Journal of Modern Physics, 2021, 12, 1418-1434
https://www.scirp.org/journal/jmp
ISSN Online: 2153-120X
ISSN Print: 2153-1196

Direct Relativistic Extension of the Madelung-de-Broglie-Bohm Reformulations of Quantum Mechanics and Quantum Hydrodynamics

Arquimedes Ruiz-Columbié[1], Hira Farooq[2], Luis Grave de Peralta[2,3*]

[1]Wind Energy Program, Texas Tech University, Lubbock, TX, USA
[2]Department of Physics and Astronomy, Texas Tech University, Lubbock, TX, USA
[3]Nano Tech Center, Texas Tech University, Lubbock, TX, USA
Email: *luis.grave-de-peralta@ttu.edu

How to cite this paper: Ruiz-Columbié, A., Farooq, H. and de Peralta, L.G. (2021) Direct Relativistic Extension of the Madelung-de-Broglie-Bohm Reformulations of Quantum Mechanics and Quantum Hydrodynamics. *Journal of Modern Physics*, **12**, 1418-1434.
https://doi.org/10.4236/jmp.2021.1210085

Received: July 18, 2021
Accepted: August 6, 2021
Published: August 9, 2021

Copyright © 2021 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/
Open Access

Abstract

The basic equations of the non-relativistic quantum mechanics with trajectories and quantum hydrodynamics are extended to the relativistic domain. This is achieved by using a Schrödinger-like equation, which describes a particle with mass and spin-0 and with the correct relativistic relation between its linear momentum and kinetic energy. Some simple but instructive free particle examples are discussed.

Keywords

Quantum Mechanics, Bohm Quantum Mechanics, Quantum Hydrodynamics, Relativistic Quantum Mechanics

1. Introduction

In 1927, shortly after E. Schrödinger published a seminal paper containing his celebrated equation [1], E. Madelung dared an interpretation showing that the Schrödinger equation can be transformed into two equations that mimic the continuity and the Euler equations of hydrodynamics [2]. The Euler equation is a particular case of the Navier-Stokes equation [3]. Such hydrodynamic interpretation is now considered a forebear of the de Broglie-Bohm Pilot Wave Theory [4] [5] [6] [7], although germs of this theory were ventured in 1924 by L. de Broglie [8]. The process followed by Madelung consisted in expressing the Schrödinger solution in an exponential form which led to the two abovementioned equations, one for the amplitude and another for the phase. Those ideas were later retaken by D. Bohm [4] [5]. Consequently, most of the work related to the Madelung-de-Broglie-Bohm reformulation of quantum mechanics and quantum hydrodynamics applies to particles moving slowly respect to the speed of light. A fully relativistic quantum mechanics with trajectories was recently formulated [9]; however, it lacks the relative simplicity of the non-relativistic formulation. Other general approaches have been reported [10] [11], but we explore in this work an alternative methodology for extending, to the relativistic domain, the known non-relativistic quantum hydrodynamics and quantum theories with trajectories. Our approach, while having some points of contacts with previous reported approach [11], is based in a surprising wave equation which resembles the Schrödinger equation, but describes a particle with mass and spin-0 which moves through a potential V, and has the correct relativistic relation between the linear momentum p and the kinetic energy K [12]-[21]:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{(\gamma_v+1)m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t). \tag{1}$$

In Equation (1), what we call the (one-dimensional) Grave de Peralta (GP) equation for a quantum particle with mass m, $\hbar$ is the Plank constant (h) divided by 2π, and γ_v is a factor commonly found in special theory of relativity formulas (the Lorentz factor), which depends on the ratio between the squares of the particle's speed (v^2) and the speed of the light in the vacuum (c^2) [22]:

$$\gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2}$$

The basic properties of Equation (1) and its solutions, and detailed discussions of how to solve Equation (1) for some interesting potentials V, can be found in recently published works [12]-[21]. In a nutshell, solving Equation (1) requires simultaneously finding the wavefunction ψ and the square of the particle v^2, which determines the value of γ_v in Equation (2). This may look at first as an unmanageable problem; however, this is not the case in at least several interesting cases [12]-[17]. In general, Equation (1) is nonlinear; this has been discussed before [12] [16]. Nevertheless, due the formal similitude with the Schrödinger equation, Equation (1) is a useful and tractable equation. It is worth noting that Equation (1) can be rewritten in the following way [12] [14] [16]:

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi(x,t),\ \hat{H} = \hat{K} + V,\ \hat{K} = \frac{\hat{p}^2}{(\gamma_v+1)m},\ \hat{p} = -i\hbar\frac{\partial}{\partial x}. \tag{3}$$

The operator K corresponds to the (approximated) relativistic kinetic energy of the particle, thus [12] [14] [16] [19] [20]:

$$\hat{K} = \frac{\hat{p}^2}{(\gamma_v+1)m} \approx \sqrt{\hat{p}^2c^2 + m^2c^4} - mc^2. \tag{4}$$

This means that Equation (1) is well-defined, but it is advantageous to write the operator K as in Equation (1), because this results in an equation formally like the Schrödinger equation, which can then be exactly solved following similar procedures than the ones required for solving the Schrödinger equation [12]-[17] [19] [20]. It is also the striking similarity between Equation (1) and the Schrödinger equation what allowed us to extend, to the relativistic domain, the basic equations of the Madelung-de-Broglie-Bohm reformulation of quantum mechanics and quantum hydrodynamics [18]. It is worth noting that previous reports discussed the existing relationship between Equation (1) and the Klein-Gordon and Dirac equations for a free particle [12] [16] [19] [20]. From a pragmatic point of view, a Schrödinger-like equation appears to be very useful since Schrödinger-like solutions may apply. Investigating exact solutions using such an analogy might make more tractable some relativistic problems. In this work, we applied a methodology that extends already studied applications of the Schrödinger equation to the relativistic domain. This approach might become beneficial. From an epistemological point of view, the Schrödinger-like approach explored here should be considered as a "mathematical hypothesis" and the practical results must be examined as to its final test. We will assume in this work this procedural interpretation of Equation (1). Nevertheless, for self-reliance purpose, a summary of the fundamentals of the GP equation is presented in the Appendix. We hope that the scientific community, which is currently working on non-relativistic quantum mechanics theories with trajectories and quantum hydrodynamics, will recognize the simplicity of the theory presented in this work, and its potential for practical applications in relativistic quantum simulations. The rest of this work is organized in the following way. In the next Section, for the first time, a relativistic extension of the de Broglie-Bohm quantum mechanics is obtained from the relativistic but Schrödinger-like GP equation. Then, a relativistic extension of the Madelung quantum electrodynamics is presented. This is followed by five free particle examples in increasing order of complexity. Finally, the conclusions of this work are given in the Conclusions.

2. Madelung-Bohm-Like Reformulation of the GP Equation

The three-dimensional (3D) GP equation for a particle moving at relativistic speeds in a potential V is given by the following expression [14] [15] [16]:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{(\gamma_v+1)m}\nabla^2\psi + V\psi. \tag{5}$$

In general, the wavefunction (ψ), the potential, and γ_v all depend on the three spatial coordinates and the time. Due to the formal similarity between Equation (5) and the Schrödinger equation, a Madelung-Bohm-like extension of Equation (5) can be done following the same procedure commonly used for reformulating the Schrödinger equation [2] [4] [5] [6] [7]. First, we look for a solution of Equation (5) of the following form:

$$\psi(\boldsymbol{r},t) = R(\boldsymbol{r},t)\mathrm{e}^{iS(\boldsymbol{r},t)/\hbar}. \tag{6}$$

In Equation (6), R and S are the amplitude and phase fields, respectively [2] [4] [7]. Inserting Equation (6) in Equation (5) and following step by step Ref. [6], we can obtain the following equations, which extend to the relativistic domain the basic equations of the Madelung-de Broglie-Bohm quantum mechanics [6]:

$$\frac{\partial}{\partial t}S+\frac{\nabla S^2}{(\gamma_v+1)m}+[V+Q]=0,\ Q=-\frac{\hbar^2}{(\gamma_v+1)m}\frac{\nabla^2 R}{R}. \tag{7}$$

$$\frac{\partial}{\partial t}R^2+\frac{2}{(\gamma_v+1)}\nabla\cdot\left[\gamma_v R^2\left(\frac{\nabla S}{\gamma_v m}\right)\right]=0. \tag{8}$$

In Equation (7), Q is the quantum potential [6]. Clearly, $\gamma_v\approx 1$ when $v^2\ll c^2$; therefore, as it should be expected when the particle moves at low speeds, Equations (7) and (8) coincide to the well-known equations of the Madelung-de Broglie-Bohm quantum mechanics [6]. At relativistic velocities, the velocity field should now be defined such that the relation between the velocity and the linear momentum (∇S) is the correct relativistic relationship [22]:

$$\gamma_v\boldsymbol{v}=\frac{\nabla S}{m}\Rightarrow\boldsymbol{v}=\frac{1}{\gamma_v}\frac{\nabla S}{m}. \tag{9}$$

Thus, the expression between parentheses in Equation (8) is the velocity field given by Equation (9). Again, when $v^2\ll c^2$, Equation (9) coincides with the non-relativistic equation [6]. However, in general [11]:

$$\boldsymbol{v}=\frac{c}{\sqrt{(mc)^2+\nabla S^2}}\nabla S\Rightarrow\gamma_v=\frac{\sqrt{(mc)^2+\nabla S^2}}{mc}. \tag{10}$$

Therefore, when ψ is known, Equation (10) determines the velocity field and γ_v. The direction of the velocity is then perpendicular to the surfaces of constant phase (S = constant). Bohm introduced a particle's trajectory as the solution of the following differential equation and initial conditions [4] [5] [6] [7]:

$$\frac{\partial}{\partial t}\boldsymbol{r}_p(t)=\boldsymbol{v}\left(\boldsymbol{r}=\boldsymbol{r}_p(t),t\right),\ \boldsymbol{r}_p(t=0)=\boldsymbol{r}_o. \tag{11}$$

Therefore, different trajectories correspond to different initial positions of the particle. The direction of the particle's velocity is always tangent to the particle's trajectory. The particle's velocity is given by the following equation:

$$\boldsymbol{v}_p(t)=\frac{\partial}{\partial t}\boldsymbol{r}_p(t). \tag{12}$$

Equations (11) and (12) related the velocity of the Bohmian particle with the velocity of the Madelung's fluid.

3. Relativistic Quantum Hydrodynamics

Madelung did not introduce particle trajectories in his reformulation of the Schrödinger equation [2]. This was done later by Bohm [4] [5]. Madelung interpreted Equations (7) and (8) as describing a fluid with density $\rho'=m\rho$ such that:

$$\rho(\boldsymbol{r},t)=R^2(\boldsymbol{r},t). \tag{13}$$

Then using Equation (9) with $\gamma_v = 1$ allowed him to directly rewrite Equation (8) with $\gamma_v = 1$ as a continuity equation [2] [6] [7]. Proceeding in a similar way, we obtained the following extension of the Madelung's continuity equation to the relativistic domain:

$$\frac{\partial}{\partial t}\rho+\frac{2}{(\gamma_v+1)}\nabla\cdot\left[\rho(\gamma_v\boldsymbol{v})\right]=0. \tag{14}$$

As it should be expected, at non-relativistic speeds, when $\gamma_v \approx 1$ because $v^2 \ll c^2$, Equation (14) coincides with the Madelung's continuity equation [2] [6] [7]. Equation (14) was obtained from Equation (8) by identifying $m\rho$ with the density of a fluid extending through space. Likewise, as it was done by Madelung [2] [6] [7], by identifying the velocity field of this fluid with the velocity field given by Equation (9), we can obtain from Equation (7) the following equation:

$$\frac{\partial}{\partial t}(\gamma_v\boldsymbol{v})+\nabla\left[\frac{\gamma_v^2}{(\gamma_v+1)}\boldsymbol{v}\cdot\boldsymbol{v}\right]+\frac{\nabla(V+Q)}{m}=0,\ Q=-\frac{\hbar^2}{(\gamma_v+1)m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}. \tag{15}$$

If v and thus γ_v only depend on time but not on position, Equation (15) can be simplified in the following Euler-like equation:

$$\frac{\partial}{\partial t}(\gamma_v\boldsymbol{v})+\frac{2}{(\gamma_v+1)}\left[(\gamma_v\boldsymbol{v})\cdot\nabla(\gamma_v\boldsymbol{v})\right]=-\frac{\nabla(V+Q)}{m}. \tag{16}$$

As it should be expected, at non-relativistic speeds, when $\gamma_v \approx 1$, Equation (16) coincides with the Euler-like equation obtained by Madelung [2] [6] [7].

4. Plane Waves

The fluid dynamic of a classical ideal fluid flow supposes the fluid is non-viscous; the flow is steady, *i.e.*, the velocity is time independent; the fluid is incompressible, *i.e.*, the liquid density is constant; and assumes that the flow is irrotational [23]. The dynamic of an ideal fluid with density ρ', which is flowing close to the Earth's surface under the influence of the Earth gravitational potential, $U_g/m = gH$, where g is the gravitational acceleration and H is the high respect to the ocean's surface, it is given by the Bernoulli equation [23]:

$$\frac{1}{2}\rho'v^2+\rho'\frac{U_g}{m}+P=\text{constant}. \tag{17}$$

In Equation (17), P is the pressure inside of the liquid. While Equation (17) is purely classical and has no connection with Madelung fluids, it is instructive to compare the Madelung liquid, associate to a free particle "guided" by a plane wave, to a classical ideal liquid under non-gravity conditions, which dynamics is described by the Bernoulli equation with $U_g = 0$. A simple solution of the GP equation for a free particle ($V = 0$) is the plane wave, normalized in a large cube of side L, given by the following equation [12] [14]:

$$\psi = \frac{1}{\sqrt{L^3}} e^{\frac{i}{\hbar}\left(\boldsymbol{p}\cdot\boldsymbol{r} - \frac{p^2}{(\gamma_v+1)m}t\right)}. \tag{18}$$

In Equation (18), p is the magnitude of the particle's linear momentum, which can take any positive real value. Evaluating Equation (18) for $\gamma_v = 1$ gives the correct normalized plane wave when the free particle is traveling at non-relativistic speeds [6]. The surfaces of constant phase corresponding to Equation (18) are planes perpendicular to the particle's linear momentum. Note that for a given value of p, the value of γ_v get univocally determined by the equality of the following formulas for the relativistic kinetic energy [12] [14] [16]:

$$K = (\gamma_v - 1)mc^2 = \frac{p^2}{(\gamma_v+1)m} \Rightarrow \gamma_v = \frac{\sqrt{p^2 + m^2c^2}}{mc}. \tag{19}$$

Using Equations (6), (13) and (18), we can obtain:

$$R = \sqrt{\rho} = L^{-\frac{3}{2}} \Rightarrow Q \equiv 0,\ S(\boldsymbol{r},t) = \left(\boldsymbol{p}\cdot\boldsymbol{r} - \frac{p^2}{(\gamma_v+1)m}t\right). \tag{20}$$

From Equation (20) follows that the Madelung fluid associated to a free particle guided by a plane wave has constant density $\rho' = m\rho$; therefore, it is incompressible. It is also no viscous because the total force acting on it is $F = -\nabla(V+Q)/m = 0$. The velocity of this fluid and the corresponding value of γ_v can be obtained using Equations (10) and (20):

$$\boldsymbol{v} = \frac{c}{\sqrt{(mc)^2 + p^2}}\boldsymbol{p},\ \gamma_v = \frac{\sqrt{(mc)^2 + p^2}}{mc}. \tag{21}$$

The value of γ_v given by Equations (19) and (21) are identical in this case, but as it will be shown in the next Section, this is not a general feature of the theory. The maximum possible value of the fluid speed, $v \approx c$, occurs when $\nabla S = p \gg mc$. This corresponds to $\gamma_v \gg 1$. The fluid velocity is constant; thus, this Madelung fluid is irrotational. Equation (16) reduces now to:

$$\nabla\left[\frac{(\gamma_v v)^2}{(\gamma_v+1)}\right] = \frac{\nabla\left[\frac{p^2}{(\gamma_v+1)m}\right]}{m} = 0 \Rightarrow K = \text{constant}. \tag{22}$$

Evidently, Equations (21) and (22) also gives the correct results at the non-relativistic limit. A comparison between Equation (22), evaluated for $\gamma_v = 1$, and the Bernoulli equation (Equation (17) with $U_g = 0$) shows that there is not pressure in the Madelung fluid associated to a free quantum particle guided by a plane wave. From Equations (11), (12), and (21) follow that the Bohmian paths of a free quantum particle associated to a plane wave are given by the following equation [6] [11]:

$$\boldsymbol{r}_p(t) = \boldsymbol{r}_o + \frac{c}{\sqrt{(mc)^2 + p^2}}\boldsymbol{p}t. \tag{23}$$

Evidently, Equation (23) also gives the correct result for a particle moving at non-relativistic speeds. In Equation (23), the initial position of the particle lies everywhere in space. Like for free classical particles moving at non-relativistic speeds, these Bohmian paths are therefore uniform, rectilinear, and perpendicular to the planes of constant phase of the wave. This is because in this case the quantum potential is null, thus $\nabla Q = 0$ [6]. Also note that Equations (14) to (16) are fulfilled because both ρ and v are constant.

5. Standing Waves

A simple but interesting case, where Q is not null, occurs when a free quantum particle is in the superposition state formed by two plane waves, which are both solutions of Equation (5) with $V = 0$ but are traveling in opposite directions along the x-axis with the same value of p:

$$\psi(x,t) = \frac{1}{\sqrt{2L^3}}\left[\mathrm{e}^{i(kx - w_k t)} + \mathrm{e}^{i(-kx - w_k t)}\right] = \frac{2}{\sqrt{2L^3}}\cos(kx)\mathrm{e}^{-iw_k t},$$
$$k = \frac{p}{\hbar},\ w_k = \frac{p^2}{(\gamma_v + 1)m\hbar}. \tag{24}$$

The speeds of the Madelung fluids associated to either one of these two plane waves are the same and given by Equation (23), but the corresponding velocities point to opposite directions; therefore, γ_v is also the same for each wave when individually considered. Consequently, the standing wave given by Equation (24) is also a solution of Equation (5) with $V = 0$, and with the same value of γ_v than for each of the plane waves components. For the standing wave:

$$R(x) = \sqrt{\rho(x)} = \frac{2}{\sqrt{2L^3}}\cos(kx)$$
$$\Rightarrow Q = -\frac{\hbar^2}{(\gamma_v + 1)m}\frac{\frac{\mathrm{d}^2}{\mathrm{d}x^2}R(x)}{R(x)} = \frac{\hbar^2 k^2}{(\gamma_v + 1)m} = \frac{p^2}{(\gamma_v + 1)m}, \tag{25}$$
$$S(t) = -\frac{p^2}{(\gamma_v + 1)m}t.$$

From Equations (24) and (25) follows that the period of the $\cos^2(kx)$ density distribution is inverse proportional to p. The Madelung fluid associated to a free particle guided by a standing wave does not have a constant density; therefore, it is compressible, thus, it does not behave like a classical ideal fluid flow. The wavelength of the standing wave, λ, is inverse proportional to p. From Equation (25) also follows that $\nabla S = 0$; therefore, from Equation (10) follows that the velocity of this fluid is zero and $\gamma_v = 1$, which is different than the γ_v value corresponding to each superposing plane wave. Consequently, the Bohmian particle associated to a standing wave is at rest. In Equation (25), Q is equal to the relativistic kinetic energy of the free particle, which is constant; therefore, $\nabla S = 0$, thus this Madelung fluid is no viscous. Equations (14) to (16) are now fulfilled because ρ does not depend on time and $v = 0$. A comparison of the results obtained in this example for a standing wave, to the results obtained in the previous Section for a plane wave, illustrates the well-known nonlocality properties of the theories resulting from the Madelung-de-Broglie-Bohm reformulation of the Schrödinger equation. The superposition of plane waves, which are solutions of the same GP equation for a free particle, modifies the properties of the corresponding Madelung fluid, and then the Bohmian trajectories of the guided particle.

6. Quasi-Standing Waves

In this Section we will consider a wavefunction of Equation (5) with $V = 0$, which is a slightly variation of Equation (24):

$$\psi(x,t) = \frac{1}{\sqrt{2L^3}}\left[e^{i(kx - w_k t)} + e^{i(-k'x - w_{k'} t)}\right],$$
$$k' = \frac{p+\Delta p}{\hbar},\ w_{k'} = \frac{(p+\Delta p)^2}{(\gamma_v+1)m\hbar},\ \Delta p \ll p. \tag{26}$$

In Equation (26), k and w_k are given by Equation (24). Consequently, the first term of the wavefunction in Equation (26) is a solution of Equation (5) with $V = 0$. However, the second term is not because there is, in the denominator of $w_{k'}$, the same value of γ_v than for w_k. Nevertheless, from Equation (19) follows that the value of γ_v corresponding to $(p + \Delta p)$ is approximately equal to the value corresponding to p in two situations. First, $\gamma_v \approx 1$ at the non-relativistic limit when $p \ll mc$. Second, $\gamma_v \approx p/mc$ at the ultra-relativistic limit $p \gg mc$; therefore, $(p+\Delta p)/mc \approx p/mc$ when $\Delta p \ll mc$. Consequently, at these two limits ψ given by Equation (26) is approximately a solution of Equation (5) with $V = 0$. We will call here, a quasi-standing wave, to the wavefunction given by Equation (26) at these two limits. After some straightforward algebraic steps for transforming Equation (26) in a form like Equation (6), we obtained the following results:

$$\nabla S = \frac{-\Delta p}{2} \Rightarrow v = \frac{-c\Delta p}{\sqrt{4m^2c^2 + (\Delta p)^2}} \Rightarrow \gamma_v = \frac{1}{\sqrt{1 - \frac{(\Delta p)^2}{4m^2c^2 + (\Delta p)^2}}}. \tag{27}$$

And:

$$\rho(x,t) = \frac{2}{L^3}\cos^2\left[k_b(x - v_b t)\right],\ k_b = \frac{\Delta p}{2\hbar},\ v_b = 2\frac{\frac{1}{2}\left[p - (p+\Delta p)\right]}{(\gamma_v+1)m}$$
$$\Rightarrow Q = \frac{(\Delta p)^2}{4(\gamma_v+1)m}. \tag{28}$$

In Equation (27), ΔS points to the negative direction of the axis x. As it should be expected, Equations (27) and (28) reduces to Equations (25) when $\Delta p = 0$. The density of the Madelung fluid associated to a free particle guided by a quasi-standing wave resembles a "standing wave" that is drifting without dispersion, in the direction of the plane wave associated with the linear momentum $p + \Delta p$, with speed $v_b \approx v \approx -\Delta p/2m$ when $\Delta p \ll mc$. It can be easily checked out that the quasi-standing wave given by Equation (26) satisfies Equation (5) with $V = 0$, and with γ_v given by Equation (27). From Equations (27) and (28) also follows that Equations (7), (8), and (14) to (16) are satisfied. Note that $\gamma_v \approx 1$ for every p when $\Delta p \ll mc$; consequently, the Bohmian particle associated to a quasi-standing wave moves like a classical particle even at the ultra-relativistic limit.

7. Beats

In this Section we will consider another wavefunction of Equation (5) with $V = 0$, which can be obtained from Equation (26) after substituting $-k'$ by $+k'$. Equation (26) corresponds to the superposition of two plane waves with slightly different values of p traveling in opposite directions. Here we will consider what happens when the two waves travel in the same direction. In this case, we obtained the following results:

$$\nabla S = p + \frac{\Delta p}{2} \Rightarrow v = \frac{c\left(2p + \Delta p\right)}{\sqrt{4m^2c^2 + \left(2p + \Delta p\right)^2}} \Rightarrow \gamma_v = \frac{1}{2\sqrt{-\frac{m^2c^2}{4m^2c^2 + \left(2p + \Delta p\right)^2}}}. \quad (29)$$

And:

$$\rho(x,t) = \frac{2}{L^3}\cos^2\left[k_b\left(x - v_b t\right)\right],\ k_b = \frac{\Delta p}{2\hbar},\ v_b = 2\frac{\frac{1}{2}\left[p + \left(p + \Delta p\right)\right]}{\left(\gamma_v + 1\right)m}$$
$$\Rightarrow Q = \frac{\left(\Delta p\right)^2}{4\left(\gamma_v + 1\right)m}. \quad (30)$$

In Equation (29), ΔS points to the positive direction of the axis x. Note that k_b does not depend on p but is proportional to Δp. Therefore, as it should be expected, Equations (29) and (30) reduce when $\Delta p = 0$ to Equations (20) and (21), which correspond to the first example of a single plane wave discussed in Section 4. The Madelung fluid now flows without dispersion in the same direction than the plane waves, and at the average speed of both waves. The factor of 2 at the front of Equation (30) for v_b is because a $\cos^2(ax - bt)$ shaped wave travels at twice the speed than a $\cos(ax - bt)$ shaped one. The corresponding Bohmian paths are uniform and rectilinear at both non-relativistic and relativistic values of v_b. This result suggests the following very interesting possibility: a free ultra-relativistic quantum particle could be associated to a Gaussian pulse, which is formed by a superposition of plane waves traveling in the same direction with similar values of p, and thus could be a solution of Equation (5) with $V = 0$. Such a Gaussian pulse would travel with no dispersion at the average relativistic speed of all the plane waves forming the Gaussian pulse.

8. Ultra-Relativistic Gaussian Wave-Packets

Gaussian wave-packets are often considered the quantum entity that closest resemble a classical particle [24] [25]. A Gaussian pulse describes a quantum particle for which the uncertainty relation between its position (Δx) and linear momentum (Δp) has its minimum value $\Delta x \Delta p = 1/2\ \hbar$. Non-relativistic Gaussian wave-packets are formed by a superposition of plane waves which are solutions of the Schrödinger equation for a free particle [24]. There is a non-linear relationship between the angular frequency (w) and the wavenumber (k) of the plane waves which are solution of the Schrödinger equation [24] [26]:

$$w = \frac{K}{\hbar} = \frac{\frac{p^2}{2m}}{\hbar} = \frac{\hbar}{2m}k^2. \tag{31}$$

The non-lineal dispersion of the Schrödinger equation determines that the phase velocities $v_{ph} = w/k$ of different plane waves are different. Consequently, Gaussian wave-packets which are solution of the Schrödinger equation deform when propagate [24]. This differentiates a Gaussian pulse associated to a quantum particle from a free classical particle that travels without deforming. Relativistic Gaussian pulses corresponding to a free particle with mass and spin-0 can be formed by superposing plane waves, which are solutions of the Klein-Gordon equation [25] [27] [28]. The dispersion of the Klein-Gordon equation also is non-lineal [27] [28]:

$$w = \frac{E}{\hbar} = \frac{\pm\sqrt{p^2c^2 + m^2c^4}}{\hbar} = \frac{\pm\sqrt{\hbar^2k^2c^2 + m^2c^4}}{\hbar}. \tag{32}$$

The non-linearity of Equation (32) determines that Gaussian pulses which are solutions of the Klein-Gordon equation also deforms when propagate. Moreover, Equation (32) admits solutions with negative kinetic energy values, which results in additional difficulties when describing the propagation of these Gaussian wave-packets [25] [27] [28]. These difficulties disappear when using Equation (5) with $V = 0$ for describing a free particle. This is because Equation (4) implies the following dispersion relation:

$$w = \frac{K}{\hbar} = \frac{+\sqrt{p^2c^2 + m^2c^4} - mc^2}{\hbar} = \frac{mc^2\left[\sqrt{\left(\frac{\hbar k}{mc}\right)^2 + 1} - 1\right]}{\hbar}. \tag{33}$$

As it should be expected, in the non-relativistic limit $p = \hbar k \ll mc$, Equation (31) can be obtained from Equation (33) by approximating the square root in Equation (33) by the first two terms of the corresponding series in powers of k/mc. Moreover, in the ultra-relativistic limit $k \gg mc$, Equation (33) becomes:

$$w = \frac{K}{\hbar} \approx ck \Rightarrow v_{ph} = \frac{w}{k} \approx c. \tag{34}$$

Therefore, one should expect that all the ultra-relativistic plane waves which are solutions of Equation (5) with $V = 0$ propagates with the same phase velocity and thus, the ultra-relativistic Gaussian pulses formed by a superposition of these plane waves should propagate without deformation. This means that ultra-relativistic Gaussian wave-packets which are solutions of Equation (5) with $V = 0$ behave more like free classical particles than the non-relativistic ones. In the ultra-relativistic limit, Equation (4) can be approximate by:

$$\hat{K} \approx \hat{p}c = -i\hbar c\nabla. \tag{35}$$

Consequently, the ultra-relativistic limit of Equation (5) with $V = 0$ is:

$$i\hbar\frac{\partial}{\partial t}\psi = -i\hbar c\nabla\psi. \tag{36}$$

A plane wave solution of Equation (36), which is a kind of Weyl equation for spin-0 particles, normalized in a large region of length L and traveling along the x-axis, is given by the following expression:

$$\psi(x,t) = \frac{1}{\sqrt{L}}\mathrm{e}^{i(kx-wt)} = \frac{1}{\sqrt{L}}\mathrm{e}^{\frac{i}{\hbar}(px-Kt)} = \frac{1}{\sqrt{L}}\mathrm{e}^{ik(x-ct)}. \tag{37}$$

The phase velocity of this plane wave is for any value of k, $v_{ph} = c$. Equation (36) is lineal, thus any wave-packet formed by a superposition of plane waves given by Equation (37) is a solution of Equation (36) and propagates without dispersion (deformation). Specifically, this occurs for the ultra-relativistic Gaussian pulse:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\varphi(k)\mathrm{e}^{ik(x-ct)}\mathrm{d}k,\ \ \varphi(k)\approx 0\ \text{ when } |\hbar k| < mc. \tag{38}$$

In Equation (38):

$$\varphi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\psi(x,t=0)\mathrm{e}^{-ikx}\mathrm{d}x = \sqrt{\frac{\sigma}{\sqrt{\pi}}}\mathrm{e}^{-\frac{\sigma^2}{2}(k-\langle k\rangle)^2}. \tag{39}$$

For obtaining Equation (39), we assumed $\psi(x,t=0)$ is a Gaussian wave-packet that at $t = 0$ is peaked at $x = 0$, and it is moving along the x-axis with average momentum $\langle p\rangle = \hbar\langle k\rangle$:

$$\psi(x,t=0) = \left[\frac{1}{\sqrt{\sigma\sqrt{\pi}}}\mathrm{e}^{-\frac{1}{2\sigma^2}x^2}\right]\mathrm{e}^{i\langle k\rangle x}. \tag{40}$$

Using Equations (39) and (40), we found the expression corresponding to an ultra-relativistic Gaussian wave-packet that propagates along the x-axis at speed c without deformation:

$$\psi(x,t) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}}\mathrm{e}^{-\frac{1}{2\sigma^2}(x-ct)^2}\mathrm{e}^{i\langle k\rangle(x-ct)}. \tag{41}$$

Therefore, for ultra-relativistic Gaussian pulses:

$$R(x,t) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}}\mathrm{e}^{-\frac{1}{2\sigma^2}(x-ct)^2},\ \ S(x,t) = \hbar\langle k\rangle(x-ct). \tag{42}$$

From Equation (42), we can obtain the density of the Madelung's quantum fluid associated to an ultra-relativistic Gaussian pulse:

$$\rho(x,t) = R^2(x,t) = \frac{1}{\sigma\sqrt{\pi}}\mathrm{e}^{-\frac{1}{\sigma^2}(x-ct)^2}. \tag{43}$$

This means ultra-relativistic Gaussian wave-packets are density pulses in the

Madelung's quantum hydrodynamic description. Equation (43) describes a Gaussian density pulse propagating along the *x*-axis at speed *c*, and which peak is at any given time at $x = ct$. There is a characteristics quantum potential (Q) in the Madelung-de Broglie-Bohm reinterpretation of the Schrödinger equation and its relativistic extension, which can be evaluated using Equations (42) and (43):

$$\begin{aligned} Q(x,t) &= -\frac{\hbar^2}{(\gamma_v+1)m}\frac{\nabla^2 R}{R} = -\frac{\hbar^2}{(\gamma_v+1)m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \\ &\approx -\frac{4\hbar^2}{\sigma^4 m}\frac{mc}{\langle p\rangle}(x-ct)^2 + \frac{2\hbar^2}{\sigma^2 m}\frac{mc}{\langle p\rangle}. \end{aligned} \tag{44}$$

The corresponding quantum force (F_Q) acting over the ultra-relativistic Gaussian wave-packet is:

$$F_Q(x,t) = -\frac{\partial Q(x,t)}{\partial x} = \frac{8\hbar^2}{\sigma^4}\frac{mc}{\langle p\rangle}(x-ct). \tag{45}$$

The quantum force is always null at the pulse's pick. The Gaussian density pulse is produced by the action of pairs of compressing quantum forces, which are equidistance from the peak and have the same magnitude but opposite directions. Note that the net quantum force over the Gaussian pulse is null, which corresponds with the propagation of the pulse with constant velocity. Particles have trajectories in the de Broglie-Bohm reinterpretation of the Schrödinger equation. The velocity of a particle associated to an ultra-relativistic Gaussian pulse can be computed from the phase field $S(x, t)$ using the following equation:

$$v = \frac{c}{\sqrt{(mc)^2 + \left(\frac{\partial S}{\partial x}\right)^2}}\frac{\partial S}{\partial x} \approx c. \tag{46}$$

I.e., a particle associated to an ultra-relativistic Gaussian pulse describes the same trajectory than the peak of the pulse.

9. Conclusion

Madelung, de Broglie, and Bohm reformulated the Schrödinger equation. In this way, they founded the non-relativistic quantum hydrodynamics and quantum mechanics with trajectories. Following a similar procedure, we reformulated the GP equation. In this way, we extended quantum hydrodynamics and quantum mechanics with trajectories to the relativistic domain. As it should be expected, we showed that at non-relativistic energies, the resulting equations coincide with the well-known non-relativistic equations. As a proof-of-concept demonstration of the potential practical value of the formulated theory, we discussed some simple but instructive free particle problems.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Schrodinger, E. (1926) *Annals of Physics*, **79**, 361-376, 489-527. https://doi.org/10.1002/andp.19263840602

[2] Madelung, E. and Zeit, F. (1927) *Zeitschrift für Physik*, **40**, 322-326. https://doi.org/10.1007/BF01400372

[3] Landau, L.D. and Lifshitz, E.M. (1987) Fluid Mechanics. Pergamon Press, Oxford.

[4] Bohm, D. (1952) *Physical Review*, **85**, 166-179. https://doi.org/10.1103/PhysRev.85.166

[5] Bohm, D. (1953) *Physical Review*, **89**, 319. https://doi.org/10.1103/PhysRev.89.319.2

[6] Holland, P.R. (1993) The Quantum Theory of Motion: An Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511622687

[7] Wyatt, R.E. (2005) Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics. Springer, New York.

[8] de Broglie, L. (1924) *Philosophical Magazine*, **47**, 446-458. https://doi.org/10.1080/14786442408634378

[9] Poirier, B. (2016) *Journal of Physics*, **701**, Article ID: 012013. https://doi.org/10.1088/1742-6596/701/1/012013

[10] Struyve, W. and Valentine, A. (2008) *Journal of Physics A*: *Mathematical and Theoretical*, **42**, Article ID: 035301. https://doi.org/10.1088/1751-8113/42/3/035301

[11] Khodagholizadeh, J., Kazemi, J. and Babazadeh, A. (2014) Relativistic Bohmian Mechanics. https://arxiv.org/abs/1405.3822v2

[12] Grave de Peralta, L. (2020) *Journal of Modern Physics*, **11**, 196-213. https://doi.org/10.4236/jmp.2020.112012

[13] Grave de Peralta, L. (2020) *Journal of Modern Physics*, **11**, 788-802. https://doi.org/10.4236/jmp.2020.116051

[14] Grave de Peralta, L. (2020) *Results in Physics*, **18**, Article ID: 103318. https://doi.org/10.1016/j.rinp.2020.103318

[15] Grave de Peralta, L. (2020) *Scientific Reports*, **10**, Article No. 14925. https://doi.org/10.1038/s41598-020-71505-w

[16] Grave de Peralta, L. (2020) *European Journal of Physics*, **41**, Article ID: 065404. https://doi.org/10.1088/1361-6404/aba7dc

[17] Grave de Peralta, L., Poirier, B. and Poveda, L.A. (2021) Direct Relativistic Extension of the Schrödinger Equation. *APS March Meeting*, 15-19 March 2021, U71-00233.

[18] Columbié, R. and Grave de Peralta, L. (2021) Direct Relativistic Extension of the Madelung-de Broglie-Bohm Theories. *APS March Meeting*, 15-19 March 2021, U71-00234.

[19] Grave de Peralta, L. (2021) *European Journal of Physics*, **42**, Article ID: 055404. https://doi.org/10.1088/1361-6404/ac0ecc

[20] Grave de Peralta, L. and Farooq, H. (2021) *Journal of Modern Physics*, **12**, 1145-1159. https://doi.org/10.4236/jmp.2021.128068

[21] Poveda, L.A., Ruiz Columbié, A. and Grave de Peralta, L. (2021) Relativistic Corrections to the Diósi-Penrose Model. http://arxiv.org/abs/2103.00994

[22] Christodeulides, C. (2016) The Special Theory of Relativity. Springer, New York.

[23] Walker, J., Halliday, D. and Resnick, R. (2014) Fundamentals of Physics. 10th Edition, Wiley, Hoboken.

[24] Powell, J.L. and Crasemann, B. (1961) Quantum Mechanics. Addison-Wesley, Boston.

[25] Tsai, H.M. and Poirier, B. (2015) *Journal of Physics*: *Conference Series*, **701**, Article ID: 012013. https://doi.org/10.1088/1742-6596/701/1/012013

[26] Griffiths, D.J. (1995) Introduction to Quantum Mechanics. Prentice Hall, Englewood Cliffs.

[27] Strange, P. (1998) Relativistic Quantum Mechanics: With Applications in Condensed Matter and Atomic Physics. Cambridge University Press, New York. https://doi.org/10.1017/CBO9780511622755

[28] Greiner, W. (1990) Relativistic Quantum Mechanics: Wave Equations. Spring-Verlag, New York.

[29] Alberto, P., Fiolhais, C. and Gil, V.M.S. (1996) *European Journal of Physics*, **17**, 19. https://doi.org/10.1088/0143-0807/17/1/004

Appendix

In Special Theory of Relativity, the relevant Lorentz invariant magnitude is the four-component momentum given by the following equation [22] [27] [28]:

$$P^{\mu}=\left(\frac{E}{c},p_x,p_y,p_z\right). \tag{A1}$$

The magnitude of the four-component vector is the relativistic invariant *mc*. Therefore, a relativistic quantum theory for a free spin-0 particle of mass *m* can be formally obtained from first quantization of the Lorentz-invariant relation between the particle's energy and the three-component momentum:

$$\sqrt{\frac{E^2}{c^2}-\boldsymbol{p}^2}=mc,\quad \boldsymbol{p}^2=p_x^2+p_y^2+p_z^2. \tag{A2}$$

Equation (A.2) can be rewritten the following way:

$$E=mc^2\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}=\gamma_v mc^2,\quad \gamma_v=\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}. \tag{A3}$$

Combining Equation (4) and (A.3), we obtain the Lorentz-covariant equation:

$$E=\frac{\boldsymbol{p}^2}{(\gamma_v+1)m}+mc^2,\quad \gamma_v=\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}. \tag{A4}$$

Making the following formal first-quantization substitutions in Equation (A.4):

$$E\to\hat{H}=i\hbar\frac{\partial}{\partial t},\quad \boldsymbol{p}\to\hat{p}=-i\hbar\nabla. \tag{A5}$$

We can obtain the following Lorentz-covariant wave equation:

$$i\hbar\frac{\partial}{\partial t}\Omega=-\frac{\hbar^2}{(\hat{\gamma}_v+1)m}\nabla^2\Omega+mc^2\Omega,\quad \hat{\gamma}_v=\sqrt{1+\frac{\hat{p}^2}{m^2c^2}}. \tag{A6}$$

Now, it is well known in quantum mechanics that applying a constant energy shift to the Hamiltonian gives rise to an immaterial time-evolving phase factor in the solution wavefunction. Therefore, in order to obtain a more Schrödinger-like result, we can remove the rest-energy contribution from Equation (10) above, by replacing Ω as follows:

$$\Omega=\psi \mathrm{e}^{-i\frac{mc^2}{\hbar}t}. \tag{A7}$$

Thus, obtaining the Poirier-Grave de Peralta (PGP) equation for a free spin-0 particle of mass *m* [19]:

$$i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{(\hat{\gamma}_v+1)m}\nabla^2\psi,\quad \hat{\gamma}_v=\sqrt{1+\frac{\hat{p}^2}{m^2c^2}}. \tag{A8}$$

The Poveda's formalism consists in parametrizing Equation (A.8) by making [19] [21]:

$$\hat{\gamma}_v\to\gamma_v=\sqrt{1+\frac{\langle\psi|\hat{p}^2|\psi\rangle}{m^2c^2}}. \tag{A9}$$

This formalizes the approach initially followed by Grave de Peralta for avoiding the use of the square root operator γ_v [12] [13] [14] [15] [16]. Equation (A.9) defines the parameter γ_v in terms of the average value, on the state ψ, of the square of the well-defined linear momentum operator. This allows for rewritten Equation (A.8) as the GP equation for a free spin-0 particle of mass:

$$i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{(\gamma_v+1)m}\nabla^2\psi,\quad \gamma_v=\sqrt{1+\frac{\langle\psi|\hat{p}^2|\psi\rangle}{m^2c^2}}. \tag{A10}$$

In a similar way, we can obtain Equation (5). Therefore, when γ_v does not explicitly depend on the time, we can look for a solution of Equation (5) of the form:

$$\psi(\boldsymbol{r},t)=\chi(\boldsymbol{r})e^{-i\frac{E'}{\hbar}t},\quad E'=E+V-mc^2=K+V. \tag{A11}$$

Then obtaining the time-independent GP equation for a spin-0 particle of mass m which is moving through a time-independent potential V:

$$\left[\frac{\hat{p}^2}{(\gamma_v+1)m}+V\right]\chi=E'\chi,\quad \gamma_v=\sqrt{1+\frac{\langle\chi|\hat{p}^2|\chi\rangle}{m^2c^2}}. \tag{A12}$$

Equation (A.12) can be rewritten in the following way:

$$\hat{K}_{Sch}\chi=\frac{\gamma_v+1}{2}(E'-V)\chi,\ \hat{K}_{Sch}=\frac{\hat{p}^2}{2m},\ \gamma_v=\sqrt{1+\frac{\langle\chi|\hat{p}^2|\chi\rangle}{m^2c^2}}. \tag{A13}$$

Clearly, when:

$$\langle\chi|\hat{p}^2|\chi\rangle\ll m^2c^2. \tag{A14}$$

The time-independent Schrödinger equation is obtained as a limit case of Equation (A.13):

$$\hat{K}_{Sch}\chi_{Sch}=(E'-V)\chi_{Sch}. \tag{A15}$$

For several important problems, solving Equation (A.13) reduces to solving an effective time-independent Schrödinger equation. For instance, this occurs for problems with stepwise constant potentials [12] [14] [16]. In each spatial region where V is constant ($V = V_o$), $\chi=\chi_{Sch}$. Consequently, solving the time-independent GP equation reduces to solving the following effective time-independent Schrödinger equation:

$$\hat{K}_{Sch}\chi_{Sch}=\varepsilon\chi_{Sch}. \tag{A16}$$

Equation (A.16) can be solved with no knowledge of the value of γ_v; therefore, after Equation (A.16) is solved, γ_v can then be calculated as:

$$\gamma_v=\sqrt{1+\frac{2}{mc^2}\langle\chi_{Sch}|\hat{K}_{Sch}|\chi_{Sch}\rangle}=\sqrt{1+\frac{2}{mc^2}\varepsilon}. \tag{A17}$$

Finally, the sum of the kinetic and potential energies of the particle is:

$$E'=(\gamma_v-1)mc^2+V_o=\left(\sqrt{1+\frac{2}{mc^2}\varepsilon}-1\right)mc^2+V_o. \tag{A18}$$

Alternatively, E' can be calculated as:

$$E' = \frac{2\varepsilon}{1+\sqrt{1+\frac{2}{mc^2}\varepsilon}} + V_o. \tag{A19}$$

It is worth noting the energy values calculated using Equations (A.18) and (A.19) are in excellent correspondence with the energies obtained using the Klein-Gordon and Dirac equations [20] [27] [28]. For instance, **Figure 1** shows a comparison of the energies calculated using the Grave de Peralta approach (continuous lines), the Schrödinger equation (dashed lines), and the Dirac equation (solid dots) [29], for a particle in a one-dimensional infinite well of width L [12] [14] [17] [21]. Even for relativistic energy values larger than 2 mc^2, there is an excellent correspondence between the energies calculated using the GP and the Dirac equations. More importantly, there is a large class of problems where $\chi \neq \chi_{Sch}$, but the formal similitude between Equation (A.13) and Equation (A.15) facilitates solving Equation (A.13) using similar procedures than the ones used for solving Equation (A.15). For instance, the energies of Hydrogen atom were calculated using the Grave de Peralta approach [13] [15] [20]. A good correspondence was obtained with the positive energies calculated using the Klein-Gordon and Dirac equations [13] [15] [20].

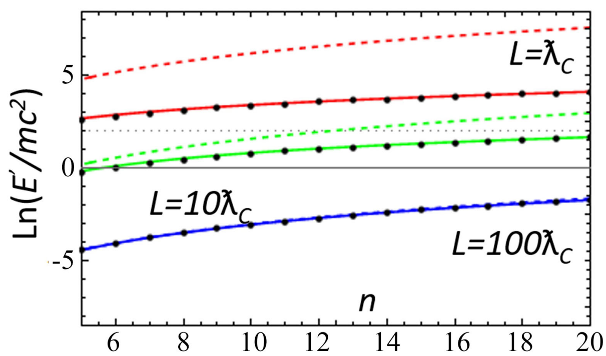

Figure 1. Comparison of the dependence on n (energy level) of the energies calculated using the GP approach (continuous line) [17] [21], the Schrödinger equation (dashed lines), and the Dirac equation [29] (solid dots) for three different widths (L) of a one-dimensional infinite well. λ_C is the (reduced) Compton wavelength associated to a particle of mass m. The tenues horizontal dotted line corresponds to $E' = 2\ mc^2$.