\mathsf{A}][\boldsymbol{x},\boldsymbol{y}],\\&[\mathsf{A}\boldsymbol{x},\boldsymbol{y}]+[\boldsymbol{x},\mathsf{A}\boldsymbol{y}]+[\boldsymbol{x},\boldsymbol{y}]\mathsf{A}=\langle\mathsf{A}\rangle[\boldsymbol{x},\boldsymbol{y}],\end{aligned}\tag{B.10}$$

or equivalently to the last

$$[\mathsf{A}\boldsymbol{x},\boldsymbol{y}]+[\boldsymbol{x},\mathsf{A}\boldsymbol{y}]=[\boldsymbol{x},\boldsymbol{y}]\left(\langle\mathsf{A}\rangle\mathsf{I}-\mathsf{A}\right).\tag{B.11}$$

All these mathematical identities may be also derived by means of the Levi-Civita pseudo-tensors. It is also clear that analogous identities can be derived for the action of operators onto the vectors to the left similar to (B.4) and (B.5) which we do not write down here.

We make a short remark to vector and volume products. It is very convenient to define $[\boldsymbol{y}]$ as an antisymmetric covariant second-rank pseudo-tensor to the contravariant vector $\boldsymbol{y}$. Then one can use identities for vector and volume products by displacement of the squared brackets such as[10]

$$\begin{aligned}&[\boldsymbol{y}]\boldsymbol{z}=[\boldsymbol{y},\boldsymbol{z}],\quad \boldsymbol{x}[\boldsymbol{y}]=[\boldsymbol{x},\boldsymbol{y}],\\&\boldsymbol{x}[\boldsymbol{y}]\boldsymbol{z}=\boldsymbol{x}[\boldsymbol{y},\boldsymbol{z}]=[\boldsymbol{x},\boldsymbol{y}]\boldsymbol{z}=[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}].\end{aligned}\tag{B.12}$$

The second line written with contra- and covariant indices and with the Levi-Civity pseudo-tensor ε_{jkl} written is for example

$$[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]=\varepsilon_{jkl}x^{j}y^{k}z^{l}=x^{j}\left(\varepsilon_{jkl}y^{k}\right)z^{l}=x^{j}[\boldsymbol{y}]_{jl}z^{l}=x^{j}[\boldsymbol{y},\boldsymbol{z}]_{j}=[\boldsymbol{x},\boldsymbol{y}]_{l}z^{l},\tag{B.13}$$

and $[\boldsymbol{y}]_{jl}\equiv\varepsilon_{jkl}y^{k}$ is the antisymmetric pseudo-tensor to vector y^{k}. An "antisymmetric" operator $[\boldsymbol{y}]_{l}^{i}$ which transforms vectors (or pseudo-vectors, depending on the kind of $\boldsymbol{y}$) can be made from $[\boldsymbol{y}]_{jk}$ only for Euclidean or pseudo-Euclidean spaces which possess a symmetrical metric tensor $g_{jk}=g_{kj},\left(g^{ij}g_{jk}=\delta_{k}^{i}\right)$ by $[\boldsymbol{y}]_{l}^{i}\equiv g^{ij}[\boldsymbol{y}]_{jl}$. With all this which may be presented in more precise form I made good experience for a long period of scientific work.

Appendix C. Algebra to a Special Operator for Bi-Anisotropic Media

We consider the algebra to the following three-dimensional operator L which

[10]This is likely also the intention when Fyodorov writes instead of my $[\boldsymbol{y}]$ the form $\boldsymbol{y}^{\times}$ with the consequence for the vector product "$[\boldsymbol{yz}]=\boldsymbol{y}^{\times}\boldsymbol{z}$" (similar to "$\boldsymbol{y}\times\boldsymbol{z}$" (Gibbs); besides, F. writes vector products also with squared brackets but without a comma between the vectors) however, with the disadvantage that it works only to the right onto vectors.

we need to obtain, in particular, to the calculation of the determinant of the operator for a general bi-anisotropic medium

$$\mathsf{L} \equiv \frac{\mathsf{A}\boldsymbol{x}\cdot\tilde{\boldsymbol{x}}\mathsf{A}-(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})\mathsf{A}}{|\mathsf{A}|}+\mathsf{B} \equiv \mathsf{M}+\mathsf{B}, \tag{C.1}$$

where A and B are general three-dimensional operators. It is a little more general than we need it since, in principal, we use in the main text only the special case of equality $\tilde{\boldsymbol{x}}=\boldsymbol{x}$ of the vectors $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$. With M we abbreviate the operator

$$\mathsf{M} \equiv \frac{\mathsf{A}\boldsymbol{x}\cdot\tilde{\boldsymbol{x}}\mathsf{A}-(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})\mathsf{A}}{|\mathsf{A}|}. \tag{C.2}$$

The determinant $|\mathsf{L}|$ of the operator L can be calculated by

$$|\mathsf{L}| = |\mathsf{M}|+\langle\overline{\mathsf{M}}\mathsf{B}\rangle+\langle\mathsf{M}\overline{\mathsf{B}}\rangle+|\mathsf{B}|, \tag{C.3}$$

where overlining an operator means the transition to the complementary operator (see Section 2). This formula is known [9] [10] and surely some others and is easily to obtain by coordinate-invariant calculation. Thus we have first to consider some algebra of the more complicate part M in (C.1).

First we have to calculate the powers M^2 and M^3 and their traces that is straightforward to make and that we do not write down. In particular, we find for the invariants of M

$$\begin{aligned}
\langle\mathsf{M}\rangle &= \frac{\tilde{\boldsymbol{x}}\mathsf{A}^2\boldsymbol{x}-\langle\mathsf{A}\rangle\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x}}{|\mathsf{A}|},\\
[\mathsf{M}] &= \frac{(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})\tilde{\boldsymbol{x}}\boldsymbol{x}}{|\mathsf{A}|},\\
|\mathsf{M}| &= 0,
\end{aligned} \tag{C.4}$$

that means the determinant of M is vanishing which simplifies the application of the formula (C.3). That the determinant $|\mathsf{M}|$ is vanishing follows also because the operator M possesses the eigenvalue $\mu=0$ to right-hand eigenvector $\boldsymbol{x}$ and to left-hand eigenvector $\tilde{\boldsymbol{x}}$. For the complementary operator $\overline{\mathsf{M}}$ to M using (A.3) we find then

$$\overline{\mathsf{M}} = \frac{(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})\boldsymbol{x}\cdot\tilde{\boldsymbol{x}}}{|\mathsf{A}|},\quad \langle\overline{\mathsf{M}}\rangle=[\mathsf{M}]=\frac{(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})\tilde{\boldsymbol{x}}\boldsymbol{x}}{|\mathsf{A}|}, \tag{C.5}$$

and, furthermore, it is easy to check

$$\overline{\mathsf{M}}\mathsf{M}=\mathsf{M}\overline{\mathsf{M}}=|\mathsf{M}|\mathsf{I}=0. \tag{C.6}$$

Now, according to the formula (C.3) with the explicit form (C.5) of $\overline{\mathsf{M}}$ and $|\mathsf{M}|=0$ we find

$$\begin{aligned}
|\mathsf{L}| &= \frac{(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})(\tilde{\boldsymbol{x}}\mathsf{B}\boldsymbol{x})-\langle\mathsf{A}\overline{\mathsf{B}}\rangle(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})+\tilde{\boldsymbol{x}}\mathsf{A}\overline{\mathsf{B}}\mathsf{A}\boldsymbol{x}+|\mathsf{A}||\mathsf{B}|}{|\mathsf{A}|}\\
&= \frac{(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})(\tilde{\boldsymbol{x}}\mathsf{B}\boldsymbol{x})-\tilde{\boldsymbol{x}}\left(\langle\mathsf{A}\overline{\mathsf{B}}\rangle\mathsf{A}-\mathsf{A}\overline{\mathsf{B}}\mathsf{A}\right)\boldsymbol{x}+|\mathsf{A}\mathsf{B}|}{|\mathsf{A}|}.
\end{aligned} \tag{C.7}$$

This agrees with the unessentially more special results of Fyodorov [9] (Equations (36.9), (36.10)) which, however, were calculated for the operator $\mathsf{B}^{-1}\mathsf{L}$ (in our notation) from multiple vector products which brings an additional factor $|\mathsf{B}|$ into the denominator of (C.7) but since he applies this immediately to the wave equation for which the determinant has to vanish he can omit this factor $|\mathsf{AB}|$ in the denominator in Equation (36.10). The numerator in (C.7) is symmetric with respect to permutation of the operators $\mathsf{A} \rightleftharpoons \mathsf{B}$. This can be achieved using a general operator identity from which results the following equivalent representation

$$|\mathsf{L}| = \frac{(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})(\tilde{\boldsymbol{x}}\mathsf{B}\boldsymbol{x}) - \tilde{\boldsymbol{x}}\left(\overline{\mathsf{AB}} + \overline{\mathsf{BA}} - [\mathsf{AB}]\mathsf{I} - \left([\mathsf{A}]\overline{\mathsf{B}} + [\mathsf{B}]\overline{\mathsf{A}} - [\mathsf{A}][\mathsf{B}]\mathsf{I}\right)\right)\boldsymbol{x} + |\mathsf{AB}|}{|\mathsf{A}|}, \quad \text{(C.8)}$$

which though a little longer in the numerator but shows this property. The mentioned operator identity is

$$\begin{aligned}[]
[\mathsf{A}]\overline{\mathsf{B}} = {} & \mathsf{ABAB} + \mathsf{AB}^2\mathsf{A} + \mathsf{BABA} - \langle\mathsf{B}\rangle\mathsf{ABA} - \langle\mathsf{AB}\rangle(\mathsf{AB} + \mathsf{BA}) \\
& + \left(\langle\mathsf{AB}\rangle\langle\mathsf{B}\rangle - \langle\mathsf{AB}^2\rangle\right)\mathsf{A} + [\mathsf{AB}]\mathsf{I},
\end{aligned} \quad \text{(C.9)}$$

and can be derived from a more general operator identity with 3 operators A, 1 operator B and 1 operator C in each sum term and in symmetric way which generalizes the Cayley-Hamilton identity by substitutions and specialization $\mathsf{C} = \mathsf{B}$. A derivation by means of the Levi-Civita pseudo-tensors is also possible. Since it is long we do not derive it here but hope to find opportunity to do this in future. Finally, we give here the other two invariants of L which are

$$\begin{aligned}
\langle\mathsf{L}\rangle &= \frac{\left(\tilde{\boldsymbol{x}}\mathsf{A}^2\boldsymbol{x} - \langle\mathsf{A}\rangle\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x}\right) + |\mathsf{A}|\langle\mathsf{B}\rangle}{|\mathsf{A}|}, \\
[\mathsf{L}] &= \frac{(\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})(\tilde{\boldsymbol{x}}\boldsymbol{x}) - \left(\tilde{\boldsymbol{x}}\mathsf{ABA}\boldsymbol{x} - \langle\mathsf{B}\rangle\tilde{\boldsymbol{x}}\mathsf{A}^2\boldsymbol{x} + \left(\langle\mathsf{A}\rangle\langle\mathsf{B}\rangle - \langle\mathsf{AB}\rangle\right)\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x}\right) + |\mathsf{A}|[\mathsf{B}]}{|\mathsf{A}|},
\end{aligned} \quad \text{(C.10)}$$

which are not symmetric in A and B and the complementary operator to L is

$$\begin{aligned}
\overline{\mathsf{L}} = \frac{1}{|\mathsf{A}|}\Big\{ & (\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x})\,\boldsymbol{x}\cdot\tilde{\boldsymbol{x}} + \mathsf{A}\boldsymbol{x}\cdot\tilde{\boldsymbol{x}}\mathsf{AB} + \mathsf{BA}\boldsymbol{x}\cdot\tilde{\boldsymbol{x}}\mathsf{A} - \tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x}(\mathsf{AB} + \mathsf{BA}) \\
& - \langle\mathsf{B}\rangle\mathsf{A}\boldsymbol{x}\cdot\tilde{\boldsymbol{x}}\mathsf{A} + \tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x}\left(\langle\mathsf{B}\rangle\mathsf{A} + \langle\mathsf{A}\rangle\mathsf{B}\right) - \left(\tilde{\boldsymbol{x}}\mathsf{A}^2\boldsymbol{x}\right)\mathsf{B} \\
& - \left(\tilde{\boldsymbol{x}}\mathsf{ABA}\boldsymbol{x} - \langle\mathsf{B}\rangle\tilde{\boldsymbol{x}}\mathsf{A}^2\boldsymbol{x} + \left(\langle\mathsf{A}\rangle\langle\mathsf{B}\rangle - \langle\mathsf{AB}\rangle\right)\tilde{\boldsymbol{x}}\mathsf{A}\boldsymbol{x}\right)\mathsf{I}\Big\} + \overline{\mathsf{B}}, \quad \langle\overline{\mathsf{L}}\rangle = [\mathsf{L}].
\end{aligned} \quad \text{(C.11)}$$

Appendix D. Is the Notion "Negative Refraction" Useful in Geometric and Wave Optics?

In connection with the paper of Pendry [22] and the mass of reactions to it I will express here also my thoughts though I did not follow this development from the beginning and cannot exclude that similar thoughts are already discussed in literature.

In wave optics in the treatment of beams we have a mean value $\boldsymbol{k}_0$ of the wave vector and a mean value ω_0 of the frequency and both together with the complex conjugate part but well separated are involved in the main factors $e^{\pm i(k_0 r - \omega_0 t)}$. From the wave vector one may form the refraction vector $\boldsymbol{n}$ by the definition

$$\boldsymbol{k} \equiv \frac{\omega}{c}\boldsymbol{n}, \quad |\boldsymbol{n}|^2 = \boldsymbol{n}^2, \quad \Rightarrow \quad \boldsymbol{k} = \frac{\omega}{c}|\boldsymbol{n}|\frac{\boldsymbol{n}}{|\boldsymbol{n}|} = |\boldsymbol{k}|\frac{\boldsymbol{n}}{|\boldsymbol{n}|}, \tag{D.1}$$

where both signs of $|\boldsymbol{n}|$ (refraction index; in general it is complex) can be chosen without something changing at the wave. Only the direction described by the unit vector $\frac{\boldsymbol{n}}{|\boldsymbol{n}|}$ in connection with the product $\boldsymbol{k} = |\boldsymbol{k}|\frac{\boldsymbol{n}}{|\boldsymbol{n}|}$ is invariant but not one of the two factors alone. If one change the sign of only one factor then one describes a wave propagating in the opposite direction and for isotropic media in no other than of these directions.

The second important fact is that under reflection and refraction of a beam at a surface $\boldsymbol{N}\boldsymbol{r} = a$ with the normal unit vector $\boldsymbol{N}$ the tangential components $\bar{\boldsymbol{k}} \equiv \left[\boldsymbol{N}\left[\boldsymbol{k}, \boldsymbol{N}\right]\right]$ of all coupled wave vectors $\boldsymbol{k}$

$$\boldsymbol{k} = \left[\boldsymbol{N}, \left[\boldsymbol{k}, \boldsymbol{N}\right]\right] + \boldsymbol{k}\boldsymbol{N} \cdot \boldsymbol{N} \equiv \bar{\boldsymbol{k}} + \boldsymbol{k}\boldsymbol{N} \cdot \boldsymbol{N}, \quad \left(\boldsymbol{N}^2 = 1\right), \tag{D.2}$$

in both media have to be the same and only the normal components $(\boldsymbol{k}\boldsymbol{N})\boldsymbol{N}$ can be different with both possible signs and the same is true for the refraction vectors $\boldsymbol{n}$. Even both signs of the normal component of the wave vector in the second medium are possible one as the refracted beam from the incident beam in the first medium and the second as an incident beam in the second medium which generates the refracted beam in the first medium which is identical in its direction with the reflected beam from the incident beam in the first medium. Both these beams possess a group velocity for isotropic media in the same direction as the wave vector independent on positive or negative signs of the electric permittivity and the magnetic permeability and whether or not the wave vectors are real or complex quantities. This excludes the possibility of refraction as drawn in Figure 1 in [22]. Only in case that the group velocity is in opposite direction to the wave or refraction vector in second medium we have in our **Figure 5** a right-hand picture which is somehow similar to mentioned of Pendry. This, however, is not connected with negative $\varepsilon(\omega)$ and $\mu(\omega)$ but merely with an active medium with all its problems as discussed (e.g., super-luminal velocities, super-diffraction of beams) and is hardly realizable. The situation changes but not basically if the medium is anisotropic and, obviously, this is not meant.

Wave vectors and equally refraction vectors possess only a modulus and a certain direction in space but not a certain sign of each separately and the notion in the heading seems to me not being really useful.