

Minimum Parametrization of the Cauchy Stress Operator

J.-F. Pommaret 

CERMICS, Ecole des Ponts Paris Tech, France

Email: jean-francois.pommaret@wanadoo.fr

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Abstract

When $\mathcal{D} : \xi \rightarrow \eta$ is a linear differential operator, a “direct problem” is to find the generating compatibility conditions (CC) in the form of an operator $\mathcal{D}_1 : \eta \rightarrow \zeta$ such that $\mathcal{D}\xi = \eta$ implies $\mathcal{D}_1\eta = 0$. When \mathcal{D} is involutive, the procedure provides successive first order involutive operators $\mathcal{D}_1, \dots, \mathcal{D}_n$ when the ground manifold has dimension n , a result first found by M. Janet as early as in 1920, in a footnote. However, the link between this “Janet sequence” and the “Spencer sequence” first found by the author of this paper in 1978 is still not acknowledged. Conversely, when \mathcal{D}_1 is given, a more difficult “inverse problem” is to look for an operator $\mathcal{D} : \xi \rightarrow \eta$ having the generating CC $\mathcal{D}_1\eta = 0$. If this is possible, that is when the differential module defined by \mathcal{D}_1 is torsion-free, one shall say that the operator \mathcal{D}_1 is parametrized by \mathcal{D} and there is no relation in general between \mathcal{D} and \mathcal{D}_2 . The parametrization is said to be “minimum” if the differential module defined by \mathcal{D} has a vanishing differential rank and is thus a torsion module. The solution of this problem, first found by the author of this paper in 1995, is still not acknowledged. As for the applications of the “differential double duality” theory to standard equations of physics (*Cauchy* and Maxwell equations can be parametrized while *Einstein* equations cannot), we do not know other references. When $n = 1$ as in control theory, the fact that controllability is a “built in” property of a control system, amounting to the existence of a parametrization and thus not depending on the choice of inputs and outputs, even with variable coefficients, is still not acknowledged by engineers. The parametrization of the *Cauchy* stress operator in arbitrary dimension n has nevertheless attracted, “separately” and without any general “guiding line”, many famous scientists (G.B. Airy in 1863 for $n = 2$, J.C. Maxwell in 1863, G. Morera and E. Beltrami in 1892 for $n = 3$, A. Einstein in 1915 for $n = 4$). The aim of this paper is to solve the minimum parametrization problem in arbitrary dimension and to apply it through effective methods that could even

be achieved by using computer algebra. Meanwhile, we prove that all these works are using the *Einstein* operator which is self-adjoint and not the *Ricci* operator, a fact showing that the *Einstein* operator, which cannot be parametrized, has already been exhibited by Beltrami more than 20 years before *Einstein*. As a byproduct, they are all based on the same confusion between the so-called *div* operator induced from the *Bianchi* operator \mathcal{D}_2 and the *Cauchy* operator which is the formal adjoint of the Killing operator \mathcal{D} parametrizing the Riemann operator \mathcal{D}_1 for an arbitrary n . We prove that this purely mathematical result deeply questions the origin and existence of gravitational waves. We also present the similar motivating situation met in the study of contact structures when $n = 3$. Like the Michelson and Morley experiment, it is thus an open historical problem to know whether *Einstein* was aware of these previous works or not, but the comparison needs no comment.

Keywords

Differential Operator, Differential Sequence, Killing Operator, Riemann Operator, *Bianchi* Operator, *Cauchy* Operator, Electromagnetism, Elasticity, General Relativity, Gravitational Waves

1. Introduction

We start recalling the basic tools from the formal theory of systems of partial differential (PD) equations and differential modules needed in order to understand and solve the parametrization problem presented in the abstract. As these new tools are difficult and not so well known, we advise the interested reader to follow them step by step on the explicit motivating examples illustrating this paper, in particular the example of the system of infinitesimal Lie equations defining contact transformations when $n = 3$. The main difficulty for the reader not familiar with these new tools is that certain concepts are evident in one framework but not at all in the other and conversely. Considering the single input/single output (SISO) classical control system $\dot{y} - \dot{u} = 0$ with standard notations for ordinary differential (OD) equations, we notice that both y and u can be given arbitrarily *separately* but that the new quantity $z = \dot{y} - u$ *cannot* as it *must* satisfy the autonomous OD equation $\dot{z} = 0$ that, of course, cannot be controlled. This is the reason for which a controllable system cannot surely provide such elements called “*torsion elements*” in module theory. The fact that the controllability just amounts to the lack of any torsion element or, equivalently, to the possibility to parametrize the control system, is left to the reader as a tricky exercise leading to compare with the basic system $\dot{y} - u = 0$ which is controllable and can be simply parametrized by the only arbitrary potential y through the formula $\dot{y} = u, y = y$. As we shall see, the surprising fact is that the lack of torsion elements (or the generating ones) can only be tested by the possibility to parametrize the given equations (or to “measure” how it *cannot* be pa-

rametrized) and no other classical method can work.

In Section 2, once we shall have found the possibility to parametrize the system or the corresponding operator, that is to say once we know that the corresponding differential module is torsion-free, we shall discover that many possible parametrizations may exist. The idea will then be to modify the formal test in order to compute the minimum number of potentials needed and to find a constructive way to obtain *at least* one such minimum parametrization.

In Section 3, we shall study with more details the parametrization problems that are existing in continuum mechanics for an arbitrary dimension n of the ground manifold, the case $n = 2$ allowing to understand why the *Airy* operator is just the formal adjoint of the *Riemann* operator, the case $n = 3$ allowing to compare the results respectively obtained by Beltrami, Maxwell and Morera, the case $n = 4$ allowing to understand why such a parametrization problem is leading to the self-adjoint *Einstein* operator and why the so-called “gravitational waves” operator is nothing else than the formal adjoint of the *Ricci* operator without any reference to *Einstein* equations.

In Section 4, we shall finally add a few unexpected results coming from the use of the symbol sequences existing for certain generic covectors.

1.1. System Theory

If X is a manifold of dimension n with local coordinates $(x) = (x^1, \dots, x^n)$, we denote as usual by $T = T(X)$ the *tangent bundle* of X , by $T^* = T^*(X)$ the *cotangent bundle*, by $\wedge^r T^*$ the *bundle of r -forms* and by $S_q T^*$ the *bundle of q -symmetric tensors*. More generally, let E be a *vector bundle* over X with local coordinates (x^i, y^k) for $i = 1, \dots, n$ and $k = 1, \dots, m$ simply denoted by (x, y) , *projection* $\pi: E \rightarrow X: (x, y) \rightarrow (x)$ and changes of local coordinate $\bar{x} = \varphi(x), \bar{y} = A(x)y$. We shall denote by E^* the vector bundle obtained by inverting the matrix A of the changes of coordinates, exactly like T^* is obtained from T . We denote by $f: X \rightarrow E: (x) \rightarrow (x, y = f(x))$ a *global section* of E , that is a map such that $\pi \circ f = id_X$ but local sections over an open set $U \subset X$ may also be considered when needed. Under a change of coordinates, a section transforms like $\bar{f}(\varphi(x)) = A(x)f(x)$ and the changes of the derivatives can also be obtained with more work. We shall denote by $J_q(E)$ the *q -jet bundle* of E with local coordinates $(x^i, y^k, y_i^k, y_{ij}^k, \dots) = (x, y_q)$ called *jet coordinates* and sections $f_q: (x) \rightarrow (x, f^k(x), f_i^k(x), f_{ij}^k(x), \dots) = (x, f_q(x))$ transforming like the sections $j_q(f): (x) \rightarrow (x, f^k(x), \partial_i f^k(x), \partial_{ij} f^k(x), \dots) = (x, j_q(f)(x))$ where both f_q and $j_q(f)$ are over the section f of E . For any $q \geq 0$, $J_q(E)$ is a vector bundle over X with projection π_q while $J_{q+r}(E)$ is a vector bundle over $J_q(E)$ with projection $\pi_q^{q+r}, \forall r \geq 0$.

DEFINITION 1.1.1: A *linear system* of order q on E is a vector sub-bundle $R_q \subset J_q(E)$ and a *solution* of R_q is a section f of E such that $j_q(f)$ is a section of R_q . With a slight abuse of language, the set of local solutions will be denoted by $\Theta \subset E$.

Let $\mu = (\mu_1, \dots, \mu_n)$ be a multi-index with *length* $|\mu| = \mu_1 + \dots + \mu_n$, *class* i if $\mu_1 = \dots = \mu_{i-1} = 0, \mu_i \neq 0$ and $\mu + 1_i = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_n)$. We set $y_q = \{y_\mu^k \mid 1 \leq k \leq m, 0 \leq |\mu| \leq q\}$ with $y_\mu^k = y^k$ when $|\mu| = 0$. If E is a vector bundle over X and $J_q(E)$ is the q -jet bundle of E , then both sections $f_q \in J_q(E)$ and $j_q(f) \in J_q(E)$ are over the section $f \in E$. There is a natural way to distinguish them by introducing the *Spencer operator*

$$d : J_{q+1}(E) \rightarrow T^* \otimes J_q(E) \text{ with components } (df_{q+1})_{\mu,i}^k(x) = \partial_i f_\mu^k(x) - f_{\mu+1_i}^k(x).$$

The kernel of d consists of sections such that

$f_{q+1} = j_1(f_q) = j_2(f_{q-1}) = \dots = j_{q+1}(f)$. Finally, if $R_q \subset J_q(E)$ is a *system* of order q on E locally defined by linear equations $\Phi^\tau(x, y_q) \equiv a_k^{\tau\mu}(x) y_\mu^k = 0$ and local coordinates (x, z) for the parametric jets up to order q , the *r-prolongation* $R_{q+r} = \rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$ is locally defined when $r = 1$ by the linear equations $\Phi^\tau(x, y_q) = 0$,

$$d_i \Phi^\tau(x, y_{q+1}) \equiv a_k^{\tau\mu}(x) y_{\mu+1_i}^k + \partial_i a_k^{\tau\mu}(x) y_\mu^k = 0 \text{ and has symbol}$$

$g_{q+r} = R_{q+r} \cap S_{q+r} T^* \otimes E \subset J_{q+r}(E)$ if one looks at the *top order terms*. If $f_{q+1} \in R_{q+1}$ is over $f_q \in R_q$, differentiating the identity $a_k^{\tau\mu}(x) f_\mu^k(x) \equiv 0$ with respect to x^i and subtracting the identity $a_k^{\tau\mu}(x) f_{\mu+1_i}^k(x) + \partial_i a_k^{\tau\mu}(x) f_\mu^k(x) \equiv 0$, we obtain the identity $a_k^{\tau\mu}(x) (\partial_i f_\mu^k(x) - f_{\mu+1_i}^k(x)) \equiv 0$ and thus the restriction $d : R_{q+1} \rightarrow T^* \otimes R_q$. More generally, we have the restriction:

$$\begin{aligned} d : \wedge^s T^* \otimes R_{q+1} &\rightarrow \wedge^{s+1} T^* \otimes R_q : (f_{\mu,l}^k(x) dx^l) \\ &\rightarrow \left((\partial_i f_{\mu,l}^k(x) - f_{\mu+1_i,l}^k(x)) dx^i \wedge dx^l \right) \end{aligned} \tag{1}$$

using standard multi-index notation for exterior forms, namely $I = \{i_1 < i_2 < \dots < i_r\}$, $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_r} \in \wedge^r T^*$ for a finite basis, and one can easily check that $d \circ d = 0$. The restriction of $-d$ to the symbol is called the *Spencer map* $\delta : \wedge^s T^* \otimes g_{q+1} \rightarrow \wedge^{s+1} T^* \otimes g_q$ and $\delta \circ \delta = 0$ similarly, leading to the algebraic δ -cohomology $H_{q+r}^s(g_q)$ [1]-[7].

DEFINITION 1.1.2: A system R_q is said to be *formally integrable* when all the equations of order $q+r$ are obtained by r prolongations *only*, $\forall r \geq 0$ or, equivalently, when the projections $\pi_{q+r}^{q+r+s} : R_{q+r+s} \rightarrow R_{q+r}$ are epimorphisms $\forall r, s \geq 0$.

Finding an intrinsic test has been achieved by D.C. Spencer in 1970 [7] along coordinate dependent lines sketched by M. Janet in 1920 [8]. The next procedure providing a *Pommaret basis* and where *one may have to change linearly the independent variables if necessary*, is intrinsic even though it must be checked in a particular coordinate system called δ -regular [1] [4] [9].

- *Equations of class n* : Solve the maximum number β_q^n of equations with respect to the jets of order q and class n . Then call (x^1, \dots, x^n) *multiplicative variables*.
- *Equations of class $i \geq 1$* : Solve the maximum number β_q^i of *remaining* equations with respect to the jets of order q and class i . Then call (x^1, \dots, x^i) *multiplicative variables* and (x^{i+1}, \dots, x^n) *non-multiplicative variables*.
- *Remaining equations of order $\leq q-1$* : Call (x^1, \dots, x^n) *non-multiplicative*

variables.

In actual practice, we shall use a *Janet tabular* where the multiplicative “variables” are in upper left position while the non-multiplicative variables are represented by dots in lower right position.

DEFINITION 1.1.3: A system of PD equations is said to be *involutive* if its first prolongation can be obtained by prolonging its equations only with respect to the corresponding multiplicative variables. In that case, we may introduce the characters $\alpha_q^i = m \frac{(q+n-i-1)!}{(q-1)!(n-i)!} - \beta_q^i$ for $i = 1, \dots, n$ with $\alpha_q^1 \geq \dots \geq \alpha_q^n \geq 0$ and we have $\dim(g_q) = \alpha_q^1 + \dots + \alpha_q^n$ while $\dim(g_{q+1}) = \alpha_q^1 + \dots + n\alpha_q^n$.

REMARK 1.1.4: As long as the *Prolongation/Projection* (PP) procedure has not been achieved in order to get an involutive system, *nothing* can be said about the CC (fine examples can be found in [6] and the recent [10]). A proof that the second order system defined by *Einstein* equations is involutive has been given by J. Gasqui in 1982 but this paper cannot be applied to the minimum parametrizations that need specific δ -regular coordinates as we shall see [11].

When R_q is involutive, the linear differential operator $\mathcal{D} : E \rightarrow J_q(E) \rightarrow J_q(E)/R_q = F_0$ of order q is said to be *involutive*. Introducing the *Janet* bundles:

$$F_r = \wedge^r T^* \otimes J_q(E) / \left(\wedge^r T^* \otimes R_q + \delta \left(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E \right) \right) \tag{2}$$

we obtain the canonical linear *Janet* sequence (introduced in [1], p 185 + p 391):

$$0 \rightarrow \Theta \rightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \dots \xrightarrow{\mathcal{D}_n} F_n \rightarrow 0 \tag{3}$$

where each other operator, induced by the *Spencer* operator, is first order involutive and generates the *compatibility conditions* (CC) of the preceding one. Similarly, introducing the *Spencer* bundles:

$$C_r = \wedge^r T^* \otimes R_q / \delta \left(\wedge^{r-1} T^* \otimes g_{q+1} \right) \tag{4}$$

we obtain the canonical linear *Spencer* sequence also induced by the *Spencer* operator:

$$0 \rightarrow \Theta \rightarrow C_0 \xrightarrow{J_q} C_1 \xrightarrow{D_1} C_2 \xrightarrow{D_2} \dots \xrightarrow{D_n} C_n \rightarrow 0 \tag{5}$$

1.2. Module Theory

Let K be a *differential field* with n commuting derivations $(\partial_1, \dots, \partial_n)$ and consider the ring $D = K[d_1, \dots, d_n] = K[d]$ of differential operators with coefficients in K with n commuting formal derivatives satisfying $d_i a = ad_i + \partial_i a$ in the operator sense. If $P = a^\mu d_\mu \in D = K[d]$, the highest value of $|\mu|$ with $a^\mu \neq 0$ is called the *order* of the operator P and the ring D with multiplication $(P, Q) \rightarrow P \circ Q = PQ$ is filtered by the order q of the operators. We have the *filtration* $0 \subset K = D_0 \subset D_1 \subset \dots \subset D_q \subset \dots \subset D_\infty = D$. As an algebra, D is generated by $K = D_0$ and $T = D_1/D_0$ with $D_1 = K \oplus T$ if we identify an element $\xi = \xi^i d_i \in T$ with the vector field $\xi = \xi^i(x) \partial_i$ of differential geometry, but

with $\xi^i \in K$ now. It follows that $D = {}_D D_D$ is a *bimodule* over itself, being at the same time a left D -module by the composition $P \rightarrow QP$ and a right D -module by the composition $P \rightarrow PQ$. We define the *adjoint* functor $ad : D \rightarrow D^{op} : P = a^\mu d_\mu \rightarrow ad(P) = (-1)^{|\mu|} d_\mu a^\mu$ and we have $ad(ad(P)) = P$ both with $ad(PQ) = ad(Q)ad(P), \forall P, Q \in D$. Such a definition can be extended to any matrix of operators by using the transposed matrix of adjoint operators (see [5] [9] [12] [13] [14] [15] [16] for more details and applications to control theory or mathematical physics).

Accordingly, if $y = (y^1, \dots, y^m)$ are differential indeterminates, then D acts on y^k by setting $d_i y^k = y_i^k \rightarrow d_\mu y^k = y_\mu^k$ with $d_i y_\mu^k = y_{\mu+1_i}^k$ and $y_0^k = y^k$. We may therefore use the jet coordinates in a formal way as in the previous section. Therefore, if a system of OD/PD equations is written in the form $\Phi^\tau \equiv a_k^{\tau\mu} y_\mu^k = 0$ with coefficients $a \in K$, we may introduce the free differential module $Dy = Dy^1 + \dots + Dy^m \simeq D^m$ and consider the differential *module of equations* $I = D\Phi \subset Dy$, both with the residual *differential module*

$M = Dy/D\Phi$ or D -module and we may set $M = {}_D M$ if we want to specify the ring of differential operators. We may introduce the formal *prolongation* with respect to d_i by setting $d_i \Phi^\tau \equiv a_k^{\tau\mu} y_{\mu+1_i}^k + (\partial_i a_k^{\tau\mu}) y_\mu^k$ in order to induce maps $d_i : M \rightarrow M : \bar{y}_\mu^k \rightarrow \bar{y}_{\mu+1_i}^k$ by residue with respect to I if we use to denote the residue $Dy \rightarrow M : y^k \rightarrow \bar{y}^k$ by a bar like in algebraic geometry. However, for simplicity, we shall not write down the bar when the background will indicate clearly if we are in Dy or in M . As a byproduct, the differential modules we shall consider will always be *finitely generated* ($k = 1, \dots, m < \infty$) and *finitely presented* ($\tau = 1, \dots, p < \infty$). Equivalently, introducing the *matrix of operators* $\mathcal{D} = (a_k^{\tau\mu} d_\mu)$ with m columns and p rows, we may introduce the morphism $D^p \xrightarrow{\mathcal{D}} D^m : (P_\tau) \rightarrow (P_\tau \Phi^\tau)$ over D by acting with D on the left of these row vectors while acting with \mathcal{D} on the right of these row vectors by composition of operators with $im(\mathcal{D}) = I$. The *presentation* of M is defined by the exact cokernel sequence $D^p \rightarrow D^m \rightarrow M \rightarrow 0$. We notice that the presentation only depends on K, D and Φ or \mathcal{D} , that is to say never refers to the concept of (explicit local or formal) solutions. It follows from its definition that M can be endowed with a *quotient filtration* obtained from that of D^m which is defined by the order of the jet coordinates y_q in $D_q y$. We have therefore the *inductive limit* $0 \subseteq M_0 \subseteq M_1 \subseteq \dots \subseteq M_q \subseteq \dots \subseteq M_\infty = M$ with $d_i M_q \subseteq M_{q+1}$ and $M = DM_q$ for $q \gg 0$ with prolongations $D_r M_q \subseteq M_{q+r}, \forall q, r \geq 0$.

DEFINITION 1.2.1: An exact sequence of morphisms finishing at M is said to be a *resolution* of M . If the differential modules involved apart from M are free, that is isomorphic to a certain power of D , we shall say that we have a *free resolution* of M .

Having in mind that K is a left D -module with the action $(D, K) \rightarrow K : (d_i, a) \rightarrow \partial_i a$ and that D is a bimodule over itself, we have only two possible constructions:

DEFINITION 1.2.2: We may define the *right* (care!) differential module

$hom_D(M, D)$.

DEFINITION 1.2.3: We define the *system* $R = hom_K(M, K)$ and set $R_q = hom_K(M_q, K)$ as the *system of order q*. We have the *projective limit* $R = R_\infty \rightarrow \dots \rightarrow R_q \rightarrow \dots \rightarrow R_1 \rightarrow R_0$. It follows that $f_q \in R_q : y_\mu^k \rightarrow f_\mu^k \in K$ with $a_k^{\mu} f_\mu^k = 0$ defines a *section at order q* and we may set $f_\infty = f \in R$ for a *section of R*. For an arbitrary differential field K , *such a definition has nothing to do with the concept of a formal power series solution (care)*.

PROPOSITION 1.2.4: When M is a left D -module, then R is also a left D -module.

Proof: As D is generated by K and T as we already said, let us define:

$$(af)(m) = af(m), \quad \forall a \in K, \forall m \in M$$

$$(\xi f)(m) = \xi f(m) - f(\xi m), \quad \forall \xi = a^i d_i \in T, \forall m \in M$$

In the operator sense, it is easy to check that $d_i a = ad_i + \partial_i a$ and that $\xi \eta - \eta \xi = [\xi, \eta]$ is the standard bracket of vector fields. We finally get $(d_i f)_\mu^k = (d_i f)(y_\mu^k) = \partial_i f_\mu^k - f_{\mu+1_i}^k$ and thus recover *exactly* the Spencer operator of the previous section though *this is not evident at all*. We also get $(d_i d_j f)_\mu^k = \partial_{ij} f_\mu^k - \partial_i f_{\mu+1_j}^k - \partial_j f_{\mu+1_i}^k + f_{\mu+1_i+1_j}^k \Rightarrow d_i d_j = d_j d_i, \forall i, j = 1, \dots, n$ and thus $d_i R_{q+1} \subseteq R_q \Rightarrow d_i R \subset R$ induces a well defined operator

$R \rightarrow T^* \otimes R : f \rightarrow dx^i \otimes d_i f$. This operator has been first introduced, up to sign, by F.S. Macaulay as early as in 1916 but this is still not acknowledged [17]. For more details on the Spencer operator and its applications, the reader may look at [15] [18] [19] [20] [21].

Q.E.D.

DEFINITION 1.2.5: With any differential module M we shall associate the *graded module* $G = gr(M)$ over the polynomial ring $gr(D) \simeq K[\chi]$ by setting $G = \bigoplus_{q=0}^\infty G_q$ with $G_q = M_q / M_{q-1}$ and we get $g_q = G_q^*$ where the *symbol* g_q is defined by the short exact sequences:

$$0 \rightarrow M_{q-1} \rightarrow M_q \rightarrow G_q \rightarrow 0 \Leftrightarrow 0 \rightarrow g_q \rightarrow R_q \rightarrow R_{q-1} \rightarrow 0$$

We have the short exact sequences $0 \rightarrow D_{q-1} \rightarrow D_q \rightarrow S_q T \rightarrow 0$ leading to $gr_q(D) \simeq S_q T$ and we may set as usual $T^* = hom_K(T, K)$ in a coherent way with differential geometry.

The two following definitions, which are well known in commutative algebra, are also valid (with more work) in the case of differential modules (see [5] for more details or the references [9] [22] [23] [24] for an introduction to homological algebra and diagram chasing).

DEFINITION 1.2.6: The set of elements

$t(M) = \{m \in M \mid \exists 0 \neq P \in D, Pm = 0\} \subseteq M$ is a differential module called the *torsion submodule* of M . More generally, a module M is called a *torsion module* if $t(M) = M$ and a *torsion-free module* if $t(M) = 0$. In the short exact sequence $0 \rightarrow t(M) \rightarrow M \rightarrow M' \rightarrow 0$, the module M' is torsion-free. Its defining module of equations I' is obtained by adding to I a representative basis of $t(M)$ set up to zero and we have thus $I \subseteq I'$.

DEFINITION 1.2.7: A differential module F is said to be *free* if $F \simeq D^r$ for some integer $r > 0$ and we shall *define* $rk_D(F) = r$. If F is the biggest free differential module contained in M , then M/F is a torsion differential module and $hom_D(M/F, D) = 0$. In that case, we shall *define* the *differential rank* of M to be $rk_D(M) = rk_D(F) = r$. Accordingly, if M is defined by a linear involutive operator of order q , then $rk_D(M) = \alpha_q^n$.

PROPOSITION 1.2.8: If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of differential modules and maps or operators, we have

$$rk_D(M) = rk_D(M') + rk_D(M'').$$

In the general situation, let us consider the sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ of modules which may not be exact and define $B = im(f) \subseteq Z = ker(g) \Rightarrow H = Z/B$.

LEMMA 1.2.9: The kernel of the induced epimorphism $coker(f) \rightarrow coim(g)$ is isomorphic to H .

Proof: It follows from a snake chase in the commutative and exact diagram where $coim(g) \simeq im(g)$:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & & & H \\
 & & & & & & \downarrow \\
 & 0 & & 0 & & & \downarrow \\
 & \downarrow & & \downarrow & & & \downarrow \\
 0 & \rightarrow & B & \rightarrow & M & \rightarrow & coker(f) \rightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \rightarrow & Z & \rightarrow & M & \xrightarrow{g} & coim(g) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & H & & 0 & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

Q.E.D.

In order to conclude this section, we may say that the main difficulty met when passing from the differential framework to the algebraic framework is the “*inversion*” of arrows. Indeed, when an operator is injective, that is when we have the exact sequence $0 \rightarrow E \xrightarrow{\mathcal{D}} F$ with $dim(E) = m, dim(F) = p$, like in the case of the operator $0 \rightarrow E \xrightarrow{J_q} J_q(E)$, on the contrary, using differential modules, we have the epimorphism $D^p \xrightarrow{\mathcal{D}} D^m \rightarrow 0$. The case of a formally surjective operator, like the *div* operator, described by the exact sequence $E \xrightarrow{\mathcal{D}} F \rightarrow 0$ is now providing the exact sequence of differential modules

$$0 \rightarrow D^p \xrightarrow{\mathcal{D}} D^m \rightarrow M \rightarrow 0 \text{ because } \mathcal{D} \text{ has no CC.}$$

2. Parametrization Problem

In this section, we shall set up and solve the minimum parametrization problem by comparing the differential geometric approach and the differential algebraic

approach. In fact, both sides are essential because certain concepts, like “torsion”, are simpler in the module approach, as we already said, while others, like “involution” are simpler in the operator approach. However, the reader must never forget that the “extension modules” or the “side changing functor” are pure product of *differential homological algebra* with no system counterpart. Also, the close link existing between “differential duality” and “adjoint operator” may not be evident at all, even for people quite familiar with mathematical physics [5] [12] [15] [16].

Let us start with a given linear differential operator $\eta \xrightarrow{\mathcal{D}_1} \zeta$ between the sections of two given vector bundles F_0 and F_1 of respective fiber dimension m and p . Multiplying the equations $\mathcal{D}_1 \eta = \zeta$ by p test functions λ considered as a section of the adjoint vector bundle $ad(F_1) = \wedge^n T^* \otimes F_1^*$ and integrating by parts, we may introduce the adjoint vector bundle $ad(F_0) = \wedge^n T^* \otimes F_0^*$ with sections μ in order to obtain the adjoint operator $\mu \xleftarrow{ad(\mathcal{D}_1)} \lambda$, writing on purpose the arrow backwards, that is from right to left. As any operator is the adjoint of another operator because $ad(ad(\mathcal{D})) = \mathcal{D}$, we may *decide* to denote by $\nu \xleftarrow{ad(\mathcal{D})} \mu$ the generating CC of $ad(\mathcal{D}_1)$ by introducing a vector bundle E with sections ξ and its adjoint $ad(E) = \wedge^n T^* \otimes E^*$ with sections ν . We have thus obtained the formally exact differential sequence:

$$\nu \xleftarrow{ad(\mathcal{D})} \mu \xleftarrow{ad(\mathcal{D}_1)} \lambda$$

and its formally exact adjoint sequence:

$$\xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta$$

providing a parametrization *if and only if* \mathcal{D}_1 generates the CC of \mathcal{D}_1 . Such a situation may not be satisfied but we shall assume it from now on because otherwise \mathcal{D}_1 *cannot* be parametrized according to the *double differential duality* test, for example in the case of the *Einstein* equations [25] [26] [27] or the extension to the conformal group and other Lie groups of transformations [15] [19] [20] [21] [28] [29] [30]. Nevertheless, for the interested reader only, we provide the following key result on which this procedure is based (see [5] [12] [13] [14] [16] for more details):

THEOREM 2.1: If M is a differential module, we have the exact sequence of differential modules:

$$0 \rightarrow t(M) \rightarrow M \xrightarrow{\varepsilon} hom_D(hom_D(M, D), D) \tag{6}$$

where the map ε is defined by $\varepsilon(m)(f) = f(m), \forall m \in M, f \in hom_D(M, D)$. Moreover, if N is the differential module defined by $ad(\mathcal{D})$, then $t(M) = ext_D^1(N, D)$.

In order to pass to the differential module framework, let us introduce the free differential modules $D\xi \simeq D^l, D\eta \simeq D^m, D\zeta \simeq D^p$. We have similarly the ad-

joint free differential modules $D\nu \simeq D^l, D\mu \simeq D^m, D\lambda \simeq D^p$, because $\dim(ad(E)) = \dim(E)$ and $\text{hom}_D(D^m, D) \simeq D^m$. Of course, in actual practice, *the geometric meaning is totally different* because we have volume forms in the dual framework. We have thus obtained the formally exact sequence of differential modules:

$$D^p \xrightarrow{\mathcal{D}_1} D^m \xrightarrow{\mathcal{D}} D^l$$

and the formally exact adjoint sequence:

$$D^p \xleftarrow{ad(\mathcal{D}_1)} D^m \xleftarrow{ad(\mathcal{D})} D^l$$

The procedure with 4 steps is as follows in the operator language:

- STEP 1: Start with the formally exact *parametrizing sequence* already constructed by differential biduality. We have thus $\text{im}(\mathcal{D}) = \ker(\mathcal{D}_1)$ and the corresponding differential module M_1 defined by \mathcal{D}_1 is torsion-free by assumption.
- STEP 2: Construct the adjoint sequence which is also formally exact by assumption.
- STEP 3: Find a *maximum* set of differentially independent CC $ad(\mathcal{D}'): \mu \rightarrow \nu'$ among the generating CC $ad(\mathcal{D}): \mu \rightarrow \nu$ of $ad(\mathcal{D}_1)$ in such a way that $\text{im}(ad(\mathcal{D}'))$ is a maximum free differential submodule of $\text{im}(ad(\mathcal{D}))$ that is any element in $\text{im}(ad(\mathcal{D}))$ is differentially algebraic over $\text{im}(ad(\mathcal{D}'))$.
- STEP 4: Using differential duality, construct $\mathcal{D}' = ad(ad(\mathcal{D}'))$.

It remains to prove that \mathcal{D}_1 generates the CC of \mathcal{D}' in the following diagram:

$$\begin{array}{ccccccc}
 \boxed{4} & & \xi' & & & & \\
 & & \uparrow \searrow & & & & \\
 & & \xi & \xrightarrow{\mathcal{D}} & \eta & \xrightarrow{\mathcal{D}_1} & \zeta & \boxed{1} \\
 & & & & & & & (7) \\
 & & & & \nu & \xleftarrow{ad(\mathcal{D})} & \mu & \xleftarrow{ad(\mathcal{D}_1)} & \lambda & \boxed{2} \\
 & & & & \uparrow \swarrow & & & & & \\
 \boxed{3} & & \nu' & & & & & & & \\
 & & \swarrow \uparrow & & & & & & & \\
 & & 0 & & 0 & & & & &
 \end{array}$$

PROPOSITION 2.2: \mathcal{D}' is a minimum parametrization of \mathcal{D}_1 .

Proof: Let us denote the number of potentials ξ by l (respectively ξ' by l'), the number of unknowns η by m and the number of given equations ζ by p . As $ad(\mathcal{D}')$ has no CC by construction, then $ad(\mathcal{D}'): \mu \rightarrow \nu'$ is a formally surjective operator. On the differential module level, we have the injective operator $ad(\mathcal{D}'): D^{l'} \rightarrow D^m$ because there are no CC. Applying $\text{hom}_D(\bullet, D)$ or duality, we get an operator $D^m \rightarrow D^{l'}$ with a cokernel which is a torsion module because it has rank $l' - \text{rk}_D(\mathcal{D}') = l' - \text{rk}_D(ad(\mathcal{D}')) = l' - l = 0$.

However, in actual practice as will be seen in the contact case, things are not so simple and we shall use the following commutative and exact diagram of differential modules based on a long *ker/coker* long exact sequence (compare to [31] and [32]):

$$\begin{array}{ccccccc}
 0 \rightarrow \ker(ad(\mathcal{D})) \rightarrow & D^l & \xrightarrow{ad(\mathcal{D})} & D^m & \rightarrow \text{coker}(ad(\mathcal{D})) \rightarrow 0 \\
 & \searrow & & \nearrow & \\
 & & L & & \\
 & \nearrow & \uparrow & \searrow & \\
 & 0 & D'' & 0 & \\
 & & \uparrow & & \\
 & & 0 & &
 \end{array} \tag{8}$$

Setting $L = D^l / \ker(ad(\mathcal{D}))$ and introducing the biggest free differential module $D'' \subseteq L$ we have $rk_D(D'') = rk_D(L) \leq rk_D(D^l) \Rightarrow l' \leq l$, we may define the injective (*care*) operator $ad(\mathcal{D}')$ by the composition of monomorphisms $D'' \rightarrow L \rightarrow D^m$ where the second is obtained by picking a basis of D'' , lifting it to D^l and pushing it to D^m by applying $ad(\mathcal{D})$. We notice that L can be viewed as the differential module defined by the generating CC of $ad(\mathcal{D})$ that could also be used as in [31].

Then we have $ad(\mathcal{D}') \circ ad(\mathcal{D}_1) = ad(\mathcal{D}_1 \circ \mathcal{D}') = 0 \Rightarrow \mathcal{D}_1 \circ \mathcal{D}' = 0$ and thus \mathcal{D}_1 is surely among the CC of \mathcal{D}' . Therefore, the differential sequence $\xi' \xrightarrow{\mathcal{D}'} \eta \xrightarrow{\mathcal{D}_1} \zeta$ on the operator level or the sequence $D^p \xrightarrow{\mathcal{D}_1} D^m \xrightarrow{\mathcal{D}'} D''$ on the differential module level may not be exact and we can thus apply the previous Lemma. Changing slightly the notations, we have now

$B = im(\mathcal{D}_1) = \ker(\mathcal{D}) \subseteq \ker(\mathcal{D}') = Z$. But we have also $rk_D(B) = m - rk_D(\mathcal{D})$, $rk(Z) = m - rk_D(\mathcal{D}') \Rightarrow rk_D(H) = rk_D(\mathcal{D}) - rk_D(\mathcal{D}') = 0$ by construction.

Taking into account the previous Lemma, we may set $coim(\mathcal{D}_1) = M_1 \subseteq D'$ by assumption and consider $im(\mathcal{D}') = M'_1 \subseteq D''$ in order to obtain the short exact sequence of differential modules $0 \rightarrow H \rightarrow M_1 \rightarrow M'_1 \rightarrow 0$. As H is a torsion module and the differential module M_1 defined by \mathcal{D}_1 is torsion-free by assumption, the only possibility is that $H = 0$ and thus $im(\mathcal{D}_1) = \ker(\mathcal{D}')$, that is \mathcal{D}' is a minimum parametrization of \mathcal{D}_1 with $l' \leq l$ potentials.

Q.E.D.

EXAMPLE 2.3: Contact transformations

With $m = n = 3, K = \mathbb{Q}(x^1, x^2, x^3) = \mathbb{Q}(x)$, we may introduce the so-called *contact* 1-form $\alpha = dx^1 - x^3 dx^2$. The system of infinitesimal Lie equations defining the infinitesimal contact transformations is obtained by eliminating the factor $\rho(x)$ in the equations $\mathcal{L}(\xi)\alpha = \rho\alpha$ where \mathcal{L} is the standard Lie derivative. This system is thus only generated by η^1 and η^2 below but is not involutive and one has to introduce η^3 defined by the first order CC:

$$\zeta \equiv \partial_3 \eta^1 - \partial_2 \eta^2 - x^3 \partial_1 \eta^2 + \eta^3 = 0$$

in order to obtain the following involutive system with two equations of class 3

and one equation of class 2, a result leading to $\beta_1^3 = 2, \beta_1^2 = 1, \beta_1^1 = 0$:

$$\begin{cases} \eta^3 \equiv \partial_3 \xi^3 + \partial_2 \xi^2 + 2x^3 \partial_1 \xi^2 - \partial_1 \xi^1 = 0 \\ \eta^2 \equiv \partial_3 \xi^1 - x^3 \partial_3 \xi^2 = 0 \\ \eta^1 \equiv \partial_2 \xi^1 - x^3 \partial_2 \xi^2 + x^3 \partial_1 \xi^1 - (x^3)^2 \partial_1 \xi^2 - \xi^3 = 0 \end{cases} \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & \bullet \\ \hline \end{array}$$

The characters are thus $\alpha_1^3 = 3 - 2 = 1 < \alpha_1^2 = 3 - 1 = 2, \alpha_1^1 = 3 - 0 = 3$ with sum equal to $1 + 2 + 3 = 6 = \dim(g_1) = 3 \times 3 - 3$. In this situation, if M is the differential module defined by this system or the corresponding operator \mathcal{D} , we know that $rk_{\mathcal{D}}(M) = \alpha_1^3 = 1 = 3 - 2 = rk_{\mathcal{D}}(D\xi) - rk_{\mathcal{D}}(\mathcal{D})$. Of course, a differential transcendence basis for \mathcal{D} can be the operator $\mathcal{D}' : \xi \rightarrow \{\eta^2, \eta^3\}$ but, in view of the CC, we may equally choose any couple among $\{\eta^1, \eta^2, \eta^3\}$ and we obtain $rk_{\mathcal{D}}(\mathcal{D}') = rk_{\mathcal{D}}(\mathcal{D}) = 2$ in any case, but now \mathcal{D}' is formally surjective, contrary to \mathcal{D} . The same result can also be obtained directly from the unique CC or the corresponding operator \mathcal{D}_1 defining the differential module M_1 . Finally, we have $rk_{\mathcal{D}}(M_1) = 3 - 1 = 2 = rk_{\mathcal{D}}(D\eta) - rk_{\mathcal{D}}(\mathcal{D}_1)$ and we check that we have indeed $rk_{\mathcal{D}}(M) + rk_{\mathcal{D}}(M_1) = 1 + 2 = 3 = rk_{\mathcal{D}}(D\xi)$.

It is well known that such a system can be parametrized by the injective parametrization (see [2] and [3] for more details and the study of the general dimension $n = 2p + 1$):

$$-x^3 \partial_3 \phi + \phi = \xi^1, -\partial_3 \phi = \xi^2, \partial_2 \phi + x^3 \partial_1 \phi = \xi^3 \Rightarrow \xi^1 - x^3 \xi^2 = \phi$$

It is however not so well known and quite striking that such a parametrization can be recovered independently by using the parametrization of the differential module defined by $\eta^1 = 0$ with potentials ξ^1 and ξ^2 while setting:

$$(\xi^1, \xi^2) \rightarrow \xi^3 = \partial_2 \xi^1 - x^3 \partial_2 \xi^2 + x^3 \partial_1 \xi^1 - (x^3)^2 \partial_1 \xi^2$$

Taking into account the differential constraint $\eta^2 \equiv \partial_3 \xi^1 - x^3 \partial_3 \xi^2 = 0$, that is $\xi^2 = -\partial_3(\xi^1 - x^3 \xi^2)$ and substituting in $\eta^3 = 0$, we get no additional constraint. We finally only need to modify the potentials while “defining” now $\phi = \xi^1 - x^3 \xi^2 = \bar{\xi}^1$ as before.

The associated differential sequence is:

$$\begin{aligned} 0 \rightarrow \phi \xrightarrow{\mathcal{D}_{-1}} \xi \xrightarrow{\mathcal{D}} \eta \xrightarrow{\mathcal{D}_1} \zeta \rightarrow 0 \\ 0 \rightarrow 1 \rightarrow 3 \rightarrow 3 \rightarrow 1 \rightarrow 0 \end{aligned}$$

with Euler-Poincaré characteristic $1 - 3 + 3 - 1 = 0$ but is *not* a Janet sequence because \mathcal{D}_{-1} is not involutive, its completion to involution being the trivially involutive operator $j_1 : \phi \rightarrow j_1(\phi)$.

Introducing the ring $D = K[d_1, d_2, d_3] = K[d]$ of linear differential operators with coefficients in the differential field K , the corresponding differential module $M \simeq D$ is projective and even free, thus torsion-free or 0-pure, being defined by the split exact sequence of free differential modules:

$$0 \rightarrow D \xrightarrow{\mathcal{D}_1} D^3 \xrightarrow{\mathcal{D}} D^3 \xrightarrow{\mathcal{D}_{-1}} D \rightarrow 0$$

We let the reader prove as an exercise that the adjoint sequence:

$$0 \leftarrow \theta \xleftarrow{ad(\mathcal{D}_{-1})} \nu \xleftarrow{ad(\mathcal{D})} \mu \xleftarrow{ad(\mathcal{D}_1)} \lambda \leftarrow 0$$

$$0 \leftarrow 1 \leftarrow 3 \leftarrow 3 \leftarrow 1 \leftarrow 0$$

starting from the Lagrange multiplier λ is also a split exact sequence of free differential modules.

We finally prove that the situation met for the contact structure is *exactly* the same as the one that we shall meet in the metric structure, namely that one can identify \mathcal{D}_{-1} not with \mathcal{D}_1 of course but with $ad(\mathcal{D}_1)$. For this, let us modify the “basis” linearly by setting $(\bar{\xi}^1 = \xi^1 - x^3 \xi^2, \bar{\xi}^2 = \xi^2, \bar{\xi}^3 = \xi^3)$ and suppressing the bar for simplicity, we obtain the new injective parametrization:

$$\phi = \xi^1, -\partial_3 \phi = \xi^2, \partial_2 \phi + x^3 \partial_1 \phi = \xi^3$$

and may eliminate ϕ in order to consider the new involutive system, renumbering the equations through a cyclic permutation of $(1, 2, 3)$:

$$\begin{cases} \eta^1 \equiv \partial_3 \xi^3 + \partial_2 \xi^2 + x^3 \partial_1 \xi^2 - \partial_1 \xi^1 = 0 \\ \eta^3 \equiv \partial_3 \xi^1 + \xi^2 = 0 \\ \eta^2 \equiv \partial_2 \xi^1 + x^3 \partial_1 \xi^1 - \xi^3 = 0 \end{cases} \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & \bullet \\ \hline \end{array}$$

with the unique first order CC defining \mathcal{D}_1 :

$$\zeta \equiv \partial_3 \eta^2 - \partial_2 \eta^3 - x^3 \partial_1 \eta^3 + \eta^1 = 0$$

Multiplying by λ and integrating by parts, we obtain for $ad(\mathcal{D}_1)$:

$$\eta^1 \rightarrow \lambda = \mu^1, \eta^2 \rightarrow -\partial_3 \lambda = \mu^2, \eta^3 \rightarrow \partial_2 \lambda + x^3 \partial_1 \lambda = \mu^3$$

obtaining therefore $\mathcal{D}_{-1} = ad(\mathcal{D}_1) \Leftrightarrow \mathcal{D}_1 = ad(\mathcal{D}_{-1})$ *exactly*.

As for $\mathcal{D}^\xi = \eta$, we obtain the formal operator matrix:

$$\begin{pmatrix} -d_1 & d_2 + x^3 d_1 & d_3 \\ d_2 + x^3 d_1 & 0 & -1 \\ d_3 & 1 & 0 \end{pmatrix}$$

Similarly, for $ad(\mathcal{D})$ we obtain the formal operator matrix:

$$\begin{pmatrix} d_1 & -(d_2 + x^3 d_1) & -d_3 \\ -(d_2 + x^3 d_1) & 0 & 1 \\ -d_3 & -1 & 0 \end{pmatrix}$$

and finally discover that $ad(\mathcal{D}) = -\mathcal{D}$, a striking result showing that both operators have the same CC and parametrization even though \mathcal{D} is *not* self-adjoint.

3. Einstein Equations

Linearizing the *Ricci* tensor ρ_{ij} over the Minkowski metric ω , we obtain the usual second order homogeneous *Ricci* operator $\Omega \rightarrow R$ with 4 terms:

$$2R_{ij} = \omega^{rs} (d_{rs} \Omega_{ij} + d_{ij} \Omega_{rs} - d_{ri} \Omega_{sj} - d_{sj} \Omega_{ri}) = 2R_{ji} \tag{9}$$

$$tr(R) = \omega^{ij} R_{ij} = \omega^{ij} d_{ij} tr(\Omega) - \omega^{ru} \omega^{sv} d_{rs} \Omega_{uv} \tag{10}$$

We may define the *Einstein* operator by setting $E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij}tr(R)$ and obtain the 6 terms [33]:

$$2E_{ij} = \omega^{rs} \left(d_{rs}\Omega_{ij} + d_{ij}\Omega_{rs} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri} \right) - \omega_{ij} \left(\omega^{rs}\omega^{uv}d_{rs}\Omega_{uv} - \omega^{ru}\omega^{sv}d_{rs}\Omega_{uv} \right) \tag{11}$$

We have the (locally exact) differential sequence of operators acting on sections of vector bundles where the order of an operator is written under its arrow:

$$T \xrightarrow[1]{Killing} S_2T^* \xrightarrow[2]{Riemann} F_1 \xrightarrow[1]{Bianchi} F_2 \tag{12}$$

$$n \xrightarrow{D} n(n+1)/2 \xrightarrow{D_1} n^2(n^2-1)/12 \xrightarrow{D_2} n^2(n^2-1)(n-2)/24$$

Our purpose is now to study the differential sequence onto which its right part is projecting:

$$S_2T^* \xrightarrow[2]{Einstein} S_2T^* \xrightarrow[1]{div} T^* \rightarrow 0$$

$$n(n+1)/2 \rightarrow n(n+1)/2 \rightarrow n \rightarrow 0$$

and the following adjoint sequence where we have set [15] [19] [20] [31] [34]:

$$Cauchy = ad(Killing), Beltrami = ad(Riemann), Lanczos = ad(Bianchi)$$

$$ad(T) \xleftarrow{Cauchy} ad(S_2T^*) \xleftarrow{Beltrami} ad(F_1) \xleftarrow{Lanczos} ad(F_2) \tag{13}$$

In this sequence, if E is a vector bundle over the ground manifold X with dimension n , we may introduce the new vector bundle $ad(E) = \wedge^n T^* \otimes E^*$ where E^* is obtained from E by inverting the transition rules exactly like T^* is obtained from T . We have for example $ad(T) = \wedge^n T^* \otimes T^* \simeq \wedge^n T^* \otimes T \simeq \wedge^{n-1} T^*$ because T^* is isomorphic to T by using the metric ω . The 10×10 *Einstein* operator matrix is induced from the 10×20 *Riemann* operator matrix and the 10×4 *div* operator matrix is induced from the 20×20 *Bianchi* operator matrix. We advise the reader not familiar with the formal theory of systems or operators to follow the computation in dimension $n = 2$ with the 1×3 *Airy* operator matrix, which is the formal adjoint of the 3×1 *Riemann* operator matrix, and $n = 3$ with the 6×6 Beltrami operator matrix which is the formal adjoint of the 6×6 *Riemann* operator matrix which is easily seen to be self-adjoint up to a change of basis.

With more details, we have:

- $n = 2$: The stress equations become $d_1\sigma^{11} + d_2\sigma^{12} = 0, d_1\sigma^{21} + d_2\sigma^{22} = 0$. Their second order parametrization $\sigma^{11} = d_{22}\phi, \sigma^{12} = \sigma^{21} = -d_{12}\phi, \sigma^{22} = d_{11}\phi$ has been provided by George Biddell *Airy* in 1863 [35] and is well known [5]. We get the second order system:

$$\begin{cases} \sigma^{11} \equiv d_{22}\phi = 0 \\ -\sigma^{12} \equiv d_{12}\phi = 0 \\ \sigma^{22} \equiv d_{11}\phi = 0 \end{cases} \begin{bmatrix} 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \end{bmatrix}$$

which is involutive with one equation of class 2, 2 equations of class 1 and it is easy to check that the 2 corresponding first order CC are just the *Cauchy* equations. Of course, the *Airy* function (1 term) has absolutely nothing to do with the perturbation of the metric (3 terms). With more details, when ω is the Euclidean metric, we may consider the only component:

$$\begin{aligned} \text{tr}(R) &= (d_{11} + d_{22})(\Omega_{11} + \Omega_{22}) - (d_{11}\Omega_{11} + 2d_{12}\Omega_{12} + d_{22}\Omega_{22}) \\ &= d_{22}\Omega_{11} + d_{11}\Omega_{22} - 2d_{12}\Omega_{12} \end{aligned}$$

Multiplying by the *Airy* function ϕ and integrating by parts, we discover that:

$$\text{Airy} = \text{ad}(\text{Riemann}) \Leftrightarrow \text{Riemann} = \text{ad}(\text{Airy})$$

in the following differential sequences:

$$\begin{array}{ccccccc} & & \text{Killing} & \text{Riemann} & & & \\ & & 2 & \rightarrow 3 & \rightarrow 1 & \rightarrow 0 & \\ & & & 1 & & 2 & \\ & & & & & & \\ & & & & \text{Cauchy} & \text{Airy} & \\ & & & & 0 \leftarrow 2 & \leftarrow 3 & \leftarrow 1 \\ & & & & & 1 & 2 \end{array}$$

- $n = 3$: It is more delicate to parametrize the 3 PD equations:

$$\begin{aligned} d_1\sigma^{11} + d_2\sigma^{12} + d_3\sigma^{13} &= 0, \\ d_1\sigma^{21} + d_2\sigma^{22} + d_3\sigma^{23} &= 0, \\ d_1\sigma^{31} + d_2\sigma^{32} + d_3\sigma^{33} &= 0 \end{aligned}$$

A direct computational approach has been provided by Eugenio Beltrami in 1892 [36] [37], James Clerk Maxwell in 1870 [38] and Giacinto Morera in 1892 [37] [39] by introducing the 6 *stress functions* $\phi_{ij} = \phi_{ji}$ in the *Beltrami parametrization*. The corresponding system:

$$\left\{ \begin{array}{l} \sigma^{11} \equiv d_{33}\phi_{22} + d_{22}\phi_{33} - 2d_{23}\phi_{23} = 0 \\ -\sigma^{12} \equiv d_{33}\phi_{12} + d_{12}\phi_{33} - d_{13}\phi_{23} - d_{23}\phi_{13} = 0 \\ \sigma^{22} \equiv d_{33}\phi_{11} + d_{11}\phi_{33} - 2d_{13}\phi_{13} = 0 \\ \sigma^{13} \equiv d_{23}\phi_{12} + d_{12}\phi_{23} - d_{22}\phi_{13} - d_{13}\phi_{22} = 0 \\ -\sigma^{23} \equiv d_{23}\phi_{11} + d_{11}\phi_{23} - d_{12}\phi_{13} - d_{13}\phi_{12} = 0 \\ \sigma^{33} \equiv d_{22}\phi_{11} + d_{11}\phi_{22} - 2d_{12}\phi_{12} = 0 \end{array} \right. \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline 1 & 2 & \bullet \\ \hline 1 & 2 & \bullet \\ \hline 1 & 2 & \bullet \\ \hline \end{array}$$

is involutive with 3 equations of class 3, 3 equations of class 2 and no equation of class 1. The three characters are thus

$$\alpha_2^3 = 1 \times 6 - 3 = 3 < \alpha_2^2 = 2 \times 6 - 3 = 9 < \alpha_2^1 = 3 \times 6 - 0 = 18 \quad \text{and we have}$$

$$\dim(g_2) = \alpha_2^1 + \alpha_2^2 + \alpha_2^3 = 18 + 9 + 3 = 30$$

$$= \dim(S_2T^* \otimes S_2T^*) - \dim(S_2T^*) = 6 \times 6 - 6 \quad [1]. \text{ The 3 CC are describing}$$

the stress equations which admit therefore a parametrization... but without any geometric framework, in particular without any possibility to imagine that the above second order operator is *nothing else but* the *formal adjoint* of the *Riemann operator*, namely the (linearized) *Riemann* tensor with $n^2(n^2 - 1)/2 = 6$ independent components when $n = 3$ [31]. Breaking the canonical form of the six equations which is associated with the Janet tabular, we may rewrite the Bel-

trami parametrization of the *Cauchy* stress equations as follows, after exchanging the third row with the fourth row, keeping the ordering $\{(11) < (12) < (13) < (22) < (23) < (33)\}$:

$$\begin{pmatrix} d_1 & d_2 & d_3 & 0 & 0 & 0 \\ 0 & d_1 & 0 & d_2 & d_3 & 0 \\ 0 & 0 & d_1 & 0 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & d_{33} & -2d_{23} & d_{22} \\ 0 & -d_{33} & d_{23} & 0 & d_{13} & -d_{12} \\ 0 & d_{23} & -d_{22} & -d_{13} & d_{12} & 0 \\ d_{33} & 0 & -2d_{13} & 0 & 0 & d_{11} \\ -d_{23} & d_{13} & d_{12} & 0 & -d_{11} & 0 \\ d_{22} & -2d_{12} & 0 & d_{11} & 0 & 0 \end{pmatrix} \equiv 0$$

as an identity where 0 on the right denotes the zero operator. However, if Ω is a perturbation of the metric ω , the standard implicit summation used in continuum mechanics is, when $n = 3$:

$$\begin{aligned} \sigma^{ij} \Omega_{ij} &= \sigma^{11} \Omega_{11} + 2\sigma^{12} \Omega_{12} + 2\sigma^{13} \Omega_{13} + \sigma^{22} \Omega_{22} + 2\sigma^{23} \Omega_{23} + \sigma^{33} \Omega_{33} \\ &= \Omega_{22} d_{33} \phi_{11} + \Omega_{33} d_{22} \phi_{11} - 2\Omega_{23} d_{23} \phi_{11} + \dots \\ &\quad + \Omega_{23} d_{13} \phi_{12} + \Omega_{13} d_{23} \phi_{12} - \Omega_{12} d_{33} \phi_{12} - \Omega_{33} d_{12} \phi_{12} + \dots \end{aligned}$$

because *the stress tensor density σ is supposed to be symmetric*. Integrating by parts in order to construct the adjoint operator, we get:

$$\begin{aligned} \phi_{11} &\rightarrow d_{33} \Omega_{22} + d_{22} \Omega_{33} - 2d_{23} \Omega_{23} \\ \phi_{12} &\rightarrow d_{13} \Omega_{23} + d_{23} \Omega_{13} - d_{33} \Omega_{12} - d_{12} \Omega_{33} \end{aligned}$$

and so on, obtaining therefore the striking identification:

$$Riemann = ad(Beltrami) \Leftrightarrow Beltrami = ad(Riemann)$$

between the (linearized) *Riemann* tensor and the Beltrami parametrization.

Taking into account the factor 2 involved by multiplying the second, third and fifth row by 2, we get the new 6×6 operator matrix with rank 3:

$$\begin{pmatrix} 0 & 0 & 0 & d_{33} & -2d_{23} & d_{22} \\ 0 & -2d_{33} & 2d_{23} & 0 & 2d_{13} & -2d_{12} \\ 0 & 2d_{23} & -2d_{22} & -2d_{13} & 2d_{12} & 0 \\ d_{33} & 0 & -2d_{13} & 0 & 0 & d_{11} \\ -2d_{23} & 2d_{13} & 2d_{12} & 0 & -2d_{11} & 0 \\ d_{22} & -2d_{12} & 0 & d_{11} & 0 & 0 \end{pmatrix}$$

clearly providing a self-adjoint operator.

Surprisingly, the Maxwell parametrization is obtained by keeping $\phi_{11} = A, \phi_{22} = B, \phi_{33} = C$ while setting $\phi_{12} = \phi_{23} = \phi_{31} = 0$ in order to obtain the system:

$$\begin{cases} \sigma^{11} \equiv d_{33}B + d_{22}C = 0 & \boxed{1 \ 2 \ 3} \\ \sigma^{22} \equiv d_{33}A + d_{11}C = 0 & \boxed{1 \ 2 \ 3} \\ -\sigma^{23} \equiv d_{23}A = 0 & \boxed{1 \ 2 \ \bullet} \\ \sigma^{33} \equiv d_{22}A + d_{11}B = 0 & \boxed{1 \ 2 \ \bullet} \\ -\sigma^{13} \equiv d_{13}B = 0 & \boxed{1 \ \bullet \ \bullet} \\ -\sigma^{12} \equiv d_{12}C = 0 & \boxed{1 \ \bullet \ \bullet} \end{cases}$$

However, *this system may not be involutive* and no CC can be found “*a priori*” because the coordinate system is surely not δ -regular. Indeed, effecting the linear change of coordinates $\bar{x}^1 = x^1, \bar{x}^2 = x^2, \bar{x}^3 = x^3 + x^2 + x^1$ and taking out the bar for simplicity, we obtain the new involutive system:

$$\begin{cases} d_{33}C + d_{13}C + d_{23}C + d_{12}C = 0 & 1 & 2 & 3 \\ d_{33}B + d_{13}B = 0 & 1 & 2 & 3 \\ d_{33}A + d_{23}A = 0 & 1 & 2 & 3 \\ d_{23}C + d_{22}C - d_{13}C - d_{13}B - d_{12}C = 0 & 1 & 2 & \bullet \\ d_{23}A - d_{22}C + d_{13}B + 2d_{12}C - d_{11}C = 0 & 1 & 2 & \bullet \\ d_{22}A + d_{22}C - 2d_{12}C + d_{11}C + d_{11}B = 0 & 1 & 2 & \bullet \end{cases}$$

and it is easy to check that the 3 CC obtained just amount to the desired 3 stress equations when coming back to the original system of coordinates. However, the three characters are different as we have now

$\alpha_2^3 = 3 - 3 = 0 < \alpha_2^2 = 2 \times 3 - 3 = 3 < \alpha_2^1 = 3 \times 3 - 0 = 9$ with sum equal to $\dim(g_2) = 6 \times 3 - 6 = 18 - 6 = 12$. We have thus a minimum parametrization that cannot be parametrized again.

Again, *if there is a geometrical background, this change of local coordinates is hiding it totally*. Moreover, we notice that the stress functions kept in the procedure are just the ones on which d_{33} is acting. The reason for such an apparently technical choice is related to very general deep arguments in the theory of differential modules that will only be explained at the end of the paper.

The Morera parametrization is obtained similarly by keeping now $\phi_{23} = L, \phi_{13} = M, \phi_{12} = N$ while setting $\phi_{11} = \phi_{22} = \phi_{33} = 0$, namely:

$$\begin{cases} d_{23}L = 0 \\ d_{33}N - d_{13}L - d_{23}M = 0 \\ d_{13}M = 0 \\ d_{22}M - d_{23}N - d_{12}L = 0 \\ d_{11}L - d_{12}M - d_{13}N = 0 \\ d_{12}N = 0 \end{cases}$$

Using now the same change of coordinates as the one already done for the Maxwell parametrization, we obtain the following system with 3 equations of (full) class 3 and 3 equations of class 2 in the Pommaret basis corresponding to the Janet tabular:

$$\begin{cases} d_{33}N + d_{23}N + d_{13}N + d_{12}N = 0 & 1 & 2 & 3 \\ d_{33}M + d_{13}M = 0 & 1 & 2 & 3 \\ d_{33}L + d_{23}L = 0 & 1 & 2 & 3 \\ (d_{23}N + d_{23}M - d_{23}L) + (d_{13}N - d_{13}M + d_{13}L) + d_{12}N = 0 & 1 & 2 & \bullet \\ 2d_{23}M + (d_{13}N - d_{13}M - d_{13}L) + d_{12}M - d_{11}L = 0 & 1 & 2 & \bullet \\ d_{22}M + (d_{12}N - d_{12}M - d_{12}L) + d_{11}L = 0 & 1 & 2 & \bullet \end{cases}$$

After elementary but tedious computations (*that could not be avoided!*), one can prove that the 3 CC corresponding to the 3 dots are effectively satisfied and

that they correspond to the 3 *Cauchy* stress equations which are therefore parametrized. The parametrization is thus provided by an involutive operator defining a torsion module because the character α_2^3 is vanishing in δ -regular coordinates, just like before for the Maxwell parametrization. *We have thus another minimum parametrization that cannot be parametrized again.* Of course, such a result could not have been understood by Beltrami in 1892 because the work of Cartan could not be adapted easily as it is using the language of exterior forms and the work of Janet only appeared in 1920 with no explicit reference to involution because only Janet bases are used [8] while the Pommaret bases have only been introduced in 1978 [1].

On a purely computational level, we may also keep only $\{\phi_1, \phi_{12}, \phi_{22}\}$ and obtain the different involutive system with the same characters and, *in particular*, $\alpha_2^3 = 0$:

$$\begin{cases} \sigma^{11} \equiv d_{33}\phi_{22} = 0 & 1 & 2 & 3 \\ -\sigma^{12} \equiv d_{33}\phi_{12} = 0 & 1 & 2 & 3 \\ \sigma^{22} \equiv d_{33}\phi_{11} = 0 & 1 & 2 & 3 \\ \sigma^{13} \equiv d_{23}\phi_{12} - d_{13}\phi_{22} = 0 & 1 & 2 & \bullet \\ -\sigma^{23} \equiv d_{23}\phi_{11} - d_{13}\phi_{12} = 0 & 1 & 2 & \bullet \\ \sigma^{33} \equiv d_{22}\phi_{11} + d_{11}\phi_{22} - 2d_{12}\phi_{12} = 0 & 1 & 2 & \bullet \end{cases}$$

So far, we have thus obtained three explicit local minimum parametrizations of the *Cauchy* stress equations with $n(n-1)/2 = 3$ stress potentials but there may be others [21].

- $n = 4$: It just remains to explain the relation of the previous results with *Einstein* equations. The first surprising link is provided by the following technical proposition:

PROPOSITION 3.1: The Beltrami parametrization is just described by the *Einstein* operator when $n = 3$. The same confusion existing between the *Bianchi* operator and the *Cauchy* operator has been made by both *Einstein* and Beltrami because the *Einstein* operator and the Beltrami operator are self-adjoint in arbitrary dimension $n \geq 3$, contrary to the *Ricci* operator.

Proof: The number of components of the *Riemann* tensor is $\dim(F_1) = n^2(n^2 - 1)/12$. We have the combinatorial formula $n^2(n^2 - 1)/12 - n(n+1)/2 = n(n+1)(n+2)(n-3)/12$ expressing that the number of components of the *Riemann* tensor is always greater or equal to the number of components of the *Ricci* tensor whenever $n > 2$. Also, we have shown in many books [1] [2] [3] [4] [15] [19] or papers [10] [21] [29] [30] that the number of *Bianchi* identities is equal to $n^2(n^2 - 1)(n-2)/24$, that is 3 when $n = 3$ and 20 when $n = 4$. Of course, it is well known that the *div* operator, induced as CC of the *Einstein* operator, has n components in arbitrary dimension $n \geq 3$.

Accordingly, when $n = 3$ we have $n^2(n^2 - 1)/12 = n(n+1)/2 = 6$ and it only remains to prove that the *Einstein* operator reduces to the Beltrami operator *and not just to the Ricci operator*. The following formulas can be found in any textbook on general relativity:

Hence the difference can only be seen when $\omega_{i \neq j} = 0$. In our situation with $n = 3$ and the Euclidean metric for simplicity, we have:

$$\begin{aligned}
 2R_{12} &= 2E_{12} = (d_{11} + d_{22} + d_{33})\Omega_{12} + d_{12}(\Omega_{11} + \Omega_{22} + \Omega_{33}) \\
 &\quad - (d_{11}\Omega_{12} + d_{12}\Omega_{22} + d_{13}\Omega_{23}) - (d_{12}\Omega_{11} + d_{22}\Omega_{12} + d_{23}\Omega_{13}) \\
 &= d_{33}\Omega_{12} + d_{12}\Omega_{33} - d_{13}\Omega_{23} - d_{23}\Omega_{13} \\
 2R_{11} &= (d_{11} + d_{22} + d_{33})\Omega_{11} + d_{11}(\Omega_{11} + \Omega_{22} + \Omega_{33}) \\
 &\quad - 2(d_{11}\Omega_{11} + d_{12}\Omega_{12} + d_{13}\Omega_{13}) \\
 &= (d_{22} + d_{33})\Omega_{11} + d_{11}(\Omega_{22} + \Omega_{33}) - 2(d_{12}\Omega_{12} + d_{13}\Omega_{13}) \\
 tr(R) &= (d_{11}\Omega_{22} + d_{11}\Omega_{33} + d_{22}\Omega_{11} + d_{22}\Omega_{33} + d_{33}\Omega_{11} + d_{33}\Omega_{22}) \\
 &\quad - 2(d_{12}\Omega_{12} + d_{13}\Omega_{13} + d_{23}\Omega_{23}) \\
 -2E_{11} &= d_{22}\Omega_{33} + d_{33}\Omega_{22} - 2d_{23}\Omega_{23}
 \end{aligned}$$

In the light of modern differential geometry, comparing these results with the works of both Maxwell, Morera, Beltrami and *Einstein*, it becomes clear that they have been confusing the *div* operator induced from the *Bianchi* operator with the *Cauchy* operator. However, it is also clear that they both obtained a possibility to parametrize the *Cauchy* operator by means of 3 arbitrary potential like functions in the case of Maxwell and Morera, 6 in the case of Beltrami who explains the previous choices, and 10 in the case of *Einstein*. Of course, as they were ignoring that the *Einstein* operator was self-adjoint whenever $n \geq 3$, they did not notice that we have $Cauchy = ad(Killing)$ and they were unable to compare their results with the *Airy* operator found as early as in 1870 for the same mechanical purpose when $n = 2$. To speak in a rough way, the situation is similar to what could happen in the study of contact structures if one should confuse \mathcal{D}_{-1} with \mathcal{D}_1 [29]. Finally, using Theorem 2.1 or Proposition 2.2, we can choose a differential transcendence basis with $n(n-1)/2$ potentials that can be indexed by $\phi_{ij} = \phi_{ji}$ with $i < j$ or $1 \leq i, j \leq n-1$ or even $2 \leq i, j \leq n$ when the dimension $n \geq 2$ is arbitrary (see [2] or [40] for more details on differential algebra).

Q.E.D.

REMARK 3.2: In the opinion of the author of this paper who is not a historian of sciences but a specialist of mathematical physics interested in the analogy existing between *electromagnetism* (EM), *elasticity* (EL) and *gravitation* (GR) by using the conformal group of space-time (see [3] [10] [28] [29] [30] [41] [42] [43] [44] for related works), it is difficult to imagine that *Einstein* could not have been aware of the works of Maxwell and Beltrami on the foundations of EL and tensor calculus. Indeed, not only they were quite famous when he started his research work but it must also be noticed that the Mach-Lippmann analogy [45] [46] [47] [48] was introduced at the same time (see [3] and [49] for more details on the field-matter couplings and the phenomenological law discovered by... Maxwell too). The main idea is that classical variational calculus using a Lagran-

gian formalism must be only considered as the basic scheme of a more general and powerful “*duality theory*” that only depends on new purely mathematical tools, namely “*group theory*” and “*differential homological algebra*” (see [4] or [15] for the theory and [21] for the applications).

The two following crucial results, still neither known nor acknowledged today, are provided by the next proposition and corresponding corollary [40]:

PROPOSITION 3.3: The *Cauchy* operator can be parametrized by the formal adjoint of the *Ricci* operator (4 terms) and the *Einstein* operator (6 terms) is thus useless. The so-called gravitational waves equations are thus nothing else than the formal adjoint of the linearized *Ricci* operator.

Proof: The *Einstein* operator $\Omega \rightarrow E$ is defined by setting

$$E_{ij} = R_{ij} - \frac{1}{2} \omega_{ij} \text{tr}(R) \quad \text{that we shall write } Einstein = C \circ Ricci \quad \text{where}$$

$C : S_2 T^* \rightarrow S_2 T^*$ is a symmetric matrix only depending on ω , which is invertible whenever $n \geq 3$. *Surprisingly*, we may also introduce the *same* linear transformation $C : \Omega \rightarrow \bar{\Omega} = \Omega - \frac{1}{2} \omega \text{tr}(\Omega)$ and the unknown composite operator $\mathcal{X} : \bar{\Omega} \rightarrow \Omega \rightarrow E$ in such a way that $Einstein = \mathcal{X} \circ C$ where \mathcal{X} is defined by (see [33], 5.1.5 p 134):

$$2E_{ij} = \omega^{rs} d_{rs} \bar{\Omega}_{ij} - \omega^{rs} d_{ri} \bar{\Omega}_{sj} - \omega^{rs} d_{sj} \bar{\Omega}_{ri} + \omega_{ij} \omega^{ru} \omega^{sv} d_{rs} \bar{\Omega}_{uv}$$

Now, introducing the test functions λ^{ij} , we get:

$$\begin{aligned} \lambda^{ij} E_{ij} &= \lambda^{ij} \left(R_{ij} - \frac{1}{2} \omega_{ij} \text{tr}(R) \right) \\ &= \left(\lambda^{ij} - \frac{1}{2} \lambda^{rs} \omega_{rs} \omega^{ij} \right) R_{ij} = \bar{\lambda}^{ij} R_{ij} \end{aligned}$$

Integrating by parts while setting as usual $\square = \omega^{rs} d_{rs}$, we obtain:

$$\left(\square \bar{\lambda}^{rs} + \omega^{rs} d_{ij} \bar{\lambda}^{ij} - \omega^{sj} d_{ij} \bar{\lambda}^{ri} - \omega^{ri} d_{ij} \bar{\lambda}^{sj} \right) \Omega_{rs} = \sigma^{rs} \Omega_{rs} \tag{14}$$

Moreover, suppressing the “bar” for simplicity, we have:

$$\begin{aligned} d_r \sigma^{rs} &= \omega^{ij} d_{rij} \lambda^{rs} + \omega^{rs} d_{rij} \lambda^{ij} - \omega^{sj} d_{rij} \lambda^{ri} - \omega^{ri} d_{rij} \lambda^{sj} \\ &= 0 \end{aligned}$$

As *Einstein* is a self-adjoint operator (contrary to the *Ricci* operator), we have the identities:

$$\begin{aligned} ad(Einstein) &= ad(C) \circ ad(\mathcal{X}) \Rightarrow Einstein = C \circ ad(\mathcal{X}) \\ \Rightarrow ad(\mathcal{X}) &= Ricci \Rightarrow \mathcal{X} = ad(Ricci) \end{aligned}$$

Indeed, $ad(C) = C$ because C is a symmetric matrix and we know that $ad(Einstein) = Einstein$. Accordingly, the operator $ad(Ricci)$ parametrizes the *Cauchy* equations, *without any reference* to the *Einstein* operator which has no mathematical origin, in the sense that it cannot be obtained by any diagram chasing. The three terms after the *Dalembert* operator factorize through the *di-*

vergence operator $d_i \lambda^i$. We may thus add the *differential constraints* $d_i \lambda^i = 0$ *without any reference to a gauge transformation* in order to obtain a (minimum) *relative parametrization* (see [14] and [18] for details and explicit examples). When $n = 4$ we finally obtain the adjoint sequences:

$$\begin{array}{ccccccc} & & \xrightarrow{\text{Killing}} & & \xrightarrow{\text{Ricci}} & & \\ & 4 & \rightarrow & 10 & \rightarrow & 10 & \\ & & \xleftarrow{\text{Cauchy}} & & \xleftarrow{\text{ad(Ricci)}} & & \\ 0 & \leftarrow & 4 & \leftarrow & 10 & \leftarrow & 10 \end{array}$$

without any reference to the *Bianchi* operator and the induced *div* operator.

Finally, using Theorem 2.1 or Proposition 2.2, we may choose a differential transcendence basis made by $\{\lambda^{ij} \mid i < j\}$ or $\{\lambda^{ij} \mid 1 < i, j < n-1\}$ or even $\{\lambda^{ij} \mid 2 < i, j < n\}$ when the dimension $n \geq 2$ is arbitrary (see again [2] or [40] for more details on differential algebra).

Q.E.D.

COROLLARY 3.4: The differential module N defined by the *Ricci* or the *Einstein* operator is not torsion-free and cannot therefore be parametrized. Its torsion submodule is generated by the 10 components of the *Weyl* operator that are *separately* killed by the *Dalembert* operator.

Proof: In order to avoid using extension modules, we present the 5 steps of the *double differential duality test* in this framework:

Step 1: Start with the *Einstein* operator $\mathcal{D}_1 : 10 \xrightarrow{\text{Einstein}} 10$.

Step 2: Consider its formal adjoint: $\text{ad}(\mathcal{D}_1) : 10 \xleftarrow{\text{Einstein}} 10$.

Step 3: Compute the generating CC, namely the *Cauchy* operator:
 $\text{ad}(\mathcal{D}) : 4 \xleftarrow{\text{Cauchy}} 10$.

Step 4: Consider its formal adjoint: $\mathcal{D} = \text{ad}(\text{ad}(\mathcal{D})) : 4 \xrightarrow{\text{Killing}} 10$.

Step 5: Compute the generating CC, namely the *Riemann* operator:
 $\mathcal{D}'_1 : 10 \xrightarrow{\text{Riemann}} 20$.

With a slight abuse of language, we have the direct sum $\text{Riemann} = \text{Ricci} \oplus \text{Weyl}$ with $20 = 10 + 10$. It follows from differential homological algebra that the 10 additional CC in \mathcal{D}'_1 that are *not* in \mathcal{D}_1 , are generating the torsion submodule $t(N)$ of the differential module N defined by the *Einstein* or *Ricci* operator. In general, if K is a differential field with commuting derivations $\partial_1, \dots, \partial_n$, we may consider the ring $D = K[d_1, \dots, d_n] = K[d]$ of differential operators with coefficients in K and it is known that $\text{rk}_D(\mathcal{D}) = \text{rk}_D(\text{ad}(\mathcal{D}))$ for any operator matrix \mathcal{D} with coefficients in K . In the present situation, as the Minkowski metric has coefficients equal to $0, 1, -1$, we may choose the ground differential field to be $K = \mathbb{Q}$. Hence, there exist operators \mathcal{P} and \mathcal{Q} such that we have an identity:

$$\mathcal{P} \circ \text{Weyl} = \mathcal{Q} \circ \text{Ricci}$$

One may also notice that $\text{rk}_D(\text{Einstein}) = \text{rk}_D(\text{Ricci})$ with:

$$rk_D(Einstein) = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2},$$

$$rk_D(Riemann) = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$$

The differential ranks of the *Einstein* and *Riemann* operators are thus equal, but *this is a pure coincidence* because $rk_D(Einstein)$ has only to do with the *div* operator induced by contracting the *Bianchi* operator, while $rk_D(Riemann)$ has only to do with the classical *Killing* operator and the fact that the corresponding differential module is a torsion module because we have a Lie group of transformations having $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ parameters (translations + rotations). Hence, as the *Riemann* operator is a direct sum of the *Weyl* operator and the *Einstein* or *Ricci* operator according to the previous theorem, each component of the *Weyl* operator must be killed by a certain operator whenever the *Einstein* or *Ricci* equations in vacuum are satisfied. *It is not at all evident* that we have $\mathcal{P} = \square$ acting on each component of the *Weyl* operator. A direct tricky computation can be found in ([49], p 206), ([50], exercise 7.7) and ([15], p 95). With more details, we may start from the long exact sequence:

$$0 \rightarrow \Theta \rightarrow 4 \xrightarrow{Killing} 10 \xrightarrow{Riemann} 20 \xrightarrow{Bianchi} 20 \rightarrow 6 \rightarrow 0$$

This resolution of the set of *Killing* vector fields is *not a Janet* sequence because the *Killing* operator is not involutive as it is an operator of finite type with symbol of dimension $n(n-1)/2 = 6$ and one should need one prolongation for getting an involutive operator with vanishing second order symbol. Splitting the *Riemann* operator we get the commutative and exact diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 10 & \rightarrow & 16 & \rightarrow & 6 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow \uparrow & & \downarrow & & \parallel & & \\
 4 & \xrightarrow{Killing} & 10 & \xrightarrow{Riemann} & 20 & \xrightarrow{Bianchi} & 20 & \rightarrow & 6 & \rightarrow & 0 \\
 & & \parallel & & \downarrow \uparrow & & \downarrow & & \downarrow & & \\
 & & 10 & \xrightarrow{Einstein} & 10 & \xrightarrow{div} & 4 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
 \end{array}$$

Passing to the module point of view, we have the long exact sequence:

$$0 \rightarrow D^6 \rightarrow D^{20} \xrightarrow{Bianchi} D^{20} \xrightarrow{Riemann} D^{10} \xrightarrow{Killing} D^4 \rightarrow M \rightarrow 0$$

which is a resolution of the *Killing* differential module $M = coker(Killing)$ and we check that we have indeed the vanishing of the *Euler-Poincaré characteristic* $6 - 20 + 20 - 10 + 4 = 0$. Accordingly, we have

$$N' = coker(Riemann) \simeq im(Killing) \subset D^4 \text{ and thus } N' \text{ is torsion-free with}$$

$rk_D(N') = 4 - 0 = 4 = n$ because $rk_D(M) = 0$.

We have the following commutative and exact diagram where $N = coker(Einstein)$:

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & 0 \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & t(N) \\
 & & & & & & & & & & \downarrow \\
 & & & & 0 & & 0 & & 0 & & \downarrow \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & 0 & \rightarrow & D^4 & \xrightarrow{div} & D^{10} & \xrightarrow{Einstein} & D^{10} & \rightarrow & N & \rightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \parallel & & \downarrow \\
 0 & \rightarrow & D^6 & \rightarrow & D^{20} & \xrightarrow{Bianchi} & D^{20} & \xrightarrow{Riemann} & D^{10} & \rightarrow & N' & \rightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & D^6 & \rightarrow & D^{16} & \rightarrow & D^{10} & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
 \end{array}$$

If L is the kernel of the epimorphism $N \rightarrow N'$, it is a torsion module because $rk_D(L) = rk_D(N) - rk_D(N') = 4 - 4 = 0$. We have thus $L \subseteq t(N)$ in the following commutative and exact diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 0 & \rightarrow & L & \rightarrow & t(N) \\
 & & \downarrow & & \downarrow \\
 0 & \rightarrow & N & = & N & \rightarrow & 0 \\
 & & \downarrow & & \downarrow \\
 & & N' & \rightarrow & N/t(N) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
 \end{array}$$

where $N/t(N)$ is a torsion-free module by definition. A snake chase allows to prove that the cokernel of the monomorphism $L \rightarrow t(N)$ is isomorphic to the kernel of the induced epimorphism $N' \rightarrow N/t(N)$ and must be therefore, at the same time, a torsion module because $rk_D(L) = rk_D(t(N)) = 0$ and a torsion-free module because $N' \subset D^4$, a result leading to a contradiction unless it is zero and thus $L = t(N)$. A snake chase in the previous diagram allows to exhibit the long exact connecting sequence:

$$0 \rightarrow D^6 \rightarrow D^{16} \rightarrow D^{10} \rightarrow t(N) \rightarrow 0$$

It must be noticed that one cannot find canonical morphisms between the classical and conformal resolutions constructed similarly because we recall that, for $n = 4$ (only), the CC of the Weyl operator are of order 2 and not 1 like the Bianchi CC for the Riemann operator (see [37] for a computer algebra check-

ing !). However, it follows from the last theorem that the short exact sequence $0 \rightarrow D^{10} \rightarrow D^{20} \rightarrow D^{10} \rightarrow 0$ splits with $D^{20} \simeq D^{10} \oplus D^{10}$ but the existence of a canonical lift $D^{20} \rightarrow D^{10} \rightarrow 0$ in the above diagram does not allow to split the right column and thus $N \neq N' \oplus t(N)$ as N' is not even free. Hence, one can only say that the space of solutions of *Einstein* equations in vacuum contains the generic solutions of the *Riemann* operator which are parametrized by arbitrary vector fields. As for the torsion elements, we have $t(N) = \text{coker}(D^{16} \rightarrow D^{10})$ and we may thus represent them by the components of the *Weyl* tensor, killed by the Dalembertian. This module interpretation may thus question the proper origin and existence of gravitational waves because the *div* operator on the upper left part of the diagram has *strictly nothing to do* with the *Cauchy* = *ad* (*Killing*) operator which cannot appear *anywhere* in this diagram.

Q.E.D.

COROLLARY 3.5: More generally, when \mathcal{D} is a Lie operator of *finite type*, that is when $[\Theta, \Theta] \subset \Theta$ under the ordinary bracket of vector fields or $[R_q, R_q] \subset R_q$ under the bracket of Lie algebroids and $g_{q+r} = 0$ for r large enough, then the Spencer sequence is locally isomorphic to the tensor product of the Poincaré sequence for the exterior derivative by a finite dimensional Lie algebra. It is thus formally exact both with its adjoint sequence. As it is known that the extension modules do not depend on the resolution used, this is the reason for which not only the *Cauchy* operator can be parametrized but also the *Cosserat couple-stress equations* $\text{ad}(\mathcal{D}_1)$ can be parametrized by $\text{ad}(\mathcal{D}_2)$, a result not evident at all (see [41] and [43] for explicit computations).

REMARK 3.6: A similar situation is well known for the *Cauchy-Riemann* equations when $n = 2$. Indeed, any infinitesimal complex transformation ξ must be solution of the linear first order homogeneous system $\xi_2^2 - \xi_1^1 = 0, \xi_2^1 + \xi_1^2 = 0$ of infinitesimal Lie equations though we obtain $\xi_{11}^1 + \xi_{22}^1 = 0, \xi_{11}^2 + \xi_{22}^2 = 0$, that is ξ^1 and ξ^2 are *separately* killed by the second order *Laplace* operator $\Delta = d_{11} + d_{22}$.

REMARK 3.7: A similar situation is also well known for the wave equations for the EM field F in electromagnetism. Indeed, starting with the first set of *Maxwell* equations $dF = 0$ and using the *Minkowski* constitutive law in vacuum with electric constant ϵ_0 and magnetic constant μ_0 such that $\epsilon_0 \mu_0 c^2 = 1$ for the second set of *Maxwell* equations, a standard tricky differential elimination allows to avoid the *Lorenz* (no “t”) gauge condition for the EM potential and to obtain *directly* $\square F = 0$ (see [15] and [40] for the details).

Using computer algebra or a direct checking with the ordering $11 < 12 < 13 < 22 < 23 < 33$, we obtain:

$$E_{33} = \omega^{44} d_{44} \Omega_{33} + \text{lower terms}$$

$$E_{23} = \omega^{44} d_{44} \Omega_{23} \dots$$

We have therefore the following Janet tabular:

1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	4
1	2	3	•
1	2	3	•
1	2	3	•
1	2	3	•

we are in the position to compute the characters of the *Einstein* operator but a similar procedure could be followed with the *Ricci* operator. We obtain at once:

$$\begin{aligned} \beta_2^4 = 6 &\Rightarrow \alpha_2^4 = (10 \times 1) - 6 = 4 \\ \beta_2^3 = 4 &\Rightarrow \alpha_2^3 = (10 \times 2) - 4 = 16 \\ \beta_2^2 = 0 &\Rightarrow \alpha_2^2 = (10 \times 3) - 0 = 30 \\ \beta_2^1 = 0 &\Rightarrow \alpha_2^1 = (10 \times 4) - 0 = 40 \end{aligned}$$

a result leading to $dim(g_2) = \alpha_2^1 + \alpha_2^2 + \alpha_2^3 + \alpha_2^4 = 90$ and $dim(g_3) = \alpha_2^1 + 2\alpha_2^2 + 3\alpha_2^3 + 4\alpha_2^4 = 164$ along with the long exact sequences:

$$\begin{aligned} 0 \rightarrow g_2 \rightarrow S_2 T^* \otimes S_2 T^* \rightarrow S_2 T^* \rightarrow 0 \\ 0 \rightarrow g_3 \rightarrow S_3 T^* \otimes S_2 T^* \rightarrow T^* \otimes S_2 T^* \rightarrow T^* \rightarrow 0 \end{aligned}$$

Now, we have by definition $div = (d_1, d_2, d_3, d_4)$ and $div \circ Einstein = (0, 0, 0, 0)$.

As the *Einstein* operator is a self-adjoint 10×10 operator matrix up to a change of basis [27], we obtain therefore, with a slight abuse of language, $det(Einstein) = 0$ because:

$$\begin{pmatrix} d_1 & d_2 & d_3 & d_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & d_2 & d_3 & d_4 & 0 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 & 0 & 0 & d_2 & d_3 & d_4 \\ 0 & 0 & 0 & d_1 & 0 & 0 & d_2 & 0 & d_3 & d_4 \end{pmatrix} \begin{pmatrix} E_{11} \\ E_{12} \\ E_{13} \\ E_{14} \\ E_{22} \\ E_{23} \\ E_{24} \\ E_{33} \\ E_{34} \\ E_{44} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \end{pmatrix}$$

a result not evident at first sight that must be compared with the Poincaré situation when $n = 3$:

$$(d_1 \ d_2 \ d_3) \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} = (0 \ 0 \ 0)$$

4. Symbol Sequences

A way to study the formal properties of a linear differential operator $\mathcal{D} : E \rightarrow F = F_0$ of order q between the sections of two vector bundles over a manifold X with dimension n is to consider the symbol of \mathcal{D} at the covector $\chi \in T^*(X)$, namely the map $\sigma_\chi(\mathcal{D})$ obtained from the leading terms of order q . With more details, if \mathcal{D} is described by the operator matrix $(a_k^{\mu} d_\mu)$ with $0 \leq |\mu| \leq q$ and $\chi = \chi_i dx^i \in T^*$, then $\sigma_\chi(\mathcal{D}) : E \rightarrow F_0$ is described by the polynomial matrix $(a_k^{\mu} \chi_\mu)$ with $|\mu| = q$. When \mathcal{D} is formally integrable, then χ is said to be *characteristic* if $\sigma_\chi(\mathcal{D})$ fails to be injective and we may introduce the characteristic set V as usual. When \mathcal{D} is involutive, it is known after the work of M. Janet (1920) that one can construct *first order* operators $\mathcal{D}_1, \dots, \mathcal{D}_n$, each one generating the compatibility conditions (CC) of the preceding one. However, the symbol maps $\sigma_\chi(\mathcal{D}_i) : F_{i-1} \rightarrow F_i$ for $i = 1, \dots, n$ may not provide an exact sequence and any non-zero covector is characteristic for these operators. A comparison of the operators involved in a control system with $n = 1$, in the study of the contact transformations with $n = 3$ and of the linearized *Einstein* equations with $n = 4$ proves that the preceding definition must be conveniently refined by saying that a covector is *static* if the matrix $\sigma_\chi(\mathcal{D})$ fails to have its maximum generic rank and the previous symbol sequence is exact otherwise and we may introduce the *static set* W as in [1] [4] [5]. What we have done with the Janet sequence can also be done with the Spencer sequence that only involves first order operators induced by the Spencer operator. A close link with differential homological algebra and the corresponding differential extension modules can also be established and illustrated [14]. These new results are adding doubts to the ones we have already expressed on the origin and existence of gravitational waves as follows.

First of all, we must distinguish two cases:

- $\max_\chi rk(\sigma_\chi(\mathcal{D})) < m \Leftrightarrow \alpha_q^n > 0$: Any covector is characteristic and we have $W \subset V = T^*$ with a slight abuse of language. This is the situation for each of the operators $\mathcal{D}_1, \dots, \mathcal{D}_n$.
- $\max_\chi rk(\sigma_{chi}(\mathcal{D})) = m \Leftrightarrow \alpha_q^n = 0$: The symbol map $\sigma_\chi(\mathcal{D})$ fails to be injective if and only if all the $m \times m$ submatrices have vanishing determinants and we have $W = V \subset T^*$.

PROPOSITION 4.1: If \mathcal{D} is involutive, the symbol sequence of the Janet sequence:

$$0 \rightarrow \ker(\sigma_\chi(\mathcal{D})) \rightarrow E \xrightarrow{\sigma_\chi(\mathcal{D})} F_0 \xrightarrow{\sigma_\chi(\mathcal{D}_1)} \dots \xrightarrow{\sigma_\chi(\mathcal{D}_n)} F_n \rightarrow 0 \tag{15}$$

is exact if and only if $\chi \notin W$.

Proof: Using the Janet tabulars for \mathcal{D} and \mathcal{D}_1 with $dim(E) = m$, we get:

$$\begin{aligned} dim(im(\sigma_\chi(\mathcal{D}))) &= rk(\sigma_\chi(\mathcal{D})) = m - \alpha_q^n(\mathcal{D}) = \beta_q^n(\mathcal{D}) \\ &= \alpha_1^n(\mathcal{D}_1) = dim(ker(\sigma_\chi(\mathcal{D}_1))) \end{aligned}$$

and so on.

Q.E.D.

When $n = 3$, coming back to the Beltrami operator, that is the *Einstein* operator which is known to be involutive, we may change the local coordinates as we did for the Maxwell and Morera parametrizations. With

$\bar{\chi} = d\bar{x}^3 = dx^1 + dx^2 + dx^3 = (1,1,1)$, then $\ker(\sigma_{\bar{\chi}}(\mathcal{D}))$ is defined by:

$$\begin{cases} \phi_{22} + \phi_{33} - 2\phi_{23} = 0 \\ \phi_{12} + \phi_{33} - \phi_{23} - \phi_{13} = 0 \\ \phi_{11} + \phi_{33} - 2\phi_{13} = 0 \\ \phi_{12} + \phi_{23} - \phi_{13} - \phi_{22} = 0 \\ \phi_{11} + \phi_{23} - \phi_{13} - \phi_{12} = 0 \\ \phi_{11} + \phi_{22} - 2\phi_{12} = 0 \end{cases}$$

Multiplying the fourth equation by 2 and substituting the first, the third and the sixth, we get:

$$(\phi_{11} + \phi_{22}) + (\phi_{22} + \phi_{33}) - (\phi_{11} + \phi_{33}) - 2\phi_{22} = 0$$

whenever $2\phi_{23} = \phi_{22} + \phi_{33}$, $2\phi_{13} = \phi_{11} + \phi_{33}$, $2\phi_{12} = \phi_{11} + \phi_{22}$. This unexpected result, which has only to do with the use of δ -regular local coordinates, is proving that the choice of the stress potentials has *strictly nothing to do* with the complementary cancellations respectively adopted by Maxwell or Morera according to Beltrami because, in both cases, $\bar{\chi} \notin V = W$ but $\chi = dx^3 = (0,0,1) \in V = W$. This result can be extended to an arbitrary dimension.

5. Conclusion

After teaching elasticity for 25 years to high level students in some of the best french civil engineering schools, the author of this paper still keeps in mind one of the most fascinating exercises that he has set up. The purpose was to explain why a dam made with concrete is *always* vertical on the water-side with a slope of about 42 degrees on the other free side in order to obtain a minimum cost and the auto-stability under cracking of the surface under water (see the introduction of [5] for more details). Surprisingly, the main tool involved is the approximate computation of the *Airy* function inside the dam. The author discovered at that time that no one of the other teachers did know that the *Airy* parametrization is nothing else than the adjoint of the linearized *Riemann* operator used as generating CC for the deformation tensor by any engineer. Being involved in General Relativity (GR) at that time, it took him 25 years (1970-1995) to prove that the *Einstein* equations could not be parametrized [25] [26] [27]. However, nobody is a prophet in his own country and it is only recently that he discovered that GR could be considered as a way to parametrize the *Cauchy* operator. It follows that *exactly the same confusion* has been done by Maxwell, Morera, Beltrami and *Einstein* because, in all these cases, the operator considered is self-adjoint. As a byproduct, the variational formalism cannot allow to discover it as no engineer could have had in mind to confuse the deformation tensor with its CC in the

Lagrangian used for finite elements computations. It is thus an open historical problem to know whether *Einstein* knew any one of the previous works done as all these researchers were quite famous at the time he was active. In our opinion at least, the comparison of the various parametrizations described in this paper needs no comment as we have only presented *facts, just facts*.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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