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What Do Bell-Tests Prove? A Detailed Critique of Clauser-Horne-Shimony-Holt Including Counterexamples

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Abstract

Many future directions of scientific endeavors depend on quantum theory and the precise interpretation and significance of the entanglement of quantum-particles. This interpretation depends in turn on the physical meaning of so called Bell-tests that are mostly performed using entangled photons and randomly switched polarizers to measure their polarization at distant locations. This paper presents a detailed critique of the well known theory of Bell tests given by Clauser-Horne-Shimony-Holt (CHSH). It is demonstrated that several important steps of the CHSH derivations contain serious inaccuracies of the underlying physics and probability theory and even a calculus error. As a consequence, the Bell-CHSH theory cannot be used to demonstrate extreme and opposite interpretations of entanglement such as super-luminal influences or alternatively super-determinism that cast aspersions on Einstein's concepts of locality and separability.

Keywords

Bell Theorem, Clauser-Horne-Shimomy-Holt, Einstein-Podolsky-Rosen

1. Introduction

If one asks for a list of significant problems in branches of current science, one is bound to find pointers toward developments of quantum mechanics that include the concept of quantum entanglement. Quantum computing, quantum teleportation and even topics in biology are using quantum entanglement for their basic considerations. The concept of quantum entanglement was defined and described by Schroedinger and has been shown to represent a main feature of quantum theory itself.

Quantum entanglement of isolated and distant pairs has been experimentally demonstrated by Kocher and Commins [1] and has also been discussed by Nordén [2], with a perspective on questions that arose around Bell's theorem.

The most intriguing questions related to the Bell theorem emerged from measurements by Aspect [3] and Zeilinger and coworkers [4]. They have suggested links of quantum entanglement to superluminal influences based on their work; a daring suggestion that also has given rise to the astounding proposition of quantum teleportation.

The basic question of the possible existence of superluminal influences is actually contained in many a scientific theory, for example that of Newton, and has seen many discussions and opposing views (of Leibnitz, in Newtons case). Since the event of Einstein's relativity, ideas of superluminal influences became unthinkable, but have been revived in the last decades based on Bell's Theorem and the experimental Bell-tests. Recently a sensational Bell-test has been performed by Zeilinger and related groups [4] involving photons from billions of years past as well as the "free will" of fellow researchers [5].

To be sure, no direct proof has ever been provided by any experiments. We do not have any measurement equipment available that can claim to transmit information faster than the speed of light in vacuum and thus we cannot measure superluminal influences in one single shot, as is well known from Einstein's special theory of relativity. The recent ideas of superluminal influences are based on the statistical results underlying the Aspect-Zeilinger type of experiments. These researchers claim that the absolute randomness and ultrafast switching between their distant measurements provides the definite proof for their assertions. This fact has even been discussed in popular TV-shows such as "Einstein's Quantum Riddle"—NOVA, which describes the Einstein-Podolsky-Rosen-type (EPR) experiments that are the basis for the work of Aspect [3], Zeilinger [4] [5] and related groups.

I show in this paper, that among other factors there are certain aspects of the random switching that render Bell-tests questionable. In fact, such switching was not included in any direct way into the theories that describe Bell tests, but has only been recommended to the experimenters in order to support Einstein locality. To demonstrate these facts, I discuss in great detail and in fairly elementary mathematical terms, the important theoretical framework of Clauser, Horne, Shimony and Holt (CHSH) [6].

Each of the following sections starts with verbatim-quotes or at least extremely faithful descriptions of the most relevant definitions, assumptions, inequalities and corollaries given in the original CHSH work. The criticisms and comments follow.

It is demonstrated that the main results of CHSH are based on assumptions lacking generality and accuracy and that their mathematical and physical steps contain several serious problems and even a calculus error. Their results are further invalidated by their lack of use of modern probability theory and particularly the work of Vorob'ev [7], which demonstrates that their inequality (1b) does not follow from their inequality (1a) without extensive conditions that CHSH were not aware of. CHSH inequality (1b) provides the important connection of the CHSH work to the results of quantum mechanics and to the experimental results.

My criticism (of the CHSH theorem that has been scrutinized for more than 50 years) may appear too harsh and even suspect to some, because physically speaking, the theorem is based on innocuous and plausible assumptions. Furthermore, the theorem is undoubtedly correct within certain mathematical frameworks. The physical consequences of the theorem, however, are such that only the extremes of super-luminal effects and/or super-determinism appear as possible explanations. This fact opens the question whether the physical assumptions to derive the theorem are really that innocuous and whether the mathematical premises correspond to the physics of actual experiments (see also [8] [9] and references therein).

The seriousness of this question is demonstrated below by explicit counter-examples to the CHSH theory, including one that invalidates the reasoning in the TV-show "Einstein's Quantum Riddle"—NOVA.

2. Connection of CHSH to Bell's Theorem and EPR Experiments

2.1. CHSH Quotation

"—Consider an ensemble of correlated pairs of particles moving so that one enters apparatus I_a and the other apparatus II_b, where a and b are adjustable apparatus parameters. In each apparatus a particle must select one of two channels labeled +1 and −1. Let the result of these selections be represented by $A(a)$ and $B(b)$, each of which equals ±1 according as the first or second channel is selected."

2.2. Criticism

To work within the aims of Bell-CHSH, we are bound to use Einstein-type space-time (or, in approximation, space and time) physics. Thus, Bell-CHSH have replaced the operators and eigenvalues of quantum theory by functions $A(a,\ldots),B(b,\ldots)$ with the "$\ldots$" being not yet fully specified, but later related to the information λ from a source (see below).

The choice of two integer numbers to define two channels and the codomain of the corresponding functions $A,B=\pm 1$ is innocuous and contradiction-free if indeed only one pair of apparatus parameters (fixed a,b) is in discussion. However, the identical choice of +1 or −1 for all possible equipment (apparatus) settings and in particular the 4 different equipment settings used by CHSH (see below) represents an inaccuracy that leads to inconsistencies, because channels pointing into different directions are denoted by the same integer. The application of integer algebra leads, in fact, to direct contradictions, as is shown in expression (5) below.

The inaccuracy of CHSH (and Bell) to deal with only 2 equipment settings, while the bulk of their paper deals with 2 settings in each station, is further aggravated by the random switching between the settings in each station: a, a' in I_a and b, b' in II_b. (Please note that we are using throughout the symbols a, a' and b, b' for the polarizer or magnet settings, as is usual in more recent publications. CHSH have used different symbols in their original paper.) The random switching had been suggested to the experimenters already by Bell and was realized by Aspect and coworkers [3] and in a sensational way, involving photons from billions years past, by Rauch and coworkers [4].

However, the possibility of random switching has not been included into the mathematical formalism of the theories of Bell and CHSH. From the viewpoint of probability theory, they needed to introduce random variables j, j' with the possible outcomes $j = a, a'$ and $j' = b, b'$ in the respective stations. Thus, it is nontrivial to write the functions A, B in terms of these random variables and to equate all possible outcomes to ±1: $A(j, \ldots) = \pm 1$ and $B(j', \ldots) = \pm 1$, because the +1 or −1 may then indeed describe channels that point in different directions, depending on the actual setting outcome. That inconsistency is hidden in the Bell-CHSH-type theories, because they only deal with the actual outcomes $a, a'; b, b'$ and leave the switching reserved for the experimenters.

We, therefore, ask why different geometric equipment-arrangements need only be described by two integer numbers and follow their algebra? In the authors' opinion, this oversimplification has been accepted by almost everyone, because of the illusion that we deal with something analog to the eigenvalues in quantum mechanics, while the Bell and CHSH formalism must not be quantum mechanical as emphasized by both Bell and CHSH (see next section below).

Experiments with polarized photons are usually characterized by descriptions such as [*horizontal*, *a*, …] instead of just +1 and [*vertical*, *a*, …] instead of −1 for a given setting *a* and analogous notations for the other settings. Thus, the experimenters imply a connection between the definition of *horizontal* or *vertical* to the equipment setting directions *a* (or *b* etc.). A more precise theoretical notation would, therefore, use $horizontal^a$ and $vertical^a$ and similar notations (such as $horizontal^b$) for the other settings, if one wishes to recognize the fact that the polarizer direction co-determines what is meant by *horizontal* and *vertical*.

Wigner [10] did notice and correct part of this problem. He and later d'Espagnat [11] have made use of group theory and a more general codomain for the functions A, B. They counted the *equal* and *not-equal* outcomes in the two stations and produced an inequality involving the counted numbers of *equal* vs *not-equal* results for a variety of different equipment setting pairs. The Wigner-d'Espagnat procedure is more general than that of Bell and CHSH. However, they did not account for all consequences of the random switching: As explained, one needs to have a consistent understanding what the expressions *horizontal* and *vertical* mean for the entire experimental system and for all pair-measurements. The Wigner-d'Espagnat inequality deals with 3 pair-observations. Such an experiment has not been performed yet and their approach leads to contradictions as discussed in reference [12].

Strictly speaking, one may only use the judgements "*equal*" and "*not-equal*" for parallel polarizers or Stern-Gerlach magnets in both stations. It is still possible to introduce different definitions such as *horizontal*^{a} and *vertical*$^{b'}$ in the respective separate stations. However, a precise notation would then also use superscripts such as *equal*$^{a,b'}$ or *not-equal*$^{a,b'}$ and similar superscripts for other settings. The use of superscripts from both stations is completely acceptable, because the notions of *equal* and *not-equal* can only be introduced after merging the data from both stations. Thus, one cannot exclude probability measures for the results *equal* and *not-equal* that are functions of the polarizer-settings of both stations and even of the angle between those settings. The standard objection that experimenters Alice and Bob in the respective stations know nothing about each other does simply not apply to the theoretician who deals with the merged data from both EPRB stations and compares them.

Similarities with relativity theory do surface here. If we have two spaceships with pilots Alice and Bob, respectively, who know absolutely nothing of each other, no correlation of any physical processes in the spaceships can ever be found. The times displayed by their respective clocks are unknown until investigated, for example, by light-beams tracking the spaceships; alternatively the clocks may be compared when the spaceships are brought together. Only in these ways may their clocks be correlated and found to depend on the difference of their velocity-histories (see also [12]), which represent, of course, a nonlocal piece of information.

As done with the velocities in relativity, one may also fix the polarizer in one station and vary the angle between the polarizers settings (see **Appendix** and references [13] as well as [14]).

Summarizing, one can state that the Bell-CHSH choice of $A, B = \pm 1$, implying that the function values can be handled as two integer numbers for all possible equipment settings, oversimplifies the actual physical facts of the Einstein-Podolsky-Rosen (EPR) type experiments. The extensive repair work of Wigner and d'Espagnat represents a major improvement, but did not demonstrate a general validity of the CHSH work, particularly not for pair measurements with multiple random polarizer settings on both sides (as shown in detail in sections below).

3. Information from the Source

3.1. CHSH Quotation

> "—Suppose now that the statistical correlation of $A(a)$ and $B(b)$ is due to information carried by and localized within each particle, and that at some time in the past the particles constituting one pair were in contact and communication regarding this information. The information, which emphatically is not quantum mechanical, is part of the content of a set of hidden variables, denoted collectively by λ. The results of the selections are

> then to be deterministic functions $A(a,\lambda)$ and $B(b,\lambda)$. Locality reasonably requires $A(a,\lambda)$ to be independent of the parameter *b* and $B(b,\lambda)$ to be likewise independent of a, since (sic) the two selections may occur at an arbitrary great distance from each other."

3.2. Criticism

We only note a minor inaccuracy: λ is defined as a "set of hidden variables" and certainly may be used as such. However, in most of what follows in the CHSH paper, the same symbol λ is used for the possible values of that hidden variable, which represent Einstein's elements of physical reality. This twofold meaning of λ has led to some confusions in the literature. We attempt to present all the following explanations in a way that avoids confusion. However, we still continue to use λ with the dualistic meaning that it has in the CHSH paper (variable as well as value of variable), in order not to deviate too much from their notation.

4. Involvement of Probability

4.1. CHSH Quotation

> "—Finally, since the pair of particles is generally emitted by a source in a manner physically independent of the adjustable parameters *a* and *b*, we assume that the normalized probability distribution $\rho(\lambda)$ characterizing the ensemble is independent of *a* and *b*."

4.2. Criticism

The possible equipment settings a,b are still regarded as "adjustable parameters" that somehow also encompass the random switching. However a,b as well as all other equipment settings are used in the equations below as given. As mentioned, the introduction of random variables $j = a, a'$; $j' = b, b'$ is a necessity if random switching ought to be included. Furthermore we need to introduce a probabilistic tool that accounts for the physical fact that measurements with different setting pairs in any given station cannot occur simultaneously. The kinematics of random switching and any dynamics of the many body physics involved in the interactions between incoming particles and measurement equipment (see also reference [15]), can generally not be described by the functions and probability density as introduced by Bell-CHSH. For example, the possible values of λ may be all different and, therefore each value λ interacts exclusively with a single pair of polarizer settings.

The only way to cover such a situation consistently with a classical probability theory, such as the framework of Kolmogorov, is by use of stochastic processes [16] [17]. Thus, a general treatment of the random switching of setting pairs requires the introduction of a time-like variable *t* and a two dimensional vector stochastic process of the kind $\left[A(a,\lambda,t), B(b,\lambda,t)\right]$. The probabilistic Bell-CHSH approach is not general enough, because it cannot include such stochastic processes [17].

This latter fact becomes visible in the immediately following mathematical steps of CHSH. They use in all their algebraic expressions the identical value of λ, but now for different pair-functions that correspond to different pair measurements as well as different polarizer setting-pairs. This procedure has been justified in two ways. First, by counterfactual reasoning of the kind "had we measured with a different setting pair, we would have encountered the same λ. We discuss this way in our criticism following the mathematical steps and the second justification in more detail afterwards.

5. Mathematical Steps of the CHSH Derivation; All with Identical λ

5.1. CHSH Quotation

Defining the correlation function

$$P(a,b)=\int_{\Gamma} A(a,\lambda)B(b,\lambda)\rho(\lambda)\mathrm{d}\lambda, \tag{1}$$

where Γ is the total λ space, we have

$$\left|P(a,b)-P(a,b')\right|$$

$$\leq \int_{\Gamma}\left|A(a,\lambda)B(b,\lambda)-A(a,\lambda)B(b',\lambda)\right|\rho(\lambda)\mathrm{d}\lambda \tag{2}$$

$$=\int_{\Gamma}\left|A(a,\lambda)B(b,\lambda)\right|\left[1-B(b,\lambda)B(b',\lambda)\right]\rho(\lambda)\mathrm{d}\lambda \tag{3}$$

$$=\int_{\Gamma}\left[1-B(b,\lambda)B(b',\lambda)\right]\rho(\lambda)\mathrm{d}\lambda \tag{4}$$

5.2. Critizism

5.2.1. Failure of Counterfactual Reasoning

The steps from Equation (2) to Equation (4) use the axioms of integer numbers as well as the concept of the absolute value that have no obvious validity for the more general outcomes of *horizontal* and *vertical* and "products" of them. This fact does not matter for Equation (1), because we may adopt a convention to just subtract the numbers of *equal* and *not-equal* products as Wigner did. However, the algebraic steps from Equations (2) to (4) lead to the product:

$$B(b,\lambda)B(b',\lambda) \tag{5}$$

that now exhibits identical λs for two elements of reality that must be, generally and physically speaking, different, because both functions B symbolize measurements in the same station for different settings. Here we encounter a spectacular failure of the counterfactual reasoning and the assumption that the same λ may be used in the mathematical expressions above. Expression (5) is definitely a red flag for the basic assumptions that govern the domain and co-domain of the Bell-CHSH functions and their products.

5.2.2. Reordering the Possible Outcomes

Bell, CHSH and their followers have attempted to circumvent some criticism of the counterfactual arguments in the following way, which agrees also with the main reasoning in references [3] [4] [5] as well as many textbooks:

Because λ represents the possible outcome-value of measurements, the claim is made that one can reorder these possible outcomes in such a way that about all the actual outcomes may be arranged in quadruples, each with the same value λ_n to obtain the following inequality:

$$\begin{aligned} -2 \le\ & A(a,\lambda_n)B(b,\lambda_n) - A(a,\lambda_n)B(b',\lambda_n) \\ & + A(a',\lambda_n)B(b,\lambda_n) + A(a',\lambda_n)B(b',\lambda_n) \le +2. \end{aligned} \tag{6}$$

where $n = 1,2,3,\cdots,N$, with N being a large number. Inserting all possible values of ±1 for the functions A, B supplies immediate verification. CHSH correctly have deduced from the random switching that Γ, the set of all possible λ_n, must be independent of the equipment-settings, because of Einstein locality. They maintain, as the majority of experts do, that this fact justifies the reordering of data into a large number of quadruples each with the same outcome-value λ_n. We will see below that this type of reordering indeed leads to an inequality that is only slightly different from the one derived by CHSH. As and aside, the possibility of reordering is actually mathematically only guaranteed for countable values of λ_n. If the variable λ represents a continuum of some form, reordering may not be a valid procedure [16]. However, for our current purpose this problem may be ignored, because of additional reasons that supersede these finer points.

The additional reasons will be discussed in detail when connecting the CHSH-Bell-type inequalities to quantum theory and the actual experiments after inequality (11) below. These reasons follow from the findings of the mathematician Vorob'ev and are also explained in the **Appendix**. We first proceed, however, with the derivation of the CHSH inequalities.

6. The CHSH Inequalities

6.1. Remark on a Calculus Error of CHSH

CHSH use the following assumptions and equations to derive their original inequality from the inequality derived in Equations (*1*) - (*4*).

CHSH assume $P(a',b) = 1-\delta$ with $0 \le \delta \le 1$ and deduce

$$\int_{\Gamma_-} \rho(\lambda)\,\mathrm{d}\lambda = \frac{\delta}{2} \tag{7}$$

with Γ_- representing the set of all λ for which $A(a',\lambda) = -B(b,\lambda)$. In addition CHSH imply

$$\delta = -2\int_{\Gamma_-} A(a',\lambda)B(b',\lambda)\rho(\lambda)\,\mathrm{d}\lambda, \tag{8}$$

Criticism

Equation (8) is only valid if also $P(a',b') = 1-\delta$. CHSH assume Γ_- to be independent of the setting-pairs, which is incorrect. In this way they derive the following inequality.

6.2. The Original CHSH Inequality

"And therefore

$$\left|P(a,b)-P(a,b')\right|\leq 2-P(a',b)-P(a',b')" \tag{9}$$

6.3. Comment

Although a calculus-error was made to derive this original CHSH inequality (9), the error is largely inconsequential, because it is removed by the additional precondition that $P(a',b')=1-\delta$, which is just more restrictive. This original CHSH inequality does, however, still suffer from the problems with the product (5) given above.

The nowadays mostly used variation of the CHSH inequality is not encumbered by the latter problem and is only somewhat "weaker" than (9):

$$\left|P(a,b)-P(a,b')+P(a',b)+P(a',b')\right|\leq 2. \tag{10}$$

This inequality follows immediately from inequality (6) by summation over all n with $1\leq n\leq N$. It does involve the reordering discussed above.

7. Connection to the Result of Quantum Theory and Experiment

7.1. CHSH Quotation

CHSH made the connection of their inequality to quantum theory by noting that $P(a,b)$ is a function of $b-a$. This function depends on the actual entanglement (pair-correlations) and is typically given by $-\cos(b-a)$ and similar functions for the pairs a,b', a',b and a',b', which thus are functions of the angle Δ between the polarizer settings; a result that is obtained by quantum theory and also corroborated experimentally. CHSH further note that Equation (9) can be written by using only three differences resulting in three numbers that commonly are called Bell- or CHSH-angles. This fact follows from the cyclical arrangement of the CHSH polarizer settings, which means that three setting pairs fully determine the fourth. CHSH arbitrarily chose $\alpha=b-a$, $\beta=b'-b$ and $\gamma=b-a'$ to obtain CHSH Equation (1b):

$$\left|P(\alpha)-P(\alpha+\beta)\right|\leq 2-P(\gamma)-P(\beta+\gamma) \tag{11}$$

7.2. Criticism

Seen from the viewpoint of rotational invariance, the angles between the polarizer settings indeed appear as the important physical variables, while the setting directions for themselves do not [14]. Thus, the introduction of the Bell-CHSH angles (or more generally scalar vector products) is indeed a necessity in order to properly compare Bell-CHSH-type theories with quantum theory and with actual measurements and experiments. This fact has, however, far reaching consequences that invalidate inequality (11).

7.3. Vorob'ev's Necessary Condition

Vorob'ev's work [7] represents not just another way to prove CHSH-Bell-type inequalities for functions on a common probability space. Its main corollary is that no constraint of the CHSH-Bell-type can be derived or proven without the existence of what Vorob'ev calls a combinatorial topological cyclicity such as the one just described above in terms of the cyclical arrangement of the polarizer settings.

The term "cyclicity" expresses the precise appearance and recurrence of the equipment settings in the different terms of inequalities such as (10), which according to Vorob'ev are a sine qua non for the constraints that can be deduced in form of inequalities or otherwise. As a consequence of the cyclicity only three of the *P*s in inequality (10) (e.g. $P(a,b), P(a,b'), P(a',b)$) may be chosen freely within their given codomain $-1 \le P \le +1$, the fourth, $P(a',b')$, cannot be freely chosen, which is the exact reason for the constraints that are expressed by the CHSH inequality.

CHSH were obviously not aware of the work of Vorob'ev [7], who proved in general terms that the validity of inequality (11) is conditional to the cyclicity of the equipment settings that label the expectation values (correlation functions (1)) in the inequalities (9) and (10). The specialization of Vorob'ev's more general work to Bell-CHSH inequalities has been discussed in in detail in references [17] [18]. A short review of the significance of Vorob'evs findings for the validity (or lack of validity) of inequality (11) is given in **Appendix**.

7.4. Removal of the Vorob'ev Cyclicity

This cyclicity, necessary for any constraint on the expectation values, is usually not mentioned in discussions of Bell-type and CHSH-type inequalities, because most of these discussions stop at inequality (10) and do not continue to describe the link with quantum theory and experiment in any detail. Inequality (10) contains the cyclicity automatically. When the step to use Bell- and CHSH-angles is taken, this is no longer the case and the inequality (11) is then invalid for the following reasons.

The choice of the three CHSH angles α, β, γ to obtain inequality (11) is arbitrary and a multitude of other choices could have been made to obtain inequalities different from (11). Their choice also does not guarantee the cyclicity of the original equipment settings. In fact, going back from inequality (11) to inequality (10), one can choose for each term an infinity (cardinality of the real numbers) of non-cyclical setting arrangements $\left[a'', b''; a''', b'''; a^{iv}, b^{iv}; a^{v}, b^{v}\right]$ that express the same four angles:

$$b'' - a'' = b - a;\ \ b''' - a''' = b' - a;\ \ b^{iv} - a^{iv} = b - a' \ \text{ and } \ b^{v} - a^{v} = b' - a'. \quad (12)$$

For these modified equipment settings, a CHSH inequality does not exist, because of the lacking cyclicity.

8. Completely Random Polarizer Settings and Counterexamples to Inequality (11)

The claim of using completely random polarizer settings in references [4] [5] and other works of Aspect, Zeilinger, Giustina and coworkers, is misleading. This fact can be seen with particular clarity in the TV-show "Einstein's Quantum Riddle"—NOVA, which creates the illusion that two absolutely random polarizer settings are used on two respective islands and the strong correlation of the measurement outcomes have, therefore, only one explanation: instantaneous influences from one island to the other, just the kind of influences that Einstein called "spooky". In an attempt to appear entirely convincing, the randomness of the setting-choices is derived from photons of billion years past [4] and from the "free will" of collaborating researchers [5]. This illusion of randomness, however, can only be created for a small number of polarizer settings, two in the published cases. These two settings are, however, carefully pre-chosen and pre-determined such that the polarizers on the two islands always exhibit one of the four desired Bell-CHSH angles; independent of the random switching. The relative position of the polarizers to each other is, thus, not random at all.

Based on the findings in the section "Removal of the Vorob'ev cyclicity", it will be shown immediately that measurements with large numbers of random polarizer settings leave only negligibly few setting combinations for which inequality (11) remains indeed valid and we will discuss in the remarks at the end of this section how even these few exceptions may be dealt with to avoid any vestige of hints toward instantaneous influences between islands.

Note that the following counterexamples to the validity of inequality (11) do not contest the independence of the set Λ of all λs from the polarizer settings. The counterexamples are merely based on the use of a much larger number of random settings and in some cases on the fact that the correlation functions P are by law of nature invariant to rotations of the polarizer-pairs involved in any specific pair-measurement, while the Vorob'ev cyclicity represents a mathematical abstraction that is not subject to laws of nature.

8.1. Counterexample 1

First, use a large number of completely random polarizer settings in both measurement stations (wings). In this case we encounter, with only negligible probability, measurement-pairs that exhibit CHSH angles. We may select these measurements-pairs in order to form a CHSH-quadrupel. However, even among these selected quadruples, it is only a negligible number that can be arranged into a Vorob'ev cyclicity. Thus the necessary cyclicity that validates (11) is rarely encountered (with probability close to 0). All of this follows immediately from the algebra discussed in the subsection "Removal of the Vorob'ev cyclicity". We further discuss in the remarks below, how even this negligible set of measurements that form a Vorob'ev cyclicity may be dealt with.

8.2. Counterexample 2

Second, use again a large number (instead of just two as the Aspect- and Zeilinger-groups do) of completely random polarizer settings in one of the EPRB wings and use corresponding settings in the other wing that conserve (with equal frequency of occurrence) the 4 CHSH angles between the two polarizer directions and thus the 4 pair correlations *P*. The overwhelming majority of the corresponding CHSH-quadruples will again not exhibit any Vorob'ev cyclicity as explained above and may thus violate inequality (11). Large numbers of such measurements may indeed be performed by using the techniques of Giustina *et al.* [19] that include electro optical modulators. One may then ask the question why such measurements that exhibit a greater randomness than those of Aspect-Zeilinger-Giustina (who have only two different but also correlated settings in each wing), are not constrained by the CHSH or Bell-type inequalities and do, therefore, not require any instantaneous influences for their explanation, while the reported Aspect-Zeilinger-Giustina-measurements supposedly do. We also see that the Aspect-Zeilinger-Giustina claim of using completely random measurements in both wings is incorrect and certainly misleading.

As mentioned, two polarizer settings are indeed switched randomly on each island in the NOVA—TV-show and in references [4] [5]. However, the polarizer settings on the two islands are correlated by predetermined choice of the CHSH angles. The experimenters would have discovered the importance of this fact, had they attempted to use many random settings in each station. This "randomness" limited by the chosen CHSH angles still guarantees the independence of Λ but not the validity of inequality (11).

It is informative in this context, to consider how the equipment settings for the measurements in ordinary space are entering the operators that act on the Hilbert-state-vectors of quantum theory. For EPRB experiments with spin $\frac{1}{2}$ particles, we require a tensor product of two Pauli matrices acting on the tensor product of spinors. One Pauli matrix (related to the measurements performed with apparatus I) may be chosen corresponding to an arbitrarily defined coordinate system that describes an arbitrary magnet-setting in ordinary space. The second Pauli matrix (related to apparatus II) is not entirely arbitrary but must account for the experimentally given Bell-CHSH angle. These facts demonstrate the Achilles heel of the work of Bell and CHSH: The validity of their inequality requires a particular cyclicity of the equipment settings in ordinary space. The choice of the quantum operators in relation to these equipment settings is, on the other hand, to a large extent arbitrary and requires only the conservation of certain vector products that result in the same Bell-CHSH angles but do not necessarily correspond to cyclical equipment settings. This fact opens the possibility of violations of the Bell-CHSH type inequalities.

8.3. Counterexample 3

As another counterexample that violates (11), consider the use of a fixed setting in one measurement station (wing) and arbitrary random settings in the other, as described in more detail in **Appendix**. This experimental arrangement permits us to obtain setting pairs that have the same Bell-CHSH angles but no cyclicity that imposes any restrictions whatsoever [14] [17] [20]. An explicit example of this kind may also be implemented on two computers (one representing the fixed setting and the other and arbitrary polarizer setting that may also be switched rapidly) and has been presented by the author [12]. As described in the **Appendix**, such measurements were indeed performed by Kwiat and coworkers and violate (11), which is no surprise, however, because no Vorob'ev cyclicity is involved.

8.4. Counterexample 4

It is important to note that the quantum results for the pair correlations P as well as the corresponding experimental results are invariant under rotations in ordinary three-dimensional space. The specific Vorob'ev cyclicity used by CHSH for the four measurement pairs is not invariant under rotations of the separate setting pairs. In addition, the four-pair inequality (6) represents, as mentioned above, a mathematical construct that includes reordering of the λs and does not follow any natural symmetry-law. This latter fact permits the removal of the Vorob'ev cyclicity by the following plausible counterexample.

Link the frame of reference to the distant stars, and consider the rotation of the polarizer settings with the rotation of the earth. This rotation will also remove the cyclical arrangement of the terms in (6) and, therefore, any CHSH constraint, while it will leave the pair correlations P unchanged. The NOVA—TV-show does not discuss the rotation of the earth relative to the stars.

8.5. What If the Vorob'ev Cyclicity Exists?

We may ask ourselves what it means when the quadruple-cyclicity indeed exists for some subset of the data or even for about all of them?

Counterexample 3 relates to the most precise EPRB-measurements ever performed to the authors' knowledge. However, due to the fixing of the polarizer setting on one side, the cyclicity is completely removed and inequality (11) is, therefore, not valid.

Counterexample 4 applies directly to the Aspect-Zeilinger-type of experiments. Rotation of the polarizer settings does certainly remove some of the cyclicity. It does also leave the possibility for some remaining cyclical or almost cyclical rearrangements of CHSH quadruples. However, a typical experimental run lasts for about one hour, which leads to significant angular changes in a system rotating with Earth. That rotation (or any other relevant rotation) together with the need of reordering the λs and the need to use consistent definitions of *horizontal* and *vertical* (as emphasized in Subsection 2.2) may lead to considerable violations of the cyclicity. Inequalities (11) and (9) have, thus, lost their theorem-character and become rather a multivariate Monte Carlo approximation problem. This fact means, in turn, that the experiments and measurements need not only be investigated with respect to how much they deviate from the originally fixed upper limit 2, but need also be investigated with respect to their, often significant, deviation from the results of quantum theory that have usually been ignored.

There are important (with respect to clarification and explanation) experiments that have not yet been performed. Assume, for example, that we have 4 Aspect-Zeilinger-type experiments with four setting pairs $a,b;a,b';a',b$ and a',b' and also four sources of entangled pairs that may all be run simultaneously. Select from the runs of the 4 pair-experiments the subset C of data for which indeed the simultaneous quadruple measurement-outcomes are equal for all measurements with equal polarizer settings. This subset C fulfills then inequalities (9) as well as (11). The quantum result for this subset must obviously also fulfill the inequalities because otherwise it would contradict algebra and the measurements (see [21]).

Last but not least, I would like to comment about the fact that for the experiments of the Aspect-Zeilinger-type, the "closure of the cyclical arrangement is not guaranteed by the simultaneity of the measurements of CHSH quadruples, but only involves sequential measurements of pairs collected with synchronized clock-pair-readings. For such sequential pair-measurements, many variations of possible violations have been suggested in the past. For example, the two-dimensional vector stochastic processes from Subsection 4.2 remove, at least in principle, any cyclicity, because the pair measurements are performed at different times and any dependency of the relative pair-outcomes on measurement times will remove the cyclicity. Violations are particularly easy to construct, when only small subsets of the totality of measurements can be arranged in cyclical form, as is the case for all of the above counterexamples.

The possibility of geometric phases that also prevent the closure of the cyclicity (as shown in **Figure A1** of the **Appendix**) was recently added as an additional option to the previously existing extensive list of possible violations [14] [20].

In summary, the reported violations of CHSH inequality (1b) (identical to (11) above), are actually no violations at all for all the experiments and measurements that do not exhibit a Vorob'ev cyclicity as clearly shown in counterexample 3. The measurements by Kwiat and coworkers (see **Appendix**) represent the most accurate Bell-test yet performed. However, these measurements exhibit no cyclicity and, therefore, are not subject to any constraint.

Furthermore, the CHSH constraints are, generally speaking, diminished as demonstrated in counterexamples 1, 2, 4, because of the limited presence of cyclicities that are a necessary condition for CHSH inequality 1b. This fact adds many more possibilities to the already previously published deficiencies of CHSH-Bell-type inequalities when applied to actual experiments (see also the work of Kupczynski [22]).

9. Conclusion

The well known CHSH results [6] are expressed by their inequalities (1a) and

(1b) and by the somewhat modified inequalities (10) and (11). It has been shown that none of these inequalities is beyond reproach and that particularly inequality (11) that connects the CHSH framework to actual experiments is violated by straightforward counterexamples. I conclude that the derivations of the CHSH inequalities and all similar inequalities, including that of Wigner, as well as their application to actual EPRB experiments, are highly questionable from both a mathematical and physical point of view and must certainly not be used to deny Einstein's views of physical locality and separability.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Kocher, C.A. and Commins, E.D. (1967) Polarization Correlation of Photons Emitted in an Atomic Cascade. *Physical Review Letters*, **18**, 575-577. https://doi.org/10.1103/PhysRevLett.18.575

[2] Nordén, B. (2016) Quantum Entanglement: Facts and Fiction—How Wrong Was Einstein after All. *QRB Discovery*, **49**, 1-12. https://doi.org/10.1017/S0033583516000111

[3] Aspect, A., Dalibard, J. and Roger, J. (1982) Experimental Test of Bell's Inequalities Using Time-Varying Analyzers. *Physical Review Letters*, **49**, 1804-1807. https://doi.org/10.1103/PhysRevLett.49.1804

[4] Rauch, D., *et al.* (2018) Cosmic Bell Test Using Random Measurement Settings from High-Redshift Quasars. *Physical Review Letters*, **121**, Article ID: 080403. https://doi.org/10.1103/PhysRevLett.121.080403

[5] The Big Bell Test Collaboration (2018) Challenging Local Realism with Human Choices. *Nature*, **557**, 212-216. https://doi.org/10.1038/s41586-018-0085-3

[6] Clauser, J.F., Horne, M.A., Shimony, A. and Holt, R.A. (1969) Proposed Experiment to Test Local Hidden-Variable Theories. *Physical Review Letters*, **23**, 880-884. https://doi.org/10.1103/PhysRevLett.23.880

[7] Vorob'ev, N.N. (1962) Consistent Families of Measures and Their Extension. *Theory of Probability and Its Applications*, **7**, 147-163. https://doi.org/10.1137/1107014

[8] Khrennikov, A. and Alodjants, A. (2019) Classical (Local and Contextual) Probability Model for Bohm-Bell Type Experimants: No-Signaling as Independence of Random Variables. *Entropy*, **21**, 157. https://doi.org/10.3390/e21020157

[9] Khrennikov, A. (2020) Has the CHSH-Inequality Any Relation to the EPR-Argument? *Quantum Bio-Informatics*, **5**, 87-92. https://doi.org/10.1142/9789811217838_0008

[10] Wigner, E.P. (1970) On Hidden Variables and Quantum Mechanical Probabilities. *American Journal of Physics*, **38**, 1005-1009. https://doi.org/10.1119/1.1976526

[11] d'Espagnat, B. (1979) The Quantum Theory and Reality. *Scientific American*, **241**, 158-181. https://doi.org/10.1038/scientificamerican1179-158

[12] Hess, K. (2018) Bell's Theorem and Instantaneous Influences at a Distance. *Journal of Modern Physics*, **9**, 1573-1590. https://doi.org/10.4236/jmp.2018.98099

[13] Hess, K. (2020) Kolmogorov's Probability Spaces for "Entangled" Data-Subsets of EPRB Experiments: No Violation of Einstein's Separation Principle. *Journal of Modern Physics*, **11**, 683-702. https://doi.org/10.4236/jmp.2020.115044

[14] Oaknin, D.H. (2020) Bell's Theorem Revisited: Geometric Phases in Gauge Theories. *Frontiers in Physics*, **8**, 142-162. https://doi.org/10.3389/fphy.2020.00142

[15] Schatten, K.H. (2021) A Physical Origin for Quantum Entanglement and Probabilistic Behaviors. *Journal of Modern Physics*, **12**, 50-58. https://doi.org/10.4236/jmp.2021.121005

[16] Hess, K. and Philipp, W. (2004) Breakdown of Bell's Theorem for Certain Objective Local Parameter Spaces. *PNAS*, **101**, 1799-1805. https://doi.org/10.1073/pnas.0307479100

[17] Hess, K., Philipp, W. and Aschwanden, M. (2006) What Is Quantum Information? *International Journal of Quantum Information*, **4**, 585-625. https://doi.org/10.1142/S0219749906002080

[18] Hess, K. and Philipp, W. (2005) The Bell Theorem as a Special Case of a Theorem of Bass. *Foundations of Physics*, **35**, 1749-1767. https://doi.org/10.1007/s10701-005-6520-y

[19] Giustina, M., *et al.* (2015) Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons. *Physical Review Letters*, **115**, Article ID: 250401.

[20] Oaknin, D.H. and Hess, K. (2020) On the Role of Vorob'ev Cyclicities and Berry's Phase in the EPR Paradox and Bell Tests.

[21] Khrennikov, A. (2014) CHSH Inequality: Quantum Probabilities as Classical Conditional Probabilities. *Foundations of Physics*, **45**, 711-725. https://doi.org/10.1007/s10701-014-9851-8

[22] Kupczynski, M. (2020) Is the Moon There If Nobody Looks: Bell Inequalities and Physical Reality. *Frontiers in Physics*, **8**, 273. https://doi.org/10.3389/fphy.2020.00273

[23] Kwiat, P.G., Waks, E., White, A.G., Appelbaum, I. and Eberhard, P.H. (1999) Ultrabright Source of Polarization-Entangled Photons. *Physical Review A*, **60**, 773-776. https://doi.org/10.1103/PhysRevA.60.R773

Appendix

The function products in all Bell-CHSH-type inequalities contain a cyclicity that the mathematician Vorob'ev [7] identified as an absolutely necessary condition to obtain the constraint imposed by these inequalities given a common probability space for all random variables. This cyclicity is illustrated in **Figure A1** that shows a quadrangle who's vertices represent the functions (random variables) in the CHSH inequality. The reader may imagine arbitrary distorted forms of the quadrangle and relate, symbolically, joint pair probability-distributions to the length of the lines between the functions. Vorobe'v noted (in a much more general way) that the arbitrary prescription of joint pair probability-distributions for three pairs does not permit complete freedom to choose the joint distribution of the last fourth pair. This fact is the direct reason for the constraints that Bell-CHSH-inequalities introduce, as shown in detail in [18]. No constraints exists without the Vorob'ev cyclicity.

It turns out that the cyclicity may rather easily be removed in view of Equation (11), because it is not the single polarizer settings that are important for the statistics of the outcomes but only the angle between them.

The most straightforward removal of the Vorob'ev cyclicity is accomplished by the experimental choice of a fixed polarizer setting in one wing as shown in **Figure A2**. For purposes of illustration, imagine again that the different lengths of the lines between the functions relate to the joint pair probability-distributions. Because they may now all be different and because of the lack of a cyclicity, there exists no constraint in form of a CHSH inequality (see also [20]).

The fact that such a rather straightforward removal of the cyclicity removes also all restrictions that the CHSH inequality and all other related inequalities impose, has received little attention by the majority of researchers in this field. Instead they have concentrated on other conditions to derive the inequalities,

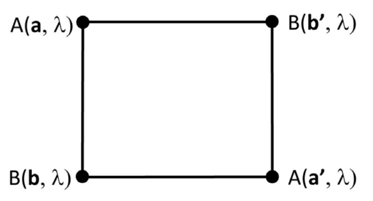

Figure A1. Symbolized Vorob'ev cyclicity for CHSH-type inequalities.

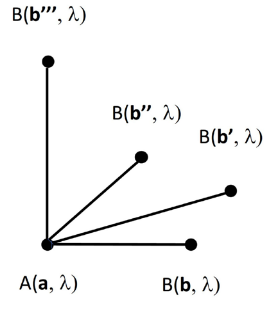

Figure A2. Removed Vorob'ev cyclicity for CHSH-type inequalities.

conditions such as "Bell locality" or the "complete randomness" and the "free will" related to setting choices, conditions that are neither necessary nor sufficient for validations or violations of the inequalities.

Experiments corresponding to this precise situation, experiments that violate the CHSH inequality with extremely high precision (probably the highest yet achieved), have been performed by Kwiat and coworkers [23] (see their **Figure A2(a)**). These experimenters have indeed fixed the polarizer-setting in one wing and varied that in the other, which amounts to varying the angle Δ between the polarizer directions. The graph of their data can be covered with great precision by a $\cos(2\Delta)$ graph which represents the quantum result for entangled photons.

The agreement with the quantum result, corresponding to the very high visibility of $V = 99.6 \pm 0.3\%$ is, however, not surprising. There exists, in this case, no valid CHSH inequality that would put any constraints on the measurement results. The explicit proof of this fact is straightforward: just insert the settings of **Figure A2** into inequalities (10) or (11), while conserving the CHSH angles and notice that the sum of all function pairs may now violate the CHSH inequalities, because the cyclicity has vanished.

It is important to realize that the wing with variable polarizer setting may also be subjected to rapid switching in this particular experiment. Rapid switching in both wings from positive to negative polarizer direction may be performed in addition, all without changing any expectation values (the correlations P).

Rapid switching of the Kwiat and coworker experiment would be useful to test Einstein's separation principle. It also would guarantee, now in a consistent way, the independence of the sets Λ from the polarizer-setting-pairs. Therefore, conspiracy-theory loopholes would be excluded as they have been excluded in the work that uses human and cosmic random number generators [4] [5]. Now, however, no constraints exist, because no Vorob'ev cyclicity exists in the measurement arrangements and no loophole needs to be closed to start with and ordinary random number generators as well as no switching at all are expected to give the same results.