

Quantum Violation of Bell's Inequality: A Misunderstanding Based on a Mathematical Error of Neglect

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Abstract

This paper intends to show how the fabled violation of Bell's inequality by the probabilistic specifications of quantum mechanics derives from a mathematical error, an error of neglect. I have no objection to the probabilities specified by quantum theory, nor to the inequality itself as characterized in the formulation of Clauser, Horne, Shimony, and Holt. Designed to assess consequences of Einstein's principle of local realism, the inequality pertains to a linear combination of four polarization products *on the same pair of photons* arising in a gedankenexperiment. My assessment displays that in this context, the summands of the relevant CHSH quantity $s(\lambda)$ inhere four symmetric functional relations which have long been neglected in analytic considerations. Its expectation $E[s(\lambda)]$ is not the sum of four "marginal" expectations from a joint distribution, as quantum theory explicitly avoids such a specification. Rather, I show that $E[s(\lambda)]$ has four distinct representations as the sum of *three* expectations of polarization products plus the expectation of a fourth which is restricted to equal a function value determined by the other three. Analysis using Bruno de Finetti's fundamental theorem of prevision (FTP) yields only a bound for $E(s)$ within $(1.1213, 2]$, surely not $2\sqrt{2}$ at all as is commonly understood. I exhibit slices of the 4-dimensional polytope of joint P_{++} probabilities actually motivated by quantum theory at the four stipulated angle settings, as it passes through 3-dimensional space. Bell's inequality is satisfied everywhere within the convex hull of extreme distributions cohering with quantum theoretic specifications, even while in keeping with local realism. Aspect's proposed "estimation" of $E(s)$ near to $2\sqrt{2}$ is based on polarization products from different photon pairs that do not have embedded within them the functional relations inhering in the rele-

vant gedankenexperiment. When one actively embeds the restrictions into Aspect's estimation procedure, it yields an estimate of 1.7667, although this is not and cannot be definitive. While my analysis supports the subjectivist construction of probability as clarifying issues relevant to the interpretation of quantum theory, the error resolved herein is purely mathematical. It pertains to the reconsideration of Bell violation irrespective of one's attitude toward the meaning of probability.

Keywords

Bell Inequality Defiance, CHSH Formulation, Fundamental Theorem of Probability, Probability Bounds, 4-Dimensional Cuts

1. Introduction

As brash as this may sound, claims that probabilistic specifications of quantum mechanics are inconsistent with local realism and defy Bell's inequality are just plain wrong. This may be difficult to accept, depending on how wedded one is to the outlook that gives rise to defiance of the inequality as it is widely understood. In the eyes of the professional physics community, the matter is now closed. The eminent journal *Nature* [1] flamboyantly announced the "Death by experiment for local realism" as an introduction to its publication of ambitious experimental results achieved at the Technical University of Delft. These were proclaimed to have closed simultaneously all seven loopholes that had been suggested as possible explanations for the purported violations of the inequality. My claim is that the touted violation of the inequality derives from a mathematical mistake, an error of neglect. Moreover, consequences of the error run through both the analytical development of the defiance structure and the statistical assessment of relevant matters. Its recognition relies only on a basic understanding of functions of many variables and on standard features of applied linear algebra. This presentation is prepared so to be read not only by physicists but by any sophisticated reader who has followed this issue at least at the level of popular description of scientific activity and who is not put off by equations per se.

The defiance of Bell's inequality is one of the seminal results underlying the understanding of quantum theory as it has developed over the past half century. More than seven thousand references to pertinent discussion can be found on Google Scholar, and I do not intend to review the corpus. Most physicists regard the inequality as a condition on freely determined expectation values and their estimates from physical experiments, calculated according to the QM formalism. I consider this viewpoint to be mistaken, for reasons I shall detail herein. However, while surely in a minority camp of objectors, my analysis can be situated within a literature of credible researchers whose perspective I share. The recent review article of Kupczynski [2] surveys and references more than eighty technical publications in this vein, both longstanding and recent, notably recognized contributions of Fine

[3], Hess [4], Hess and Philipp [5], Khrennikov [6], and Kupczynski herself. The review provides many more references from these and other researchers. Of course many of these have been contested, particularly in the perspective of Mermin [7]. Nonetheless, I believe the assessment of the situation that I provide is truly novel, and augments the reservations that have been proposed within the dissenting camp.

It is clear in his own writings that John Bell himself was puzzled by the implications of his inequality [8] [9] [10]. He suspected that something was wrong with the understanding that when coupled with local realism the probabilities of quantum mechanics seem to defy its structure, and he expressed undying confidence that this error would be discovered in due time. I am proposing that I have found the error he sought. I accept all probabilistic assertions supported by quantum theory, and I shall exhibit their implied satisfaction of the inequality bounds which they are widely supposed to defy.

I do not contest the experimental results of the Delft group, nor any of the related experimentation which has followed from the pathbreaking initial work of Alain Aspect and his group [11] [12]. I do contest the inferences they are purported to support. In this note I will first review the derivation of the inequality in the context to which it applies, featuring its relation to Einstein's principle of local realism. The review will focus on the CHSH form of the inequality to which Aspect's optical experimentation is considered to be relevant. Identifying the neglected functional relations that are involved in a thought experiment on a single pair of photons, I will show that claims to quantum defiance of the inequality are mistaken, and show how to derive the actual implications of quantum theory for the probabilities under consideration. Further will be shown why Aspect's computations (and all subsequent extensions) proposed to exhibit empirical confirmation of the inequality defiance are ill considered, and how they ought to be adjusted. This demonstration relies on the computational mechanics of Bruno de Finetti's fundamental theorem of probability. The results are displayed both algebraically and geometrically.

2. The Physical Setup of Four Experiments Providing Context for Bell's Inequality in CHSH Form: A 16-D Problem

We shall review the setup of an optical variant of Bell's experiment, designed by Alain Aspect in the 1980's to take advantage of a formulation of the problem proposed by Clauser, Horne, Shimony, and Holt [13]. The original discussions of the inequality violation were couched in terms of observations of spins of paired electrons. Although specific algebraic details differ for the two types of experimental situation, the conclusions reached would be identical.

An experiment is conducted on a pair of photons traveling in opposite directions along an axis, z , from a common source. As seen in **Figure 1**, the direction one of the photons travels toward detector A on the left is directly opposite to the direction its paired photon travels toward detector B on the right: $\mathbf{z}_A = -\mathbf{z}_B$. At the end of their respective journeys, each of the paired photons engages polarizing material that either allows it to pass through or to be deflected.

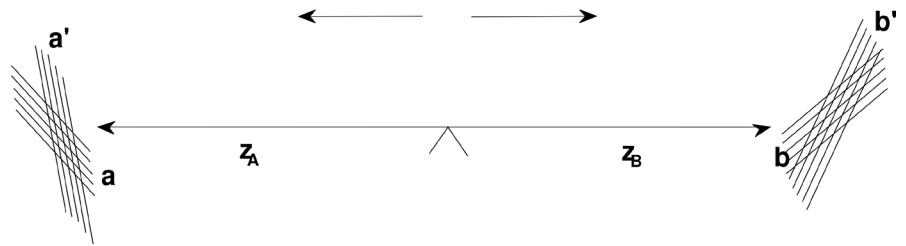


Figure 1. Polarizing material is aligned at the detection stations of A and B , each with two possible choices of direction in the (x, y) dimension perpendicular to the z direction of the incoming photon: direction \mathbf{a} or \mathbf{a}' at station A , and \mathbf{b} or \mathbf{b}' at station B .

The detection of a photon that passes through the polarizer is designated by denoting the numerical value of $A = +1$, while the detection of a photon as blocked is designated by the value of $A = -1$. The polarizer addressed by photon A is oriented in a variable direction \mathbf{a}^* in the (x, y) plane perpendicular to \mathbf{z}_A . It can be set in either of two specific directions designated as \mathbf{a} and \mathbf{a}' in the experimental setups we shall consider. Similarly, the direction of the polarizer met by the photon at station B can be set at either \mathbf{b} or \mathbf{b}' in its (x, y) plane. Depending on the specific pair of polarization directions $\mathbf{a}^*, \mathbf{b}^*$ chosen for any particular experiment, we shall observe the paired values of either $(A(\mathbf{a}), B(\mathbf{b}))$, or $(A(\mathbf{a}), B(\mathbf{b}'))$, or $(A(\mathbf{a}'), B(\mathbf{b}))$, or $(A(\mathbf{a}'), B(\mathbf{b}'))$. Since the observations of the A and the B photon detections can each equal either $+1$ or -1 whatever the angle pairing might be, the chosen observation pair $(A(\mathbf{a}^*), B(\mathbf{b}^*))$ will equal one of the four possibilities $(+, +), (+, -), (-, +)$, or $(-, -)$, where the needless numeric values of 1 are suppressed in each designated pairing.

Experimental choices of the two polarizing directions yield a specific *relative angle* between them at stations A and B in any given experiment. Using Aspect's notation that parentheses around a pair of directions denotes the relative angle between them, the experimental detection angle settings $(\mathbf{a}^*, \mathbf{z}_A)$ and $(\mathbf{b}^*, \mathbf{z}_B)$ imply the *relative angle* between polarizers at stations A and B in the (x, y) dimension as $(\mathbf{a}^*, \mathbf{b}^*)$. Bell's inequality is relevant to this context in which the two photon polarization directions can be paired at any one of four distinct relative angles, denoted by the parenthetic pairs $(\mathbf{a}, \mathbf{b}), (\mathbf{a}, \mathbf{b}'), (\mathbf{a}', \mathbf{b}),$ or $(\mathbf{a}', \mathbf{b}')$.

In order to view the relative angles we are talking about, mentally we would need to swing the (x, y) plane around by 180° as it is viewed by the photon directed to station A , and superimpose it on the (x, y) plane as it is viewed by the photon directed to station B . In this manner we can understand the size and meaning of the relative angles *between* the various values of polarization orientations \mathbf{a}^* and \mathbf{b}^* as seen here in **Figure 2**, which follows.

The theory of quantum mechanics motivates specification of probabilities for the four observable outcome possibilities of the polarization experiment as depending on the *relative angle* $(\mathbf{a}^*, \mathbf{b}^*)$ between the direction vectors of the

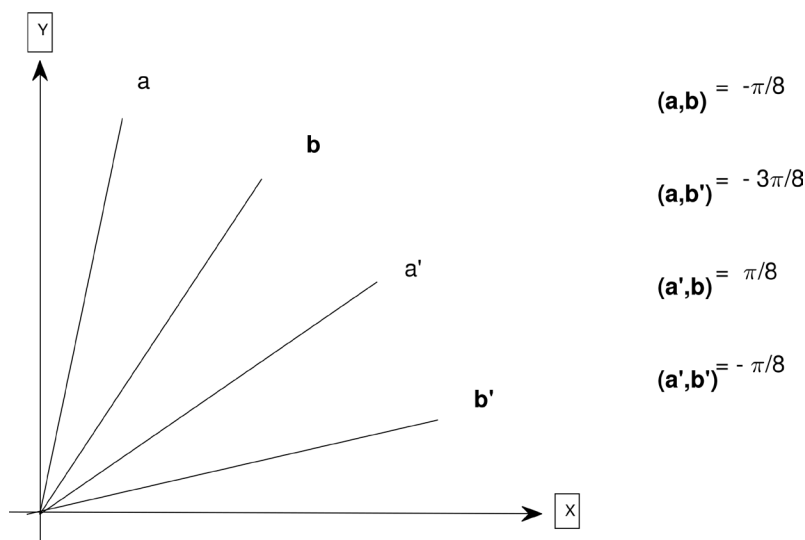


Figure 2. Directional vectors of the polarization angle settings at the observation stations *A* and *B*, viewed in a common axis orientation. The specific relative angle size settings displayed are the extreme violation values, a feature to be discussed.

polarizers at stations *A* and *B*. For any such relative angle pairing, the probabilities specified by quantum theory for the four possible experimental observations $\{++,+-,-+,--\}$ are

$$\begin{aligned}
 &P\left[(A(\mathbf{a}^*)=+1)(B(\mathbf{b}^*)=+1)\right] \\
 &= P\left[(A(\mathbf{a}^*)=-1)(B(\mathbf{b}^*)=-1)\right] = \frac{1}{2}\cos^2(\mathbf{a}^*,\mathbf{b}^*), \\
 &\text{and} \\
 &P\left[(A(\mathbf{a}^*)=+1)(B(\mathbf{b}^*)=-1)\right] \\
 &= \left[(A(\mathbf{a}^*)=-1)(B(\mathbf{b}^*)=+1)\right] = \frac{1}{2}\sin^2(\mathbf{a}^*,\mathbf{b}^*).
 \end{aligned} \tag{1}$$

For efficiency in what follows, we shall denote the four probabilities appearing in Equations (1) by P_{++}, P_{--}, P_{+-} , and P_{-+} when the pertinent angle setting is evident.

These four probabilities surely sum to equal 1, because the sum of $\cos^2 + \sin^2$ of any angle equals 1. A few properties of the joint probability mass function (pmf) they compose should be noticed. Firstly, the four probabilities can be specified by the value of any one of them. The equations (1) stipulate that no matter what the relative angle $(\mathbf{a}^*, \mathbf{b}^*)$ may be, the values of $P_{++} = P_{--}$, and $P_{+-} = P_{-+}$. Since the four probabilities do sum to 1 then, the specification of P_{++} as the value p , for example, implies that the pmf vector $[P_{++}, P_{--}, P_{+-}, P_{-+}]$ would be $[p, p, (1-2p)/2, (1-2p)/2]$. Another implication of this feature is that the probabilities for the paired detection outcomes depend only on the *product* of the two measurements. For both outcomes $++$ and $--$ yield a product of +1 and both outcomes $+-$ and $-+$ yield a product of -1. Thus, the QM-motivated distribution for the experimental value of the polarization *product* $A(\mathbf{a}^*)B(\mathbf{b}^*)$ is specified by $P[A(\mathbf{a}^*)B(\mathbf{b}^*)=+1] = \cos^2(\mathbf{a}^*, \mathbf{b}^*)$ and

$$P[A(\mathbf{a}^*)B(\mathbf{b}^*) = -1] = \sin^2(\mathbf{a}^*, \mathbf{b}^*).$$

As will be important to recognize in what follows, the expected value (first moment) of this distribution for the detection product is

$$\begin{aligned} E[A(\mathbf{a}^*)B(\mathbf{b}^*)] &= (+1)\cos^2(\mathbf{a}^*, \mathbf{b}^*) + (-1)\sin^2(\mathbf{a}^*, \mathbf{b}^*) \\ &= \cos^2(\mathbf{a}^*, \mathbf{b}^*) - \sin^2(\mathbf{a}^*, \mathbf{b}^*) = \cos 2(\mathbf{a}^*, \mathbf{b}^*) \end{aligned} \tag{2}$$

according to standard double angle formulas. It is worthwhile reminding right here that “the expected value of a probability distribution” is the “first moment” of the distribution. Geometrically, it is the point of balance of the probability mass function weights when they are positioned in space at the places where the possible observations to which they pertain might occur. It is a property of a probability distribution for the outcome of a specific single observable variable. A final peculiarity of Equation (2) to be useful far down the road in this explication is that the expectation value $E[A(\mathbf{a}^*)B(\mathbf{b}^*)]$ can also be represented as

$$E[A(\mathbf{a}^*)B(\mathbf{b}^*)] = 2\cos^2(\mathbf{a}^*, \mathbf{b}^*) - 1 = 4P_{++}(\mathbf{a}^*, \mathbf{b}^*) - 1. \tag{3}$$

For the value of $\sin^2(\mathbf{a}^*, \mathbf{b}^*)$ appearing in the final line of Equation (2) can also be written as $1 - \cos^2(\mathbf{a}^*, \mathbf{b}^*)$. Enough of this for now.

Secondly, again no matter what the relative angle $(\mathbf{a}^*, \mathbf{b}^*)$ may be, the marginal probability that the detection observation of the photon equals +1 at either observation station is equal to 1/2. For the standard margining equation for the result of a paired experiment yields

$$\begin{aligned} P[(A(\mathbf{a}^*) = +1)] &= P[(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = +1)] + P[(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = -1)] \\ &= \frac{1}{2}\cos^2(\mathbf{a}^*, \mathbf{b}^*) + \frac{1}{2}\sin^2(\mathbf{a}^*, \mathbf{b}^*) = 1/2 \end{aligned} \tag{4}$$

This result codifies a touted feature of physical processes at quantum scales of magnitude, that the photon behaviours of particle pairs are understood to be *entangled*. Since the probability for the joint photon behaviour

$P[(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = +1)]$ does not factor into the product of their marginal probabilities $P[(A(\mathbf{a}^*) = +1)]$ and $P[(B(\mathbf{b}^*) = +1)]$, the conditional distribution for either one of these events depends on the context of the conditioning behaviour:

$$P[(A(\mathbf{a}^*) = +1) | (B(\mathbf{b}^*) = +1)] = \cos^2(\mathbf{a}^*, \mathbf{b}^*) \neq P[(A(\mathbf{a}^*) = +1)] = 1/2, \tag{5}$$

and $P[(A(\mathbf{a}^*) = +1) | (B(\mathbf{b}^*) = -1)] = \sin^2(\mathbf{a}^*, \mathbf{b}^*)$, which is different still.

We have concluded what we need to say at the moment about the prescriptions of quantum theory relevant to quantum polarization behaviour of a single pair of prepared photons. Before proceeding to the specification of Bell’s inequality, we need to address three issues: what quantum theory professes *not* to say

on account of the uncertainty principle, the relevance of the principle of local realism, and a proposal regarding supplementary variables that may impinge on the experimental results.

3. The Uncertainty Principle: What Quantum Theory Disavows

The problem of quantum physics relevant to Bell's inequality concerns the consideration of what might happen in imagined designs of physical situations that are impossible to instantiate. In our developments to this point, we have identified a physical experiment on a pair of photons, polarizing them at one of four exclusive possible relative angle pairings, (\mathbf{a}, \mathbf{b}) , $(\mathbf{a}, \mathbf{b}')$, $(\mathbf{a}', \mathbf{b})$, and $(\mathbf{a}', \mathbf{b}')$. We could perform this polarization experiment on a specific pair of photons at any one of these angle pairings. Moreover, we could perform the experiment at all four angle settings if we generated different pairs of photons to engage each of them. However, we cannot perform all four experiments simultaneously *on the same pair of photons*. The theory of quantum mechanics recognizes this fact explicitly, avowing the uncertainty principle which abnegates all claims regarding the outcome of an experiment that cannot possibly be observed. Not only would any such claim necessarily evade empirical corroboration, but the theoretical algebraic mechanism that is used to identify the quantum probabilities for the results of the polarization experiment embeds this impossibility into its protocol which we shall not detail here. In a word, two measurements of a physical system are recognised to be jointly observable only if the product of the matrix operators that characterise them commutes. Each possible polarization observation pair at a relative angle $(\mathbf{a}^*, \mathbf{b}^*)$ is characterised by its own matrix operator H . Since we have four possible experimental designs under consideration, codified by the paired angle settings (\mathbf{a}, \mathbf{b}) , $(\mathbf{a}, \mathbf{b}')$, $(\mathbf{a}', \mathbf{b})$, and $(\mathbf{a}', \mathbf{b}')$, there are four distinct matrix operators, denoted by $H_{(\mathbf{a}, \mathbf{b})}$, $H_{(\mathbf{a}, \mathbf{b}')}$, $H_{(\mathbf{a}', \mathbf{b})}$, and $H_{(\mathbf{a}', \mathbf{b}')}$, which codify our experimental measurement possibilities. It is a simple matter to determine algebraically that no two of these proposed matrices commute.

All this is to say that the technical manipulations of mathematical quantum theory instantiate formally just what we knew to begin with... that we cannot simultaneously perform the measurement observation of the polarization products at *both* angle settings (\mathbf{a}, \mathbf{b}) and $(\mathbf{a}, \mathbf{b}')$ on the *same pair of photons*, not to speak of $(\mathbf{a}', \mathbf{b})$ and $(\mathbf{a}', \mathbf{b}')$ as well. Although quantum theoretical intrigue allows us to assert probabilities such as $P[(A(\mathbf{a}), B(\mathbf{b})) = (+1, -1)]$ or whatever, it abstains from any prognostication of the form $P[(A(\mathbf{a}), B(\mathbf{b})) = (+1, -1) \text{ and } (A(\mathbf{a}), B(\mathbf{b}')) = (+1, +1)]$. This would be a probability assertion regarding simultaneous outcomes of a jointly unobservable pair of events.

Well, who would want to? We shall now find out. Although impossible to perform, we are surely permitted to think about what might happen if we could perform such simultaneous experiments. Enter the realm of a gedankenexperiment.

4. The Principle of Local Realism and Its Relevance to Bell

A feature that will be found crucial to the touted violation of Bell's inequality is that it pertains to experimental results supposedly conducted with a *single photon pair* at all four angle settings. This is very clear in the memorial article of Aspect [14] and many assessments of the inequality that properly recognize this. An example would be the article of Adenier [15]. Many discussants do not. How did such a context for the experiment arise? When the probabilistic pronouncements of quantum theory were formalized, Einstein among others was puzzled by the fact that the conditional probability for the outcome of the experiment at station *A* depends on both the angle at which the experiment is conducted at station *B* and on the outcome of that experiment. This matter is codified by the conditional probabilities we have seen in Equations (5). This entanglement of seemingly unrelated physical processes was deemed by Einstein to be a matter of "spooky action at a distance". Along with Podolsky and Rosen [16] he proposed a solution to this enigma, positing that there must be some other factors relevant to what might be happening at the polarizer stations *A* and *B* which would account for photon detections that are found to arise. As yet unspecified in the theory, he considered such factors to identify unknown values of "supplementary variables". It was proposed that the probabilities inherent in the results of quantum theory must be representations of scientific uncertainty about the action of these other variables on the two photons at their respective stations. This was their proposed way of accounting for the spooky action at a distance: the "state" of a photon in a polarization experiment, along with the condition of its attendant supplementary variables, involves its disposition to respond to the experiment at any and every one of its relative angle settings.

However, there is one aspect of the matter upon which Einstein wanted to insist: this was termed "the principle of local realism". Although it is central to matters under consideration in this problem, the applicable formulation of this principle, its meaning, and its relevance to the CHSH formulation of Bell's inequality (which we are soon to address) have been matters of contention. In unembellished form, the locality principle merely asserts that physical mechanics engaging at some particular location are not influenced by physical conditions arising in another unconnected locale far removed in space. What this would mean precisely for a gedanken scenario such as we will be considering is a matter of published discussion, notably among Mermin [17], Hess and Philipp [18], and Mermin [19], though many more have been involved. The discussion concerns whether the principle of locality alone is sufficient to establish a factorization that is involved in the CHSH form of Bell's inequality, and how the principle might need to be extended.

Whatever the precise form of its motivation relative to locality, the defiance of Bell's inequality derives from a specific and precise mathematical condition that we will recognize in its development. Fair enough, quantum theory does stipulate the probability for photon behaviour at station *A* with its polarizer direction

\mathbf{a} as depending on whether the polarizer direction at B is set at \mathbf{b} or \mathbf{b}' and on what happens there. However, in any specific instance of the paired experiments conducted at a relative polarization angle (\mathbf{a}, \mathbf{b}) , if the measurement observation at A happened to equal $A(\mathbf{a}, \mathbf{b}) = +1$, say, then in this instance the measurement at A would have to be the same in any simultaneous gedankenobservation, no matter whether the direction setting at station B were \mathbf{b} or \mathbf{b}' . That is to say, if the polarization observation $A(\mathbf{a}) = +1$ in a particular experiment on a pair of photons measure in the paired angle design (\mathbf{a}, \mathbf{b}) , then the value of $A(\mathbf{a})$ would also have to equal $+1$ in a companion experiment on the same pair of photons if the polarization directions were set in the angle pairing $(\mathbf{a}, \mathbf{b}')$. The physical processes occurring at station A in any experimental run, uncertain though they may well be, are unrelated to those occurring far away at B .

Actually, our mathematical exposition of the probabilistic specifications of quantum theory has already deferred to such an understanding. We have been denoting the photon detection value at station A merely by $A(\mathbf{a})$ rather than denoting it by $A(\mathbf{a}, \mathbf{b})$, even before we have now introduced consideration of this principle of local realism. In the context of locality, the importance of such simplification of the notation was stressed explicitly by Aspect [14]. In fact we had no need to denote the paired direction at B in our notation for $A(\mathbf{a})$ earlier, because we can only do the experiment on a specific photon at one specific possibility pair determining the angle pairing $(\mathbf{a}^*, \mathbf{b}^*)$. So we have merely denoted the observed results as $A(\mathbf{a})$ and $B(\mathbf{b})$, or as $A(\mathbf{a})$ and $B(\mathbf{b}')$. Nonetheless, the QM probabilities of Equation (1) stipulated that each of the paired results of the experiments does depend jointly on the *relative* angle between the two polarization directions.

Despite this notational deference, we should now recognize and expressly declare that this principle of local realism is based upon a claim that lies outside the bounds of matters addressed by the theory of quantum physics. For, as we have noted, it is impossible to make a measurement of both the photon detection product $A(\mathbf{a})B(\mathbf{b})$ and the product $A(\mathbf{a})B(\mathbf{b}')$ on the same pair of photons. So quantum theory explicitly disavows addressing this matter directly, though it is surely a matter of relevance to the interpretation of quantum theoretic prescriptions.

We are ready to conclude this Section by proposing an experimental measurement that lies at the heart of Bell's inequality. We are not yet ready to assess it, nor to explain its relevance to the principle of local realism, but we shall merely air it now for viewing. Peculiar, it is considered to be the result of a gedankenexperiment.

Consider a pair of photons to be ejected toward stations A and B at which the pair of polarizers can be directed in any of the four relative angles we have described. According to the detection of whether the photons pass through the polarizers or are deflected by them, Bell's inequality pertains to an experimental

quantity defined by the equation

$$s \equiv A(\mathbf{a})B(\mathbf{b}) - A(\mathbf{a})B(\mathbf{b}') + A(\mathbf{a}')B(\mathbf{b}) + A(\mathbf{a}')B(\mathbf{b}') \quad (6)$$

Mathematically, we would refer to this quantity s as a linear combination of four polarization detection products. Any one of the four terms that determine the value of s could be observed in an experiment on a pair of prepared photons. Before we explain why this quantity is of interest, we should recognize right here only that we *can* observe the value of this quantity s if we are to conduct four component experiments on four distinct pairs of photons, each ejected toward stations A and B with the polarizers directed at a different relative angle pairing. However, we *cannot* observe the value of s if it were meant to pertain to all four experiments being conducted simultaneously on the same pair of photons. It just cannot be done, and quantum theory is very explicit about having nothing directly to say about its value. If we are to consider the value of s in such an experimental design, it could only be as the result of a “thought experiment”. Enough said for now.

Why would we even be interested by such a “gedankenexperiment” as its perpetrators called it, and what does the supposition of “hidden variables” have to do with the matter?

5. Einstein’s Proposal of Hidden Variables Relevant to the Matter

The famous paper [16] which addressed these matters presented ideas that had been brewing for many years [20]. It is now widely known merely as the EPR proposal. The ideas were opposed to those of others who were proclaiming that the experimental and theoretical discoveries of QM support the view that at its fundamental level of particulate matter, the behaviour of Nature is random, and that quantum theory had identified its probabilistic structure. Convinced that “he (the old one) is not playing dice with the universe”, EPR proposed that the formulation of quantum theory is incomplete, and quantum probabilities represent our uncertainty about the influence of unspecified supplementary variables. The article stimulated a fury of healthy discussion and argument that I shall not summarize here. Well documented both in the professional journals of physics and in literature of popular science, the discussion has featured considerations of the collapse of a quantum system when subject to observation that disturbs it, the non-locality of quantum processes, and esoteric formulations of the “many worlds” view of quantum theory. What matters for my presentation here is that Einstein’s views were widely relegated as a quirky peculiar sideline, and the recognition of randomness as a fundamental feature of quantum activity came to the forefront of theoretical physics.

Enter John Bell. Interested in a reconsideration of Einstein’s view, he began his research with an idea to re-establish its validity as a contending interpretation of what we know. However, he was surprised to find this programme at an impasse when he discovered that if the principle of local realism is valid then the

probabilistic specifications of quantum theory which we have described above seem to defy a simple requirement of mathematical probabilities. In the context of a hidden variables interpretation of the matter, this seemed to require that the principle of local realism must be rejected. Reported in a pair of articles [8] [9], these results too stimulated a continuation of the flurry which has lasted through the 2015 publication in *Nature* of their apparently definitive substantiation by the research at the Delft University of Technology.

The specification of Bell's inequality can take many forms. The context in which it is addressed in the remainder of my exposition here was presented in an article by Clauser, Horne, Shimony, and Holt [13], commonly referred to as the CHSH formulation. This was the form that attracted still another principal investigator in this story, Alain Aspect. A young experimentalist, he wondered how could such a monumental result of quantum physics pertain only to a thought experiment, devoid of actual physical experimental confirmation. He thought to have devised an experimental method that could confirm or deny the defiance of Bell's inequality. My assessment of his empirical work follows directly from his clear and thoughtful explanation of the situation [14] reported to a conference organized to memorialize Bell's work. My notation is largely the same as Aspect's. I adjust only the notation for expectation of a random variable to the standard form of $E(X)$, replacing his notation of $\langle X \rangle$ which has become standard in mathematical physics in the context of bra-ket notation which I avoid. Here is how it works.

5.1. Explicit Construction of s with Hidden Variables

Hidden variables theory proposes that the quantity s which we have introduced in Equation (6) should be considered to derive from a physical function of unobserved and unknown hidden variables, whose values might be codified by the vector λ , viz.,

$$s(\lambda) \equiv A(\lambda, \mathbf{a})B(\lambda, \mathbf{b}) - A(\lambda, \mathbf{a})B(\lambda, \mathbf{b}') + A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b}) + A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b}'), \quad (7)$$

for $\lambda \in \Lambda$. The variable designated by λ here could be a vector of any number of components identifying unknown features of the experimental setup that are relevant to the outcome of the experiment in any specific instantiation. The set designated by Λ is meant to represent the space of possible values of these hidden variables. The status of these variables in the context of any particular experiment is supposed only to depend on the state of the photon pair and its surrounds, independent of the angle setting $(\mathbf{a}^*, \mathbf{b}^*)$ at which the polarizers are directed. According to the deterministic outlook underlying physical theory relying on hidden variables, if we could only know the values of these unspecified variables at the time of any experimental run and have a complete theoretical understanding of their relevance to the polarization behaviour of the photon pair, then we would know what would be the values of the polarization incidence detection of the photon pair at any one or all of the possible angle settings.

Now the personalist subjective theory of probability (apparently subscribed to by Einstein, and surely by Bruno de Finetti and by me) specifies that any individual's uncertain knowledge of the values of observable but unknown quantities could be representable by a probability density over its space of possibilities. Aspect denotes such a density in this situation by $\rho(\lambda)$. For any proponent of quantum probabilities it might well be presumed to be "rotationally invariant" over the full 360° of angles at which the photon may be fluttering toward the polarizer. That is to say, the probabilities for the possible values of the supplementary variables do not depend on the angular direction in (x, y) dimensions of the photons along z axes heading toward stations A and B , rather only on the size of the angle.

Since we avowedly have no idea of what these hidden variables might be, much less what their numerical values may be relevant to any specific experimental run, we can only ponder the "expected value" of $s(\lambda)$ with respect to the distribution specified by $\rho(\lambda)$. The feature of rotational invariance implies that this expectation is the same no matter what be the rotational angle at which the photons flutter relative to their (x, y) plane detections. Let's write this expectation equation down:

$$E[s(\lambda)] = E[A(\lambda, \mathbf{a})B(\lambda, \mathbf{b})] - E[A(\lambda, \mathbf{a})B(\lambda, \mathbf{b}')] + E[A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b})] + E[A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b}')] \quad (8)$$

where expectation is assessed with respect to the density $\rho(\lambda)$, yielding then more simply

$$E[s] = E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{b}')] + E[A(\mathbf{a}')B(\mathbf{b})] + E[A(\mathbf{a}')B(\mathbf{b}')] \quad (9)$$

as an expectation relative to the random polarization products at these various angles. Equation (8) follows directly from Equation (7) because a rule of probability says that the expectation of any linear combination of random quantities equals the same linear combination of their expectations. Fortunately, we have already reported in Equation (5) that the probabilities of quantum theory identify the expected value of any polarization product at the variable relative polarization angle $(\mathbf{a}^*, \mathbf{b}^*)$ as $E[A(\mathbf{a}^*)B(\mathbf{b}^*)] = \cos 2(\mathbf{a}^*, \mathbf{b}^*)$. So we are ready to proceed.

5.2. Finally, Bell's Inequality

We have now arrived at a place we can state precisely what Bell's inequality says. There is just a little more specificity to detail before we soon will have it. However, I should alert you that there is a little tic in the understanding of Equation (8) to which we shall return after we learn how the inequality is currently understood to be defied by quantum theory. But on the face of it, the validity of Equation (8) is plain as day.

Now re-examining Equation (7), it is apparent that it can be factored into a simplified form:

$$\begin{aligned}
s(\lambda) &\equiv A(\lambda, \mathbf{a})B(\lambda, \mathbf{b}) - A(\lambda, \mathbf{a})B(\lambda, \mathbf{b}') \\
&\quad + A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b}) + A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b}'), \quad \text{for } \lambda \in \Lambda, \\
&= A(\lambda, \mathbf{a})[B(\lambda, \mathbf{b}) - B(\lambda, \mathbf{b}')] + A(\lambda, \mathbf{a}')[B(\lambda, \mathbf{b}) + B(\lambda, \mathbf{b}')], \text{ and alternatively} \\
&= B(\lambda, \mathbf{b})[A(\lambda, \mathbf{a}) + A(\lambda, \mathbf{a}')] - B(\lambda, \mathbf{b}')[A(\lambda, \mathbf{a}) - A(\lambda, \mathbf{a}')]. \quad (9)
\end{aligned}$$

It is important to notice that once again, in performing this simple factorization of the components $A(\lambda, \mathbf{a})$ and $A(\lambda, \mathbf{a}')$ in this second line, we have implicitly presumed the principle of local realism (and perhaps even more according to the interpretations considered in the discussions among Mermin, Hess, Philipp and others to which we have alluded). For when we consider the first two summands of the first line, $A(\lambda, \mathbf{a})B(\lambda, \mathbf{b})$, and $A(\lambda, \mathbf{a})B(\lambda, \mathbf{b}')$, we should notice that the value of $A(\lambda, \mathbf{a})$ in that first term is evaluated in an experiment at which the paired polarization angle is (\mathbf{a}, \mathbf{b}) , whereas in the second term from which it is factored it is evaluated in an experiment at the relative polarization angle $(\mathbf{a}, \mathbf{b}')$. It is the principle of local realism or its extension, extraneous to any claims of quantum theory, that provides the observed value of $A(\lambda, \mathbf{a})$ must be identical in these two conditions which are impossible to instantiate together. It is only under the condition of this assertion that we would be able to factor this term out of the two expressions. The same goes for the factorization of $A(\lambda, \mathbf{a}')$. This is not a source of any worry. I am merely mentioning this so that we are all aware of what is going on. The same feature of supposition is pertinent to the alternative factorization of the terms $B(\lambda, \mathbf{b})$ and $B(\lambda, \mathbf{b}')$ in the third line from the terms of the first line.

Having arrived at this factorization, it will now take just a little thought to recognize that if the value of the quantity s is supposed to be determined from a thought experiment on a single pair of photons, then the numerical value of s can equal only either $+2$ or -2 . Of course, if we were to calculate the value of s from performing four component experiments with *four different pairs* of photons (something we can actually do), then the four component product values might each then equal either -1 or $+1$, so the value of s might equal any of $\{-4, -2, 0, +2, +4\}$. However, in such a case the factorization we performed in Equation (9) would not be permitted. For each of the observed detection products appearing in the first line would pertain to a different pair of photons whose multiplicands would be free to equal either $+1$ or -1 as prescribed by experiment. The same possibilities would be accessible if the principle of local realism were not valid. However, if the value of s were to be calculated from the results of a thought experiment on the same pair of photons, then its possibilities would be limited according to local realism (and perhaps its extension) merely to $\{-2, +2\}$. Here is how to recognize this.

Suppose the values of $B(\lambda, \mathbf{b})$ and $B(\lambda, \mathbf{b}')$ were both observed to equal $+1$. Then the first term in the factored form of the second line must equal $A(\lambda, \mathbf{a})[B(\lambda, \mathbf{b}) - B(\lambda, \mathbf{b}')] = 0$; and furthermore, the second term in the factored representation would then be $A(\lambda, \mathbf{a}')[B(\lambda, \mathbf{b}) + B(\lambda, \mathbf{b}')] = 2A(\lambda, \mathbf{a}')$. The factor

$A(\lambda, \mathbf{a}')$ equaling either $+1$ or -1 would then be multiplied by the factor $[B(\lambda, \mathbf{b}) + B(\lambda, \mathbf{b}')]]$ which would equal the number $+2$. Thus, the value of s could equal only either -2 or $+2$. Alternatively, suppose that the values of $B(\lambda, \mathbf{b})$ and $B(\lambda, \mathbf{b}')$ are both observed to equal -1 . Then by a similar argument the value of the first factored expression would again equal 0 and the second expression would equal either -1 or $+1$ multiplied now by -2 . Again the computed result of the value of s could equal only -2 or $+2$. I leave it to the reader to confirm the same result for the possible values of s if the values of $B(\lambda, \mathbf{b})$ and $B(\lambda, \mathbf{b}')$ were observed to equal either -1 and $+1$ respectively, or $+1$ and -1 respectively.

The conclusion is indisputable. If the principle of local realism holds, then the value of s that would be instantiated as a result of a thought experiment on the same pair of photons in all four polarization angle settings can equal only -2 or $+2$. Thus, the expected value $E(s)$ deriving from *any* coherent probability distribution over the four values of the component paired polarization experiments would have to be a number between -2 and $+2$. Stated algebraically and simply, without all the provisos explaining its content, Bell's inequality is the requirement that $-2 \leq E(s) \leq +2$.

Well, what do the probabilities of quantum theory imply for the value of $E(s)$? The answer universally presumed to be correct by proponents of the Bell violation is that when the design of the four experiments on a single pair of photons is constructed at a particular array of angle settings that we shall soon identify, then $E(s) = 2\sqrt{2} = 2.8284$ to four decimal places, a number that exceeds $+2$. (I shall present the details in the next paragraph.) Moreover, the experimental results of Aspect, as well as the more sophisticated experimentation of succeeding decades, is understood to corroborate this result to many decimal places. I will soon explain how this result is derived as well. However, I will insist on also showing you that not only is this theoretical derivation wrong, but that the calculations used to corroborate this result from experimental evidence are misdirected. Nonetheless, there is nothing at all wrong with the experimental results, which are what they are.

5.3. The Mistaken Violation of Bell's Inequality

It turns out that Bell's inequality is *not* deemed to be defied at *every* four-plex of possible experimental angle settings that we have characterised generically as (\mathbf{a}, \mathbf{b}) , $(\mathbf{a}, \mathbf{b}')$, $(\mathbf{a}', \mathbf{b})$, or $(\mathbf{a}', \mathbf{b}')$. At some paired directional settings of the polarizers it seems not to be defied at all. Among other pairings at which it seems to be defied, it is apparently defied more strongly at some pairings than at others. Aspect had thought that if we were to find experimental evidence of the defiance, we should try to find it at the angle pairings for which the theoretical defiance is the most extreme. It is a matter of simple calculus of extreme values to discover that the most extreme violation of the equality should occur at the angle settings $(\mathbf{a}, \mathbf{b}) = -\pi/8$, $(\mathbf{a}, \mathbf{b}') = -3\pi/8$, $(\mathbf{a}', \mathbf{b}) = \pi/8$, or $(\mathbf{a}', \mathbf{b}') = -\pi/8$.

(The angle measurements are expressed here in terms of their polar representations. In terms of degrees, the angle $-\pi/8 = -22.5^\circ$, while $-3\pi/8 = -67.5^\circ$, and $+\pi/8 = +22.5^\circ$.) You may wish to examine our **Figure 2** and notice that the angles between the various polarization directions we depicted there correspond to these relative angles. For the record, doubling these angles yields the values of $\pm\pi/4 = \pm 45^\circ$ and $-3\pi/4 = -135^\circ$ in these instances. And why does that matter?...

Recall Equation (8) and the ensuing sentences. Evaluating $E(s)$ according to this equation at the four angle settings just mentioned requires evaluating the summand component expectations. Each of them in the form

$E[A(\mathbf{a}^*)B(\mathbf{b}^*)] = \cos 2(\mathbf{a}^*, \mathbf{b}^*)$, these would then be

$$E[A(\lambda, \mathbf{a})B(\lambda, \mathbf{b})] = \cos 2(\mathbf{a}, \mathbf{b}) = \cos(-\pi/4) = 1/\sqrt{2}, \quad (10a)$$

$$E[A(\lambda, \mathbf{a})B(\lambda, \mathbf{b}')] = \cos 2(\mathbf{a}, \mathbf{b}') = \cos(-3\pi/4) = -1/\sqrt{2}, \quad (10b)$$

$$E[A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b})] = \cos 2(\mathbf{a}', \mathbf{b}) = \cos(\pi/4) = 1/\sqrt{2}, \quad (10c)$$

$$\text{and } E[A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b}')] = \cos 2(\mathbf{a}', \mathbf{b}') = \cos(-\pi/4) = 1/\sqrt{2}, \quad (10d)$$

then apparently yielding

$$E[s(\lambda)] = 1/\sqrt{2} - (-1/\sqrt{2}) + 1/\sqrt{2} + 1/\sqrt{2} = 4/\sqrt{2} = 2\sqrt{2}.$$

Voila! The expected value of s apparently equals $2\sqrt{2} \approx 2.8284$, a real number outside of the interval $[-2, +2]$, defying Bell's inequality! What could be more simple, direct, and stunning?

The answer is seen most simply by constructing and then examining a matrix, which in the jargon of the operational subjective theory of probability is called "the realm matrix of possible observation values" that could result from the performance of the gedankenexperiment in CHSH form. I will display this entire matrix on the next page, in a partitioned form of its full extension as it pertains to every aspect of the problem we shall discuss. Then we shall discuss it, piece by piece. I should mention here that while the name "realm matrix of possibilities" has arisen from within the operational subjective construction of the theory of probability, the matrix itself is merely a well-defined matrix of numbers that can be understood and appreciated by any experimentalist, no matter what may be your personal views about the foundations of probability. In the jargon of quantum physics it might be called the ensemble matrix of possible observation vectors.

6. A Neglected Functional Dependence

In specifying the QM motivated expectation $E[s(\lambda)]$ as they do in our Equation (8), Aspect/Bell fail to recognize a *symmetric functional dependence* among the values of the four proposed polarization products composing $s(\lambda)$ as defined in Equation (7), when it is meant to correspond to the result of the 4-ply thought-experiment on the same pair of photons. Perhaps surprisingly, the achieved values of any three *products* of the paired polarization indicators imply a unique value for the fourth product. We now engage to substantiate this claim.

6.1. The Realm Matrix of Experimental Quantities

Consider the realm matrix of all quantities relevant to the observations that might be made in the proposed 4-ply gedankenexperiment on a pair of photons under investigation. On the left side of the realm equation is written the name $\mathbf{R}(\mathbf{X})$, where \mathbf{X} is a partitioned vector of names of every quantity that will be relevant to the outcome of the experiment and what quantum theory asserts about it. You will already recognize those appearing in the first two partitioned sections. On the right side of the realm equation appears a matrix whose columns exhaustively identify the values of these partitioned quantities that could possibly result from conducting the gedankenexperiment. We shall discuss them in turn.

$$\mathbf{R} \begin{pmatrix} A(\mathbf{a}) \\ B(\mathbf{b}) \\ A(\mathbf{a}') \\ B(\mathbf{b}') \\ \text{*****} \\ A(\mathbf{a})B(\mathbf{b}) \\ A(\mathbf{a})B(\mathbf{b}') \\ A(\mathbf{a}')B(\mathbf{b}) \\ A(\mathbf{a}')B(\mathbf{b}') \\ \text{*****} \\ A(\mathbf{a}')B(\mathbf{b}') \\ \text{*****} \\ \Sigma_{/(a,b)} \\ \Sigma_{/(a,b')} \\ \Sigma_{/(a',b)} \\ \Sigma_{/(a',b')} \\ \text{*****} \\ s(\lambda) \\ S_{A/B(a',b')} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \text{*****} \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ \text{*****} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ \text{*****} \\ 3 & -1 & -1 & -1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & -1 & -1 & -1 & 3 \\ 3 & 1 & -1 & 1 & 1 & -1 & -3 & -1 & -1 & -3 & -1 & 1 & 1 & -1 & 1 & 3 \\ 3 & -1 & 1 & 1 & -1 & -1 & -3 & 1 & 1 & -3 & -1 & -1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 1 & -1 & -1 & 1 & -3 & -1 & -1 & -3 & 1 & -1 & -1 & 1 & 1 & 3 \\ \text{*****} \\ 2 & 2 & -2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & 2 & 2 \\ 2 & 4 & 0 & 2 & 2 & 0 & 0 & -2 & -4 & -2 & -2 & 0 & 0 & -2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The sixteen columns of four-dimensional vectors in the first partitioned block exhaustively list all the speculative 4×1 vectors of observation values that could possibly arise among the four experimental detections of photons at the four angles of polarizer pairings. In order to observe the detection *products* at the four relative angles $A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b}),$ and $A(\mathbf{a}')B(\mathbf{b}')$, we would surely have to observe each of the four *multiplicands* involved in their specification: $A(\mathbf{a}), B(\mathbf{b}), A(\mathbf{a}')$, and $B(\mathbf{b}')$. Since each of these observation values might equal only either -1 or $+1$, there are sixteen possibilities of the 4-dimensional result of the 4-ply experiment. There are no presumptions made about these prospective quantity values: neither whether they “exist” or not prior to the conduct of the experiment at all, nor even whether they exist in any form after

the experiment is conducted. We have merely made a list of what we could possibly observe if indeed we were capable of conducting the proposed gedankenexperiment on the same pair of photons. The observation vector would have to equal one of the 16 columns appearing in the top bank of the partitioned realm matrix.

Every other component quantity in the columns displayed in subsequent blocks of the realm matrix is computed via some function of these possibilities. Notice once again that the “exhaustiveness” of this list presupposes the principle of local realism, specifying for example that the value of $A(\mathbf{a})$ identifying whether the photon passes through the polarizer at A or not, would be the same no matter whether the polarizer at which the paired photon engages station B is set at direction \mathbf{b} or at \mathbf{b}' .

To begin the completion of the realm matrix, the second block of components identifies the four designated *products* of the paired polarization indicators that yield the value of the quantity s as it is simply defined in Equation (6). The first row of this second block, identifying the product $A(\mathbf{a})B(\mathbf{b})$, is the componentwise product of the first two rows of the first block. The second row of this block, identifying the product $A(\mathbf{a})B(\mathbf{b}')$, is the componentwise product of the first and fourth rows of the first block, and so on. This second block lists exhaustively all the combinations of polarization products that we could possibly observe in the conduct of our gedankenexperiment. Examine any one of these columns of products, checking that in fact the value of each product in that column is equal to the product of the corresponding multiplicands appearing in the column directly above it.

The first item to notice about this realm matrix is that, whereas the sixteen columns of the first block of polarization observations are distinct, the second block contains only *eight* distinct column vectors. Columns 9 through 16 in block two of the realm matrix reproduce columns 1 through 8 in reverse order. Moreover, examining the first *three* rows of this second block more closely, it can be recognized that the first eight columns of these rows exclusively exhaust the simultaneous measurement possibilities for the three product quantities they identify. These are the eight vectors of the cartesian product $\{+1, -1\}^3$, which are repeated in columns nine through sixteen in reverse order. Together, what these two observations mean is that the fourth product quantity in this second block of vector components is derivable as a function of the first three. What is more, any one of the product quantities identified in block two is determined by the same computational function of the other three! This is what I meant earlier when alluding that the photon detection products in the gedankenexperiment have embedded within them four symmetric functional relations. This can be seen by examining the columns of the *fourth* block of the matrix, which we shall do shortly.

The third block of the realm matrix contains only a single row, corresponding to a quantity we designate as $\mathcal{A}(\mathbf{a}')\mathcal{B}(\mathbf{b}')$. This quantity takes values only of

± 1 , but it is logically independent of the product quantities appearing in the first three rows of block two. This is the quantity that Aspect/Bell think they are assessing when they freely specify the quantum expectations for all four angle settings as they do, seemingly defying Bell's inequality. We denote its name with calligraphic type to distinguish it from the actual polarization product

$A(\mathbf{a}')B(\mathbf{b}')$ whose functional relation to the other three products we are now identifying. Peculiar, this singular component of the fourth partition block is not an "Alice and Bob" observation quantity, but rather an "Aspect/Bell" imagined quantity. It is logically independent of the first three "Alice and Bob" products. This is to say that whatever values these products may be, the value of $A(\mathbf{a}')B(\mathbf{b}')$ may equal $+1$ in the appropriate row among the first eight columns, or it may equal -1 in the corresponding column among the second eight. However, it does *not* represent the photon detection product $A(\mathbf{a}')B(\mathbf{b}')$ in the four imagined experiments on a single photon pair.

6.2. Specifying the Functional Form via Block Four

Quantities in the fourth block of the realm matrix are designated with the names $\Sigma_{/(a,b)}$, $\Sigma_{/(a,b')}$, $\Sigma_{/(a',b)}$, and $\Sigma_{/(a',b')}$. These quantities are defined by sums of column elements in *those rows of the second block* that are *not marked* behind the slash in the notational subscript. For examples,

$$\Sigma_{/(a,b)} \equiv A(\mathbf{a})B(\mathbf{b}') + A(\mathbf{a}')B(\mathbf{b}) + A(\mathbf{a}')B(\mathbf{b}'), \text{ and}$$

$$\Sigma_{/(a,b')} \equiv A(\mathbf{a})B(\mathbf{b}) + A(\mathbf{a}')B(\mathbf{b}) + A(\mathbf{a}')B(\mathbf{b}').$$

The quantities $\Sigma_{/(a',b)}$ and $\Sigma_{/(a',b')}$ are defined similarly.

Next to notice is that the fourth row of the *second* matrix block, corresponding to $A(\mathbf{a}')B(\mathbf{b}')$, has an entry of 1 if and only if the fourth row of the *fourth* block, corresponding to $\Sigma_{/(a',b')}$, has an entry of -1 or $+3$ in the same column. When that entry is $+1$ or -3 , the corresponding entry of the second block is -1 . What this recognition does is to identify the functional relation of the fourth polarization product to the first three polarization products, viz.,

$$\begin{aligned} A(\mathbf{a}')B(\mathbf{b}') &= \mathbf{G} [A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a}')B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}')] \\ &\equiv \left(\Sigma_{/(a',b')} = -1 \text{ or } +3 \right) - \left(\Sigma_{/(a',b')} = +1 \text{ or } -3 \right). \end{aligned} \quad (11)$$

Here and throughout this note I am using indicator notation in which parentheses surrounding a mathematical statement that might be true and might be false signifies the number 1 when the interior statement is true, and signifies 0 when it is false.

Some eyeball work is required to recognize the functional relationship (11) by examining the final row of block two and of block four together. It may take even more concentration to recognize that this very same functional rule identifies each of the other three polarization products as a function of the other three as well! The four product quantities $A(\cdot)B(\cdot)$ are related by four symmetric functional relationships, each of them being calculable via the same functional

rule applied to the other three! This surprising recognition identifies the source of the Aspect/Bell error in assessing the QM-motivated expectation for $s(\lambda)$ in the way they do.

It is surely true that $E[s(\lambda)]$ equals a linear combination of four expectations of polarization products, as specified in Equation (8). Moreover, if the definition of $s(\lambda)$ in Equation (6) were understood to represent the combination of observed products from experiments on four distinct pairs of photons, then the possible values of $s(\lambda)$ would span the integers $\{-4, -2, 0, 2, 4\}$; the expectation of each product $E[A(\mathbf{a}^*)B(\mathbf{b}^*)]$ would equal $-1/\sqrt{2}$ or $+1/\sqrt{2}$ as appropriate to the angle $(\mathbf{a}^*, \mathbf{b}^*)$; and $E[s(\lambda)]$ would equal $2\sqrt{2}$ as proposed by Aspect/Bell. This involves no violation of any probabilistic inequality at all, and there is no suggestion of mysterious activity of quantum mechanics.

However, when it is proposed that the paired polarization experiments at all four considered angles pertain to the same photon pair, then each of the products is restricted to equal the specified function value of the other three that we identified explicitly for $A(\mathbf{a}')B(\mathbf{b}')$ in Equation (11) as $\Sigma_{(\mathbf{a}', \mathbf{b}')}$ via the function $\mathbf{G}[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a}')B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}')]]$. In this context, Aspect's expected quantity would be representable equivalently by any of the following equations:

$$\begin{aligned}
 E[s(\lambda)] &= E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{b}')] + E[A(\mathbf{a}')B(\mathbf{b})] \\
 &\quad + E\{\mathbf{G}[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b})]\} \\
 &= E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{b}')] + E[A(\mathbf{a}')B(\mathbf{b}')] \\
 &\quad + E\{\mathbf{G}[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b}']]\} \\
 &= E[A(\mathbf{a})B(\mathbf{b})] + E[A(\mathbf{a}')B(\mathbf{b})] + E[A(\mathbf{a}')B(\mathbf{b}')] \\
 &\quad - E\{\mathbf{G}[A(\mathbf{a})B(\mathbf{b}), A(\mathbf{a}')B(\mathbf{b}), A(\mathbf{a}')B(\mathbf{b}']]\} \\
 &= -E[A(\mathbf{a})B(\mathbf{b}')] + E[A(\mathbf{a}')B(\mathbf{b})] + E[A(\mathbf{a}')B(\mathbf{b}')] \\
 &\quad + E\{\mathbf{G}[A(\mathbf{a})B(\mathbf{b}'), A(\mathbf{a}')B(\mathbf{b}), A(\mathbf{a}')B(\mathbf{b}']]\}
 \end{aligned} \tag{12}$$

The symmetries imposed on this problem would yield an identical result in each case, which would surely *not* yield $2\sqrt{2}$ at all. *This is the mathematical error of neglect to which the title of this current exposition alludes.* What might the symmetries yield?

The functional relation we have exposed in Equation (11) is *not* linear. If it were, then the specification of an expectation for its arguments would imply the expectation value for the function value. As it is not, the specification of expectation values for the arguments only imply *bounds* on any cohering expectation value for the fourth. These numerical bounds can be computed using a theorem due to Bruno de Finetti which he first presented at his famous lectures at the Institute Henri Poincaré in 1935. He named it only in his swansong text [21]. It was first characterized in the form of a linear programming problem by Bruno and Gilio [22], and has appeared in various forms in recent decades. Among them are presentations in dual form by Whittle [23] [24] using standard formalist notation and objectivist concepts. We shall review the content of de Finetti's

theorem shortly, and then examine its relevance to assessing the expectation of $s(\lambda)$ motivated by considerations of quantum mechanics. We need first to air some further brief remarks about the final block of the realm matrix.

6.3. The Remaining Block of Quantities and Their Realm Components

The first row of block five of the realm matrix merely identifies the values of $s(\lambda)$ associated with the polarization observation possibilities enumerated in the columns of block one. Each component of this row is computed from the corresponding column of block two according to the defining Equation (1). It is evident that every entry of this row is either -2 or $+2$. This corresponds to the argument we have made following the factorization Equation (9) in Section 5. The second row of this block pertains to a quantity denoted as $s_{\mathcal{A}/\mathcal{B}(\mathbf{a}',\mathbf{b}')}$. Its value is defined similarly to Equation (6), but its final summand is specified as the Aspect/Bell quantity $\mathcal{A}(\mathbf{a}')\mathcal{B}(\mathbf{b}')$ rather than the actual polarization product quantity $A(\mathbf{a}')B(\mathbf{b}')$ that appears in this equation defining $s(\lambda)$. Again peculiar, its realm can be seen to include the elements $\{-4, -2, 0, 2, 4\}$ whereas the realm of $s(\lambda)$ includes only $\{-2, 2\}$. The fact that the possibilities for $s_{\mathcal{A}/\mathcal{B}(\mathbf{a}',\mathbf{b}')}$ include both -4 and $+4$ is what makes it not surprising that the expectation of *this quantity* is $2\sqrt{2}$ as pronounced by proponents of the Aspect/Bell analysis.

The third row of block five is merely an accounting device, denoting that the “sure” quantity, 1, is equal to 1 no matter what the observed results of the four imagined optic experiments of Aspect/Bell might be. Its relevance will become apparent when the need arises to apply de Finetti’s fundamental theorem to quantum assertions.

It is time for a rest and an interlude. It is a mathematical interlude whose complete understanding relies only on your knowledge of some basic methods of linear algebra. If you would like a slow didactic introduction to the subject, my best suggestion is to look at Chapter 2.10 of my book [25]. You may even wish to start in Section 2.7. Another purely computational presentation appears in the article of Capotorti *et al.* [26], Section 4. I will make another attempt here in a brief format, merely to keep this current exposition self-contained. What does the fundamental theorem of probability say?

7. The Relevance of the Fundamental Theorem of Probability

In brief, the fundamental theorem says that if you can specify expectation values for any vector of quantities whatsoever, then the rules of probability provide numerical bounds on a cohering expectation for any other quantity you would like to assess. These can be computed from the compilation of a linear programming routine. If the expectations you have specified are incoherent (meaning self-contradictory) among themselves, then the linear programming problems they motivate have no solution. This theorem is immediately relevant to

our situation here in which we have identified quantum-theory-motivated expectations for any three of the four detection products that determine the value of s for the gedankenexperiment. We wish to find the bounds on the cohering expectation for the fourth detection product which is restricted to equal a function value determined by these three. A discursive pedagogical introduction is available in Lad [[25], Section 2.10]. In brief, here is how the theorem works.

Suppose you have identified the expectations for N quantities, and you are wondering what you might assert as the expectation for another one, call it the $(N + 1)^{st}$. What you should do to assess your sensible possibilities is firstly to construct the realm matrix of possible values for the vector of all $(N + 1)$ quantities. Let's call the vector $\mathbf{X}_{N+1} = (X_1, X_2, \dots, X_N, X_{N+1})^T$, and call its realm matrix then $\mathbf{R}(\mathbf{X}_{N+1})$. In general it will look something like the realm matrix we have just constructed for various aspects of our gedankenexperiment. It will have $N + 1$ rows, and some number K columns. Just as an example, the realm matrix we have already constructed happens to have $(N + 1) = 16$ rows and $K = 16$ columns. (Mind you, we have not yet specified expectation for the first N components of the quantity vector to which this realm applies, but let's not let that deter us. I am merely suggesting here an example of a realm matrix that could be considered to have $(N + 1)$ rows. Let's continue with the general abstract specification.)

Now any such vector of quantities can be expressed as the product of its realm matrix with a particular vector of events. The matrix equation, displayed in a form that partitions the final row, would look like this:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_N \\ *** \\ X_{N+1} \end{pmatrix} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,(K-1)} & x_{1,K} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,(K-1)} & x_{2,K} \\ x_{3,1} & x_{3,2} & \cdots & x_{3,(K-1)} & x_{3,K} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{N,1} & x_{N,2} & \cdots & x_{N,(K-1)} & x_{N,K} \\ *** & ** & ** & ** & ** \\ x_{(N+1),1} & x_{(N+1),2} & \cdots & x_{(N+1),(K-1)} & x_{(N+1),K} \end{pmatrix} \begin{pmatrix} (\mathbf{X}_{N+1} = \mathbf{x}_1) \\ (\mathbf{X}_{N+1} = \mathbf{x}_2) \\ (\mathbf{X}_{N+1} = \mathbf{x}_3) \\ \vdots \\ \vdots \\ (\mathbf{X}_{N+1} = \mathbf{x}_{(K-1)}) \\ (\mathbf{X}_{N+1} = \mathbf{x}_K) \end{pmatrix}.$$

On the left of this equation is the column vector of the quantity observations under consideration. To the right of the equality comes firstly the $(N + 1) \times K$ realm matrix whose K columns list all the possible columns of numbers that could possibly result as the observation vector. These K columns, each of which has $(N + 1)$ components, correspond to vectors denoted as $\mathbf{x}_{\cdot 1}, \mathbf{x}_{\cdot 2}, \mathbf{x}_{\cdot 3}, \dots, \mathbf{x}_{\cdot (K-1)}$, and $\mathbf{x}_{\cdot K}$. (The initial subscripted dot denotes that this is a whole column of numbers. The number that follows the dot denotes which of the columns of the matrix it is we are talking about.) This matrix is multiplied by the final $K \times 1$ column vector of events that identify whether the quantity vector \mathbf{X}_{N+1} turns out upon observation to be the first, the second, ..., or the K^{th} of these listed columns. We shall denote this vector by $\mathbf{Q}(\mathbf{X}_{N+1})$, and call it "the partition vector generated by \mathbf{X}_{N+1} ". One and only one of its component events will equal 1 and the rest

will equal 0. But we do not know which of them is the 1, because we do not know which column of possibilities in the realm matrix will be the one that represents the observed outcome of the vector of quantities \mathbf{X}_{N+1} .

We can represent this matrix equation more concisely and in a useful form by writing it in an abbreviated partitioned form:

$$\begin{pmatrix} \mathbf{X}_N \\ X_{N+1} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \mathbf{X}_N \\ X_{N+1} \end{pmatrix} \mathbf{Q}(\mathbf{X}_{(N+1)}).$$

The payoff from constructing this matrix structure is that now every row of this partitioned equation has on its left-hand side the unknown value of a quantity, X_i . On the right-hand side in that row appears a list of the possible values of that quantity, each multiplied in a linear combination with the events that denote whether each of them is indeed the value of this quantity (in the context of the observed values of the other quantities shown in that column as well). Each row of this equation specifies how a different one of the quantities under consideration equals a linear combination of events. We have heard of that before. The expectation of a linear combination equals the same linear combination of expectations for those events, which would be their probabilities if we could specify values for them. This tells us that we can evaluate an expectation operator on this partitioned equation to yield the result that

$$E \begin{pmatrix} \mathbf{X}_N \\ X_{N+1} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \mathbf{X}_N \\ X_{N+1} \end{pmatrix} P[\mathbf{Q}(\mathbf{X}_{(N+1)})].$$

Well, we have not mentioned anything about probability specifications appearing in the vector $P[\mathbf{Q}(\mathbf{X}_{(N+1)})]$ on the right-hand side of this equality. The only restrictions of probability are that these must be non-negative numbers that sum to 1, since the vector $\mathbf{Q}(\mathbf{X}_{(N+1)})$ constitutes a partition. We have mentioned only that expectations have been identified for the first N components of the vector on the left-hand side, $E(\mathbf{X}_N)$. Yet we can compute something important on the basis of this realization. The linearity of this equation ensures that the implied value for the expectation of the final unspecified component $E(X_{N+1})$ must lie within a specific interval. It is computable as the

minimum and maximum values of $\mathbf{R}(X_{N+1})\mathbf{q}_K$

subject to the linear restrictions $\mathbf{R}(\mathbf{X}_N)\mathbf{q}_K = E(\mathbf{X}_N)$,

as required of the expectations that we have presumed to be specified,

and where the components of \mathbf{q}_K must be non-negative and must sum to 1.

Such a computation is provided by the procedures of a linear programming problem. The “solutions” to these linear programming problems are the vectors \mathbf{q}_{\min} and \mathbf{q}_{\max} that yield these minimum and maximum values for $E(X_{N+1})$ subject to these constraints. The final row vector identifying $E(X_{N+1})$ whose extreme values we seek is called “the objective function” of the problems. Its coefficients are the partitioned final row of the general realm matrix we identify as $\mathbf{R}(X_{N+1})$. Notice that that X is not bold. It represents merely the final quan-

tity in the column vector \mathbf{X}_{N+1} . The coefficients vector of the objective function is the final row vector of the realm matrix.

Here are the specific details appropriate to our gedankenexperiment.

$$\mathbf{E} \begin{pmatrix} 1 \\ A(\mathbf{a})B(\mathbf{b}) \\ A(\mathbf{a})B(\mathbf{b}') \\ A(\mathbf{a}')B(\mathbf{b}) \\ A(\mathbf{a}')B(\mathbf{b}') \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix} \mathbf{q}_8 \quad (13)$$

I have listed the order of the quantities in the vector at left to begin with the sure quantity, 1, which equals 1 no matter what happens in the gedankenexperiment. There follow the four summands of the CHSH quantity s , of which we have noticed that each one of them is restricted in the gedankenexperiment to equal a function value of the other three. That is why there are only eight columns in their realm matrix, as opposed to sixteen columns in the expansive realm matrix we have already examined. As to the components of the vector \mathbf{q}_8 at the right of the right-hand side, notice that quantum theory says nothing at all about these, individually. Each of them would equal the probability that the 4-ply gedankenexperiment would yield detection products designated by a specific column of the realm matrix. However, these would involve the joint detection of photon products in four distinct measurements that are known to be incompatible. On account of the generalised uncertainty principle, quantum theory eschews specification of such probabilities. Nonetheless, for any individual photon detection product in a specific experimental design, denoted on the left-hand side of the equation, quantum theory does specify an expectation value of either $1/\sqrt{2}$ or $-1/\sqrt{2}$, as we have recognized. Since these four products are not all free to equal +1 or -1 at the same time, we may assert expectation values for any three of them, and use linear programming computations to find the cohering bounds on the expectation of the fourth that would accompany them, yielding bounds on the expectation Equation (13).

7.1. The Result: Quantum Theory Identifies Restrictions on the Valuation of \mathbf{q}_8

This is what we find. The columns of the matrix below display the computed results of the paired \mathbf{q}_{\min} and \mathbf{q}_{\max} vectors corresponding to four linear programming problems. Each of them determines a bound on an expected function value that appears in one of the four forms of the expectation value $E[s(\lambda)]$ which we displayed in Equation (12). The first pair of columns, for example, identify the fifth row of the matrix in Equation (13) as the objective function, $E[A(\mathbf{a}', \mathbf{b}')]]$, constrained by QM-specified values of the expectations of the first four rows. The second pair of columns identify the fourth row of (13) as the objective function constrained by QM specifications of expectations for rows 1, 2, 3, and 5, and display the appropriate solution vectors; and so on.

$$\begin{pmatrix} & \min(\mathbf{a}', \mathbf{b}') & \max(\mathbf{a}', \mathbf{b}') & \min(\mathbf{a}', \mathbf{b}) & \max(\mathbf{a}', \mathbf{b}) & \min(\mathbf{a}, \mathbf{b}') & \max(\mathbf{a}, \mathbf{b}') & \min(\mathbf{a}, \mathbf{b}) & \max(\mathbf{a}, \mathbf{b}) \\ q_1 & 0 & 0.1464 & 0 & 0.1464 & 0.5607 & 0.7803 & 0 & 0.1464 \\ q_2 & 0.7803 & 0.5607 & 0 & 0.1464 & 0.1464 & 0 & 0 & 0.1464 \\ q_3 & 0.0732 & 0 & 0.0732 & 0 & 0 & 0.0732 & 0 & 0 \\ q_4 & 0 & 0.1464 & 0.7803 & 0.5607 & 0.1464 & 0 & 0 & 0.1464 \\ q_5 & 0 & 0.1464 & 0 & 0.1464 & 0.1464 & 0 & 0.7803 & 0.5607 \\ q_6 & 0.0732 & 0 & 0 & 0 & 0 & 0.0732 & 0.0732 & 0 \\ q_7 & 0.0732 & 0 & 0.0732 & 0 & 0 & 0 & 0.0732 & 0 \\ q_8 & 0 & 0 & 0.0732 & 0 & 0 & 0.0732 & 0.0732 & 0 \end{pmatrix}$$

Each of these column vectors resides in 8-dimensional space, providing a coherent assessment of probabilities for the constituent event vector $\mathbf{Q}(\mathbf{X}_8)$, without specifying precise probabilities for any of them. In fact, quantum theory denies itself the capability of identifying such probabilities precisely. We will discuss this feature further, below. However the results of the linear programming computations can and do specify possibilities for what might be specified in a way that would cohere with what quantum theory can and does tell us. The columns of this matrix identify some of them. In fact, these columns display extreme values of what are possible. Any convex (linear) combination of them would cohere with quantum theory as well. Thus, geometrically the columns constitute vertices of a polytope of quantum-theory-supported possibilities for $P[\mathbf{Q}(\mathbf{X}_8)]$. This polytope is called “the convex hull” of these vectors. However, although we have found eight of them, the rank of the matrix of all of them is only four! That is, these eight-dimensional vectors all reside within a four-dimensional subspace of a unit-simplex. Why is quantum theory not more specific in specifying the expectation of Bell’s quantity $E(s)$? We shall delay this discussion until we have clarified what we have learned from these results of $\mathbf{q}_{\min}(\mathbf{a}^*, \mathbf{b}^*)$ and $\mathbf{q}_{\max}(\mathbf{a}^*, \mathbf{b}^*)$.

7.2. Implied Bounds on Expected Detection Products and on $E(s) \in (1.1213, 2]$

According to the prescription of Equation (12), each of these \mathbf{q}_8 vectors appearing in Section 7.1 would identify a vertex of another polytope of cohering expectation vectors for the components of the CHSH quantity s . Followed at bottom by the expectation values $E(s)$ they imply, these are

$$\begin{pmatrix} E[A(\mathbf{a})B(\mathbf{b})] & 0.7071 & 0.7071 & 0.7071 & 0.7071 & 0.7071 & 0.7071 & -1.0000 & -0.1213 \\ E[A(\mathbf{a})B(\mathbf{b}')] & -0.7071 & -0.7071 & -0.7071 & -0.7071 & 0.1213 & 1.0000 & -0.7071 & -0.7071 \\ E[A(\mathbf{a}')B(\mathbf{b})] & 0.7071 & 0.7071 & -1.0000 & -0.1213 & 0.7071 & 0.7071 & 0.7071 & 0.7071 \\ E[A(\mathbf{a}')B(\mathbf{b}')] & -1.0000 & -0.1213 & 0.7071 & 0.7071 & 0.7071 & 0.7071 & 0.7071 & 0.7071 \\ E[s] & 1.1213 & 2.0000 & 1.1213 & 2.0000 & 2.0000 & 1.1213 & 1.1213 & 2.0000 \end{pmatrix}$$

In any of these columns appear three values of $E[A(\mathbf{a}^*)B(\mathbf{b}^*)]$ specifications supported by quantum theory, and a fourth value which is either a lower

bound or upper bound on any cohering expectation for the fourth. (By the way, 0.7071 is the value of $1/\sqrt{2}$ to four decimal places.) At the bottom of the column is the value of $E(s)$ that would correspond to these four. The vectors of the four $E[A(\mathbf{a}^*)B(\mathbf{b}^*)]$ values are the vertices of the four-dimensional space of QM-supported expectation values of the gedankenexperiment, and the value of $E(s)$ listed at bottom would be a quantum-theory-permitting assessment of $E(s)$, Bell's quantity. All of their convex combinations lie within Bell's reputed bounds of $[-2, +2]$. There is more to be said about this, but let us first address now the question of why quantum theory leaves four dimensions of freedom unaccounted for in its prescriptions.

7.3. Why are There Four Free Dimensions to the QM Specification of $E(s)$?

Let's just get down to it, without any prelude. Quantum theory specifies precise values for outcome probabilities of the photon pair detections at any choice of three angle settings of the gedankenexperiment. Consider for example the polarization detection probabilities at the angles (\mathbf{a}, \mathbf{b}) , $(\mathbf{a}, \mathbf{b}')$, and $(\mathbf{a}', \mathbf{b})$. These have been identified in our Equation (1), while the corresponding expectations of detection products appeared in Equation (2). If quantum theory were to specify a complete distribution for the outcome of this gedankenexperiment, it would have to specify eight probabilities. These would involve three corresponding to detection events at any one of the polarization angles, also jointly at any two of the three detection angles, and also at all three of the detection angles. But according to the uncertainty principle discussed in Section 3, the theory eschews commitments regarding the latter four of these probabilities: neither

$$\begin{aligned}
 &P\{[(A(\mathbf{a})=+1)(B(\mathbf{b})=+1)][(A(\mathbf{a})=+1)(B(\mathbf{b}')=+1)]\}, \text{ nor} \\
 &P\{[(A(\mathbf{a})=+1)(B(\mathbf{b})=+1)][(A(\mathbf{a}')=+1)(B(\mathbf{b})=+1)]\}, \text{ nor} \\
 &P\{[(A(\mathbf{a})=+1)(B(\mathbf{b})=+1)][(A(\mathbf{a}')=+1)(B(\mathbf{b})=+1)]\}, \text{ nor} \\
 &P\{[(A(\mathbf{a})=+1)(B(\mathbf{b})=+1)][(A(\mathbf{a})=+1)(B(\mathbf{b}')=+1)] \\
 &\quad [(A(\mathbf{a}')=+1)(B(\mathbf{b})=+1)]\}
 \end{aligned}$$

For each of these would amount to claims regarding the joint outcomes of incompatible measurements, characterised by Hermitian matrix operators that do not commute. Quantum theory explicitly avoids such claims. That leaves four dimensions of the eight-dimensional pmf over the four detection products unspecified... explicitly! That is why quantum theory allows four unspecified dimensions to the expectations it provides regarding the four polarization products on the same pair of photons.

Perhaps this comment does need a little bit more explication. You will need to

view Equation (13) while reading the following remarks. They concern assertions that quantum theory does allow us to make, and those that it doesn't. Recall that we are considering a linear programming problem in which quantum expectations are asserted for the polarization products at the angle settings (\mathbf{a}, \mathbf{b}) , $(\mathbf{a}, \mathbf{b}')$, and $(\mathbf{a}', \mathbf{b})$, and investigating coherent bounds for expectation of the product at the setting $(\mathbf{a}', \mathbf{b}')$. Notice firstly that quantum theory does allow us to, and indeed insists that we assert

$$\begin{aligned} E[A(\mathbf{a})B(\mathbf{b})] &= q_1 + q_2 + q_3 + q_4 - q_5 - q_6 - q_7 - q_8 \\ &= 1/\sqrt{2} \end{aligned}$$

Examining the corresponding columns of the realm matrix seen in (13), it is evident that these involve assertions regarding the outcomes of $(A(\mathbf{a})B(\mathbf{b}) = +1)$ and $(A(\mathbf{a})B(\mathbf{b}) = -1)$ irrespective of the values of $A(\mathbf{a})B(\mathbf{b}')$ and $A(\mathbf{a}')B(\mathbf{b})$. For each of these events involve an outcome of the product $A(\mathbf{a})B(\mathbf{b})$ summed over all four possible joint outcomes of the products $A(\mathbf{a})B(\mathbf{b}')$ and $A(\mathbf{a}')B(\mathbf{b})$. So these latter two incompatible observations would be irrelevant to the assertion of this expectation. The same feature would pertain to the required assertions of $E[A(\mathbf{a})B(\mathbf{b}')]$ and $E[A(\mathbf{a}')B(\mathbf{b})]$ which are involved in the first LP problem. Neither of these involves any concomitant assertions regarding observations incompatible with them. On the other hand, an assertion of a probability for the joint occurrence of two pairs of polarization observations, such as

$P\left\{\left[(A(\mathbf{a}) = +1)(B(\mathbf{b}') = +1)\right]\left[(A(\mathbf{a}') = +1)(B(\mathbf{b}) = +1)\right]\right\}$ for example, would require specifications of the sum $q_1 + q_3$. Examining Equation (13) makes clear that it is only columns 1 and 3 of the matrix in which this joint event is instantiated. Asserting a specific value for the sum $q_1 + q_3$ would necessarily entail assessments of joint probabilities for incompatible events. The same would be true of any of the other three probabilities regarding joint events for which quantum theory eschews assessment.

If one were to claim, as do the reigning proponents of Bell violations, that the probabilities of quantum theory support the valuation of $E[s(\lambda)] = 2\sqrt{2}$ according to the derivation that concluded our Section 5, that would be just plain wrong. Full stop.

Our next project is an amusing one, of actually envisaging the 4-dimensional polytope of quantum probabilities relevant to the gedankenexperiment. This will be achieved by passing the 4-dimensional quantum polytope we have identified through the 3-dimensional space in which we live. By this method we can view it, just as the inhabitants of 2-dimensional space in Abbott's amusing story of *Flatland* [27] viewed the sphere passing through their lower dimensional world. It suddenly appeared as a point, which gradually expanded to circles of increasing diameter, and then diminished until they suddenly disappeared again. Let's view what we can of our 4-dimensional quantum polytope in this way.

7.4. Transforming the Expectation Polytope into Quantum Probabilities

The expected photon detection products displayed in Section 7.2 can be transformed into P_{++} probabilities by applying the transformation

$P_{++}(\mathbf{a}^*, \mathbf{b}^*) = [E(\mathbf{a}^*, \mathbf{b}^*) + 1]/4$ of Equation (3) to the eight vertices. This yields the vertices of another polytope in the space of the probability vector

$[P_{++}(\mathbf{a}, \mathbf{b}), P_{++}(\mathbf{a}, \mathbf{b}'), P_{++}(\mathbf{a}', \mathbf{b}), P_{++}(\mathbf{a}', \mathbf{b}')]'$ displayed below:

$$\begin{pmatrix} P_{++}(\mathbf{a}, \mathbf{b}) & 0.4268 & 0.4268 & 0.4268 & 0.4268 & 0.4268 & 0.4268 & 0 & 0.2197 \\ P_{++}(\mathbf{a}, \mathbf{b}') & 0.0732 & 0.0732 & 0.0732 & 0.0732 & 0.2803 & 0.5000 & 0.0732 & 0.0732 \\ P_{++}(\mathbf{a}', \mathbf{b}) & 0.4268 & 0.4268 & 0 & 0.2197 & 0.4268 & 0.4268 & 0.4268 & 0.4268 \\ P_{++}(\mathbf{a}', \mathbf{b}') & 0 & 0.2197 & 0.4268 & 0.4268 & 0.4268 & 0.4268 & 0.4268 & 0.4268 \end{pmatrix}.$$

7.5. And Now Viewing It !... as It Passes through Our Space

The convex hull of the 4-D column vectors shown in Section 7.4 can be visualized through a sequence of 3-D intersections it affords with slices perpendicular to any one of its axes. **Figure 3** displays such a sequence of slices perpendicular to the $P_{++}(\mathbf{a}', \mathbf{b}')$ axis at values increasing from 0 to 0.4268. When

$P_{++}(\mathbf{a}', \mathbf{b}') = 0$, the intersection of the slice identifies only a single vertex point (0.0732, 0.4268, 0.4268) which appears in the subplot (1,1). See also column one of the matrix in Section 7.4. As the value of $P_{++}(\mathbf{a}', \mathbf{b}')$ for the slice level increases to a 0.1098 in subplot (2,1), the intersection appears as a tetrahedron. The size of the intersecting tetrahedron increases further at the probability level 0.2197 in subplot (3,1). The tetrahedrons continue to increase in size as the level of the $P_{++}(\mathbf{a}', \mathbf{b}')$ increases still further to 0.2561 in subplot (1, 2), but a corner of their intersections begins to be cut off there. This clipped portion is cut more severely from the enlarging polytope as $P_{++}(\mathbf{a}', \mathbf{b}')$ increases further, displayed in subplots (2,2) through (3,2) which is our view of the polytope when it suddenly disappears.

The symmetry of the configuration implies that slices along the other axes would create identical intersection sequences.

8. What to Make of Aspect's and Subsequent Empiricism

Taken in by the alluring derivation of Section 5.3 which ignores the symmetric functional relations among the polarization products of the gedankenexperiment, Aspect and followers were convinced that Bell's inequality has been defied, and that the theory of hidden variables must be rejected. This conclusion would support the assertion that quantum theory has identified the structure of randomness which supposedly inheres in Nature at its finest resolution. The behaviour of the photons is considered to be governed purely by a probability distribution. It remained only to devise some physical experiments that could verify the defiance of the inequality.

According to the tenets of objective probability theory and its statistical programme, probabilities are not observable quantities. What are observable are

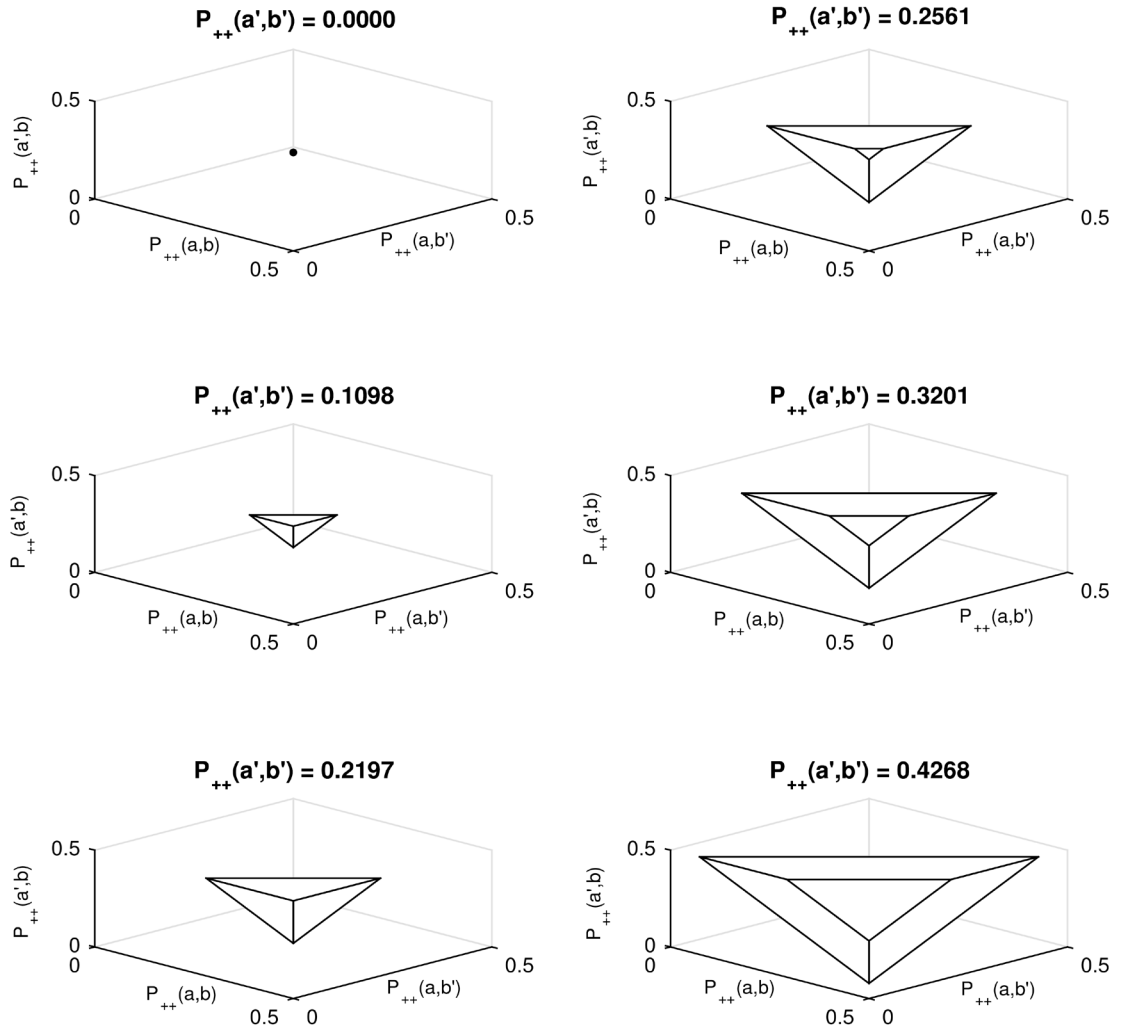


Figure 3. Sequential intersections of the 4-D convex hull of vectors $[P_{++}(\mathbf{a},\mathbf{b}), P_{++}(\mathbf{a},\mathbf{b}'), P_{++}(\mathbf{a}',\mathbf{b}), P_{++}(\mathbf{a}',\mathbf{b}')]$ with slices perpendicular to the $P_{++}(\mathbf{a}',\mathbf{b}')$ axis, at levels increasing from 0 to 0.4268 as designated in the heading of each component figure. Read the display sequentially down the first column and then down the second column.

outcomes of random variables which are generated by them. It is a matter of statistical theory to devise methods for estimating the unobservable probabilities and their implied expectations from carefully observed outcomes of the random variables they generate. Understood in this way, Equation (8) which I repeat here constitutes a structure requiring estimation if the violation of Bell's inequality is to be verified:

$$E[s(\lambda)] = E[A(\mathbf{a})B(\mathbf{b})] - E[A(\mathbf{a})B(\mathbf{b}')] + E[A(\mathbf{a}')B(\mathbf{b})] + E[A(\mathbf{a}')B(\mathbf{b}')].$$

Resorting to long respected statistical procedures, the unobservable expectations of detection products on the right-hand-side of this equation can be estimated by the generally applicable non-parametric method of moments. Supported by the probabilistic law of large numbers, its validity as an estimating

procedure stems from the 1930's.

The programme for estimating Equation (8) would proceed as follows. To estimate the first component of $E[s(\lambda)]$, which is $E[A(\mathbf{a})B(\mathbf{b})]$, one would conduct N independent polarization experiments at the angle setting (\mathbf{a}, \mathbf{b}) , and record the value of the polarization products $A(\mathbf{a})B(\mathbf{b})$ observed in each case, these being either -1 or $+1$. The average of these values would provide a method of moments estimate of the expectation $E[A(\mathbf{a})B(\mathbf{b})]$ which is common to all of these random experiments. A similar programme would be followed in estimating the other three components of $E[s(\lambda)]$.

Using the notation of Aspect [14] we would conduct N repetitions of the CHSH/Bell experiment with the relative polarizing angles set at (\mathbf{a}, \mathbf{b}) , resulting in $N_{++}(\mathbf{a}, \mathbf{b})$ observations of $(A(\mathbf{a}), B(\mathbf{b})) = (+, +)$, $N_{+-}(\mathbf{a}, \mathbf{b})$ observations of $(+, -)$, $N_{-+}(\mathbf{a}, \mathbf{b})$ observations of $(-, +)$, and $N_{--}(\mathbf{a}, \mathbf{b})$ observations of $(-, -)$. An estimated version of equation (8) would then be expressed as

$$\hat{E}[s(\lambda)] \equiv \hat{E}[A(\mathbf{a})B(\mathbf{b})] - \hat{E}[A(\mathbf{a})B(\mathbf{b}')] + \hat{E}[A(\mathbf{a}')B(\mathbf{b})] + \hat{E}[A(\mathbf{a}')B(\mathbf{b}')], \quad (14)$$

where the component estimator $\hat{E}[A(\mathbf{a})B(\mathbf{b})]$ is defined by

$$\hat{E}[A(\mathbf{a})B(\mathbf{b})] \equiv \frac{[N_{++}(\mathbf{a}, \mathbf{b}) - N_{+-}(\mathbf{a}, \mathbf{b}) - N_{-+}(\mathbf{a}, \mathbf{b}) + N_{--}(\mathbf{a}, \mathbf{b})]}{[N_{++}(\mathbf{a}, \mathbf{b}) + N_{+-}(\mathbf{a}, \mathbf{b}) + N_{-+}(\mathbf{a}, \mathbf{b}) + N_{--}(\mathbf{a}, \mathbf{b})]}, \quad (15)$$

with a similar specification for the components of $\hat{E}[s(\lambda)]$ pertaining to the relative angles $(\mathbf{a}', \mathbf{b})$, $(\mathbf{a}, \mathbf{b}')$, and $(\mathbf{a}', \mathbf{b}')$. The denominator of (15) is equal to N , the number of experiments run at this angle, merely displayed as the sum of its four component counts of $N_{\pm\pm}(\mathbf{a}, \mathbf{b})$.

The momentous results were published by Aspect *et al.* [11] [12], confirming the apparent defiance of Bell's inequality to several decimal places. Through the following three decades the experimental setup was embellished so to account for a variety of various possible loopholes tendered as an explanation.

8.1. Examining and Reassessing Aspect's Empirical Results

What are we to make of Aspect's and subsequent empirical results?

Aspect [[14], p.15], and [12] reports the estimation $\hat{E}[s(\lambda)]$ from experimental data, using the method of moments as defined in Equations (14) and (15). Of course actually, it is impossible to conduct an experiment on a single pair of photons at all four angle settings, much less conduct a sequence of such experiments. Instead, experimental sequences of observations using different photon pairs were generated at each of four angle settings. These were presumed to provide independent estimates of the four expectations as they appear in Equation (15). These independent estimates were then inserted into Equation (14), yielding Aspect's touted estimate $\hat{E}[s(\lambda)]$ near to $2\sqrt{2}$.

Although experimentation protocols have subsequently been improved to account for the challenges of possible loopholes during the following thirty years, the estimation procedures using the improved data have been the same. Results

from several of the improved protocols have been reported only in the form of so-called p-values of significance for hypothesis tests posed as to whether $E[s(\lambda)]$ exceeds 2 or not. The results have been lionized, apparently quite impressive, and deemed to be decisive.

We can now recognize the fault in Aspect's estimation procedure which allows complete liberty in all four polarization product estimations $\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$, using experimental incidence values of $N_{\pm\pm}(\mathbf{a}^*, \mathbf{b}^*)$ from many experimental runs *with different photon pairs*. Each of his experimental observations may be whatever value it happens to be at its experimental angle setting, identifying whatever value of polarization product that it does. However, if the estimation were meant to apply to the ontological understanding of $s(\lambda)$ in the gedankenexperiment within which he and Bell couch their theoretical claims, he would have to adjust this methodology. One might well pick experimental runs using three different photon pairs at any three angles one wishes, to simulate the behaviours $A(\mathbf{a}^*)B(\mathbf{b}^*)$ for any three polarization products of a single pair of photons. However, to be consistent with the Aspect/Bell problem as posed for this single pair of photons at all four relative angle settings, one then would need to compute the implied value of the polarization product observation for the fourth angle according to the functional form that we have identified in Equation (11). The same functional form connects the detection product at any one of the four angle settings to the other three.

Statistical estimation values reported by Aspect as well as those by subsequent research groups over the past thirty years have no relevance to the estimation of $E[s(\lambda)]$ as it is understood to pertain to four spin products on a single pair of photons. It is perfectly reasonable to find estimation values exceeding the bounds of $[-2, +2]$ as they have. For although these results *could reasonably pertain* to an estimate of $E[s(\lambda)]$ with $s(\lambda)$ defined as a combination of polarization products on *four different pairs of photons*, they *do not pertain* to Bell's inequality which is relevant to a 4-ply gedankenexperiment on the same pair of photons at all four angle settings. In the context to which their experimental results are appropriate, $E[s(\lambda)]$ is not bound by the Bell bounds of $[-2, +2]$, but rather by the interval $[-4, +4]$ which is unchallenged in this context.

Nonetheless, Aspect's empirical estimation programme might be adjusted to account for the symmetric functional relations that would necessarily characterise the imagined results of the gedankenexperiment. In the next subsection I shall display the unsurprising results of such an adjusted methodology. They do not suggest any defiance of Bell's inequality at all. The simulation I construct will mimic the way Aspect's data needs to be treated, recognizing his data as the result of conditionally independent experiments on distinct pairs of photons at each of the four relative angle settings of the polarizers.

8.2. Exposition by Simulation

Because Aspect's experimental observation data is not available in full, a method

for correcting his estimation procedure shall now be displayed using simulated data based on quantum theoretic specifications, along with a presentation of its numerical implications. To begin, four columns of one million (10^6) pseudo random numbers, uniform on $[0,1]$, were generated with a MATLAB routine. These were then transformed into simulated observations of paired photon polarization experiments at the four relative angles we have been studying. These transformations were performed using the QM probabilities based on calculations of $\frac{1}{2}\cos^2(\mathbf{a}^*,\mathbf{b}^*)$ and $\frac{1}{2}\sin^2(\mathbf{a}^*,\mathbf{b}^*)$ as described in our Equations (1). Each resulting simulated polarization pair was then multiplied together to yield a polarization product. In this way were created four columns of simulated observations corresponding to polarization products from one million experiments at each of the four angles: $(\mathbf{a}',\mathbf{b}'),(\mathbf{a},\mathbf{b}'),(\mathbf{a},\mathbf{b}),(\mathbf{a}',\mathbf{b})$. We shall refer to this matrix of simulated polarization products below as the SIMPROD matrix.

Aspect's estimation Equation (15) was applied to each of these columns, yielding estimates of the expected polarization product pertinent to that column, $\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$. These appear in the first row of **Table 1**. These four estimates were then inserted into Equation (14) appropriately to yield an Aspect estimate $\hat{E}[s(\lambda)] = 2.827738$, appearing in the second row of the Table under *each* of these columns. This number is quite near to $2\sqrt{2} \approx 2.828427$, as was Aspect's reported empirical estimate, proposed as an evidential violation of Bell's inequality. As we now know, the problem is that when the product observations are supposed to apply to *the same* photon pair, the observed value of the polarization product at any angle is required to be related to the product at the other three angles via the functional equation we specified in our Equation (11). The four of them may not all range freely in a gedankenexperiment, as they may in real experiments on *different* pairs of photons. Rather, they are required to be bound by the symmetric functional relation $G(\cdot, \cdot)$ that we have identified. The rows of the matrix SIMPROD do *not* respect this requirement, so the Aspect estimate $\hat{E}[s(\lambda)]$ which they produce cannot be used to estimate the expected value of $s(\lambda)$ for the gedankenexperiment. We shall now endeavor to correct this error.

The third row of **Table 1** has been generated then by first applying the function $G(\cdot, \cdot)$ to each choice of three components of the rows of the SIMPROD matrix. Each result was entered into the same row of a companion matrix of the

Table 1. Corrections to Aspect's estimate of $E[s(\lambda)]$.

	$(\mathbf{a}^*,\mathbf{b}^*)$	(\mathbf{a},\mathbf{b})	(\mathbf{a},\mathbf{b}')	(\mathbf{a}',\mathbf{b})	$(\mathbf{a}',\mathbf{b}')$
$\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$		0.707232	-0.706186	0.706840	0.707480
Aspect $\hat{E}[s]$		2.827738	2.827738	2.827738	2.827738
Functional $\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$		-0.353078	0.354348	-0.354766	-0.353934
Corrected $\hat{E}[s]$		1.767180	1.767204	1.765740	1.766964

same size, but placed into the column corresponding to the column entry that was *not* used in the evaluation of the G function. Let's call this matrix by the name SIMGEN. Next, Aspect's estimation Equation (15) was applied to each of the four columns of SIMGEN, and the result is printed in the third row of **Table 1**, labeled "Functional $\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$ ". These display estimates of $E[G(\cdot, \cdot)]$ required for estimation of the four alternative expectation equations (12). In this way we can be considered to have generated 4 times 10^6 simulated versions of the Aspect/Bell gedankenexperiment. Their component results can be taken to be any three simulation results from a row of SIMPROD along with the fourth result being the functionally generated result found in the same row and the appropriate fourth column of SIMGEN. Finally, the last row of **Table 1** presents the estimated values of $E[s(\lambda)]$ deriving from these simulated experiments. They appear as "corrected estimates", column by column, for each of which the $\hat{E}[G(\cdot, \cdot)]$ is the one appropriate to that column while the other three expected polarization products are those appropriate to the other three columns of row 1 of the Table. The elements of this row display corrected estimates of $E[s(\lambda)]$ as they should be calculated with the simulated Aspect data. Each of these four estimates is slightly different from the others. Averaging them over the four ways of generating a column of polarization products from the other three columns of simulated products would yield a "Corrected estimate" of $E[s(\lambda)]$ as 1.766772, well within the Bell bounds of $[-2, +2]$.

Based on Aspect's report of his experimental data, I feel quite sure that applying this same estimation procedure to his experimental data, considered as a simulation of the impossible gedankenexperiment, would yield a similar result.

Results on the order of this peculiar number are quite stable over repeated runs of this simulation as described. Since the theoretical analysis reported in this article yields only an interval of cohering possibilities for $E[s(\lambda)]$, this simulation leaves us with a tantalizing problem of how to account for this stable result, which is quite near to $[3/\sqrt{2} - 1/(2\sqrt{2})] \approx 1.767766952966369$. This specific result is a construct of the gratuitous independence feature embedded in the simulation results across each of the three angle pairings used to generate the function-bound simulation results. Such a feature would be highly suspect in Nature, given what we know now about quantum entanglement itself in a single experiment. I should mention that among all distributions in the polytope cohering with the prescriptions of quantum theory, the maximum entropy distribution inheres an expectation value of $E(s) = 1.1522$. Discussion of its assessment and related issues must await another forum. However, there can be no real empirical evidence on the issue, since it is impossible in principle to activate the setup of the four imagined simultaneous experiments on a single pair of photons. Thus, the physicists' long interest in the fabled gedankenexperiment.

8.3. A Comment on Empirical Work and Statistical Estimation

While Aspect's conception of statistical estimates appropriate to the photon de-

tection problem is understandable, and corrections can be made to improve its relevance to the Aspect/Bell problem, developments of statistical theory and practice during the past fifty years have surely generated superior methods for evaluating the physical theory of quantum behavior. These rely on the subjective theory of probability which, under the leadership of Bruno de Finetti and researchers adhering to his viewpoint, has gained substantial credibility from the past half-century of research in the foundations of probability and statistics. There are even some prominent physicists among its proponents, though not many. Proclaimers of inherent randomness in the physics of quantum behaviour have won the day for now, largely on the basis of the mistaken violation of Bell's inequality that we have debunked in this article. In the very least, it is apparent that calls for open access to raw data [28] from several well-known research programs that publish summary results, usually in the form of discredited p-values, need to be heeded.

9. Concluding Comments

The mathematical structure of the Aspect/Bell problem and its resolution align well with the theory of subjective probability. This viewpoint is in keeping with Einstein's interpretation of quantum mechanics, known by his famous adage that the old one does not roll dice. However, readers more comfortable with the standard realist interpretation of quantum mechanics may also consider the probabilities as ontic properties of the photons themselves without disturbing the mathematical issues we have engaged. Anyone who professes uncertain knowledge about the possible values of a quantum optical gedankenexperiment may assert whatever probabilities are deemed appropriate for the sixteen possible observation vectors displayed in block one of our experimental realm matrix. This may involve as many or as few expectations as one wishes, whether based on the accepted theory of quantum mechanics or not. These of course need to be assessed scientifically in the light of what evidence can be brought to bear. Similarly, realist proponents of quantum theory may hypothesize whatever probability values they think it prescribes. However, since the sixteen vectors of possible polarization observations listed in the realm matrix provide an exclusive and exhaustive list of possible gedankenexperiment results, the sum of these probabilities must equal 1 for anyone who makes coherent assertions. This understanding is what resolves the conundrum posed by apparent violations of Bell's inequality.

As to the characterization of the theory of hidden variables, this is another endeavour that has been misconstrued in accepted literature, largely on the basis of the mistaken understanding of the defiance of Bell's inequality which we have corrected here. I have examined this matter in a separate manuscript entitled "Resurrection of the principle of local realism and the prospects for supplementary variables." Along with a manuscript on my reassessment of Mermin's "quantum mysteries" [29], it is currently available only on my ResearchGate

page. For now I shall merely state that mathematically, the theory of supplementary variables specifies the form of a mixing density $\rho(\lambda)$ that can be made isomorphic to *any* coherent distribution over the empirical observations of polarization experiments whatsoever. It matters not whether they are the prescriptions of quantum theory or not. At any rate, no coherent distribution over observable quantities supports the defiance of Bell's inequality, whether considered to be a formalization of hidden variables theory or not.

Virtually all discussion of quantum probabilities since the original work of Bell has supported the conclusion that probabilities pertinent to quantum behaviour can violate the seemingly innocuous inequality that he identified. The mathematical error that has been discovered and reported here substantiates the end of an era of accepting this conclusion. The results we have aired will have ramifications for many published estimations based on more sophisticated experimentation as well. There are further consequences for a host of theoretical issues that have been studied and discussed in the context of a mistaken understanding. These include related notions of hidden variables, entangled particles, and information transfer. Discussions of these topics do require philosophical attention to a variety of conceptual constructs in which they are imbedded. However, the analysis of Aspect/Bell presented here has nothing to do with philosophical distinctions. It has identified a mathematical error in accepted work that must be recognized no matter what might be the philosophical positions of interested parties. Probabilistic forecasts motivated by quantum theory do not violate any laws of probability theory.

Discussions of related issues proceeding henceforth will need to begin with this new recognition. Interestingly, this resolution was suspected in some way by Bell himself, though not the analytical detail. This was clearly evident in his musings on the hidden variables question in Bell [10] which he himself had reprinted in a collection of his publications, Bell [30]. My discovery of the functional relations involved among the components of the 4-ply gedanken quantity S and the 4-D polytope of their cohering quantum theoretic distributions is truly novel.

A final reference relevant to this analysis is the article of Romano Scozzafava [31] on the role of probability in statistical physics. He discusses several issues that clarify fundamental matters in the context of the constructive mathematics of Bruno de Finetti's operational subjective statistical method.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Wiseman, H. (2015) *Nature*, **526**, 649-650. <https://doi.org/10.1038/nature15631>
- [2] Kupczynski, M. (2020) *Frontiers in Physics*, **8**, 273. <https://doi.org/10.3389/fphy.2020.00273>
- [3] Fine, A. (1982) *Physics Review Letters*, **48**, 291-295. <https://doi.org/10.1103/PhysRevLett.48.291>
- [4] Hess, K. (1970) *Einstein Was Right!* Jenny Stanford Publishing, New York.
- [5] Hess, K. and Philipp, W. (2001) *Proceedings of the National Academy of Sciences*, **98**, 14228-14233. <https://doi.org/10.1073/pnas.251525098>
- [6] Khrennikov, A. (2007) *AIP Conference Proceedings*, **962**, 121-131. <https://doi.org/10.1063/1.2827294>
- [7] Mermin, N.D. (1985) *Physics Today*, **38**, 38-47. <https://doi.org/10.1063/1.880968>
- [8] Bell, J.S. (1964) *Physics*, **1**, 195-200. <https://doi.org/10.1103/PhysicsPhysiqueFizika.1.195>
- [9] Bell, J.S. (1966) *Review of Modern Physics*, **38**, 447-452. <https://doi.org/10.1103/RevModPhys.38.447>
- [10] Bell, J.S. (1987) Introduction to the Hidden-Variable Question, 1971. In: *Speakable and Unspeakable in Quantum Mechanics. Collected Papers on Quantum Philosophy*, Cambridge University Press, Cambridge, 29-39.
- [11] Aspect, A., Grangier, P. and Roger, G. (1981) *Physical Review Letters*, **47**, 460-463. <https://doi.org/10.1103/PhysRevLett.47.460>
- [12] Aspect, A., Dalibard, J. and Roger, G. (1982) *Physical Review Letters*, **49**, 1804-1807. <https://doi.org/10.1103/PhysRevLett.49.1804>
- [13] Clauser, J.F., Horne, M.A., Shimony, A. and Holt, R.A. (1969) *Physical Review Letters*, **23**, 880-884. <https://doi.org/10.1103/PhysRevLett.23.880>
- [14] Aspect, A. (2002) Bell's Theorem: The Naive View of an Experimentalist. In: *Quantum [Un]speakables*, Springer, Berlin, 119-153. https://doi.org/10.1007/978-3-662-05032-3_9
- [15] Adenier, G. (2001) Refutation of Bell's Theorem. In: *Foundations of Probability and Physics*, World Scientific, Singapore, 29-38. https://doi.org/10.1142/9789812810809_0002
- [16] Einstein, A., Podolsky, B. and Rosen, N. (1935) *Physical Review*, **47**, 777-780. <https://doi.org/10.1103/PhysRev.47.777>
- [17] Mermin, N.D. (2003) *Europhysics Letters*, **61**, 143-147. <https://doi.org/10.1209/epl/i2003-00199-0>

- [18] Hess, K. and Philipp, W. (2004) *Europhysics Letters*, **67**, 691-692.
<https://doi.org/10.1209/epl/i2003-10306-9>
- [19] Mermin, N.D. (2004) *Europhysics Letters*, **67**, 693-694.
<https://doi.org/10.1209/epl/i2004-10102-1>
- [20] Mehra, J. (1987) *Foundations of Physics*, **17**, 461-506.
<https://doi.org/10.1007/BF01559698>
- [21] de Finetti, B. (1974, 1975) *Theory of Probability 1 (2)*. Wiley, Chichester.
https://doi.org/10.1007/978-94-010-2288-0_1
- [22] Bruno, G. and Gilio, A. (1980) *Statistica*, **40**, 337-344.
- [23] Whittle, P. (1970) *Probability*. Penguin, Harmondsworth.
- [24] Whittle, P. (1971) *Optimization under Constraint*. Wiley, London.
- [25] Lad, F. (1996) *Operational Subjective Statistical Methods: A Mathematical, Philosophical, and Historical Introduction*. Wiley, New York.
- [26] Capotorti, A., Lad, F. and Sanfilippo, G. (2007) *The American Statistician*, **61**, 132-138. <https://doi.org/10.1198/000313007X190943>
- [27] Abbott, E.A. (1884) *Flatland: A Romance of Many Dimensions*. Selley and Co., London.
- [28] Khrennikov, A. (2015) Unuploaded Experiments Have No Result.
<https://arxiv.org/abs/1505.04293>
- [29] Mermin, N.D. (1981) *Journal of Philosophy*, **78**, 397-408.
<https://doi.org/10.2307/2026482>
- [30] Bell, J.S. (1987) *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy*. Cambridge University Press, Cambridge.
- [31] Scozzafava, R. (2000) *Transport Theory and Statistical Physics*, **29**, 107-123.
<https://doi.org/10.1080/00411450008205863>