Bound States of a System of Two Fermions on Invariant Subspace

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Abstract

We consider a Hamiltonian of a system of two fermions on a three-dimensional lattice \( \mathbb{Z}^3 \) with special potential \( \hat{v} \). The corresponding Shrödinger operator \( H(k) \) of the system has an invariant subspace \( L_{123}^{-\varphi} \), where we study the eigenvalues and eigenfunctions of its restriction \( H_{123}(k) \). Moreover, there are shown that \( H_{123}(k_1,k_2,\pi) \) has also infinitely many invariant subspaces \( \mathcal{H}_{123}(n), n \in \mathbb{N} \), where the eigenvalues and eigenfunctions of eigenvalue problem

\[
H(k_1,k_2,\pi) f = z f, \quad f \in \mathcal{H}_{123}^{-n}(n)
\]

are explicitly found.

Keywords

Hamiltonian, Fermion, Bound State, Shrödinger Operator, Invariant Subspace, Total Quasi-Momentum, Eigenvalue, Birman-Schwinger Principle

1. Introduction

The nature of bound states of two-particle cluster operators for small parameter values was first studied in detail by Minlos and Mamatov [1] and then in a more general setting by Minlos and Mogilner [2]. In [3], Howland showed that the Rellich theorem on perturbations of eigenvalues does not extend to the resonance theory. Studying bound states of a two-particle system Hamiltonian \( H \) on the \( d \)-dimensional lattice \( \mathbb{Z}^d \) reduces to studying \([2] [4] [5] [6] [7]\) the eigenvalues of a family of Shrödinger operators \( H(k), k \in \mathbb{T}^d \), where \( k \) is the total quasi-momentum of a system. Moreover, eigenfunctions of \( H(k) \) are interpreted as bound states of the Hamiltonian \( H \), and eigenvalues, as the bound state...
energies. The bound states of $H$ of a system of two fermions on a one-dimensional lattice were studied in [4], a system of two bosons on a two-dimensional lattice was studied in [6], and perturbations of the eigenvalues of a two-particle Schrödinger operator on a one-dimensional lattice were studied in [8]. The finiteness of the number of eigenvalues of Shrödinger operator on a lattice was studied in the works [7] [9].

The discrete spectrum of the two-particle continuous Schrödinger operator

$$h_\lambda = -\Delta + \lambda V$$

was studied by many authors, with the conditions for the potential $V$ formulated in its coordinate representation. The condition for the finiteness of the set of negative elements of the spectrum and the absence of positive eigenvalues of $h_\lambda$ can be found in [10]. If $V \leq 0$, then the number of negative eigenvalues $N(\lambda)$ is a nondecreasing function of $\lambda \in (0, \infty)$, and each eigenvalue $z_n(\lambda)$ decreases on the half-axis $(0, \infty)$. It is known that when the coupling constant $\lambda$ decreases, the bound state energies of $h_\lambda$ tend to the boundary of the continuous spectrum (see [10]) and for some finite $\lambda$ are on the boundary. Two questions then arise: Does a bound or virtual state correspond to such a threshold state (i.e., is the corresponding wave function square-integrable)? And where do the bound states “disappear to” as $\lambda$ decreases further? The study of the first question was the subject in [11] [12]. Regarding the second question, it turns out that the bound state disappears by being absorbed into the continuous spectrum and becomes a resonance [5].

Here, we consider bound states of the Hamiltonian $\hat{H}$ (see (1)) of a system of two fermions on the three-dimensional lattice $\mathbb{Z}^3$ with the special potential $\hat{v}$ (see (5)). In other words, we study the discrete spectrum of a family of the Schrödinger operators $H(k), \ k = (k_1, k_2, k_3) \in \mathbb{T}^3$, (see (3)) corresponding to $\hat{H}$ in the invariant subspace $L^2_{123}(\mathbb{T}^3)$.

Restriction of the operator $H(k)$ in the invariant subspace $L^2_{123}(\mathbb{T}^3)$ is denoted by $H_{123}(k)$.

In the case $k = \bar{x} := (\pi, \pi, \pi)$, the operator $H(\bar{x})$ has an infinite number of eigenvalues of the form $6 - \bar{v}(n), n \in \mathbb{Z}$ and the essential spectrum consists of the single point 6. Here, the potential $\hat{v}$ is defined by (5) and $\bar{v} : \mathbb{N} \to \mathbb{R}$ is a decreasing function on $\mathbb{N}$ and $v \in \ell_2(\mathbb{N})$. These eigenvalues $z_n(\bar{x}) = 6 - \bar{v}(n), n \in \mathbb{N}$ are arranged in ascending order, $z_1(\bar{x}) < \cdots < z_n(\bar{x}) < \cdots$, and the smallest eigenvalue $z_1(\bar{x}) = 6 - \bar{v}(1)$ is threefold, $z_2(\bar{x}) = 6 - \bar{v}(2)$ is sevenfold, and the other eigenvalues $z_n(\bar{x}) = 6 - \bar{v}(n), n \geq 3$ are ninefold. All ninefold eigenvalues $z_n(\bar{x}) = 6 - \bar{v}(n), n \geq 3$ of the operator $H(\bar{x})$ are simple eigenvalues for the operator $H_{123}(\bar{x})$.

Further, we investigate eigenvalues and eigenfunctions of the restriction operator $H_{123}(k)$.

In the case $k = (k_1, k_2, \pi)$ the corresponding operator $H_{123}(k_1, k_2, \pi)$ has infinitely many invariant subspaces $\mathfrak{B}_{123}(n) := L^2(\mathbb{T}) \otimes L^2(\mathbb{T}) \otimes L^2(n), n \in \mathbb{N}$. It
is proved that the restriction $H_{123}^{a}(k_1, k_2, \pi)$ of the operator $H_{123}^{a}(k_1, k_2, \pi)$ in the invariant subspace $R_{123}(n)$ has no more than one eigenvalue. If exists, it can be calculated explicitly. For every $(k_1, k_2) \in (-\pi, \pi)^2$ the operator $H_{123}^{a}(k_1, k_2, \pi)$ has only a finite number of eigenvalues.

For any perturbation $\beta > 0$, the essential spectrum $\{6\}$ of $H(\pi)$ becomes the essential spectrum $\sigma_{es}(H(\pi - 2\beta, \pi)) = [6 - 2\sin \beta, 6 + 2\sin \beta]$. If the potential $\hat{v}$ is of the form (5), the Shrödinger equation

$$H_{123}(\pi - 2\beta, \pi, \pi) f = zf, \quad f \in R_{123}(n)$$

can be exactly solved (see Theorem 1).

The Shrödinger equations $H(\pi - 2\beta, \pi, \pi) f = zf$ and $H(\pi - 2\beta, -\pi, \pi) f = zf, \quad f \in R_{123}(n)$ with small $\beta$ are solved by using methods invariant subspaces and operator theory.

2. Description of the Hamiltonian and Expansion in a Direct Integral

The free Hamiltonian $\hat{H}_0$ of a system of two fermions on a three-dimensional lattice $Z^3$ usually corresponds to a bounded self-adjoint operator acting in the Hilbert space $L^2(Z^3 \times Z^3) := \{f \in L^2(Z^3 \times Z^3) : f(x, y) = -f(y, x)\}$ by the formula

$$\hat{H}_0 = -\frac{1}{2m} \Delta_1 - \frac{1}{2m} \Delta_2.$$

Here, $m$ is the fermion mass, which we assume to be equal to unity in what follows, $\Delta_1 = \Delta \otimes I$ and $\Delta_2 = I \otimes \Delta$, where $I$ is the identity operator, and the lattice Laplacian $\Delta$ is a difference operator that describes a translation of a particle from a side to a neighboring side,

$$(\Delta \hat{\psi})(x) = \sum_{j=1}^{3} \left[ \hat{\psi}(x + \mathbf{e}_j) + \hat{\psi}(x - \mathbf{e}_j) - 2\hat{\psi}(x) \right], \quad x \in Z^3, \quad \hat{\psi} \in L^2(Z^3),$$

where $\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)$ are unit vectors in $Z^3$. The total Hamiltonian $\hat{H}$ acts in the Hilbert space $L^2(Z^3 \times Z^3)$ and is the difference of the free Hamiltonian $\hat{H}_0$ and the interaction potential $\hat{V}_2$ of the two fermions (see [8] [13]):

$$\hat{H} = \hat{H}_0 - \hat{V}_2,$$  \hspace{1cm} (1)

where

$$\left(\hat{V}_2 \hat{\psi}\right)(x, y) = \hat{v}(x - y)\hat{\psi}(x, y), \quad \hat{\psi} \in L^2(Z^3)^2, \quad \hat{v} \in L^2(Z^3 \times Z^3).$$

Hereafter, we assume that $\hat{v} \in L^2(Z^3)$ and $\hat{v}(x) = \hat{v}(-x) \geq 0$ for all $x \in Z^3$.  \hspace{1cm} (2)

Under this condition, the Hamiltonian $\hat{H}$ is a bounded self-adjoint operator in $L^2(Z^3 \times Z^3)^2$.

We pass to momentum representation using the Fourier transform [2] [4] [7]

$$F : L^2(Z^3 \times Z^3) \rightarrow L^2(T^3 \times T^3).$$
The Hamiltonian $H = H_0 - V = F \hat{H} F^{-1}$ in the momentum representation commutes with the unitary operators $U_s, s \in \mathbb{Z}$, given by

$$(U_s f)(k_1, k_2) = \exp\left(-i(s, k_1 + k_2)\right) f(k_1, k_2), \quad f \in L^2\left(\mathbb{T}^3 \times \mathbb{T}^3\right).$$

It follows that there exist decompositions of $L^2\left(\mathbb{T}^3 \times \mathbb{T}^3\right)$ and the operators $U_s$ and $H$ into direct integrals (see [7] [9] and [10])

$L^2\left(\mathbb{T}^3 \times \mathbb{T}^3\right) = \int_{\mathbb{Z}} \oplus L^2\left(F_k\right) d\mathbf{k}, \quad U_s = \int_{\mathbb{Z}} \oplus U_s\left(k\right) d\mathbf{k}, \quad H = \int_{\mathbb{Z}} \oplus \hat{H}\left(k\right) d\mathbf{k}.$

Here,

$F_k = \{(k_1, k_2) \in \mathbb{T}^3 \times \mathbb{T}^3: k_1 + k_2 = k\}, \quad k \in \mathbb{T}^3,$

and $U_s\left(k\right)$ is an operator of multiplication by the function $\exp\left(-i(s, k)\right)$ in $L^2\left(F_k\right)$. The fiber operator $\hat{H}\left(k\right)$ of $H$ also acts in $L^2\left(F_k\right)$ and is unitarily equivalent to $H\left(k\right) := H_0\left(k\right) - V$, which is called the Shrödinger operator. This operator acts in the Hilbert space $L^2\left(\mathbb{T}^3\right):= \{f \in L^2\left(\mathbb{T}^3\right): f(-\mathbf{q}) = -f(\mathbf{q})\}$ by the formula

$$(H\left(k\right)f)(\mathbf{q}) = \varepsilon_k\left(\mathbf{q}\right) f(\mathbf{q}) - \left(2\pi\right)^{\frac{3}{2}} \int_{\mathbb{S}^2} v(\mathbf{q} - s) f(s) d\mathbf{s}. \quad (3)$$

The unperturbed operator $H_0\left(k\right)$ is an operator of multiplication by the function

$$\varepsilon_k\left(\mathbf{q}\right) = \varepsilon\left(\frac{k}{2} + \mathbf{q}\right) + \varepsilon\left(\frac{k}{2} - \mathbf{q}\right)$$

$$= 6 - 2 \cos\frac{k}{2} \cos q_1 - 2 \cos\frac{k}{2} \cos q_2 - 2 \cos\frac{k}{2} \cos q_3. \quad (4)$$

From (3) and (4), it follows that

$$H\left(k_1, k_2, k_3\right) = H\left(-k_1, k_2, k_3\right) = H\left(k_1, -k_2, k_3\right) = H\left(k_1, k_2, -k_3\right),$$

so we can assume $k_1, k_2, k_3 \in [0, \pi].$

The perturbation operator $V$ is an integral operator in $L^2\left(\mathbb{T}^3\right)$ with the kernel

$$\left(2\pi\right)^{\frac{3}{2}} v(\mathbf{q} - s) = \left(2\pi\right)^{\frac{3}{2}} (F\hat{V})(\mathbf{q} - s),$$

and belongs to the class of Hilbert-Schmidt operators $\Sigma_2$.

In this work, we consider the operator $H\left(k\right)$ with the potential $\hat{v}$ of the form

$$\hat{v}(n) = \hat{v}(n_1, n_2, n_3) = \begin{cases} \mathbb{T}\left(|n|\right), & |n_1| + |n_2| \leq 1 \\ 0, & |n_1| + |n_2| \geq 2 \end{cases} \quad (5)$$

where $|n| = |n_1| + |n_2| + |n_3|$. Supporter is in the cylinder:

$$D = \{n = (n_1, n_2, n_3) \in \mathbb{Z}^3: n_3 \in \mathbb{Z}, |n_1| + |n_2| \leq 1\}.$$

Since for every function $\hat{\psi} \in L^2\left(\mathbb{Z}^3\right)$ the equality $\hat{\psi}(\mathbf{x}, \mathbf{x}) = 0, \mathbf{x} \in \mathbb{Z}^3$ holds, then the value of the potential $\hat{v}$ at the origin can be set arbitrary, since it does not affect the result, for simplicity, we assume that $\hat{v}(0) = 0.$
The function $\mathbf{v} : \mathbb{N} \to \mathbb{R}$ in (5) is decreasing in $\mathbb{N}$, i.e.,

$$\mathbf{v}(1) > \mathbf{v}(2) > \cdots$$

and belongs to $\ell_2(\mathbb{N})$. The kernel $\mathbf{v}$, of the integral operator $V$, i.e., the Fourier transform $\mathbf{v}(\mathbf{p}) = \mathbf{\hat{v}}(\mathbf{p})$, of the potential $\mathbf{\hat{v}}$, has the form

$$\mathbf{v}(\mathbf{p}) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \sum_{\mathbf{n} \in \mathbb{Z}^3} \mathbf{\hat{v}}(\mathbf{n}) e^{i\mathbf{a} \cdot \mathbf{p}}$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \left[2\mathbf{\Psi}(1)(\cos p_1 + \cos p_2 + \cos p_3) + 2\mathbf{\Psi}(2)(\cos 2p_3 + \cos p_1 \cos p_2 + \cos p_1 \cos p_3 + 2 \cos p_2 \cos p_3) + 2\mathbf{\Psi}(n+2)(\cos (n+2)p_3 + \cos (n+1)p_3 \cos p_1 + \cos p_2) + 4\cos p_1 \cos p_2 \cos np_3\right]$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \left[2\mathbf{\Psi}(1)(\cos p_1 + \cos p_2 + \cos p_3) + 2\mathbf{\Psi}(2)(\cos 2p_3 + \cos p_1 \cos p_2 + \cos p_1 \cos p_3 + 2 \cos p_2 \cos p_3) + 2\mathbf{\Psi}(n+2)(\cos (n+2)p_3 + \cos (n+1)p_3 \cos p_1 + \cos p_2) + 4\cos p_1 \cos p_2 \cos np_3\right].$$

**Eigenvalues of the operator** $H(k)$. We note that the spectra of the operators $H_0(k)$ and $V$ are known. The operator $H_0(k)$ does not have eigenvalues, its spectrum is continuous and coincides with the range of the function $\mathbf{\varepsilon}_k$:

$$\sigma(H_0(k)) = \left[m(k), M(k)\right], \quad \text{where} \quad m(k) = \min_{q \in \mathbb{R}} \mathbf{\varepsilon}_k(q), \quad M(k) = \max_{q \in \mathbb{R}} \mathbf{\varepsilon}_k(q).$$

The spectrum of $V$ consists of the set $\{0, \mathbf{\Psi}(n), n \in \mathbb{N}\}$. Under condition (2), the operator $V$ is a Hilbert-Schmidt operator and is hence compact. By the Weyl theorem [10], the essential spectrum of $H(k)$ coincides with the spectrum of $H_0(k)$:

$$\sigma_{ess}(H(k)) = \left[m(k), M(k)\right].$$

If $k = \pi$, then the spectrum of $H(\pi) = 6I - V$ consists of eigenvalues of the form $6 - \mathbf{\Psi}(n), n \in \mathbb{N}$ and the essential spectrum is $\{6\}$. If $k_j = \pi$ (for some $j \in \{1, 2, 3\}$), then there exists a potential $\mathbf{\hat{v}}$ such that $H(k)$ has an infinite number of eigenvalues outside the continuous spectrum (see [4] [14]).

We recall some notations and known facts. For any self-adjoint operator $B$ acting in a Hilbert space $\mathbb{H}$ without an essential spectrum to the right of $\mu \in \mathbb{R}$, we let $n(\mu, B)$ denote the number of its eigenvalues to the right of $\mu$. We let $N(k, z)$ denote the number of eigenvalues of $H(k)$ to the left of $z \leq m(k)$, i.e., $N(k, z) = n(-z, -H(k))$. The number $N(k, m(k))$ in fact coincides with the number of eigenvalues outside the continuous spectrum of $H(k)$. It follows from the self-adjointness of $H(k) = H_0(k) - V$ and positivity of $V$ that

$$\sigma(H(k)) \cap (M(k), \infty) = \emptyset,$$

and hence $\sigma_{disc}(H(k)) \subset (-\infty, m(k))$. Therefore we seek only eigenvalues $z$ less than $m(k)$.

For any $k \in \mathbb{T}^3$ and $z < m(k)$, we define the integral operator

$$G(k, z) = \frac{1}{V^2} r_0(k, z) V^2,$$

where $r_0(k, z)$ is the resolvent of the unperturbed operator $H_0(k)$. Under
condition (2), the operator $V$ is positive, and we let $V^{\frac{1}{2}}$ denote the positive square root of the positive operator $V$. A solution $f$ of the Schrödinger equation

$$H(k)f = zf$$

and the fixed points $\varphi$ of $G(k,z)$ are connected by the relations

$$f = r_0(k,z)V^{\frac{1}{2}}\varphi \quad \text{and} \quad \varphi = V^{\frac{1}{2}}f.$$ 

The following proposition (the Birman–Schwinger principle) holds [9].

**Lemma 1.** The number of eigenvalues of $H(k)$ to the left of $z < m(k)$ coincides with the number of eigenvalues of $G(k,z)$ greater than unity, i.e., the equality

$$N(k,z) = n(1, G(k,z))$$

holds.

**Lemma 2.** If for some $k \in \mathbb{T}^3$ the limit operator

$$\lim_{z \to m(k)^-} G(k,z) = G(k, m(k))$$

exists and is compact, then the equality

$$N(k, m(k)) = n(1, G(k, m(k)))$$

(8)

holds.

Equality (8) states that the number of eigenvalues of $H(k)$, to the left of $m(k)$ is equal to the number of eigenvalues of $G(k, m(k))$ greater than unity.

### 3. Invariant Subspaces of $H(k)$

In this section, we study the invariant subspaces with respect to the operator $H(k)$.

Let $L^2_1(\mathbb{T}) = \{ f \in L^2(\mathbb{T}) : f(-p) = -f(p) \}$ be a subspace of the space $L^2_1(\mathbb{T})$, consisting of odd functions on $\mathbb{T} = [-\pi, \pi]$, and $L^2_2(\mathbb{T}) = \{ f \in L^2(\mathbb{T}) : f(-p) = f(p) \}$ be a subspace of $L^2_2(\mathbb{T})$, consisting of even functions on $\mathbb{T}$. In addition, we use the notation

$$L^2_{123}(\mathbb{T}^3) = L^2_1(\mathbb{T}) \otimes L^2_2(\mathbb{T}) \otimes L^2_2(\mathbb{T}), \quad L^2_{123}(\mathbb{T}^3) = L^2_1(\mathbb{T}) \otimes L^2_2(\mathbb{T}) \otimes L^2_2(\mathbb{T}).$$

Note that $L^2_{123}(\mathbb{T}^3)$ is a subspace of the space $L^6_1(\mathbb{T}^3)$. It is natural to expect the invariance of the subspace $L^2_{123}(\mathbb{T}^3)$ with respect to the operator $H(k)$. It turns out that this subspace is invariant under the operator $H(k)$, i.e. the following statement holds.

**Lemma 3.** Let the potential $\hat{\varphi}$ have the form (5). Then the subspace $L^2_{123}(\mathbb{T}^3)$ is invariant under the action of $H(k)$.

**Proof.** We prove that this subspace is invariant first with respect to $H_0(k)$, and then with respect to $V$. It follows from representation (4) that the function $\varepsilon_k$ belongs to the subspace $L^2_{123}(\mathbb{T}^3)$, and it follows from the inclusion $f \in L^2_{123}(\mathbb{T}^3)$ that $\varepsilon_k f \in L^2_{123}(\mathbb{T}^3)$. This proves that $L^2_{123}(\mathbb{T}^3)$ is invariant with respect to $H_0(k)$.

Simple calculations show that the function (see (7))
\[(I_{f^v})(p_1, p_2, p_3) = \frac{1}{(2\pi)^2} \int_\mathbb{T}^3 v(p_1 - s_1, p_2 - s_2, p_3 - s_3) f(s_1, s_2, s_3) ds_1 ds_2 ds_3\]

belongs to the subspace \(L_{123}(\mathbb{T}^3)\) for \(f \in L_{123}(\mathbb{T}^3)\). Hence, we prove the invariance of \(L_{123}(\mathbb{T}^3)\) with respect to \(V\) and it follows that \(L_{123}(\mathbb{T}^3)\) is invariant with respect to \(H(k) = H_0(k) - V\).

\(H_{123}(k)\) denotes the restriction of \(H(k)\) to the respective subspace \(L_{123}(\mathbb{T}^3)\). The action of \(H_{123}(k) := H_0(k)\) is unchanged, the unperturbed operator \(H_0(k)\) is an operator of multiplication by the function \(\varepsilon_k\). We present the formula for \(V_{123} = V|_{L_{123}(\mathbb{T}^3)}\) operator \(V\) acts on the element \(f \in L_{123}(\mathbb{T}^3)\) according to the formula

\[
(V_{123}f)(p) = \frac{1}{\pi} \sum_{n=1}^{\infty} \overline{v}(n + 2) \int_{\mathbb{T}} \sin p_1 \sin p_2 \sin np_3 \sin q_1 \sin q_2 \sin nq_3 f(q) dq.
\]

Note that for \(k = \pi\), the spectrum of \(H(\pi) = 6I - V\) consists only of the eigenvalues \(6, 6 - \overline{v}(n), n \in \mathbb{N}\) and the essential spectrum \(\{6\}\). Under condition (6) the number \(z_v(\pi) = 6 - \overline{v}(1)\) is a threefold eigenvalue of \(H(\pi)\), with the corresponding eigenfunctions

\[
\sin p_1, \sin p_2, \sin p_3,
\]

the number \(z_v(\pi) = 6 - \overline{v}(2)\) is a sevenfold eigenvalue with the corresponding eigenfunctions

\[
\sin 2p_3, \cos p_1 \sin p_2, \sin p_1 \cos p_2, \cos p_1 \sin p_3, \\
\sin p_1 \cos p_3, \cos p_2 \sin p_3, \sin p_2 \cos p_3,
\]

for each \(n \geq 3\), the number \(z_v(\pi) = 6 - \overline{v}(n)\) is a ninefold eigenvalue, and the corresponding eigenfunctions are

\[
\sin (n + 2)p_3, \sin p_1 \cos (n + 1)p_2, \sin p_2 \cos (n + 1)p_3, \\
\sin (n + 1)p_1 \cos p_1 \sin (n + 1)p_2, \sin p_2 \cos p_1 \cos p_2, \sin np_3 \cos p_1 \cos p_2, \\
\sin p_2 \cos p_1 \cos np_3, \sin p_3 \cos p_2 \cos np_3, \sin p_1 \sin p_2 \sin np_3.
\]

The number \(z_v(\pi) = 6\) is an eigenvalue of an infinite multiplicity, and the corresponding eigenfunctions are

\[
\psi_{(n_1, n_2, n_3)}(p) = \sin n_1 p_1 \sin n_2 p_2 \sin n_3 p_3, \; n_1, n_2 \geq 3.
\]

All ninefold eigenvalues \(z_v(\pi) = 6 - \overline{v}(n), n \geq 3\) of the operator \(H(\pi)\) are simple eigenvalues for the operator \(H_{123}(\pi)\), and the number \(z_v(\pi) = 6\) is an eigenvalue of an infinite multiplicity.

If the third coordinate \(k_3\) of the total quasimomentum \(k\) is equal to \(\pi\), then the operator \(H(k, k_2, \pi)\) has infinitely many invariant subspaces \(\mathcal{R}_{123}(n), n \in \mathbb{N}\).

Next, we give a description of the invariant subspace \(\mathcal{R}_{123}(n), n \in \mathbb{N}\).

The system of functions

\[
\left\{ \psi_v (q) = \frac{1}{\sqrt{\pi}} \sin nq \right\}_{n \in \mathbb{N}}
\]
is an orthonormal basis in the space \( L^2(\mathbb{T}) \). Let us denote by \( L^r(n), n \in \mathbb{N} \) the one-dimensional subspace spanned by the vector \( \psi_n^- \). The space \( L^2(\mathbb{T}) \) can be decomposed into the direct sum

\[
L^2(\mathbb{T}) = \sum_{n=1}^{\infty} L^r(n).
\]

This decomposition produces another decomposition

\[
L^2(\mathbb{T}) = \sum_{n=1}^{\infty} \left[ L^2(\mathbb{T}) \otimes L^2(\mathbb{T}) \otimes L^r(n) \right]
\]

\[
= \sum_{n=1}^{\infty} \left[ \mathcal{R}_{123}(n) \right] = \sum_{n=1}^{\infty} \otimes \mathcal{R}_{123}(n),
\]

where

\[
\mathcal{R}_{123}(n) = L^2(\mathbb{T}) \otimes L^r(n), \quad L^2(\mathbb{T}) = L^2(\mathbb{T}) \otimes L^2(\mathbb{T}).
\]

**Lemma 4.** Let the potential \( \hat{v} \) have the form (5). Then the subspace \( \mathcal{R}_{123}(n) \) is invariant under \( H_{123}(k_1, k_2, \pi) \) for any \( n \in \mathbb{N} \).

**Proof.** Let \( \left( f \psi_n^-(p_1, p_2, p_3) = f(p_1, p_2) \psi_n^-(p_3) \right) \), where \( f \in L^2(\mathbb{T}^2) \), \( \psi_n^- \in L^r(n) \) is an arbitrary element of \( \mathcal{R}_{123}(n) \). We consider the action of

\[
H_{123}(k_1, k_2, \pi) = H_0(k_1, k_2, \pi) - V_{123} \quad \text{on} \quad f \psi_n^-:
\]

\[
\left( H_0(k_1, k_2, \pi) f \psi_n^- \right)(p) = \left[ \left( 6 - 2 \cos k_1 \cos p_1 - 2 \cos k_2 \cos p_2 \right) f(p_1, p_2) \right] \psi_n^-(p_3),
\]

\[
(V_{123} f \psi_n^-)(p) = \left[ \frac{\nu(n+2)}{\pi^2} \int_{\mathbb{T}^2} \sin p_1 \sin q_1 \sin p_2 \sin q_2 f(q_1, q_2) dq_1 dq_2 \right] \psi_n^-(p_3).
\]

To obtain the last formula (10), we use the orthogonality of the system of functions \( \{ \psi_n^- \}_{n=1}^{\infty} \) in \( L^2(\mathbb{T}) \). Relations (9) and (10) imply the equality

\[
\left( H_{123}(k_1, k_2, \pi) f \psi_n^- \right)(p_1, p_2, p_3) = \left( H_0(k_1, k_2, \pi) f \psi_n^- \right)(p_1, p_2, p_3) - \left( V_{123} f \psi_n^- \right)(p_1, p_2, p_3)
\]

\[
= \left[ \left( 6 - 2 \cos k_1 \cos p_1 - 2 \cos k_2 \cos p_2 \right) f(p_1, p_2) \right] \psi_n^-(p_3)
\]

\[
- \left[ \frac{\nu(n+2)}{\pi^2} \int_{\mathbb{T}^2} \sin p_1 \sin q_1 \sin p_2 \sin q_2 f(q_1, q_2) dq_1 dq_2 \right] \psi_n^-(p_3)
\]

which completes the proof of the lemma.

We denote by \( H_{123}(k_1, k_2, \pi) \) restriction of the operator \( H_{123}(k_1, k_2, \pi) \) in the invariant subspace \( \mathcal{R}_{123}(n) \). Formula (11) shows that the restriction \( H_{123}(k_1, k_2, \pi) \) to the subspace \( \mathcal{R}_{123}(n) = L^2(\mathbb{T}^2) \otimes L^r(n) \) has the form

\[
H_{123}(k_1, k_2, \pi) = \left[ 2I + H_0(k_1, k_2) - \nu(n+2)V_{11} \right] \otimes I,
\]

where \( I \) is the identity operator and \( H_{123}^{(e)}(k) = 2I + H_0(k) - \nu(n+2)V_{11} \), \( k = (k_1, k_2) \), is a two-dimensional two-particle operator acting in \( L^2(\mathbb{T}^2) \) by
the formula

\[
\left( H_{123}^{(s)}(k)f\right)(p) = \left( 2 + \varepsilon_k(p) \right)f(p) - \frac{\nabla(n + 2)}{\pi^2} \int \sin p_1 \sin p_2 \sin q_1 \sin q_2 f(q) dq,
\]

where \( \varepsilon_k(p) = 4 - 2 \cos \frac{k}{2} \cos p_1 - 2 \cos \frac{k}{2} \cos p_2 \), and \( V_{11} \) is a one-dimensional integral operator in \( L_2(\mathbb{T}^2) \) with the kernel

\[
v(p, q) = \frac{1}{\pi^2} \sin p_1 \sin p_2 \sin q_1 \sin q_2.
\]

Studying the eigenvalues of \( H_{123}^{(s)}(k_1, k_2, \pi) \) by representations (12) reduces to studying the eigenvalues of

\[
H_{123}^{(s)}(k) = 2I + H_0(k) - \nabla(n + 2)V_{11}, k = (k_1, k_2)
\]

i.e. the three-dimensional problem reduces to the two-dimensional problem.

### 4. Eigenvalues of the Operator \( H_{123}^{(s)}(k) \)

Our main goal in this section is to study the behavior of the nondegenerate eigenvalue \( z_{n+2}(\pi) = 6 - \nabla(n + 2), n \in \mathbb{N} \) of \( H_{123}^{(s)}(\pi) \) at small perturbations \( \beta \) \((k_1 = \pi - 2\beta \) or \( k_2 = \pi - 2\beta \)), i.e. the eigenvalues of \( H_{123}^{(s)}(\pi - 2\beta, \pi, \pi) \) (or \( H_{123}^{(s)}(\pi, \pi - 2\beta, \pi) \)) at small perturbations \( \beta \). The studying of the eigenvalues of \( H_{123}^{(s)}(\pi - 2\beta, \pi, \pi) \) is reduced to study the eigenvalues of the operator \( H_{123}^{(s)}(\pi - 2\beta, \pi, \pi) \) for each fixed \( n \in \mathbb{N} \). In turn, the problem of studying the eigenvalues of the operator \( H_{123}^{(s)}(\pi - 2\beta, \pi, \pi) \) by virtue of (12) is reduced to study of the discrete spectrum of the operator

\[
H_{123}^{(s)}(\pi - 2\beta, \pi, \pi) = 2I + H_0(\pi - 2\beta, \pi, \pi) - \nabla(n + 2)V_{11}.
\]

Studying the eigenvalues of \( H_{123}^{(s)}(\pi - 2\beta, \pi, \pi) \) and \( H_{123}^{(s)}(\pi, \pi - 2\beta, \pi) \) reduces to studying the eigenvalues of \( H_{\lambda}(k) \) acting in \( L_2(\mathbb{T}^2) \) by the formula

\[
\left( H_{\lambda}(k)f\right)(p) = \varepsilon_k(p)f(p) - \frac{\lambda}{\pi^2} \int \sin p \sin q f(q) dq,
\]

where

\[
\varepsilon_k(p) = 2 - 2 \cos \frac{k}{2} \cos p.
\]

It is known that the essential spectrum of

\[
H_{\lambda}(\pi - 2\beta) = H_0(\pi - 2\beta) - \lambda V, \beta \in \left[ 0, \frac{\pi}{2} \right]
\]

consists of a segment

\[
\left[ m(\beta), M(\beta) \right], \text{ where } m(\beta) = 2 - 2 \sin \beta, M(\beta) = 2 + 2 \sin \beta.
\]

Further we give some information about the eigenvalues and eigenfunctions of the operator \( H_{\lambda}(k) \). Combining Theorem 6.3 in [6], Theorem 5.10 in [15] and Lemmas 1 and 2 we obtain the following statement about eigenvalues of the operator \( H_{\lambda}(k) \).

**Lemma 5.** Let \( \beta \in \left[ 0, \frac{\pi}{2} \right] \).
a) If $\lambda < \sin \beta$, then the operator $H_\lambda (\pi - 2\beta)$ has no eigenvalues lying outside of the essential spectrum.

b) If $\lambda = \sin \beta$, then the left edge $m(\beta)$ of essential spectrum of the operator $H_\lambda (\pi - 2\beta)$ is a resonance.

c) If $\lambda > \sin \beta$, then the operator $H_\lambda (\pi - 2\beta)$ has a unique nondegenerate eigenvalue

$$z_\lambda (\beta) = 2 - \lambda - \frac{1}{\lambda} \sin^2 \beta$$

which lying in the left of the essential spectrum with corresponding normalized eigenfunction

$$f_\lambda (p) = \frac{C_\lambda \sin p}{2 - 2\sin \beta \cos p - z_\lambda (\beta)} \in L_2^2 (\mathbb{T}).$$

Here $C_\lambda$ is the normalizing multiplicity.

The following lemma establishes a connection between the operators $H_\lambda (\pi - 2\beta)$ and $H_\lambda (k)$.

**Lemma 6.** Let the potential $\hat{v}$ have the form (5). Then:

a) The subspace $L_2 (\mathbb{T}) \otimes L (1)$ and its orthogonal complement $(L_2 (\mathbb{T}) \otimes L (1))^\perp$ are invariant under $H_{12}^{(n)} (\pi - 2\beta, \pi)$.

b) restriction of the operator $H_{12}^{(n)} (\pi - 2\beta, \pi)$ to the invariant subspace $(L_2 (\mathbb{T}) \otimes L (1))^\perp$ coincides with the unperturbed operator $H_0 (\pi - 2\beta, \pi)$.

c) restriction of the operator $H_{12}^{(n)} (\pi - 2\beta, \pi)$ to the invariant subspace $L_2 (\mathbb{T}) \otimes L (1)$ can be represented as a tensor product:

$$H_{12}^{(n)} (\pi - 2\beta, \pi) = [4I + H_0 (\pi - 2\beta - \lambda (n)V) \otimes I].$$

Here, $I$ is the identity operator, and $H_{12}^{(n)} (\pi - 2\beta) = H_0 (\pi - 2\beta) - \lambda (n)V_1$, $\lambda (n) = \mathbb{V}(n + 2)$ is a one-dimensional two-particle operator acting in $L_2 (\mathbb{T})$ by the formula (13).

This lemma is proved in the same way as the Lemma 4. In particular, part b) of the lemma implies that the operator $H_{12}^{(n)} (\pi - 2\beta, \pi)$ has no eigenfunctions in $(L_2 (\mathbb{T}) \otimes L (1))^\perp$. Thus, studying the eigenvalues of the operator $H_{12}^{(n)} (\pi - 2\beta, \pi)$ is reduced to studying eigenvalues of the operator $H_{12}^{(n)} (\pi - 2\beta) = H_0 (\pi - 2\beta - \lambda (n)V_1).

From Lemmas 5 - 6 and tensor product (15) implies the following statement regarding operator $H_{12}^{(n)} (\pi - 2\beta, \pi)$.

**Theorem 1.** Let $\beta \in \left(0, \frac{\pi}{2}\right)$ and $n \in \mathbb{N}$.

a) If $\mathbb{V}(n + 2) < \sin \beta$, then the operator $H_{12}^{(n)} (\pi - 2\beta, \pi)$ has no eigenvalues lying outside of the essential spectrum.
b) If \( \mathcal{V}(n+2) = \sin \beta \), then the left edge \( m(\beta) \) of essential spectrum of the operator \( H_{123}^{(n)}(\pi - 2\beta, \pi) \) is a resonance.

c) If \( \mathcal{V}(n+2) > \sin \beta \), then the operator \( H_{123}^{(n)}(\pi - 2\beta, \pi) \) has a unique non-degenerate eigenvalue
\[
z^{(n)}_{123}(\pi - 2\beta, \pi) = 4 + z^{(n)}_{(a)}(\beta) = 6 - \mathcal{V}(n+2) - \frac{1}{\mathcal{V}(n+2)} \sin^2 \beta, \quad (16)
\]
which lies in the left of the essential spectrum and with the corresponding normalized eigenfunction
\[
f^{(n)}_{(a)}(p_1, p_2) = f^{(n)}_{(a)}(p_1) \frac{\sin p_2}{\sqrt{\pi}} = f^{(n)}_{(a)}(p_1) \psi_1^* (p_2) \in L_2(\mathbb{T}) \otimes L_2(1),
\]
where \( f^{(n)}_{(a)} \) is the normalized eigenfunction of the operator \( H_{(a)}^{(n)}(\pi - 2\beta) \) corresponding to the eigenvalue \( z^{(n)}_{(a)}(\beta) \), the operator \( H_{(a)}^{(n)}(k) \) is defined by the formula (13).

d) The operator \( H_{123}^{(n)}(\pi - 2\beta, \pi) \) has no embedded eigenvalues in the interval \( (m(\beta), M(\beta)) \).

Similar statement is true for the operator \( H_{123}^{(n)}(\pi, \pi - 2\beta) \). The eigenvalues of the operators \( H_{123}^{(n)}(\pi, \pi - 2\beta) \) and \( H_{123}^{(n)}(\pi - 2\beta, \pi) \) are same, but eigenfunctions differ with variable replacement \( p_1 \) and \( p_2 \). In other words, the operators \( H_{123}^{(n)}(k_1, k_2) \) and \( H_{123}^{(n)}(k_2, k_1) \) are unitary equivalent. Therefore, the operators \( H_{123n}(k_1, k_2, \pi) \) and \( H_{123n}(k_2, k_1, \pi) \) are unitary equivalent too.

Similar statement can relatively be formulated for the operator \( H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta) \). For this purpose, we introduce the following notation. Through
\[
\Delta_n(\beta, z) = 1 - \mathcal{V}(n+2) \int_{\mathbb{T}^2} \frac{\sin^2 p_1 \sin^2 p_2 \, dp_1 \, dp_2}{2 + 2 \sin \beta \cos p_1 - \sin \beta \cos p_2 - z}
\]
we denote the Fredholm determinant of the operator \( I - \mathcal{V}(n+2) \mathcal{V}_n r_n(\beta, z) \), where \( r_n(\beta, z) \) is the resolvent of the operator \( 2I + H_0(\pi - 2\beta, \pi - 2\beta) \), and \( \mathcal{V}_n \) is an integral operator with the kernel
\[
v(p, q) = \frac{1}{\pi} \sin p_1 \sin p_2 \sin q_1 \sin q_2.
\]

Through \( C_{11}^n \) denote the value of the following integral:
\[
C_{11}^n = \pi \int_{\mathbb{T}^2} \sin^2 p_1 \sin^2 p_2 \, dp_1 \, dp_2 = \int_{\mathbb{T}^2} \left| \psi_1^* (p_1) \right|^2 \left| \psi_1^* (p_2) \right|^2 \, dp_1 \, dp_2 = \frac{2 \varepsilon(p)}{2 \varepsilon(p)}.
\]

Simple calculations reveal the following approximate value \( C_{11}^n \approx 0.302347 \).

**Theorem 2.** Let \( \beta \in \left(0, \frac{\pi}{2}\right] \), \( n \in \mathbb{N} \).

a) If \( \mathcal{V}(n+2) < \frac{\sin \beta}{C_{11}^n} \), then the operator \( H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta) \) has no eigenvalues lying outside of the essential spectrum.

b) If \( \mathcal{V}(n+2) = \frac{\sin \beta}{C_{11}^n} \), then the left edge \( m(\beta) = 6 - 4 \sin \beta \) of the spectrum
of the operator $H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta)$ is an eigenvalue.

c) If $\sqrt{(n + 2)} > \frac{\sin \beta}{C_{11}}$, then the operator $H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta)$ has a unique nondegenerate eigenvalue $z_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta)$ below the essential spectrum.

d) The operator $H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta)$ has no embedded eigenvalues in the interval $(m(\beta), M(\beta))$.

This theorem is proved in similar way as Lemma 5. There are some differences:

1) In Theorem 2, the eigenvalue $z_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta)$ was calculated with the accuracy of $2\beta$:

$$z_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta) = 6 - \sqrt{(n + 2)} - \frac{2}{\sqrt{(n + 2)}} \sin^2 \beta + O(\beta^4)$$

and corresponding normalized eigenfunction has the form

$$f_{123}^{(n)}(p_1, p_2) = \frac{C_s(\beta) \sin p_1 \sin p_2}{6 - 2\sin \beta \cos p_1 - 2\sin \beta \cos p_2 - z_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta)} \in L_2(T^2), \quad (17)$$

where $C_s(\beta)$ is the normalizing multiplicity.

2) Left edge $m(\beta) = 6 - 2\sin \beta$ of the essential spectrum is a resonance for the operator $H_{123}^{(n)}(\pi - 2\beta, \pi)$, but for the operator $H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta)$ the left edge $m(\beta) = 6 - 4\sin \beta$ of the essential spectrum is the eigenvalue, i.e. the equation $H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta) f = m(\beta) f$ has a non-trivial solution

$$f(p_1, p_2) = \frac{C \sin p_1 \sin p_2}{2 - \cos p_1 - \cos p_2}$$

and it belongs to $L_2(T^2)$.

5. Conclusions

1) We have shown that the operator $H_{123}^{(n)}(k_1, k_2, \pi)$ has infinitely many invariant subspaces $G_{123}^{(n)}(n), n \in \mathbb{N}$. It has been proved that if condition $\sqrt{(n + 2)} > \sin \beta$ holds then the operator $H_{123}^{(n)}(\pi - 2\beta, \pi, \pi)$ has a unique simple eigenvalue $z_{123}^{(n)}(\pi - 2\beta, \pi)$ of the form (16), otherwise, the operator has no eigenvalues outside of the essential spectrum. A similar statement holds for the operator $H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta, \pi)$.

2) Without loss of generality it can be assumed that $\sqrt{(3)} \leq 1$. Since, if $\sqrt{(3)} > 1$ then it follows from $\lim_{n \to \infty} \sqrt{(n)} = 0$ that there exists a number $m \in \mathbb{N}$ such that $\sqrt{(m + 2)} \leq 1$ and monotonicity of $\sqrt{(n)}$ implies that $\sqrt{(n)} > 1$ for $n = 3, 4, \cdots, m + 1$, and in this case, the eigenvalues $z_{123}^{(n)}(\pi - 2\beta, \pi, \pi), n = 1, 2, \cdots, m - 1$ of $H_{123}^{(n)}(\pi - 2\beta, \pi, \pi)$ exist for all $\beta \in [0, \pi/2]$.

For a fixed $\beta \in (0, \pi/2]$ there exists $m \in \mathbb{N}$ such that $\sin \beta \in (\sqrt{(m + 3)}, \sqrt{(m + 2)})$ and the operator $H_{123}^{(n)}(\pi - 2\beta, \pi, \pi)$ has $m$ non-degenerate eigenvalues outside of the essential spectrum (see Theorem 1):
\[ z_{123}^{(i)}(\pi - 2\beta, \pi, \pi) := z_{123}^{(i)}(\pi - 2\beta, \pi, \pi) = 6 - \sqrt{3} \frac{1}{\sqrt{3}} \sin^2 \beta, \]
\[ z_{123}^{(2)}(\pi - 2\beta, \pi, \pi) := z_{123}^{(2)}(\pi - 2\beta, \pi, \pi) = 6 - \sqrt{4} \frac{1}{\sqrt{4}} \sin^2 \beta, \]
\[ \vdots \]
\[ z_{123}^{(m)}(\pi - 2\beta, \pi, \pi) := z_{123}^{(m)}(\pi - 2\beta, \pi, \pi) = 6 - \sqrt{m+2} \frac{1}{\sqrt{m+2}} \sin^2 \beta. \]

The corresponding normalized eigenfunctions are of the forms:
\[ f_{123}^{(i)}(p_1, p_2, p_3) = f_{123}^{(i)}(p_1, p_2) \psi^-_1(p_3) \psi^-_1(p_3) \in L^2(\mathbb{T}) \otimes L^1(1) \otimes L^1(1), \]
\[ f_{123}^{(2)}(p_1, p_2, p_3) = f_{123}^{(2)}(p_1, p_2) \psi^-_1(p_3) \psi^-_1(p_3) \in L^2(\mathbb{T}) \otimes L^1(1) \otimes L^2(2), \]
\[ \vdots \]
\[ f_{123}^{(m)}(p_1, p_2, p_3) = f_{123}^{(m)}(p_1, p_2) \psi^-_m(p_3) \psi^-_m(p_3) \in L^2(\mathbb{T}) \otimes L^1(1) \otimes L^m(m), \]
where \( f_{123}^{(m)} \) is the normalized eigenfunction of the operator \( H_{123}^{(m)}(\pi - 2\beta) \) corresponding to the eigenvalue \( z_{123}^{(m)}(\beta) \) and the operator \( H_{123}^{(m)}(k) \) is defined by the formula (13), \( \lambda(m) = \sqrt{m+2} \).

The eigenvalues of the operators \( H_{123}^{(1)}(\pi - 2\beta, \pi, \pi) \) and \( H_{123}(\pi, \pi - 2\beta, \pi) \) are same but eigenfunctions differ with variable replacement \( p_1 \) and \( p_2 \). In other words, the operators \( H_{123}^{(1)}(\pi - 2\beta, \pi, \pi) \) and \( H_{123}(\pi, \pi - 2\beta, \pi) \) are unitary equivalent.

In the case \( \sin \beta = \sqrt{m+2} \), the left edge \( m(\beta) = 6 - 2\sin \beta \) of the essential spectrum is a resonance of the operator \( H_{123}^{(1)}(\pi - 2\beta, \pi, \pi) \) (see Theorem 1).

3) Let for some \( m \in \mathbb{N} \) the relation \( \sin \beta \in (\sqrt{m+3}, \sqrt{m+2}) \) hold then the operator \( H_{123}^{(1)}(\pi - 2\beta, \pi - 2\beta, \pi) \) has \( m \) nondegenerate eigenvalues outside the essential spectrum (see Theorem 2) and for small \( \beta \):
\[ z_{123}^{(1)}(\pi - 2\beta, \pi - 2\beta, \pi) := z_{123}^{(1)}(\pi - 2\beta, \pi - 2\beta) = 6 - \sqrt{3} \frac{2}{\sqrt{3}} \sin^2 \beta + O(\beta^4), \]
\[ z_{123}^{(2)}(\pi - 2\beta, \pi - 2\beta, \pi) := z_{123}^{(2)}(\pi - 2\beta, \pi - 2\beta) = 6 - \sqrt{4} \frac{2}{\sqrt{4}} \sin^2 \beta + O(\beta^4), \]
\[ \vdots \]
\[ z_{123}^{(m)}(\pi - 2\beta, \pi - 2\beta, \pi) := z_{123}^{(m)}(\pi - 2\beta, \pi - 2\beta) = 6 - \sqrt{m+2} \frac{2}{\sqrt{m+2}} \sin^2 \beta + O(\beta^4). \]

The corresponding normalized eigenfunctions are of the forms:
\[ f_{123}^{(1)}(p_1, p_2, p_3) = f_{123}^{(1)}(p_1, p_2) \psi^-_1(p_3) \psi^-_1(p_3) \in L^2(\mathbb{T}) \otimes L^1(1), \]
\[ f_{123}^{(2)}(p_1, p_2, p_3) = f_{123}^{(2)}(p_1, p_2) \psi^-_1(p_3) \psi^-_1(p_3) \in L^2(\mathbb{T}) \otimes L^2(2), \]
where, \( f^{(n)}_{123} \) is the normalized eigenfunction of the operator 
\( H_{123}^{(m)}(\pi - 2\beta, \pi - 2\beta) \) corresponding to the eigenvalue 
\( z^{(n)}_{123} = (\pi - 2\beta, \pi - 2\beta) \) defined by the formula (17).

In the case \( \sin \beta = \sqrt{(m + 2)C_{11}^{-1}} \), the left edge \( m(\beta) = 6 - 4\sin \beta \) of the essential spectrum is the eigenvalue of 
\( H_{123}^{(m)}(\pi - 2\beta, \pi - 2\beta, \pi) \) (see Theorem 2) 
with the corresponding eigenfunction
\[
f(p) = C \frac{\sin p_1 \sin p_2}{2 - \cos p_1 - \cos p_2} \cdot \sin mp_3 \in L^2(T^3) \otimes L^2(m).
\]

**Remark 1.** If the potential \( \hat{v} \) is even in all arguments \( p_1, p_2, p_3 \) and the condition \( \hat{v} \in \ell_2(Z^3) \) holds, then the statements of Lemmas 3 - 4 remain valid.

**Remark 2.** If \( k_3 \neq \pi \), then the subspaces \( \mathcal{H}_{123}(n), n \in \mathbb{N} \) are not invariant under the operator \( H_{123}(k_1, k_2, k_3) \).

**Acknowledgements**

This work was supported by the Grant OT-F4-66 of Fundamental Science Foundation of Uzbekistan.

**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

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