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Bound States of a System of Two Fermions on Invariant Subspace

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Abstract

We consider a Hamiltonian of a system of two fermions on a three-dimensional lattice $\mathbb{Z}^3$ with special potential $\hat{v}$. The corresponding Shrödinger operator $H(\mathbf{k})$ of the system has an invariant subspace $L_{123}^{-}(\mathbb{T}^3)$, where we study the eigenvalues and eigenfunctions of its restriction $H_{123}^{-}(\mathbf{k})$. Moreover, there are shown that $H_{123}^{-}(k_1,k_2,\pi)$ has also infinitely many invariant subspaces $\mathfrak{R}_{123}^{-}(n), n\in\mathbb{N}$, where the eigenvalues and eigenfunctions of eigenvalue problem

$$H(k_1,k_2,\pi)f = zf,\quad f\in\mathfrak{R}_{123}^{-}(n)$$

are explicitly found.

Keywords

Hamiltonian, Fermion, Bound State, Shrödinger Operator, Invariant Subspace, Total Quasi-Momentum, Eigenvalue, Birman-Schwinger Principle

1. Introduction

The nature of bound states of two-particle cluster operators for small parameter values was first studied in detail by Minlos and Mamatov [1] and then in a more general setting by Minlos and Mogilner [2]. In [3], Howland showed that the Rellich theorem on perturbations of eigenvalues does not extend to the resonance theory. Studying bound states of a two-particle system Hamiltonian H on the d-dimensional lattice $\mathbb{Z}^d$ reduces to studying [2] [4] [5] [6] [7] the eigenvalues of a family of Shrödinger operators $H(\mathbf{k}), \mathbf{k}\in\mathbb{T}^d$, where $\mathbf{k}$ is the total quasi-momentum of a system. Moreover, eigenfunctions of $H(\mathbf{k})$ are interpreted as bound states of the Hamiltonian H, and eigenvalues, as the bound state energies. The bound states of H of a system of two fermions on a one-dimensional lattice were studied in [4], a system of two bosons on a two-dimensional lattice was studied in [6], and perturbations of the eigenvalues of a two-particle Shrödinger operator on a one-dimensional lattice were studied in [8]. The finiteness of the number of eigenvalues of Shrödinger operator on a lattice was studied in the works [7] [9].

The discrete spectrum of the two-particle continuous Shrödinger operator

$$h_\lambda = -\Delta + \lambda V$$

was studied by many authors, with the conditions for the potential V formulated in its coordinate representation. The condition for the finiteness of the set of negative elements of the spectrum and the absence of positive eigenvalues of h_λ can be found in [10]. If $V \le 0$, then the number of negative eigenvalues $N(\lambda)$ is a nondecreasing function of $\lambda \in (0,\infty)$, and each eigenvalue $z_n(\lambda)$ decreases on the half-axis $(0,\infty)$. It is known that when the coupling constant λ decreases, the bound state energies of h_λ tend to the boundary of the continuous spectrum (see [10]) and for some finite λ are on the boundary. Two questions then arise: Does a bound or virtual state correspond to such a threshold state (*i.e.*, is the corresponding wave function square-integrable)? And where do the bound states "disappear to" as λ decreases further? The study of the first question was the subject in [11] [12]. Regarding the second question, it turns out that the bound state disappears by being absorbed into the continuous spectrum and becomes a resonance [5].

Here, we consider bound states of the Hamiltonian $\hat{H}$ (see (1)) of a system of two fermions on the three-dimensional lattice $\mathbb{Z}^3$ with the special potential $\hat{v}$ (see (5)). In other words, we study the discrete spectrum of a family of the Shrödinger operators $H(\mathbf{k})$, $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{T}^3$, (see (3)) corresponding to $\hat{H}$ in the invariant subspace $L_{123}^-(\mathbb{T}^3)$.

Restriction of the operator $H(\mathbf{k})$ in the invariant subspace $L_{123}^-(\mathbb{T}^3)$ is denoted by $H_{123}^-(\mathbf{k})$.

In the case $\mathbf{k} = \vec{\pi} := (\pi,\pi,\pi)$, the operator $H(\vec{\pi})$ has an infinite number of eigenvalues of the form $6 - \hat{v}(\mathbf{n}), \mathbf{n} \in \mathbb{Z}^3$ and the essential spectrum consists of the single point 6. Here, the potential $\hat{v}$ is defined by (5) and $\overline{v} : \mathbb{N} \to \mathbb{R}$ is a decreasing function on $\mathbb{N}$ and $\overline{v} \in \ell_2(\mathbb{N})$. These eigenvalues $z_n(\vec{\pi}) = 6 - \overline{v}(n), n \in \mathbb{N}$ are arranged in ascending order, $z_1(\vec{\pi}) < \cdots < z_n(\vec{\pi}) < \cdots$, and the smallest eigenvalue $z_1(\vec{\pi}) = 6 - \overline{v}(1)$ is threefold, $z_2(\vec{\pi}) = 6 - \overline{v}(2)$ is sevenfold, and the other eigenvalues $z_n(\vec{\pi}) = 6 - \overline{v}(n), n \ge 3$ are ninefold. All ninefold eigenvalues $z_n(\vec{\pi}) = 6 - \overline{v}(n), n \ge 3$ of the operator $H(\vec{\pi})$ are simple eigenvalues for the operator $H_{123}^-(\vec{\pi})$.

Further, we investigate eigenvalues and eigenfunctions of the restriction operator $H_{123}^-(\mathbf{k})$.

In the case $\mathbf{k} = (k_1, k_2, \pi)$ the corresponding operator $H_{123}^-(k_1, k_2, \pi)$ has infinitely many invariant subspaces $\mathfrak{R}_{123}^-(n) := L_2^-(\mathbb{T}) \otimes L_2^-(\mathbb{T}) \otimes L^-(n), n \in \mathbb{N}$. It is proved that the restriction $H^{-}_{123n}(k_1,k_2,\pi)$ of the operator $H^{-}_{123}(k_1,k_2,\pi)$ in the invariant subspace $\mathfrak{R}^{-}_{123}(n)$ has no more than one eigenvalue. If exists, it can be calculated explicitly. For every $(k_1,k_2)\in(-\pi,\pi)^2$ the operator $H^{-}_{123}(k_1,k_2,\pi)$ has only a finite number of eigenvalues.

For any perturbation $\beta>0$, the essential spectrum $\{6\}$ of $H(\vec{\pi})$ becomes the essential spectrum $\sigma_{ess}(H(\pi-2\beta,\pi,\pi))=[6-2\sin\beta,6+2\sin\beta]$. If the potential $\hat{v}$ is of the form (5), the Shrödinger equation $H^{-}_{123}(\pi-2\beta,\pi,\pi)f=zf$, $f\in\mathfrak{R}^{-}_{123}(n)$ can be exactly solved (see Theorem 1).

The Shrödinger equations $H(\pi-2\beta,\pi,\pi)f=zf$ and $H(\pi-2\beta,\pi-2\beta,\pi)f=zf$, $f\in\mathfrak{R}^{-}_{123}(n)$ with small β are solved by using methods invariant subspaces and operator theory.

2. Description of the Hamiltonian and Expansion in a Direct Integral

The free Hamiltonian $\hat{H}_0$ of a system of two fermions on a three-dimensional lattice $\mathbb{Z}^3$ usually corresponds to a bounded self-adjoint operator acting in the Hilbert space $\ell_2^{as}(\mathbb{Z}^3\times\mathbb{Z}^3):=\{f\in\ell_2(\mathbb{Z}^3\times\mathbb{Z}^3): f(\mathbf{x},\mathbf{y})=-f(\mathbf{y},\mathbf{x})\}$ by the formula

$$\hat{H}_0=-\frac{1}{2m}\Delta_1-\frac{1}{2m}\Delta_2.$$

Here, m is the fermion mass, which we assume to be equal to unity in what follows, $\Delta_1=\Delta\otimes I$ and $\Delta_2=I\otimes\Delta$, where I is the identity operator, and the lattice Laplacian Δ is a difference operator that describes a translation of a particle from a side to a neighboring side,

$$(\Delta\hat{\psi})(\mathbf{x})=\sum_{j=1}^{3}\left[\hat{\psi}(\mathbf{x}+\mathbf{e}_j)+\hat{\psi}(\mathbf{x}-\mathbf{e}_j)-2\hat{\psi}(\mathbf{x})\right],\ \mathbf{x}\in\mathbb{Z}^3,\ \hat{\psi}\in\ell_2(\mathbb{Z}^3),$$

where $\mathbf{e}_1=(1,0,0),\mathbf{e}_2=(0,1,0),\mathbf{e}_3=(0,0,1)$ are unit vectors in $\mathbb{Z}^3$. The total Hamiltonian $\hat{H}$ acts in the Hilbert space $\ell_2^{as}(\mathbb{Z}^3\times\mathbb{Z}^3)$ and is the difference of the free Hamiltonian $\hat{H}_0$ and the interaction potential $\hat{V}_2$ of the two fermions (see [8] [13]):

$$\hat{H}=\hat{H}_0-\hat{V}_2, \tag{1}$$

where

$$(\hat{V}_2\hat{\psi})(\mathbf{x},\mathbf{y})=\hat{v}(\mathbf{x}-\mathbf{y})\hat{\psi}(\mathbf{x},\mathbf{y}),\quad \hat{\psi}\in\ell_2^{as}\left((\mathbb{Z}^3)^2\right):=\ell_2^{as}(\mathbb{Z}^3\times\mathbb{Z}^3).$$

Hereafter, we assume that

$$\hat{v}\in\ell_2(\mathbb{Z}^3)\ \text{ and }\ \hat{v}(\mathbf{x})=\hat{v}(-\mathbf{x})\geq 0\quad\text{for all }\mathbf{x}\in\mathbb{Z}^3. \tag{2}$$

Under this condition, the Hamiltonian $\hat{H}$ is a bounded self-adjoint operator in $\ell_2^{as}\left((\mathbb{Z}^3)^2\right)$.

We pass to momentum representation using the Fourier transform [2] [4] [7]

$$F:\ell_2^{as}(\mathbb{Z}^3\times\mathbb{Z}^3)\to L_2^{as}(\mathbb{T}^3\times\mathbb{T}^3).$$

The Hamiltonian $H = H_0 - V = F\hat{H}F^{-1}$ in the momentum representation commutes with the unitary operators $U_{\mathbf{s}}, \mathbf{s} \in \mathbb{Z}^3$, given by

$$\left(U_{\mathbf{s}} f\right)(\mathbf{k}_1, \mathbf{k}_2) = \exp\left(-i(\mathbf{s}, \mathbf{k}_1 + \mathbf{k}_2)\right) f(\mathbf{k}_1, \mathbf{k}_2), \quad f \in L_2^{as}\left(\mathbb{T}^3 \times \mathbb{T}^3\right).$$

It follows that there exist decompositions of $L_2^{as}\left(\mathbb{T}^3 \times \mathbb{T}^3\right)$ and the operators $U_{\mathbf{s}}$ and H into direct integrals (see [7] [9] and [10])

$$L_2^{as}\left(\mathbb{T}^3 \times \mathbb{T}^3\right) = \int_{\mathbb{T}^3} \oplus L_2^{as}(F_{\mathbf{k}}) \mathrm{d}\mathbf{k}, \; U_{\mathbf{s}} = \int_{\mathbb{T}^3} \oplus U_{\mathbf{s}}(\mathbf{k}) \mathrm{d}\mathbf{k}, \; H = \int_{\mathbb{T}^3} \oplus \tilde{H}(\mathbf{k}) \mathrm{d}\mathbf{k}.$$

Here,

$$F_{\mathbf{k}} = \left\{(\mathbf{k}_1, \mathbf{k}_2) \in \mathbb{T}^3 \times \mathbb{T}^3 : \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}\right\}, \quad \mathbf{k} \in \mathbb{T}^3,$$

and $U_{\mathbf{s}}(\mathbf{k})$ is an operator of multiplication by the function $\exp\left(-i(\mathbf{s}, \mathbf{k})\right)$ in $L_2^{as}(F_{\mathbf{k}})$. The fiber operator $\tilde{H}(\mathbf{k})$ of H also acts in $L_2^{as}(F_{\mathbf{k}})$ and is unitarly equivalent to $H(\mathbf{k}) := H_0(\mathbf{k}) - V$, which is called the Shrödinger operator. This operator acts in the Hilbert space $L_2^{o}\left(\mathbb{T}^3\right) := \left\{f \in L_2\left(\mathbb{T}^3\right) : f(-\mathbf{q}) = -f(\mathbf{q})\right\}$ by the formula

$$\left(H(\mathbf{k}) f\right)(\mathbf{q}) = \varepsilon_{\mathbf{k}}(\mathbf{q}) f(\mathbf{q}) - (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} v(\mathbf{q} - \mathbf{s}) f(\mathbf{s}) \mathrm{d}\mathbf{s}. \tag{3}$$

The unperturbed operator $H_0(\mathbf{k})$ is an operator of multiplication by the function

$$\begin{aligned} \varepsilon_{\mathbf{k}}(\mathbf{q}) &= \varepsilon\left(\frac{\mathbf{k}}{2} + \mathbf{q}\right) + \varepsilon\left(\frac{\mathbf{k}}{2} - \mathbf{q}\right) \\ &= 6 - 2\cos\frac{k_1}{2}\cos q_1 - 2\cos\frac{k_2}{2}\cos q_2 - 2\cos\frac{k_3}{2}\cos q_3. \end{aligned} \tag{4}$$

From (3) and (4), it follows that

$$H(k_1, k_2, k_3) = H(-k_1, k_2, k_3) = H(k_1, -k_2, k_3) = H(k_1, k_2, -k_3),$$

so we can assume $k_1, k_2, k_3 \in [0, \pi]$.

The perturbation operator V is an integral operator in $L_2^{o}\left(\mathbb{T}^3\right)$ with the kernel

$$(2\pi)^{-\frac{3}{2}} v(\mathbf{q} - \mathbf{s}) = (2\pi)^{-\frac{3}{2}} (F\hat{v})(\mathbf{q} - \mathbf{s}),$$

and belongs to the class of Hilbert-Schmidt operators Σ_2.

In this work, we consider the operator $H(\mathbf{k})$ with the potential $\hat{v}$ of the form

$$\hat{v}(\mathbf{n}) = \hat{v}(n_1, n_2, n_3) = \begin{cases} \overline{v}(|\mathbf{n}|), & |n_1| + |n_2| \le 1 \\ 0, & |n_1| + |n_2| \ge 2 \end{cases} \tag{5}$$

where $|\mathbf{n}| = |n_1| + |n_2| + |n_3|$. Supporter is in the cylinder:

$$D = \left\{\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3 : n_3 \in \mathbb{Z}, |n_1| + |n_2| \le 1\right\}.$$

Since for every function $\hat{\psi} \in \ell_2^{as}\left(\left(\mathbb{Z}^3\right)^2\right)$ the equality $\hat{\psi}(\mathbf{x}, \mathbf{x}) = 0, \mathbf{x} \in \mathbb{Z}^3$ holds, then the value of the potential $\hat{v}$ at the origin can be set arbitrary, since it does not affect the result, for simplicity, we assume that $\hat{v}(0) = 0$.

The function $\overline{v}:\mathbb{N}\to\mathbb{R}$ in (5) is decreasing in $\mathbb{N}$ *i.e.*,

$$\overline{v}(1)>\overline{v}(2)>\cdots \tag{6}$$

and belongs to $\ell_2(\mathbb{N})$. The kernel v, of the integral operator *V*, *i.e.*, the Fourier transform $v(\mathbf{p})=(F\hat{v})(\mathbf{p})$, of the potential $\hat{v}$, has the form

$$\begin{aligned} v(\mathbf{p}) &:= (F\hat{v})(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}}\sum_{\mathbf{n}\in\mathbb{Z}^3}\hat{v}(\mathbf{n})\mathrm{e}^{i(\mathbf{n},\mathbf{p})} \\ &= \frac{1}{(2\pi)^{3/2}}\Big[2\overline{v}(1)(\cos p_1+\cos p_2+\cos p_3) \\ &+2\overline{v}(2)(\cos 2p_3+2\cos p_1\cos p_2+2\cos p_1\cos p_3+2\cos p_2\cos p_3) \\ &+2\sum_{n=1}^{\infty}\overline{v}(n+2)\big(\cos(n+2)p_3+2\cos(n+1)p_3(\cos p_1+\cos p_2) \\ &+4\cos p_1\cos p_2\cos np_3\big)\Big]. \end{aligned} \tag{7}$$

Eigenvalues of the operator $H(\mathbf{k})$. We note that the spectra of the operators $H_0(\mathbf{k})$ and V are known. The operator $H_0(\mathbf{k})$ does not have eigenvalues, its spectrum is continuous and coincides with the range of the function $\varepsilon_{\mathbf{k}}$:

$$\sigma(H_0(\mathbf{k}))=[m(\mathbf{k}),M(\mathbf{k})],\ \text{where}\ m(\mathbf{k})=\min_{\mathbf{q}\in\mathbb{T}^3}\varepsilon_{\mathbf{k}}(\mathbf{q}),\ M(\mathbf{k})=\max_{\mathbf{q}\in\mathbb{T}^3}\varepsilon_{\mathbf{k}}(\mathbf{q}).$$

The spectrum of V consists of the set $\{0,\overline{v}(n),n\in\mathbb{N}\}$. Under condition (2), the operator V is a Hilbert-Schmidt operator and is hence compact. By the Weyl theorem [10], the essential spectrum of $H(\mathbf{k})$ coincides with the spectrum of $H_0(\mathbf{k})$:

$$\sigma_{ess}(H(\mathbf{k}))=[m(\mathbf{k}),M(\mathbf{k})].$$

If $\mathbf{k}=\vec{\pi}$, then the spectrum of $H(\vec{\pi})=6I-V$ consists of eigenvalues of the form $6-\overline{v}(n),n\in\mathbb{N}$ and the essential spectrum is $\{6\}$. If $k_j=\pi$ (for some $j\in\{1,2,3\}$), then there exists a potential $\hat{v}$ such that $H(\mathbf{k})$ has an infinite number of eigenvalues outside the continuous spectrum (see [4] [14]).

We recall some notations and known facts. For any self-adjoint operator B acting in a Hilbert space $\mathscr{H}$ without an essential spectrum to the right of $\mu\in\mathbb{R}$, we let $n(\mu,B)$ denote the number of its eigenvalues to the right of μ. We let $N(\mathbf{k},z)$ denote the number of eigenvalues of $H(\mathbf{k})$ to the left of $z\le m(\mathbf{k})$, *i.e.*, $N(\mathbf{k},z)=n(-z,-H(\mathbf{k}))$. The number $N(\mathbf{k},m(\mathbf{k}))$ in fact coincides with the number of eigenvalues outside the continuous spectrum of $H(\mathbf{k})$. It follows from the self-adjointness of $H(\mathbf{k})=H_0(\mathbf{k})-V$ and positivity of V that

$$\sigma(H(\mathbf{k}))\cap(M(\mathbf{k}),\infty)=\varnothing,$$

and hence $\sigma_{disc}(H(\mathbf{k}))\subset(-\infty,m(\mathbf{k}))$. Therefore we seek only eigenvalues z less than $m(\mathbf{k})$.

For any $\mathbf{k}\in\mathbb{T}^3$ and $z<m(\mathbf{k})$, we define the integral operator

$$G(\mathbf{k},z)=V^{\frac{1}{2}}r_0(\mathbf{k},z)V^{\frac{1}{2}},$$

where $r_0(\mathbf{k},z)$ is the resolvent of the unperturbed operator $H_0(\mathbf{k})$. Under condition (2), the operator V is positive, and we let $V^{\frac{1}{2}}$ denote the positive square root of the positive operator V. A solution f of the Schrödinger equation

$$H(\mathbf{k})f = zf$$

and the fixed points φ of $G(\mathbf{k},z)$ are connected by the relations

$$f = r_0(\mathbf{k},z)V^{\frac{1}{2}}\varphi \text{ and } \varphi = V^{\frac{1}{2}}f.$$

The following proposition (the Birman-Schwinger principle) holds [9].

Lemma 1. *The number of eigenvalues of* $H(\mathbf{k})$ *to the left of* $z < m(\mathbf{k})$ *coincides with the number of eigenvalues of* $G(\mathbf{k},z)$ *greater than unity, i.e., the equality*

$$N(\mathbf{k},z) = n(1,G(\mathbf{k},z))$$

holds.

Lemma 2. *If for some* $\mathbf{k} \in \mathbb{T}^3$ *the limit operator*
$\lim_{z\to m(\mathbf{k})-} G(\mathbf{k},z) = G(\mathbf{k},m(\mathbf{k}))$ *exists and is compact, then the equality*

$$N(\mathbf{k},m(\mathbf{k})) = n(1,G(\mathbf{k},m(\mathbf{k}))) \tag{8}$$

holds.

Equality (8) states that the number of eigenvalues of $H(\mathbf{k})$, to the left of $m(\mathbf{k})$ is equal to the number of eigenvalues of $G(\mathbf{k},m(\mathbf{k}))$ greater than unity.

3. Invariant Subspaces of $H(\mathbf{k})$

In this section, we study the invariant subspaces with respect to the operator $H(\mathbf{k})$.

Let $L_2^-(\mathbb{T}) = \{f \in L_2(\mathbb{T}) : f(-p) = -f(p)\}$ be a subspace of the space $L_2(\mathbb{T})$, consisting of odd functions on $\mathbb{T} = [-\pi,\pi]$, and $L_2^+(\mathbb{T}) = \{f \in L_2(\mathbb{T}) : f(-p) = f(p)\}$ be a subspace of $L_2(\mathbb{T})$, consisting of even functions on $\mathbb{T}$. In addition, we use the notation

$$L_{123}^-(\mathbb{T}^3) := L_2^-(\mathbb{T}) \otimes L_2^-(\mathbb{T}) \otimes L_2^-(\mathbb{T}), \quad L_{123}^+(\mathbb{T}^3) := L_2^+(\mathbb{T}) \otimes L_2^+(\mathbb{T}) \otimes L_2^+(\mathbb{T}).$$

Note that $L_{123}^-(\mathbb{T}^3)$ is a subspace of the space $L_2^o(\mathbb{T}^3)$. It is natural to expect the invariance of the subspace $L_{123}^-(\mathbb{T}^3)$ with respect to the operator $H(\mathbf{k})$. It turns out that this subspace is invariant under the operator $H(\mathbf{k})$, *i.e.* the following statement holds.

Lemma 3. *Let the potential* $\hat{v}$ *have the form* (5). *Then the subspace* $L_{123}^-(\mathbb{T}^3)$ *is invariant under the action of* $H(\mathbf{k})$.

Proof. We prove that this subspace is invariant first with respect to $H_0(\mathbf{k})$, and then with respect to V. It follows from representation (4) that the function $\varepsilon_{\mathbf{k}}$ belongs to the subspace $L_{123}^+(\mathbb{T}^3)$, and it follows from the inclusion $f \in L_{123}^-(\mathbb{T}^3)$ that $\varepsilon_{\mathbf{k}} f \in L_{123}^-(\mathbb{T}^3)$. This proves that $L_{123}^-(\mathbb{T}^3)$ is invariant with respect to $H_0(\mathbf{k})$.

Simple calculations show that the function (see (7))

$$(Vf)(p_1,p_2,p_3)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{T}^3} v(p_1-s_1,p_2-s_2,p_3-s_3)f(s_1,s_2,s_3)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3$$

belongs to the subspace $L_{123}^{-}(\mathbb{T}^3)$ for $f\in L_{123}^{-}(\mathbb{T}^3)$. Hence, we prove the invariance of $L_{123}^{-}(\mathbb{T}^3)$ with respect to *V*, and it follows that $L_{123}^{-}(\mathbb{T}^3)$ is invariant with respect to $H(\mathbf{k})=H_0(\mathbf{k})-V$.

$H_{123}^{-}(\mathbf{k})$ denotes the restriction of $H(\mathbf{k})$ to the respective subspace $L_{123}^{-}(\mathbb{T}^3)$. The action of $H_{0(123)}^{-}(\mathbf{k}):=H_0(\mathbf{k})$ is unchanged, the unperturbed operator $H_0(\mathbf{k})$ is an operator of multiplication by the function $\varepsilon_{\mathbf{k}}$. We present the formula for $V_{123}^{-}=V|_{L_{123}^{-}(\mathbb{T}^3)}$ operator *V* acts on the element $f\in L_{123}^{-}(\mathbb{T}^3)$ according to the formula

$$(V_{123}^{-}f)(\mathbf{p})=\frac{1}{\pi^3}\sum_{n=1}^{\infty}\overline{v}(n+2)\int_{\mathbb{T}^3}\sin p_1\sin p_2\sin np_3\sin q_1\sin q_2\sin nq_3 f(\mathbf{q})\mathrm{d}\mathbf{q}.$$

Note that for $\mathbf{k}=\vec{\pi}$, the spectrum of $H(\vec{\pi})=6I-V$ consists only of the eigenvalues $6, 6-\overline{v}(n), n\in\mathbb{N}$ and the essential spectrum $\{6\}$. Under condition (6) the number $z_1(\vec{\pi})=6-\overline{v}(1)$ is a threefold eigenvalue of $H(\vec{\pi})$, with the corresponding eigenfunctions

$$\sin p_1, \sin p_2, \sin p_3,$$

the number $z_2(\vec{\pi})=6-\overline{v}(2)$ is a sevenfold eigenvalue with the corresponding eigenfunctions

$$\sin 2p_3, \cos p_1\sin p_2, \sin p_1\cos p_2, \cos p_1\sin p_3,$$
$$\sin p_1\cos p_3, \cos p_2\sin p_3, \sin p_2\cos p_3,$$

for each $n\ge 3$, the number $z_n(\vec{\pi})=6-\overline{v}(n)$ is a ninefold eigenvalue, and the corresponding eigenfunctions are

$$\sin(n+2)p_3, \sin p_1\cos(n+1)p_3, \sin p_2\cos(n+1)p_3,$$
$$\sin(n+1)p_3\cos p_1, \sin(n+1)p_3\cos p_2, \sin np_3\cos p_1\cos p_2,$$
$$\sin p_2\cos p_1\cos np_3, \sin p_1\cos p_2\cos np_3, \sin p_1\sin p_2\sin np_3.$$

The number $z_\infty(\vec{\pi})=6$ is an eigenvalue of an infinite multiplicity, and the corresponding eigenfunctions are

$$\psi_{(n_1,n_2,n_3)}^{---}(\mathbf{p})=\sin n_1p_1\sin n_2p_2\sin n_3p_3,\ n_3\in\mathbb{N}, n_1+n_2\ge 3.$$

All ninefold eigenvalues $z_n(\vec{\pi})=6-\overline{v}(n), n\ge 3$ of the operator $H(\vec{\pi})$ are simple eigenvalues for the operator $H_{123}^{-}(\vec{\pi})$, and the number $z_\infty(\vec{\pi})=6$ is an eigenvalue of an infinite multiplicity.

If the third coordinate k_3 of the total quasimomentum $\mathbf{k}$ is equal to π, then the operator $H(k_1,k_2,\pi)$ has infinitely many invariant subspaces $\mathfrak{R}_{123}^{-}(n), n\in\mathbb{N}$.

Next, we give a description of the invariant subspace $\mathfrak{R}_{123}^{-}(n), n\in\mathbb{N}$.

The system of functions

$$\left\{\psi_n^{-}(q)=\frac{1}{\sqrt{\pi}}\sin nq\right\}_{n\in\mathbb{N}}$$

is an orthonormal basis in the space $L_2^-(\mathbb{T})$. Let us denote by $L^-(n), n\in\mathbb{N}$ the one-dimensional subspace spanned by the vector ψ_n^-. The space $L_2^-(\mathbb{T})$ can be decomposed into the direct sum

$$L_2^-(\mathbb{T})=\sum_{n=1}^{\infty}\oplus L^-(n).$$

This decomposition produces another decomposition

$$\begin{aligned}L_{123}^-(\mathbb{T}^3)&=\sum_{n=1}^{\infty}\oplus\left\{L_2^-(\mathbb{T})\otimes L_2^-(\mathbb{T})\otimes L^-(n)\right\}\\&=\sum_{n=1}^{\infty}\oplus\left\{L_{12}^-(\mathbb{T}^2)\otimes L^-(n)\right\}=\sum_{n=1}^{\infty}\oplus\mathfrak{R}_{123}^-(n),\end{aligned}$$

where

$$\mathfrak{R}_{123}^-(n):=L_{12}^-(\mathbb{T}^2)\otimes L^-(n),\quad L_{12}^-(\mathbb{T}^2)=L_2^-(\mathbb{T})\otimes L_2^-(\mathbb{T}).$$

Lemma 4. *Let the potential* $\hat{v}$ *have the form* (5). *Then the subspace* $\mathfrak{R}_{123}^-(n)$ *is invariant under* $H_{123}^-(k_1,k_2,\pi)$ for any $n\in\mathbb{N}$.

Proof. Let $(f\psi_n^-)(p_1,p_2,p_3):=f(p_1,p_2)\psi_n^-(p_3)$, where $f\in L_{12}^-(\mathbb{T}^2)$, $\psi_n^-\in L^-(n)$ is an arbitrary element of $\mathfrak{R}_{123}^-(n)$. We consider the action of $H_{123}^-(k_1,k_2,\pi)=H_0(k_1,k_2,\pi)-V_{123}^-$ on $f\psi_n^-$:

$$\begin{aligned}&\left(H_0(k_1,k_2,\pi)f\psi_n^-\right)(\mathbf{p})\\&=\left[\left(6-2\cos\frac{k_1}{2}\cos p_1-2\cos\frac{k_2}{2}\cos p_2\right)f(p_1,p_2)\right]\psi_n^-(p_3),\end{aligned}\tag{9}$$

$$\begin{aligned}&\left(V_{123}^-f\psi_n^-\right)(\mathbf{p})\\&=\left[\frac{\bar{v}(n+2)}{\pi^2}\int_{\mathbb{T}^2}\sin p_1\sin q_1\sin p_2\sin q_2 f(q_1,q_2)\,\mathrm{d}q_1\mathrm{d}q_2\right]\psi_n^-(p_3).\end{aligned}\tag{10}$$

To obtain the last formula (10), we use the orthogonality of the system of functions $\{\psi_n^-\}_{n\in\mathbb{N}}$ in $L_2^-(\mathbb{T})$. Relations (9) and (10) imply the equality

$$\begin{aligned}&\left(H_{123}^-(k_1,k_2,\pi)f\psi_n^-\right)(p_1,p_2,p_3)\\&=\left(H_0(k_1,k_2,\pi)f\psi_n^-\right)(p_1,p_2,p_3)-\left(V_{123}^-f\psi_n^-\right)(p_1,p_2,p_3)\\&=\left[\left(6-2\cos\frac{k_1}{2}\cos p_1-2\cos\frac{k_2}{2}\cos p_2\right)f(p_1,p_2)\right]\psi_n^-(p_3)\\&\quad-\left[\frac{\bar{v}(n+2)}{\pi^2}\int_{\mathbb{T}^2}\sin p_1\sin q_1\sin p_2\sin q_2 f(q_1,q_2)\,\mathrm{d}q_1\mathrm{d}q_2\right]\psi_n^-(p_3)\end{aligned}\tag{11}$$

which completes the proof of the lemma.

We denote by $H_{123n}^-(k_1,k_2,\pi)$ restriction of the operator $H_{123}^-(k_1,k_2,\pi)$ in the invariant subspace $\mathfrak{R}_{123}^-(n)$. Formula (11) shows that the restriction $H_{123n}^-(k_1,k_2,\pi)$ to the subspace $\mathfrak{R}_{123}^-(n)=L_{12}^-(\mathbb{T}^2)\otimes L^-(n)$ has the form

$$H_{123n}^-(k_1,k_2,\pi)=\left[2I+H_0(k_1,k_2)-\bar{v}(n+2)V_{11}\right]\otimes I,\tag{12}$$

where I is the identity operator and $H_{123}^{(n)}(\mathbf{k}):=2I+H_0(\mathbf{k})-\bar{v}(n+2)V_{11}$, $\mathbf{k}=(k_1,k_2)$, is a two-dimensional two-particle operator acting in $L_{12}^-(\mathbb{T}^2)$ by the formula

$$\left(H_{123}^{(n)}(\mathbf{k})f\right)(\mathbf{p})$$
$$=\left(2+\varepsilon_{\mathbf{k}}(\mathbf{p})\right)f(\mathbf{p})-\frac{\overline{v}(n+2)}{\pi^2}\int_{\mathbb{T}^2}\sin p_1\sin p_2\sin q_1\sin q_2 f(\mathbf{q})\mathrm{d}\mathbf{q},$$

where $\varepsilon_{\mathbf{k}}(\mathbf{p})=4-2\cos\frac{k_1}{2}\cos p_1-2\cos\frac{k_2}{2}\cos p_2$, and V_{11} is a one-dimensional integral operator in $L_{12}^{-}(\mathbb{T}^2)$ with the kernel

$$v(\mathbf{p},\mathbf{q})=\frac{1}{\pi^2}\sin p_1\sin p_2\sin q_1\sin q_2.$$

Studying the eigenvalues of $H_{123n}^{-}(k_1,k_2,\pi)$ by representations (12) reduces to studying the eigenvalues of

$$H_{123}^{(n)}(\mathbf{k})=2I+H_0(\mathbf{k})-\overline{v}(n+2)V_{11},\mathbf{k}=(k_1,k_2)$$

i.e. the three-dimensional problem reduces to the two-dimensional problem.

4. Eigenvalues of the Operator $H_{123}^{-}(\mathbf{k})$

Our main goal in this section is to study the behavior of the nondegenerate eigenvalue $z_{n+2}(\vec{\pi})=6-\overline{v}(n+2),n\in\mathbb{N}$ of $H_{123}^{-}(\vec{\pi})$ at small perturbations β ($k_1=\pi-2\beta$ or $k_2=\pi-2\beta$), *i.e.* the eigenvalues of $H_{123}^{-}(\pi-2\beta,\pi,\pi)$ (or $H_{123}^{-}(\pi,\pi-2\beta,\pi)$) at small perturbations β. The studying of the eigenvalues of $H_{123}^{-}(\pi-2\beta,\pi,\pi)$ is reduced to study the eigenvalues of the operator $H_{123n}^{-}(\pi-2\beta,\pi,\pi)$ for each fixed $n\in\mathbb{N}$. In turn, the problem of studying the eigenvalues of the operator $H_{123n}^{-}(\pi-2\beta,\pi,\pi)$ by virtue of (12) is reduced to study of the discrete spectrum of the operator

$$H_{123}^{(n)}(\pi-2\beta,\pi)=2I+H_0(\pi-2\beta,\pi)-\overline{v}(n+2)V_{11}.$$

Studying the eigenvalues of $H_{123}^{(n)}(\pi-2\beta,\pi)$ and $H_{123}^{(n)}(\pi,\pi-2\beta)$ reduces to studying the eigenvalues of $H_\lambda(k)$ acting in $L_2^{-}(\mathbb{T})$ by the formula

$$\begin{aligned}&\left(H_\lambda(k)f\right)(p)=\varepsilon_k(p)f(p)-\frac{\lambda}{\pi}\int_{\mathbb{T}}\sin p\sin q f(q)\mathrm{d}q,\\&\varepsilon_k(p)=2-2\cos\frac{k}{2}\cos p.\end{aligned}\tag{13}$$

It is known that the essential spectrum of $H_\lambda(\pi-2\beta)=H_0(\pi-2\beta)-\lambda V_1,\beta\in\left(0,\frac{\pi}{2}\right]$ consists of a segment $\left[m(\beta),M(\beta)\right]$, where $m(\beta)=2-2\sin\beta$, $M(\beta)=2+2\sin\beta$.

Further we give some information about the eigenvalues and eigenfunctions of the operator $H_\lambda(k)$. Combining Theorem 6.3 in [6], Theorem 5.10 in [15] and Lemmas 1 and 2 we obtain the following statement about eigenvalues of the operator $H_\lambda(k)$.

Lemma 5. *Let* $\beta\in\left(0,\frac{\pi}{2}\right]$.

a) If $\lambda < \sin\beta$, then the operator $H_\lambda(\pi - 2\beta)$ has no eigenvalues lying outside of the essential spectrum.

b) If $\lambda = \sin\beta$, then the left edge $m(\beta)$ of essential spectrum of the operator $H_\lambda(\pi - 2\beta)$ is a resonance.

c) If $\lambda > \sin\beta$, then the operator $H_\lambda(\pi - 2\beta)$ has a unique nondegenerate eigenvalue

$$z_\lambda(\beta) = 2 - \lambda - \frac{1}{\lambda}\sin^2\beta$$

which lying in the left of the essential spectrum with corresponding normalized eigenfunction

$$f_\lambda^-(p) = \frac{C_\lambda \sin p}{2 - 2\sin\beta\cos p - z_\lambda(\beta)} \in L_2^-(\mathbb{T}). \tag{14}$$

Here C_λ is the normalizing multiplicity.

d) The operator $H_\lambda(\pi - 2\beta)$ has no embedded eigenvalues in the interval $(m(\beta), M(\beta))$.

Hilbert space $L_{12}^-(\mathbb{T}^2) = L_2^-(\mathbb{T}) \otimes L_2^-(\mathbb{T})$ can be written as a direct sum:

$$L_2^-(\mathbb{T}) \otimes L_2^-(\mathbb{T}) = L_2^-(\mathbb{T}) \otimes L^-(1) \oplus \left(L_2^-(\mathbb{T}) \otimes L^-(1)\right)^\perp.$$

The following lemma establishes a connection between the operators $H_{123}^{(n)}(\pi - 2\beta, \pi)$ and $H_\lambda(k)$.

Lemma 6. *Let the potential* $\hat{v}$ *have the form* (5)*. Then:*

a) the subspace $L_2^-(\mathbb{T}) \otimes L^-(1)$ and its orthogonal complement $\left(L_2^-(\mathbb{T}) \otimes L^-(1)\right)^\perp$ are invariant under $H_{123}^{(n)}(\pi - 2\beta, \pi)$.

b) restriction of the operator $H_{123}^{(n)}(\pi - 2\beta, \pi)$ to the invariant subspace $\left(L_2^-(\mathbb{T}) \otimes L^-(1)\right)^\perp$ coinsides with the unperturbed operator $H_0(\pi - 2\beta, \pi)$.

c) restriction of the operator $H_{123}^{(n)}(\pi - 2\beta, \pi)$ to the invariant subspace $L_2^-(\mathbb{T}) \otimes L^-(1)$ can be represented as a tensor product:

$$H_{123}^{(n)}(\pi - 2\beta, \pi) = \left[4I + H_0(\pi - 2\beta) - \overline{v}(n+2)V_1\right] \otimes I. \tag{15}$$

Here, I is the identity operator, and $H_{\lambda(n)}(\pi - 2\beta) := H_0(\pi - 2\beta) - \lambda(n)V_1$, $\lambda(n) = \overline{v}(n+2)$ is a one-dimensional two-particle operator acting in $L_2^-(\mathbb{T})$ by the formula (13).

This lemma is proved in the same way as the Lemma 4. In particular, part b) of the lemma implies that the operator $H_{123}^{(n)}(\pi - 2\beta, \pi)$ has no eigenfunctions in $\left(L_2^-(\mathbb{T}) \otimes L^-(1)\right)^\perp$. Thus, studying the eigenvalues of the operator $H_{123}^{(n)}(\pi - 2\beta, \pi)$ is reduced to studying eigenvalues of the operator $H_{\lambda(n)}(\pi - 2\beta) = H_0(\pi - 2\beta) - \lambda(n)V_1$.

From Lemmas 5 - 6 and tensor product (15) implies the following statement regarding operator $H_{123}^{(n)}(\pi - 2\beta, \pi)$.

Theorem 1. *Let* $\beta \in \left(0, \frac{\pi}{2}\right]$ *and* $n \in \mathbb{N}$.

a) If $\overline{v}(n+2) < \sin\beta$, then the operator $H_{123}^{(n)}(\pi - 2\beta, \pi)$ has no eigenvalues lying outside of the essential spectrum.

b) If $\overline{v}(n+2)=\sin\beta$, then the left edge $m(\beta)$ of essential spectrum of the operator $H_{123}^{(n)}(\pi-2\beta,\pi)$ is a resonance.

c) If $\overline{v}(n+2)>\sin\beta$, then the operator $H_{123}^{(n)}(\pi-2\beta,\pi)$ has a unique non-degenerate eigenvalue

$$z_{123}^{(n)}(\pi-2\beta,\pi)=4+z_{\lambda(n)}(\beta)=6-\overline{v}(n+2)-\frac{1}{\overline{v}(n+2)}\sin^2\beta, \tag{16}$$

which lies in the left of the essential spectrum and with the corresponding normalized eigenfunction

$$f_{\lambda(n)}^{--}(p_1,p_2)=f_{\lambda(n)}^{-}(p_1)\frac{\sin p_2}{\sqrt{\pi}}=f_{\lambda(n)}^{-}(p_1)\psi_1^{-}(p_2)\in L_2^{-}(\mathbb{T})\otimes L^{-}(1),$$

where $f_{\lambda(n)}^{-}$ is the normalized eigenfunction of the operator $H_{\lambda(n)}(\pi-2\beta)$ corresponding to the eigenvalue $z_{\lambda(n)}(\beta)$, the operator $H_{\lambda(n)}(k)$ is defined by the formula (13).

d) The operator $H_{123}^{(n)}(\pi-2\beta,\pi)$ has no embedded eigenvalues in the interval $(m(\beta),M(\beta))$.

Similar statement is true for the operator $H_{123}^{(n)}(\pi,\pi-2\beta)$. The eigenvalues of the operators $H_{123}^{(n)}(\pi,\pi-2\beta)$ and $H_{123}^{(n)}(\pi-2\beta,\pi)$ are same, but eigenfunctions differ with variable replacement p_1 and p_2. In other words, the operators $H_{123}^{(n)}(k_1,k_2)$ and $H_{123}^{(n)}(k_2,k_1)$ are unitary equivalent. Therefore, the operators $H_{123n}^{-}(k_1,k_2,\pi)$ and $H_{123n}^{-}(k_2,k_1,\pi)$ are unitary equivalent too.

Similar statement can relatively be formulated for the operator $H_{123}^{(n)}(\pi-2\beta,\pi-2\beta)$. For this purpose, we introduce the following notation. Through

$$\Delta_n(\beta,z)=1-\frac{\overline{v}(n+2)}{\pi^2}\int_{\mathbb{T}^2}\frac{\sin^2 p_1\sin^2 p_2\mathrm{d}p_1\mathrm{d}p_2}{2+2(2-\sin\beta\cos p_1-\sin\beta\cos p_2)-z}$$

we denote the Fredholm determinant of the operator $I-\overline{v}(n+2)V_{11}r_0(\beta,z)$, where $r_0(\beta,z)$ is the resolvent of the operator $2I+H_0(\pi-2\beta,\pi-2\beta)$, and V_{11} is an integral operator with the kernel

$$v(\mathbf{p},\mathbf{q})=\frac{1}{\pi^2}\sin p_1\sin p_2\sin q_1\sin q_2.$$

Through C_{11}^{--} denote the value of the following integral:

$$C_{11}^{--}=\frac{1}{\pi^2}\int_{\mathbb{T}^2}\frac{\sin^2 p_1\sin^2 p_2\mathrm{d}p_1\mathrm{d}p_2}{2(2-\cos p_1-\cos p_2)}=\int_{\mathbb{T}^2}\frac{|\psi_1^{-}(p_1)|^2|\psi_1^{-}(p_2)|^2\mathrm{d}p_1\mathrm{d}p_2}{2\varepsilon(\mathbf{p})}.$$

Simple calculations reveal the following approximate value $C_{11}^{--}\approx 0.302347$.

Theorem 2. *Let* $\beta\in\left(0,\frac{\pi}{2}\right]$, $n\in\mathbb{N}$.

a) If $\overline{v}(n+2)<\frac{\sin\beta}{C_{11}^{--}}$, then the operator $H_{123}^{(n)}(\pi-2\beta,\pi-2\beta)$ has no eigenvalues lying outside of the essential spectrum.

b) If $\overline{v}(n+2)=\frac{\sin\beta}{C_{11}^{--}}$, then the left edge $m(\beta)=6-4\sin\beta$ of the spectrum of the operator $H_{123}^{(n)}(\pi-2\beta,\pi-2\beta)$ is an eigenvalue.

c) If $\overline{v}(n+2)>\frac{\sin\beta}{C_{11}^{--}}$, then the operator $H_{123}^{(n)}(\pi-2\beta,\pi-2\beta)$ has a unique nondegenerate eigenvalue $z_{123}^{(n)}(\pi-2\beta,\pi-2\beta)$ below the essential spectrum.

d) The operator $H_{123}^{(n)}(\pi-2\beta,\pi-2\beta)$ has no embedded eigenvalues in the interval $(m(\beta),M(\beta))$.

This theorem is proved in similar way as Lemma 5. There are some differences:

1) In the Theorem 2, the eigenvalue $z_{123}^{(n)}(\pi-2\beta,\pi-2\beta)$ was calculated with the accuracy of β^2:

$$z_{123}^{(n)}(\pi-2\beta,\pi-2\beta)=6-\overline{v}(n+2)-\frac{2}{\overline{v}(n+2)}\sin^2\beta+O(\beta^4)$$

and corresponding normalized eigenfunction has the form

$$f_{123}^{(n)}(p_1,p_2)=\frac{C_n(\beta)\sin p_1\sin p_2}{6-2\sin\beta\cos p_1-2\sin\beta\cos p_2-z_{123}^{(n)}(\pi-2\beta,\pi-2\beta)}\in L_{12}^{-}(\mathbb{T}^2), \tag{17}$$

where $C_n(\beta)$ is the normalizing multiplicity.

2) Left edge $m(\beta)=6-2\sin\beta$ of the essential spectrum is a resonance for the operator $H_{123}^{(n)}(\pi-2\beta,\pi)$, but for the operator $H_{123}^{(n)}(\pi-2\beta,\pi-2\beta)$ the left edge $m(\beta)=6-4\sin\beta$ of the essential spectrum is the eigenvalue, *i.e.* the equation $H_{123}^{(n)}(\pi-2\beta,\pi-2\beta)f=m(\beta)f$ has a non-trivial solution

$$f(p_1,p_2)=\frac{C\sin p_1\sin p_2}{2-\cos p_1-\cos p_2}$$

and it belongs to $L_{12}^{-}(\mathbb{T}^2)$.

5. Conclusions

1) We have shown that the operator $H_{123}^{-}(k_1,k_2,\pi)$ has infinitely many invariant subspaces $\mathfrak{R}_{123}^{-}(n), n\in\mathbb{N}$. It has been proved that if condition $\overline{v}(n+2)>\sin\beta$ holds then the operator $H_{123n}^{-}(\pi-2\beta,\pi,\pi)$ has a unique simple eigenvalue $z_{123}^{(n)}(\pi-2\beta,\pi)$ of the form (16), otherwise, the operator has no eigenvalues outside of the essential spectrum. A similar statement holds for the operator $H_{123n}^{-}(\pi-2\beta,\pi-2\beta,\pi)$.

2) Without loss of generality it can be assumed that $\overline{v}(3)\le 1$. Since, if $\overline{v}(3)>1$ then it follows from $\lim_{n\to\infty}\overline{v}(n)=0$ that there exists a number $m\in\mathbb{N}$ such that $\overline{v}(m+2)\le 1$ and monotonicity of $\overline{v}$ implies that $\overline{v}(n)>1$ for $n=3,4,\cdots,m+1$, and in this case, the eigenvalues $z_{123}^{(n)}(\pi-2\beta,\pi), n=1,2,\cdots,m-1$ of $H_{123}^{-}(\pi-2\beta,\pi,\pi)$ exist for all $\beta\in[0,\pi/2]$.

For a fixed $\beta\in(0,\pi/2]$ there exists $m\in\mathbb{N}$ such that $\sin\beta\in(\overline{v}(m+3),\overline{v}(m+2))$ and the operator $H_{123}^{-}(\pi-2\beta,\pi,\pi)$ has m nondegenerate eigenvalues outside of the essential spectrum (see Theorem 1):

$$z_{123}^{(1)}(\pi-2\beta,\pi,\pi) := z_{123}^{(1)}(\pi-2\beta,\pi) = 6-\overline{v}(3)-\frac{1}{\overline{v}(3)}\sin^2\beta,$$

$$z_{123}^{(2)}(\pi-2\beta,\pi,\pi) := z_{123}^{(2)}(\pi-2\beta,\pi) = 6-\overline{v}(4)-\frac{1}{\overline{v}(4)}\sin^2\beta,$$

$$\vdots$$

$$z_{123}^{(m)}(\pi-2\beta,\pi,\pi) := z_{123}^{(m)}(\pi-2\beta,\pi) = 6-\overline{v}(m+2)-\frac{1}{\overline{v}(m+2)}\sin^2\beta.$$

The corresponding normalized eigenfunctions are of the forms:

$$f_{123\lambda(1)}^{---}(p_1,p_2,p_3) = f_{\lambda(1)}^{-}(p_1)\psi_1^{-}(p_2)\psi_1^{-}(p_3) \in L_2^{-}(\mathbb{T})\otimes L^{-}(1)\otimes L^{-}(1),$$

$$f_{123\lambda(2)}^{---}(p_1,p_2,p_3) = f_{\lambda(2)}^{-}(p_1)\psi_1^{-}(p_2)\psi_2^{-}(p_3) \in L_2^{-}(\mathbb{T})\otimes L^{-}(1)\otimes L^{-}(2),$$

$$\vdots$$

$$f_{123\lambda(m)}^{---}(p_1,p_2,p_3) = f_{\lambda(m)}^{-}(p_1)\psi_1^{-}(p_2)\psi_m^{-}(p_3) \in L_2^{-}(\mathbb{T})\otimes L^{-}(1)\otimes L^{-}(m),$$

where, $f_{\lambda(m)}^{-}$ is the normalized eigenfunction of the operator $H_{\lambda(m)}(\pi-2\beta)$ corresponding to the eigenvalue $z_{\lambda(m)}(\beta)$ and the operator $H_{\lambda(m)}(k)$ is defined by the formula (13), $\lambda(m)=\overline{v}(m+2)$.

The eigenvalues of the operators $H_{123}^{-}(\pi-2\beta,\pi,\pi)$ and $H_{123}^{-}(\pi,\pi-2\beta,\pi)$ are same but eigenfunctions differ with variable replacement p_1 and p_2. In other words, the operators $H_{123}^{-}(\pi-2\beta,\pi,\pi)$ and $H_{123}^{-}(\pi,\pi-2\beta,\pi)$ are unitary equivalent.

In the case $\sin\beta=\overline{v}(m+2)$, the left edge $m(\beta)=6-2\sin\beta$ of the essential spectrum is a resonance of the operator $H_{123}^{-}(\pi-2\beta,\pi,\pi)$ (see Theorem 1).

3) Let for some $m\in\mathbb{N}$ the relation $\sin\beta\in\left(\overline{v}(m+3)C_{11}^{--},\overline{v}(m+2)C_{11}^{--}\right)$ hold then the operator $H_{123}^{-}(\pi-2\beta,\pi-2\beta,\pi)$ has m nondegenerate eigenvalues outside the essential spectrum (see Theorem 2) and for small β:

$$z_{123}^{(1)}(\pi-2\beta,\pi-2\beta,\pi) := z_{123}^{(1)}(\pi-2\beta,\pi-2\beta)$$
$$= 6-\overline{v}(3)-\frac{2}{\overline{v}(3)}\sin^2\beta+O(\beta^4),$$

$$z_{123}^{(2)}(\pi-2\beta,\pi-2\beta,\pi) := z_{123}^{(2)}(\pi-2\beta,\pi-2\beta)$$
$$= 6-\overline{v}(4)-\frac{2}{\overline{v}(4)}\sin^2\beta+O(\beta^4),$$

$$\vdots$$

$$z_{123}^{(m)}(\pi-2\beta,\pi-2\beta,\pi) := z_{123}^{(m)}(\pi-2\beta,\pi-2\beta)$$
$$= 6-\overline{v}(m+2)-\frac{2}{\overline{v}(m+2)}\sin^2\beta+O(\beta^4).$$

The corresponding normalized eigenfunctions are of the forms:

$$f_{123}^{(1)-}(p_1,p_2,p_3) = f_{123}^{(1)}(p_1,p_2)\psi_1^{-}(p_3) \in L_{12}^{-}(\mathbb{T}^2)\otimes L^{-}(1),$$

$$f_{123}^{(2)-}(p_1,p_2,p_3) = f_{123}^{(2)}(p_1,p_2)\psi_2^{-}(p_3) \in L_{12}^{-}(\mathbb{T}^2)\otimes L^{-}(2),$$

$$\vdots$$

$$f_{123}^{(m)-}(p_1,p_2,p_3)=f_{123}^{(m)}(p_1,p_2)\psi_m^-(p_3)\in L_{12}^-(\mathbb{T}^2)\otimes L^-(m),$$

where, $f_{123}^{(m)}$ is the normalized eigenfunction of the operator $H_{123}^{(m)}(\pi-2\beta,\pi-2\beta)$ corresponding to the eigenvalue $z_{123}^{(m)}(\pi-2\beta,\pi-2\beta)$ defined by the formula (17).

In the case $\sin\beta=\overline{v}(m+2)C_{11}^{--}$, the left edge $m(\beta)=6-4\sin\beta$ of the essential spectrum is the eigenvalue of $H_{123}^{-}(\pi-2\beta,\pi-2\beta,\pi)$ (see Theorem 2) with the corresponding eigenfunction

$$f(\mathbf{p})=\frac{C\sin p_1\sin p_2}{2-\cos p_1-\cos p_2}\cdot\sin mp_3\in L_{12}^-(\mathbb{T}^2)\otimes L^-(m).$$

Remark 1. If the potential $\hat{v}$ is even in all arguments p_1,p_2,p_3 and the condition $\hat{v}\in\ell_2(\mathbb{Z}^3)$ holds, then the statements of Lemmas 3 - 4 remain valid.

Remark 2. If $k_3\neq\pi$, then the subspaces $\mathfrak{R}_{123}^{-}(n),n\in\mathbb{N}$ are not invariant under the operator $H_{123}^{-}(k_1,k_2,k_3)$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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