

A Re-Interpretation of Quasi-Local Mass

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Abstract

Because the equivalence principle forbids local mass density, we cannot formulate general relativistic mass as an integral over mass density as in Newtonian gravity. This century-old problem was addressed forty years ago by Penrose, and many papers have since extended the concept. Currently there is no satisfactory physical understanding of the nature of quasi-local mass. In this paper I review the key issues, the current status, and propose an alternative interpretation of the problem of local mass and energy density for gravity systems from an information perspective.

Keywords

Quasi-Local Mass, Curved Space-Time, Gravitational Energy, Stress-Energy Tensor, Encoding Information in Geometry, Schwarzschild Metric

1. Introduction

Except for general relativity, classical mechanics can be understood entirely within the context of the conservation of energy and momentum. Yet every general relativity text contains discussion of the fact that the energy of gravitating systems cannot be formulated in curved space-time. A recent Centennial paper [1] by Chen, *et al.* begins by stating:

“How to give a meaningful description of energy-momentum for gravitating systems... has been an outstanding fundamental issue since Einstein began his search for gravity theory.”

The problem of how best to describe the energy-momentum and angular momentum in gravitating system suffers from the fact that *“It is known that these quantities cannot be given a local density.”* A century-long failure to solve the fundamental problem indicates confusion; the goal of this work is to provide necessary clarification of the issue.

In fact, a *second* fundamental issue surfaces in quasi-local mass, that of Ha-

miltonian-based physics, in which the dynamics of a system is described through total energy by Hamilton's equations of motion. This approach is completely equivalent to Newtonian mechanics but does not apply to general relativity. Noether, who in 1918 proved "*there is no covariant total energy-momentum density tensor for gravitating systems*", also provided the fundamental basis of most 20th century physics by linking conservation theorems to symmetry transformations. But there is no general translational symmetry in curved space, so this avenue is denied to physicists. Penrose observed, non-local gravitational field energy is formulable only in expressions used *at infinity* for an *asymptotically flat* space-time manifold. In other words, the problems of curved space are treated by backing off to a "flat space" formulation at infinity. Moreover, an attempt is made to measure the energy of a system by enclosing it with a membrane, or closed spacelike two-surface, and attaching this to an energy-momentum four vector. Little wonder that all major contributors to the quasi-local mass approach admit that there is no framework in which it is understood. As this paper was being completed, yet another approach was published in which Hamiltonians in *asymptotically flat spacetime in five spacetime dimensions*, focused on spatial infinity.

The plan of this paper is as follows:

Part I introduces coordinate systems and the underlying issue of curved space.

Part II introduces the concept of quasi-local mass, invented by Penrose forty years ago, and reviews Yau's treatment of the problem and the reason for the focus on "null infinity" and asymptotically flat space-time, then Bart's 2019 summary of the current status of quasi-local conserved quantities.

Part III develops a reinterpretation of quasi-local mass, based on an information theoretic approach to the subject.

Part I Introduction to coordinate systems and invariance

An axiom of physics is that coordinate systems have no effect on physical reality. When this is violated, physics becomes confusing. Despite that coordinates can have no effect on physics, *the equivalence principle* of general relativity yields a built-in contradiction: by transforming local coordinates the gravitational field can be banished. Weinberg [2] on the equivalence principle:

"At every space-time point in an arbitrary gravitational field it is possible to choose a 'locally inertial coordinate system' such that [locally] the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation."

Einstein based his theory of gravity on the equivalence principle, which states that choice of the proper coordinate system makes gravity disappear; the basic problem of local density of energy-momentum. The principle is only approximate, applying when tidal forces can be ignored, but is generally treated as absolute. We later introduce an absolute principle, the principle of primordial self-interaction, but first we review coordinate system issues. We begin by noting that Euclidean space is Pythagorean in that distances in space and time are un-

limited.

Pythagorean distance in Euclidean *space* and *time*:

$$ds^2 = c^2 dt^2 + dx^2 + dy^2 + dz^2 \rightarrow \infty \quad (1)$$

Euclidian four-space can thus be mapped onto all events in the universe, allowing us to label every event and relate any event to any other. These relations constitute our *physics* or *models of reality*. But what are the relations? They are intended to capture objective reality in some sense. Nozick, in *Invariance: the structure of the objective world* [3] observes that an objective fact is accessible from different angles, *i.e.*, “*an objective fact is invariant under various transformations.*”

The Galilean transformation describes a photon’s position in 4-space with time $\bar{x} = \bar{v}t$ where velocity \bar{v} can point in any direction and $|\pm\bar{v}| = c$, with $x = -ct$ representing a photon moving in the negative direction. Differentiating we have $d(x - ct) = 0$ or $d(x + ct) = 0$. After Hestenes, [4] we define geometric algebra multi-vector $X = ct + \bar{x}$ and its complement $\tilde{X} = ct - \bar{x}$ and show that $dX d\tilde{X} = c^2 dt^2 - d\bar{x}^2 = 0$ is an *invariance* that holds for all distances $d\bar{x}$ and corresponding duration dt , independent of coordinate systems. If we denote the value $dX d\tilde{X}$ by ds^2 we obtain the invariance relation for the photon equation of motion, where we use $d\bar{x}^2 = dx^2 + dy^2 + dz^2$:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0 \quad (2)$$

We interpret this as an *invariance relation* in Euclidian 4-space and call it the *Minkowski invariance*. But Minkowski did not conceive of this as an invariance relation; he believed it to be a description of “space-time”, famously stating: “*space and time by itself are doomed to fade away... only a kind of union of the two will preserve an independent reality.*” The Lorentz transformation applied to this imagined spacetime rotates space into time and time into space. In *Energy-time theory*, developed in *Physics of clocks in absolute space and time* [5] this fundamental photon-based invariance (2) is used to derive classical Hamiltonian

$$H = E = \sqrt{m_0^2 c^4 + c^2 p^2} . \quad (3)$$

If two clocks tell identical time when side-by-side, and one clock is accelerated to velocity \bar{v} , the time interval read on the moving clock, $d\tau$, will run more slowly than the time duration dt measured on the clock at rest according to

$$\frac{d\tau}{dt} = \gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}} \quad (4)$$

Derivation of this *time dilation* is based on Galilean transformations in absolute time and space and relativistic inertia $m = \gamma m_0$ with m_0 the rest mass. Per Lucas and Hodgson [6]: “*If we... retain Newtonian dynamics, and the Newtonian definition of velocity and acceleration, then we... still obtain relativistically correct results if we... allow mass to depend on the velocity.*”

In special relativity rest mass is defined in all inertial reference frames in relative motion, and the Lorentz transformation applied to Minkowski “space” and

“time”. Yet the most solid experimental fact of special relativity, *time dilation*, is reproduced *exactly* in absolute space and time. Energy-time theory does *not* yield *length contraction* which Rindler [7] says will probably *never* be tested. Nor does it yield Einstein’s *velocity addition law*, known to be violated in particle accelerators [8]. The physics of *clock slowing* in absolute time (*universal simultaneity*) and absolute space (with *preferred frame* defined by local gravity) follows from increased inertia of the clock mechanism and consequent decrease in acceleration of the restoring force common to all harmonic oscillators, by *exactly* a factor of γ . Energy-time physics (compatible with *all* relativistic experiments [9]) is based on Galilean transformation *in time and space*. *Minkowski is an invariance relation*.

Thus, Hestenes’ multi-vector formulation of time and space leads to special relativity if viewed as *Minkowski spacetime* operated on by Lorentz transformation but leads to classical physics if treated as *Minkowski invariance*. The mathematical construct representing time and space is the same; the ontological assumptions determine what one does with it.

The principle of general covariance implies that coordinates are labels of space-time events that can be assigned completely arbitrarily. The only quantities that have physical meaning, the measurables, are *invariant* under coordinate transformations.” Such invariance is used to derive the physics of absolute space and time in reference 5; the physics of *energy-time* rather than relativistic *space-time*. Yet Poisson and Will [10] state: “*All local aspects of gravity can be turned off by doing physics in a freely moving (coordinate) frame of reference. Gravity is not present in these frames.*”

In other words, one can mathematically *do away with* the local gravitational field and hence any associated energy, a problem that has never been solved. Most physicists are aware of Noether’s seminal work on symmetry and conservation, but the reason Noether began investigations was “*to clarify the issue of gravitational energy.*” She proved “*there is no covariant total energy-momentum density tensor for gravitating systems.*” MTW [11] discuss this as “*a consequence of the equivalence principle...*”. A century of effort *has not solved the problem*.

2. Coordinates and Gravity as Curved Space-Time

Einstein’s transformation of gravity into geometry followed Minkowski’s idea of spacetime, and Einstein’s own analysis of Maxwell-Hertz equations, based on his interpretation of Michelson-Morley’s experimental results. Hertzian electrodynamics *is* Galilean invariant, but Einstein believed Lorentz invariance was required. Symmetry arguments imply that Lorentz’s transformation in four-dimensional spacetime is not supported in 3-space plus absolute time, but these arguments were not advanced until long after Einstein’s relativity was cast in stone. Following the historical approach, flat space is viewed as a limit to the general curved space metric:

$$ds^2 = g_{ij} dx^i dx^j, \quad (5)$$

where g_{ij} represents the curvature on a manifold in four-space with tangent vectors i and j . The geometric meaning of g_{ij} is quite clear, however the physical meaning is debatable. Most treatments seem to identify the term with gravity, however MTW state:

“...nowhere has a precise definition of the term ‘gravitational field’ been given – nor will one be given. Many different mathematical entities are associated with gravitation: the metric, the Riemann curvature tensor, the Ricci curvature tensor, the curvature scalar, the covariant derivative, the connection coefficients, etc. ... the terms ‘gravitational field’ and ‘gravity’ refer in a vague, collective sort of way to all of these entities.”

This is backwards; a better statement of physics would be:

“... all of these entities refer in a vague, collective sort of way to ‘gravitational field’ and ‘gravity’.”

In our treatment we view the metric g_{ij} as key. Beckwith [12] states: “In general relativity metric $g_{ij}(\bar{x}, t)$ is a set of numbers associated with each point which gives the distance to neighboring points, i.e., general relativity is a classical theory.” Per Yau [13], in general relativity Einstein’s equation is obtained by taking the variation of

$$\frac{1}{16\pi} \int R + \int \mathcal{L} \quad (6)$$

where R is the scalar curvature of the space-time and \mathcal{L} is the Lagrangian of matter coupled to gravity. The gravitational interaction is described by means of space-time metric g_{ij} with signature $(-, +, +, +)$; the metric of Minkowski space-time $\mathbb{R}^{3,1}$ (the vacuum with zero matter) being

$ds^2 = g_{ij} dx^i dx^j = -dt^2 + dx^2 + dy^2 + dz^2$. The variational equation has the form

$$R_{ij} - \frac{1}{2} R g_{ij} = T_{ij}. \quad (7)$$

We interpret this equation as follows: R_{ij} is curvature on a manifold, a Cartesian creation. The term R is a mean curvature over a manifold geometry, and T_{ij} is the physical stress energy tensor that induces the local curvature. In matter-free space $T_{ij} = 0$ and local curvature is essentially given by $R_{ij}/R = g_{ij}$, with the more global curvature represented by the smooth mean term R . When local stress energy is added, the smooth curvature is locally distorted with the difference in curvature represented by Equation (7). This interpretation will be useful for understanding quasi-local mass. The first solution to Einstein’s equation is the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + dr^2 (d\theta^2 + \sin^2(\theta) d\phi^2). \quad (8)$$

This and the Kerr metric both solve the Einstein equations with $T_{ij} = 0$ (vacuum).

Part II. Penrose and “quasi-local mass”

General relativity's incompatibility with local energy density of gravity has been known since Einstein formulated his equations, *i.e.*, the paradigmatic concept of energy conservation is *not* incorporated in Einstein's theory with respect to the energy of gravity itself. Energy tensor $T^{\mu\nu}$ describes the energy density of all *non-gravitational* fields; the gravitational field contributes only non-locally to the total energy. It is formulated only at infinity for an asymptotically flat space-time, while the energy of angular momentum exhibits an origin-dependence that is problematic even at flat space infinity. Non-local gravitational energy remained an unsolved problem, so Penrose in 1982 [14] proposed that "*energy-momentum is quasi-local: i.e., it is associated with a closed 2-surface...*". Of his new approach to the problem of local density, *quasi-local mass*, he noted: "*several problems of interpretation remain to be solved.*" The expression for total energy, momentum, or angular momentum surrounded by a closed surface, led to equations that, "*being highly over-determined, have no non-trivial solutions in the general curved space-time.*"

His attempts to formulate quasi-local angular momentum in terms of "twistor space", were based on complex analytic methods to solve problems in real differential geometry. According to Murray [15], in most cases the emphasis is on the *geometry* of the problem rather than *analysis*. Physical fields on Minkowski space are encoded into complex analytic objects on twistor space via the Penrose transform, which, per Murray, is an integral transform given by Whittaker and Watson. Penrose observes that "*we seem to require an analog*" of the infinity twistor, $I^{\alpha\beta}$. Based on several suggested definitions, and without the existence of the required analog, he used a relation from standard twistor theory, $m^2 = -A_{\alpha\beta}\bar{A}_{\alpha\beta}/2$, which Tod applied to a sphere of symmetry in Schwarzschild, Reissner-Nordstrom, and Friedmann-Robertson-Walker space-times, obtaining results related to the mass that would be surrounded by a sphere in flat space, of surface area equal to that of the quasi-local boundary, Ω , immersed in a fluid of density equal to that of the model. Yet the suggested definitions are not conformally invariant so they cannot be immediately applied at null infinity. Penrose ends by asking whether a *mass positivity theorem* can be proved at the quasi-local level, and whether one can obtain an *angular momentum flux law*. His approach to the problem of null local mass density in general relativity has survived for 40 years, despite the general level of confusion as to the physical interpretation of quasi-local mass.

3. Yau's Continuation of Penrose's Quasi-Local Mass

Following Equations (6) and (8) Yau considers the dynamics of scalar field $\Phi(t, r, \theta, \phi)$ in Kerr space-time whose propagation is described by a scalar wave equation. Since the Kerr space-time has a Killing vector $\partial/\partial t$, the Lagrangian $\int |\nabla\Phi|^2$ associated to the wave equation defines a local energy density which is positive except within the ergosphere, where it is negative, causing the energy method to breakdown. "*An important question for the wave equation is whether*

the wave will decay in time if initially it is not spread out. (...) The problem of linear stability of Kerr is still open.” The decay of the solution of the scalar wave equation is a special case of the decay of solutions to the Teukolsky equation which describes the linear stability of the Kerr black hole. A remarkable property of the Kerr metric is that the Teukolsky equation is separable by the ansatz

$$e^{-i\omega t - ik\phi} R(r) \Theta(\theta) \quad (9)$$

which Chandrasekhar observed “*has the aura of the miraculous.*” This is however compatible with the separation of gravity G and gravitomagnetic C field solutions of the Heaviside equations.

Most conservation theorems in physics derive from Noether’s treatment of symmetry. For most mechanical systems the Lagrangian is invariant under translation of time and space; the resulting conserved quantity is a four-vector defining energy and linear momentum. However, most general relativity spacetimes do not admit translational symmetry, thereby frustrating the use of Noether’s theorem. This is key to understanding Penrose’s work: “*...one can define a total energy-momentum vector for an isolated physical system if there is asymptotic translational symmetry.*” When space-time is asymptotically flat, there is a space-like hypersurface which outside a compact set is diffeomorphic to \mathbb{R}^3 minus a ball and the metric g_{ij} has the form

$$g_{ij} = \delta_{ij} + \mathcal{O}\left(\frac{1}{r}\right). \quad (10)$$

In [16] I show that *the principle of self-interaction of the primordial field*, $\nabla\psi = \psi\psi$, leads to the linearized metric formulated in flat space, which is the goal of the asymptotic approach.

Yau: “*The total energy in general relativity cannot be obtained by integrating any local density along a hypersurface—the density would depend on first order differential differentiation of g_{ij} and there is a coordinate system where such quantities are zero at that point*”, per the equivalence principle, violating the axiom of physics that physical quantities must be independent of choice of coordinate system. Yau proves that certain integrals over mean curvature are non-negative, then defines quasi-local mass in terms of the curvature of a closed space-like 2-surface in space-time, subject to conditions. He considers spherical symmetric space-time foliated by orbits of $SU(2)$ and associates to an orbit the area $4\pi r^2$. The mean curvature vector of the orbit is $-2\nabla r/r$ where ∇ is with respect to the quotient Lorentzian (1, 1) metric. From curvature formulas he obtains the quasi-local mass of this orbit’s sphere

$$M = r(1 - |\nabla r|). \quad (11)$$

He compares this to Misner and Sharp’s 1964 definition [17] of mass

$$m = \frac{r}{2}(1 - |\nabla r|^2) \quad (12)$$

and obtains the relation

$$m = M - \frac{M^2}{2r} \tag{13}$$

such that at space-like infinity $M = m$. Yau closes by stating that he is still in process of deriving more properties of the quasi-local mass that he just introduced.

Yet another popular quasi-local mass is the Bartnik mass [18], defined as

$$m_B \Upsilon(\Sigma) = \frac{1}{8\pi} \int_{\Sigma} (H_0 - H) d\Sigma \tag{14}$$

where Σ bounds a space-like hypersurface Ω and H_0 is the mean curvature of the unique isometric embedding of Σ into \mathbb{R}^3 and H is the mean curvature of Σ in Ω . Recall that we interpreted the variational Equation (7) as follows: R_{ij} is curvature on a manifold, while R is a mean curvature over a manifold geometry, and T_{ij} is the physical stress energy tensor inducing local curvature. The Bartnik mass (14) represents the mean curvature H of Σ in Ω , roughly corresponding to local curvature R_{ij} in four-space while H_0 is the mean curvature of the unique isometric embedding of Σ into \mathbb{R}^3 , that is, Minkowski space, defined asymptotically. This is a merely heuristic explanation, since, as Penrose noted, “several problems of interpretation remain to be solved.”

Yau stated that the Misner and Sharp mass (12) is the same as the Hawking mass

$$m_{H(\Sigma)} = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} |H|^2 d\Sigma \right), \tag{15}$$

where Σ is a spacelike 2-surface (the boundary over the region) and H is the mean curvature vector of Σ in the spacetime. If we assume that the H_0 term in the Bartnik mass is normalized such that

$$\frac{1}{16\pi} \int_{\Sigma} |H_0|^2 d\Sigma = 1, \tag{16}$$

we can transform the Hawking mass Equation (15) to more closely resemble Bartnik mass (14):

$$m_{H(\Sigma)} = \sqrt{\frac{|\Sigma|}{(16\pi)^3}} \left(\int_{\Sigma} (|H_0|^2 - |H|^2) d\Sigma \right). \tag{17}$$

4. Quasi-Local Conserved Quantities in General Relativity

Henk Bart’s 2019 PhD dissertation [19] summarizes the current situation:

“...there exists no general framework in which a definition of quasi-local energy is sufficiently understood.”

Bart attempts to provide such a framework, noting that “in general relativity, as a consequence of the equivalence principle, a local energy momentum tensor of the gravitational field does not exist.” Yet, “curvature of space-time is understood as a result of the presence of sources of energy. Energy in general relativity certainly exists, but not in a local sense. ... defining a notion of quasi-local

energy has proven to be surprisingly difficult...at the time of this writing the community does not agree on a set of pragmatic criteria that a notional quasi-local energy would have to satisfy.”

After review of some energy basics, having to do with the equivalence of the Hamiltonian and Lagrangian formalisms, he introduces the concept of *asymptotically flat spacetimes*, while admitting that what one means by “asymptotes” is a matter of taste.

The issue, again, is that conservation of energy has been formulated in terms of Noether’s theorem in which conservation is inherently linked to symmetry – conservation of energy and momentum are linked to time translation and space translation respectively. These translational symmetries are not generally available in curved space time and thus Noether’s power tools are not available. This lies behind the attempt to asymptotically map curved space-time into flat space-time where Noether’s theorem holds. So, Bart reviews *asymptotic symmetries at null infinity*, referred to as BMS symmetry after Bondi. He notes that the isometry group of flat space-time is the 10-dimensional Poincare group consisting of the Lorentz group of rotations and boosts and the translation group and stresses the fact that the asymptotic symmetry group is not the Poincare group, as might have been expected—instead *the asymptotic symmetry group is infinite dimensional*; a Hamiltonian at null infinity does not exist.

In attempting to provide a general framework for constructing quasi-local conserved quantities, he focuses on conserved quantities associated with asymptotic symmetries at null infinity, in the manner of Wald and Zoupas [20]. Bart also chooses the Einstein-Hilbert action supplemented with the Gibbons-Hawking-York boundary term

$$S = \frac{1}{16\pi} \int_M R + \frac{1}{8\pi} \int_{\partial M} (K - K_0) \quad (18)$$

where R as usual, denotes the Ricci scalar and K is the trace of the extrinsic curvature density of the (time-like) boundary ∂M . A Hamiltonian on M is defined in terms of Noether’s charge two-form $Q[\xi]$ as follows:

$$H_x[\xi] = \int_C Q[\xi] - \xi \cdot B_x \quad (19)$$

representing a quasi-local conserved quantity defined on ∂M , with boundary conditions X . This equation is not yet well defined. B is a closed spacelike codimension two-surface in ∂M , on which H represents a quasi-local conserved quantity associated with a vector field that is tangent to ∂M . Much of Bart’s effort involves an attempt to meaningfully define B subject to conditions X . He does not in detail investigate either uniqueness or existence of the proposed boundary. He does show that the asymptotic limit of the quasi-local BMS charges (his Equation (2.78)) agree with the BMS charges constructed by Wald and Zoupas at null infinity. He concludes that the BMS charge given by his Equation (2.78) satisfies certain criteria:

- 1) It vanishes on Minkowski space-time

- 2) It asymptotes to Bondi mass at null infinity.
- 3) On round spheres in the metric it is equal to the Misner-Sharp energy.

He then puts forward the possibility that the gravitational part of the zero mode BMS charge as constructed may be a useful definition of the quasi-local energy. This is his result.

Summary of quasi-local mass

Penrose attempted to address the problem by defining “quasi-local mass” (and angular momentum), and others during the past forty years have worked on this concept because

“many important statements in general relativity make sense only with the presence of a good definition of quasi-local mass.”

It is apparent that four decades of development of the concept of quasi-local mass has produced nothing that provides a physically meaningful interpretation of local energy density; forbidden by the equivalence principle. As seen, the primary tool of conservation, Noether’s theorem, fails due to the general lack of translation symmetry in curved space-time. The approach to this has been to seek meaningful asymptotic symmetry at null infinity, although this introduces its own set of problems. In this context a Hamiltonian-like construct is defined and attempts to reproduce Noether-like results are made; nevertheless, no one today has any physically meaningful understanding of local energy density in general relativity. Baggott [21]: For Bohr, the Copenhagen interpretation obliges us to resist the temptation to ask: *but how does nature actually do that?*

“And there lies the rub: for what is the purpose of the scientific theory if not to aid our understanding of the physical world?”

It seems unlikely that any further work along the lines we have examined above will lead to such. Therefore, a re-interpretation of quasi-local energy seems appropriate. This is our focus in part III.

5. Problems with Einstein’s Equations

A non-vanishing energy-density necessarily produces gravity, represented as curvature of space-time. Recall that Einstein’s vision of gravity is *pure geometry*. His basic equation

$$G_{\mu\nu} = \kappa T_{\mu\nu}^{(m)} \quad (20)$$

specifies that stress-energy tensor $T_{\mu\nu}^{(m)}$ defines the distribution of mass-energy while $G_{\mu\nu}$ defines geometry of the *curved space-time* induced by the existence of the material stress-energy $T_{\mu\nu}^{(m)}$. The problem is to *derive the curved space geometry* from $T_{\mu\nu}^{(m)}$. Lorentz, Levi-Civita and Klein argued that the Einstein curvature tensor $G_{\mu\nu}$ was the only proper gravitational energy-momentum density; hence one should regard the Einstein equation in the form

$$-\kappa^{-1} G_{\mu\nu} + T_{\mu\nu}^{(m)} = 0 \quad (21)$$

as describing the vanishing sum of gravitation and material energy-momentum. However, this is generally *self-contradictory* if the solution is in curved space-

time and curvature is known only *after* the problem is solved; how could we possibly express the source distribution in the *not yet solved for* curvature geometry? We cannot solve for curvature without knowing the source; and *cannot* express source distribution $T_{\mu\nu}^{(m)}$ responsible for curvature in the curved space that is the object of solution. How is this problem handled in relativity? Chen *et al.* observe:

“Specifically, in the case of GR, knowing the curvature gives, via the Einstein tensor, the symmetric Hilbert energy-momentum density.”

This does *not* solve the problem. It states that, *if* the problem could be solved, *and* we knew the solution (“knowing the curvature”) then we could represent the energy-momentum density. This is *not* entirely un-akin to the observation that, “*if we had some ham, we could have ham and eggs, if we had some eggs.*” It does *not* solve the never-solved problem of gravitational energy density that induces curvature of space-time.

Physicists generally have no intuitive conception of *curved space-time*. Feynman [22] observed that gravity theory suffers because

“...one side of the equation is ... geometric, and the other side is not [geometric] ...even for very simple problems, we have no idea how to go about writing down a proper $T^{\mu\nu}$.”

Two possible solutions exist: the trivial solution is $G_{\mu\nu} = 0$; the stress-energy tensor is everywhere zero in all coordinate frames. This generally implies “flat space” but Vishwakarma has analyzed curvature in [23] in view of the fact that a proper energy-stress tensor of the gravitational field does not exist. He also discusses the Kasner solution, which I have treated in [24]. As noted in his title, this trivial solution involves “*a new paradigm in GR*”.

There is also a *nontrivial* way to avoid the paradox of solving an equation expressed in nontrivially different coordinate systems. This involves placing the center of mass of a spherically symmetric body at *the only point* common to both flat space and curved space, *the origin*, and appealing to Birkhoff’s *Shell theorem*. This inherently limits Einstein’s general relativity to *One-body solutions* of the *N-body problem*.

Heaviside [25] extended Newtonian physics to include *field energy density* in flat space, and this is recognized as iteratively equivalent to Einstein’s non-linear field equation. Feynman noted:

“*It is one of the peculiar aspects of the theory of gravitation, that it has both a field interpretation and a geometric interpretation...*”

It is not generally clear how a density-based field description is related to metric *curvature* solutions; our immediate goal is to clarify this.

Part III Information Coding of Density as Geometry

Consider the transition from *physical-field-based* physics in flat space to *curved-space-time-geometry-based* physics? Despite the common perception that *Minkowski space-time* views space as *empty*, Einstein [26] came to realize that

“There is no such thing as empty space, i.e., a space without a field. Space-time does not claim existence on its own, but only as a structural quality of the field.”

Further, *physical fields are real and have energy.* Ohanian and Ruffini [27] *“The gravitational field may be regarded as the material medium sought by Newton; the field is material because it possesses energy density.”* Gravity is not viewed as *curved space-time*, but as a *field with energy density*. In fact, special relativity is based on pretending that gravity does not exist at all; and general relativity on pretending that gravitational energy does not exist “locally” – it can always be transformed away. Yet *coordinate systems*, curved or flat, *can have no effect on physical reality*. The *logical contradiction* that changes in coordinate system can make a physically real field vanish is hidden in the *Equivalence Principle* used to derive our premier theory of gravity: *“gravity as space-time curvature”*.

Rather than view “information” as a physical entity [28], we treat information in its original *coding* perspective: there is no energy density information encoded in flat space coordinates. Coordinate systems can change with *absolutely no effect* on field energy density; and the energy distribution can change with absolutely no effect on the coordinate system. All information about the physical field is contained in the *energy density distribution*; none in the flat space coordinates!

If we remove energy density information by claiming zero local energy density, *it must be replaced somehow!* The only physical info is energy density and coordinate info, so when removing the density info we must replace it with coordinate info. This is done by *replacing constant length flat space coordinates with variable length (metric) curved space coordinates*. No information is lost; the real physical information encoded by the density distribution over flat space is replaced by abstract information in which *physical energy density is encoded as “geometry”*. According to Poisson and Will:

“the metric ... achieves two purposes: it encodes geometrical information about coordinate system, and it encodes physical information about the gravitational field.”

I would say that *the metric encodes physical information about the gravitational field as geometrical information in a coordinate system*. This clearly recognizes that the physical field energy density is ontologically real and is mathematically equivalent to the description of the real physical density encoded in the abstract formulation of curved space time!

Physical density-based reality is distinguished from *curved space-time-based reality* via inertia. In *Energy-time theory* the special relativity γ -factor applies to *inertial mass*. In *curved spacetime* real inertial force is replaced by the geodesic abstraction. However, Feynman, Weinberg, Padmanabhan and others insist that *“Curved space-time is not a necessary conception of gravity.”*

The primordial gravitational field is treated in *The Primordial Principle of Self-interaction*. Self-interaction equation $\vec{\nabla} \vec{\psi} = \vec{\psi} \vec{\psi}$ is used to derive Heavi-

side equations known to be equivalent under iterations to Einstein's field equations. A geometrical algebra multi-vector is $\vec{\psi} = \vec{G} + i\vec{C}$ where \vec{G} is the gravitational field vector and \vec{C} is the gravitomagnetic bivector and duality operator i converts \vec{C} to \vec{C} . In the absence of circulation bivector \vec{C} the solution to $\vec{\nabla}\vec{\psi} = \vec{\psi}\vec{\psi} \Rightarrow \vec{\nabla}\vec{G} = \vec{G}\vec{G}$ is $\vec{G} = 1/\vec{r}$ ($\equiv \vec{r}/r^2$). Hence $\vec{G}\vec{G} = 1/r^2$ is the energy density of the primordial field. The mass of the field inside a sphere of radius r is (modulo $4\pi/3$)

$$m \cong r^3(\rho) = r^3(\vec{G}\vec{G}) = r^3\left(\frac{1}{r^2}\right) = r \Rightarrow \frac{m}{r} \cong 1 \quad (22)$$

Thus potential $\phi = -(m/r \cong 1)$ where Newton's gravitational constant $g = 1$ and speed of light $c = 1$. Let us assume that this energy is distributed with unit density. Consider a volume defined by $dx dy dz$ where $dx=1, dy=1, dz=1$. Ignoring the sign of ϕ we define unit energy density of the primordial gravitational field with no material objects in the field. Recall that the difference in potential energy of two points is the work done in moving mass m between two points.

$$U(r) = W_{\infty r} = \int_{\infty}^r \vec{F} \cdot d\vec{r} = \int \frac{gMm}{r^2} dr = gMm \left[-\frac{1}{r} \right]_{\infty}^r = -\frac{gMm}{r} = m\phi \quad (23)$$

The gravitational energy per unit mass, ϕ describes potential energy at the point in question; it is interaction energy $m\phi$ that would exist if m units of mass were to be introduced at that point. This *test mass* m is presumed so small that its own gravitational potential need not be considered. Next add mass M to space and recall that the only point common to both flat space and curved space is the origin $(0,0,0)$, assumed coincident. The massive body M does not exist at a point but assume it is symmetrical and apply Birkhoff's *Shell theorem* to reduce M to an equivalent point mass. The gravitational energy (per unit mass) r distant from the origin is $\phi = -M/r$. The total gravitational energy is the sum of the primordial gravitational field plus the potential due to mass M located at the origin

$$\frac{m}{r} + \frac{M}{r} \Rightarrow 1 + \phi. \quad (24)$$

Unit cube $dx dy dz$ with $dx=1, dy=1, dz=1$ contains energy density $1 + \phi$. The negative sign of the potential is suppressed, but as seen in Equation (30) the term of interest is always positive, whether $(1 + \phi)$ or $-(1 + \phi)$. We normalize this density $1 + \phi \rightarrow 1$ by choosing new variable coordinates $dx' \neq 1, dy' \neq 1, dz' \neq 1$. Based on the equivalence principle one might assume that $1 + \phi \rightarrow 0$. But local density that *cannot be defined* is not the same as local energy density disappearing; but it ceases to provide *physical information*. We normalize the energy in every volume element such that $dx' dy' dz'$ always contains unit energy as seen on the right side of **Figure 1**.

In this way information has been transformed from the energy density of the field ϕ , in constant (informationless) coordinates of flat space, to the information

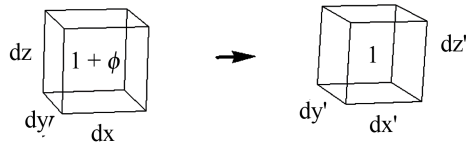


Figure 1. The primordial gravitational field has $\phi_0 = m/r = r/r = 1$ in flat space, $dx, dy, dz = 1$. Mass M at the origin adds potential energy $\phi = M/r$. We transform to a unit density (no density information) in curved spacetime as shown in the cube at right, $dx', dy', dz' \neq 1$.

of the (variable) coordinates containing a constant (informationless) energy density in curved space-time; consistent with the fundamental requirement that coordinate systems cannot affect physical reality. They label physical reality such that we all agree upon the points under discussion. The value 1 represents apparent energy density in curved space instead of the value zero seemingly implied by the equivalence principle. Thus, it is not required that energy density vanish, only that *information about the energy density must vanish*, in accordance with the statement that “*gravitational energy is not localizable*”. So, we work not to get rid of physical energy density; but to get rid of energy density *information*. Energy density is normalized such that every local region in curved space contains exactly one unit of energy in a region bounded by $dx', dy', dz' \neq 1$. Defining dx'_i to accomplish this goal transfers physical density information in flat space to curvature information of curved space. Every flat space differential has unit length; no information contained in a normalized unit of the coordinate system. In curved space, metric intervals are defined to normalize energy density of the field, removing information from the energy density, transferring it to the curved space metric.

6. Analyzing the Encoding Scheme

“Gravitational energy is not localizable” if *every local region* (in curved space-time) *contains a unit of energy indistinguishable from any other local region*, whereas the local metric at any point is distinguishable from the metric at any neighboring point. This is captured by the generalized Pythagorean for curved space-time:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \tag{25}$$

Metric $g_{\mu\nu}$ transforms energy density information contained in ϕ into the local coordinate information contained in $g_{\mu\nu}$. Whereas $dx dy dz$ coordinate differentials are independent of ϕ , the transformed coordinates are totally dependent on ϕ , *i.e.*,

$$dx' dy' dz' \equiv dx'(\phi) dy'(\phi) dz'(\phi) . \tag{26}$$

For the most general solution we force each metric-based component to satisfy the relation

$$(1 + \phi) dx'_i(\phi) = dx_i \tag{27}$$

in which case the key coordinate variable relation becomes

$$dx'_i(\phi) = \frac{dx_i}{1+\phi} \quad (28)$$

The consequent volume element becomes

$$dx'_1(\phi)dx'_2(\phi)dx'_3(\phi) \equiv \frac{dx_1}{1+\phi} \frac{dx_2}{1+\phi} \frac{dx_3}{1+\phi}. \quad (29)$$

And for any two of these dimensions, we obtain the (always positive) product:

$$dx'_i(\phi)dx'_j(\phi) \equiv \frac{dx_i}{1+\phi} \frac{dx_j}{1+\phi}. \quad (30)$$

In 3-space this is effectively a *time-slice* of Euclidian reality, $dt \equiv 0$; using convention $i, j \in \{1, 2, 3\}$, $\mu, \nu \in \{0, 1, 2, 3\}$. Four dimensional spacetime contributes to the difficulty of reconciling Einstein's theory with intuitive ideas of time and space. We consider 3-space energy density so we begin with the Pythagorean relation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. From Equations (25) and (30) we see that

$$g_{ii} = \frac{1}{(1+\phi)^2} = (1+\phi)^{-2}. \quad (31)$$

Consider Wheeler's remarks about a white dwarf star: "*It is small, but not terribly small; dense, but not terribly dense. Space-time is 'flat' within it...*" The term "flat" here means ϕ is small, approaching zero, allowing use of the binomial expansion $(1+\phi)^{-2} \cong 1-2\phi$. Yilmaz, [29] derives a metric proportional to $e^{-2\phi}$ compatible with consensus GR for which $g_{ij} = 1-2\phi$. This approximation is consistent with linearized equation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric with signature (+, -, -, -) and $h_{\mu\nu} = -2\phi$. Limiting ourselves to 3-space ($dt \equiv 0$) we obtain

$$ds^2 = \frac{-1}{1+2\phi} (dx^2 + dy^2 + dz^2). \quad (32)$$

since the binomial expansion also yields $\frac{1}{(1+\phi)^2} = \frac{1}{1+2\phi}$. Factor h_{ii} represents

diagonals term $dx_i dx_i$. In *A Primordial Space-Time Metric* the gravitomagnetic field of the dense region of field moving in the z -direction satisfies $h_{xy} - h_{yx} = 0$. This angular momentum relation causes $dx dy$ and $dy dx$ terms to cancel, yielding Equation (32). The exact solution we've been considering is based on the mass M located at the origin where spherical symmetry implies

$h_{xy} = h_{yx} = 0$ (and cyclical iterations); also, compatible with Equation (32). The alternative form of Equation (32) yields $ds^2 = -(1-2\phi)(dx^2 + dy^2 + dz^2)$. Observe that since $\phi = -gM/r$ we let $g = 1$ and then rewrite the equation as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)^{-1} (dx^2 + dy^2 + dz^2) \quad (33)$$

Inspection shows that Equation (33) is identical to the dr^2 term in Schwarzschild metric Equation (8) when the Minkowski signatures are considered. GR textbooks can be consulted for derivation of the Schwarzschild metric surrounding mass M at the origin. The normalizing encoding process derives *ex-*

actly the three-space metric representing gravitational energy density equivalent information. Rindler explains the Schwarzschild metric spatial geometry in understandable terms:

“One way to visualizing the curved 3-space like that of the Schwarzschild lattice, whose metric is given by

$$dl^2 = \frac{dr^2}{1+2\phi} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \tag{34}$$

is to pretend that it is really flat, but that its rulers behave strangely...”

In (34), we consider a *radial ruler*, [in which case $d\theta = d\varphi = 0$] and note that the rulers shrink by factor $(1+2\phi)^{1/2} \approx 1+\phi$; *exactly* the result for dx, dy and dz that we achieved in Equation (28) by transforming from actual physical energy density in flat space (constant rulers) to the equivalent information encoded as geometry of curved space (variable metric). QED.

By comprehending that we are *encoding energy-density as geometry* [30] and drawing a few simple diagrams we transform from flat space energy density (physical reality) to curved space with almost no computation and obtain the standard Schwarzschild space metric. What about the *time-time* metric, g_{00} ?

7. The Time Metric Associated with Curved 3-Space

Gravitational energy density *information* is banished in curved space by the equivalence principle, but physical information associated with the density of the field cannot be simply discarded—it must be transformed into the geometry of the variable metric coordinate systems. In this case the non-trivial solution represented by the Schwarzschild metric. The assumption of absolute space and time underlying the physics of inertial clocks is based on defining absolute time as *universal simultaneity*, hence our curved space solution holds at any time (slice) t with $dt \equiv 0$. Thus the time dimension dropped out of Schwarzschild’s metric solution reducing the problem to that of curved 3-space as derived in Section 6. Recall that the Minkowski invariant defined by Equation (2),

$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = 0$ is *invariant* because it does not vary with coordinate systems, hence $ds'^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 = 0$ and hence $ds^2 = 0 = dt^2 - dr^2 = dt'^2 - dr'^2$. Equation (28) expresses coordinate differentials as dx_i and dx'_i therefore we use $ds^2 = g^{\mu\nu} dx_\mu dx_\nu$. Hence, to obtain an expression for the time-time metric we use

$$dt'^2 = dr'^2 = \frac{dr^2}{1+2\phi} = g^{00} dx_0 dx_0 = g^{00} dt^2. \tag{35}$$

Minkowski-invariant-based energy-time theory derives from the equation of motion for photons

$$dr^2 - c^2 dt^2 = 0 \Rightarrow \frac{dr^2}{dt^2} = c^2. \tag{36}$$

Therefore, from Equation (35) we find $\frac{dr^2}{dt^2} = g^{00} (1+2\phi) = c^2$. The inverse

metric $g_{00} = 1/g^{00}$ and for $c^2 = 1$ we obtain $g^{00}g_{00} = 1$ and $g^{00}(1+2\phi) = 1$. Therefore

$$g_{00} = 1 + 2\phi \quad (37)$$

which is found to be in *exact agreement* with the Schwarzschild time-time metric, thus our flat space distribution of field energy density, $\phi(\vec{r})$ encodes energy density information as geo(metric) information of curved space and the corresponding time (the conjugate of energy).

8. Quasi-Local Mass Density in Encoded Geometry

Having derived exactly the Schwarzschild metric solution to Einstein's geometric formulation, in terms of the Heaviside equations in Euclidean space, we apply this to the interpretation of quasi-local energy distribution. The energy exists, but the density information has been normalized and now exists in the curvature metric: every $dx'dy'dz'$ cell in the geometric formulation contains one unit of gravitational field energy; thus, gravitational fields in flat space, with physically real energy density, can be formulated in curved space with normalized field energy. All density information of the field has been transferred to the coordinate system metric, such that energy is normalized in *every* volume element in curved space. Schwarzschild's solution to Einstein's equations literally falls out of this approach, implying that Feynman, Weinberg, and others are correct in their claims that curved space-time is not a necessary conception of gravity.

Heaviside's equations derive straightforwardly from a primordial self-interaction principle, which, unlike the approximate equivalence principle, is an *absolute* principle of physics—Heaviside is not a *weak field approximation* but instead is valid for *all* field strengths. Misunderstanding this has led physicists to view Einstein's curved space formulation as the *true* theory of gravity, and flat space equations as only valid when fields are weak. The two are equivalent formulations but it is the misnomer “weak field approximation” that sticks in physicist's minds.

We conclude from this treatment that reality is Euclidian, not Riemannian. Einstein adopted a nonphysical geometric approach, even though physical gravity in flat space connects geometrical gravity and curved spaces *only at one point*, the shared origin of the two coordinate systems. It is not trivial that Feynman claims “*we have no idea how to go about writing down a proper $T^{\mu\nu}$* ”. Equally significant is the claim by Will and Poisson that Heaviside-based *weak field* methods work amazingly well in strong fields.

We are thus led to a re-interpretation of quasi-local energy density. Energy density physically exists (how could it not?) but information contained in the physical field is transferred to variable coordinates as described. Local volumes of space are defined such that every volume contains the same energy as every other volume, where the volumes are defined as $dx'dy'dz'$ related to flat space volume's $dx dy dz = 1$ by Equation (28). Each side of a cube in curved space is $1/(1+M/r)$ that of a corresponding cube in flat space, leading to the situation

shown schematically in **Figure 2**, which shows the density-based volumes at 10 different equally spaced values of r . This technique is chosen because it is reasonably easy to calculate and reasonably easy to visualize. An attempt to draw the configuration at every point in space would be far more complicated and would add nothing to the comprehension of the meaning of quasi-local energy density.

Earlier we observed that Yau’s calculation of quasi-local mass in spherical symmetric space-time foliated by orbits of $SU(2)$ [Equation (11)] compared to Misner and Sharp’s [and Hawking’s] definition of mass [Equation (12)] yields relation (13)

$$m = M - \frac{M^2}{2r},$$

such that at spacelike infinity $m = M$. We now rewrite this as

$$\frac{m}{M} = 1 + \frac{\phi}{2}, \tag{38}$$

recognizing that the equality at space-like infinity corresponds (asymptotically) to the flat space density, while on the apparent horizon he obtains $M = 2m$, we ignore the factor of two and consider

$$\frac{m}{M} = 1 + \phi. \tag{39}$$

Unsurprisingly we find that quasi-local mass is formulated such that the key geometric encoding term $(1 + \phi)$ is prominent. For a concept that has resisted physical interpretation for 40 years, this is about as far as we wish to push things.

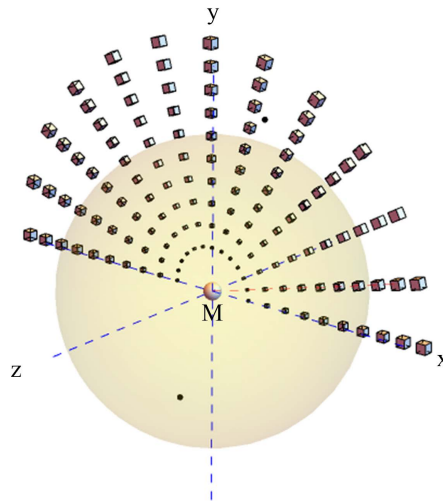


Figure 2. The graphical representation of quasi-local energy density based on the density information encoded geometrically. Every volume element in curved space holds exactly one unit of field energy. That is, the curved space volume elements are distributed in flat space to aid visualization. This approach yields *exactly* the Schwarzschild solution to Einstein’s geometric formulation. One half of one slice through the spherical volume about mass M is shown, as the full 3D image would be too “busy” and add nothing to our understanding.

9. Conclusions

The *equivalence principle-based* banishment of local energy density is so intolerable that physicists and mathematicians proposed the idea that, while energy density did not exist locally, it could somehow be integrated inside an appropriately defined curved-space boundary. Penrose proposed this concept in 1982 but noted that *several problems of interpretation remain to be solved*. Thirty years later Yau closed his treatment saying he was still in process of deriving new properties of the quasi-local mass that he just introduced. A decade later Bart noted that *currently no framework exists in which a definition of quasi-local energy is sufficiently understood*. The approach taken by these three (and others) consisted essentially of attempting to formulate energy density as proportional to curvature, with emphasis on “quasi-local” density as the difference between local curvature and mean curvature. This was associated with “*asymptotic symmetries at null infinity*” to regain translational symmetry so that Noether’s theorem could be applied, and meaningful Hamiltonians introduced. The conglomeration of concepts, none of which are physically meaningful, produced mathematical results that proved nothing, but led to hope that things could be worked out, if only we understood the problem a little better.

We here re-interpret gravity theory in such a way that the energy density of the field is physically real; its disappearance is explained by transferring density information from the field in information-less flat space coordinates to curvature information based on the Newtonian potential, with results equivalent to the Schwarzschild solution. **Figure 2** represents normalized curved space density displayed in flat space coordinates (corresponding to “null infinity”). It views each curved space volume element as the relevant boundary, inside of which a known quantity of field energy exists. The total energy is obtained by adding the relevant volume elements. The exact mathematical procedure to best accomplish this may be messy, but the physical understanding of gravitational field reality should be quite clear. Based upon our analysis of the gravitational field energy-momentum we concur with Feynman, Weinberg, and others: the concept of curved space-time is *not* necessary for gravity. Ontological arguments imply that both theories are not equally valid.

The *equivalence principle-based* disappearance of energy of the gravitational field when curved coordinate frames appear has confused general relativists since Einstein proposed his geometric approach. In *energy-time theory* the specification of field energy density in flat space determines the physics. If this information is removed, it must be replaced by some other source of information; the curved coordinates supply this information. The physical gravitational field does *not* vanish; its energy density is normalized: every corresponding volume element in curved space coordinates, $dx'dy'dz'$, contains exactly *one* unit of energy, yielding a metric *exactly* equal to Schwarzschild’s solution. Gravitational field energy does *not* vanish in this approach; the normalization effects the transfer of *gravitational field information* to curved space coordinates.

Will and Poisson note that this *flat space* approach is useful for very strong fields, but they offer no reason for this fact. *The primordial principle of self-interaction* derives Heaviside's equations with no *weak field* assumptions; the equations work for *all* field strengths and represent a complete theory of gravity. This radically differs from the prevailing view that only curved space-time equations are complete.

The latest relevant paper dealing with quasi-local mass [31] illustrates the major issue: physicists have essentially given up on physically understanding reality in favor of mathematically elaborating upon well-established problems. It investigates the asymptotic structure of Einstein gravity in *five spacetime dimensions* (which no one understands, other than mathematically) by focusing on spatial infinity and using Hamiltonian techniques.

The implications of *the primordial principle of self-interaction* and of *encoding energy-density in geometry* are that curved-space formalisms are artifacts and gravitational field theory in Euclidean space represents physical reality. An ontological shift to a density-based formalism has significant consequences.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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