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Revision of Stationary Schrödinger Equation

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Abstract

Presence of centripetal force field in space shall cause time dilation of any clock at rest therein. Therefore, duration of unit of time determined by any clock in such field is not constant but varies with location of the clock in the field. This means that speed of light in vacuo in centripetal force field is not and cannot be a true physical constant but a function of location in such field because definition of c involves a unit of time and duration of that time unit varies with location in such field. However, classical Schrödinger equation assumes *a prior* the constancy of c in field, even though this may not be the case. Therefore, it is necessary to revise the classical equation in order to comply with the law of mass-energy equivalence of Einstein hence time dilation in centripetal force field.

Keywords

Wave Mechanics, Schrödinger Equation, Gravitation

1. Introduction

Metrological analysis showed that, under the law of mass-energy equivalence (LME) [1], physical space must be finite and time dilation shall occur to unit of time defined on atomic clock (atomic time, AT) in motion therein [2]. It was also shown that, under the law of Planck on photon energy [3], in conjunction with the law of energy conservation, Planck constant is a state invariant (SIT) in centripetal force field and time dilation shall occur in such field regardless of choice/definition of energy/time [4]. Static electric field (SEF) is a centripetal force field, therefore, presence of SEF shall alter unit of AT therein. Therefore, duration of unit of AT in SEF is not SIT but state variant of the field, *i.e.*, function of location of clock defining particle at rest in the field. By definition, speed of light in vacuo (SLV) is inversely proportional to duration of unit of AT [5]. Metrological analysis showed that length of unit of length in space is SIT even under field environment

[2] [4]. Therefore, SLV is not and cannot be SIT in SEF but varies in the field. However, constancy, hence state invariance, of SLV was assumed *a prior* in Schrödinger equation [6] and general category of wave mechanic equations (WME). It is therefore necessary to revise such in order to comply with LME and state-dependent SLV.

2. The Law of Coulomb on Electrostatics

The original law of Coulomb on electrostatics (LCE) [7] for charge particles at rest in space is expressed as, under infinite space approximation (ISA),

$$\boldsymbol{F}_{k,j} = \frac{q_j q_k}{4\pi\varepsilon_0}\boldsymbol{f}_{k,j},\ \boldsymbol{f}_{k,j} \equiv \frac{\boldsymbol{r}_k - \boldsymbol{r}_j}{\left\|\boldsymbol{r}_k - \boldsymbol{r}_j\right\|^3}. \tag{1}$$

$\boldsymbol{F}_{k,j}$: Electrostatic force experienced by charge particle k at rest in field of charge particle j at rest. ε_0 : Electric constant. q : Rest charge of particle. $\boldsymbol{r}$: Location vector of charge particle.

Retaining symmetry of the original law on the ground of *interaction*, LCE is rewritten as

$$\boldsymbol{F}_{k,j,x} = \frac{q_{j,x} q_{k,x}}{\sqrt{4\pi\varepsilon_{0,j,x}}\sqrt{4\pi\varepsilon_{0,k,x}}}\boldsymbol{f}_{k,j},\ x = f, 0, e. \tag{2}$$

x : State indicator, perspective at location of associated entity at rest in SEF.

By definition [8],

$$\begin{gathered}\frac{1}{4\pi\varepsilon_0} = \frac{\alpha\hbar c}{\boldsymbol{e}^2} \quad\rightarrow\quad \frac{1}{4\pi\varepsilon_{0,x}} = \frac{\alpha_x \hbar_x c_x}{\boldsymbol{e}_x^2} \quad\rightarrow \\ \boldsymbol{F}_{k,j,x} = \sqrt{\alpha_{j,x}\hbar_{j,x}c_{j,x}}\sqrt{\alpha_{k,x}\hbar_{k,x}c_{k,x}}\frac{q_{j,x}}{\boldsymbol{e}_{j,x}}\frac{q_{k,x}}{\boldsymbol{e}_{k,x}}\boldsymbol{f}_{k,j}.\end{gathered} \tag{3}$$

α : Fine structure constant. $\hbar$: Planck constant, $\hbar \equiv h/2\pi$. c : SLV. $\boldsymbol{e}$: Elementary charge.

By definition of charge,

$$q \equiv n_q \boldsymbol{e} \quad\rightarrow\quad q_x \equiv n_{q,x}\boldsymbol{e}_x. \tag{4}$$

n_q : Numeral aspect of charge.

By the rule of metrology [4],

$$n_{q,x} = \frac{q_x}{\boldsymbol{e}_x} \subset \text{SIT} \quad\rightarrow\quad n_{q,x} = n_q \quad\rightarrow\quad q_x = n_q \boldsymbol{e}_x. \tag{5}$$

That is, numeral aspect of charge of charge particle, referred to as reduced charge, is SIT if charge of an entity and unit of the charge are in identical state. Therefore, counted in reduced form, amount of charges of any charge system is SIT. Therefore, electric neutrality of electric neutral system is SIT, *i.e.*, such system shall remain electrically neutral regardless of state of the system if charges of the system are counted in reduced form. Accordingly,

$$\boldsymbol{F}_{k,j,x} = c_{\boldsymbol{i}}\sqrt{\alpha_{j,x}\hbar_{j,x}}\sqrt{\alpha_{k,x}\hbar_{k,x}}\beta_{e,j,x}^{1/2} n_{q,j}\beta_{e,k,x}^{1/2} n_{q,k}\boldsymbol{f}_{k,j},\ \beta_{e,y,x} \equiv \frac{c_{y,x}}{c_{y,\boldsymbol{i}}},\ y \in \{j,k\}. \tag{6}$$

β_e : Referred to as Coulomb Factor. c_i : SLV as measured/defined in Rest State (RS) [2]. $c_{j,x}$: SLV as measured by particle j at rest in field of others.

Since Planck constant is SIT in SEF,

$$\hbar_x = \hbar_i \equiv \hbar \quad \to \quad \boldsymbol{F}_{k,j,x} = \hbar c_i \sqrt{\alpha_{j,x}}\sqrt{\alpha_{k,x}}\,\beta_{e,j,x}^{1/2} n_{q,j}\,\beta_{e,k,x}^{1/2} n_{q,k}\,\boldsymbol{f}_{k,j}\,. \tag{7}$$

Assumption 3: static fine structure constant is SIT,

$$\alpha_x = \alpha_i \equiv \alpha \quad \to \quad \boldsymbol{F}_{k,j,x} = \alpha\hbar c_i\,\beta_{e,j,x}^{1/2} n_{q,j}\,\beta_{e,k,x}^{1/2} n_{q,k}\,\boldsymbol{f}_{k,j}\,. \tag{8}$$

That is, it is **assumed** that outcome of measurement of static fine structure constant shall not be affected by where in a field setup for such measurement is located regardless of strength of the field as long as the field is static in Rest Frame (RF) [2] and setup for the measurement is at rest in the field. Therefore, under **Assumption 3**, static fine structure constant is identical/one and same/indifferent whether measurement of the entity is conducted in far field (field strength approaching zero) or vicinity of electric Schwarzschild Sphere (ESS) as long as field and setup for measurement are both at rest with respect to RF.

Denote

$$q^* \equiv \beta_{e,q}^{1/2} n_q \quad \to \quad \boldsymbol{F}_{k,j,x} = \alpha\hbar c_i\, q_{j,x}^* q_{k,x}^*\,\boldsymbol{f}_{k,j}\,. \tag{9}$$

q^*: Rest charge parameter of charge particle in reduced form at rest in field of others.

This is the LCE in infinite space under **Assumption 3**.

3. The Law of Coulomb Electrostatics in Finite Space

Consider a charge particle in RS in $\mathbf{S}^3$ (three dimensional finite space) with its location therein assigned as origin of RF. Then, exists SEF in association with the charge at rest in $\mathbf{S}^3$ with center of the field at the origin. Such field can be probed by test charge (effect of probe charge to field being probed is ignored/disregarded by definition of *test* charge). Thus, from Equation (2),

$$\boldsymbol{F}_{p,x} = \frac{q_{f,x} q_{p,x} f_s}{\sqrt{4\pi\varepsilon_{0,f,x}}\sqrt{4\pi\varepsilon_{0,p,x}}}\boldsymbol{e}_\varphi\,. \tag{10}$$

$\boldsymbol{F}_{p,x}$: Electrostatic force probed by probe at rest in field. $q_{f,x}$: Rest charge of particle causing the field. $\varepsilon_{0,f,x}$: Electric constant as measured at rest at location of the particle causing the field. $q_{p,x}$: Rest charge of probe at rest in the field. $\varepsilon_{0,p,x}$: Electric constant as measured at rest at location of probe in the field. s: Internal distance between charge particle and probe. f_s: Function of $\boldsymbol{s}$. $\boldsymbol{e}_\varphi$: Unit vector of $\boldsymbol{s}$ at location of probe in field in $\mathbf{S}^3$. x: State indicator.

By definition of electric field,

$$\boldsymbol{E} \equiv \frac{\boldsymbol{F}_p}{q_p} \quad \to \quad \boldsymbol{E}_{f,x} \equiv \frac{\boldsymbol{F}_{p,x}}{q_{p,x}} = \frac{q_{f,x}}{\sqrt{4\pi\varepsilon_{0,f,x}}\sqrt{4\pi\varepsilon_{0,p,x}}} f_s \boldsymbol{e}_\varphi\,. \tag{11}$$

$\boldsymbol{E}$: Electric field. q_p: Probe charge. $\boldsymbol{E}_f$: SEF associated with charge causing the field.

In classical field theories, field of an object is an intrinsic property of the object alone that is unaltered regardless of presence/absence and/or action/inaction of others. Accordingly, plurality of fields is vector additive in space. However, due to LME and state invariance of Planck constant in field, presence of charge in field of charge particle shall cause alteration of the electric constant associated with the particle hence rest charge parameter of same, therefore alteration of the field of the charge particle. Therefore, classical field is incompatible with LME, and SEF as expressed in Expression (11) also depends on others, hence plurality of the fields is not simple vector additive but interactive.

By flux conservation theorem of Gauss for SEF,

$$\oiint_{\boldsymbol{S}} \varepsilon_0 \boldsymbol{E} \cdot d\boldsymbol{S} = q \rightarrow \oiint_{\mathrm{SPS}} \sqrt{\varepsilon_{0,f,x}} \sqrt{\varepsilon_{0,p,x}} \boldsymbol{E}_{f,x} \cdot d\boldsymbol{S} = q_{f,x} . \tag{12}$$

$\boldsymbol{S}$: Any nonlocal simultaneous enclosure of simple/single connectivity. $d\boldsymbol{S}$: Areal element of the enclosure. q : Charge enclosed by the enclosure. SPS: Spherical probing sphere.

In spherical coordinates of $\mathbf{S}^3$ [2],

$$d\boldsymbol{S}_\varphi = \boldsymbol{e}_\varphi R_e^2 \sin^2 \varphi \sin \vartheta d\vartheta d\theta \rightarrow f_s = \frac{1}{R_e^2 \sin^2 \left[s / R_e \right]} . \tag{13}$$

R_e : External radius of $\mathbf{S}^3$. $d\boldsymbol{S}_\varphi$: Areal element of SPS.

Therefore, LCE in $\mathbf{S}^3$ is expressed as, for SEF centered at internal origin,

$$\boldsymbol{F}_e = \frac{\alpha \hbar c_{\boldsymbol{i}} q_s^* q_e^*}{R_e^2 \sin^2 \left[s / R_e \right]} \boldsymbol{e}_\varphi . \tag{14}$$

$\boldsymbol{F}_e$: Electrostatic force experienced by charge particle at rest in SEF. q_s^* : Rest charge parameter of charge causing the field. q_e^* : Rest charge parameter of charge particle at rest in the field. s : Internal distance between field causing charge and charge in the field.

In vicinity of the origin,

$$s \ll R_e \rightarrow \boldsymbol{F}_e \simeq \frac{\alpha \hbar c_{\boldsymbol{i}} q_s^* q_e^*}{s^2} \hat{\boldsymbol{s}} . \tag{15}$$

That is, in near field of origin (in comparison with external radius of finite space), LCE is of the familiar form, as an approximation.

4. Electron in Static Electric Field of Proton

Consider an electron at rest in field of a proton in RS at origin of RF. From Equation (15), electrostatic force experienced by the electron is, under ISA,

$$\begin{matrix} \beta_{e,p,x} = 1 \\ \beta_{e,e,x} \equiv \beta_e \end{matrix} , \begin{matrix} n_{q,p} = +1 \\ n_{q,e} = -1 \end{matrix} \rightarrow \boldsymbol{F}_{e,x} = -\frac{\alpha \hbar c_{\boldsymbol{i}} \beta_e^{1/2}}{r^2} \hat{\boldsymbol{r}} . \tag{16}$$

$\beta_{e,p,x}$: Coulomb Factor at rest at location of proton in field of others. $\beta_{e,e,x}$: Coulomb Factor at rest at location of electron in SEF of proton. $n_{q,p}$: Reduced charge of proton. $n_{q,e}$: Reduced charge of electron. $\boldsymbol{F}_{e,x}$: Electrostatic force experienced by electron at rest in SEF of proton centered at origin. $\boldsymbol{r}$: Distance between electron and proton. $\hat{\boldsymbol{r}}$: Unit vector of $\boldsymbol{r}$ at location of electron in SEF of proton.

Denote

$$r_e \equiv \frac{\alpha \hbar c_{\boldsymbol{i}}}{E_{e,\boldsymbol{i}}}, \rho \equiv \frac{r}{r_e} \rightarrow \boldsymbol{F}_{e,x} = -\frac{E_{e,\boldsymbol{i}}}{r_e} \frac{\beta_e^{1/2}}{\rho^2} \hat{\boldsymbol{\rho}} . \tag{17}$$

r_e : Characteristic length of field (CLF) of elementary charge in RS. $E_{e,i}$: Self energy of electron in RS. ρ : Distance between electron and proton, in reduced unit.

The CLF above is also known as classical electron radius [9].

Consider relocating the electron in SEF of proton in infinitesimal displacement/velocity. Work done to the electron is

$$dw_{e,x} = -\boldsymbol{F}_{e,x} \cdot d\boldsymbol{r} \quad \rightarrow \quad \frac{dw_{e,x}}{d\rho} = E_{e,\boldsymbol{i}} \frac{\beta_e^{1/2}}{\rho^2}. \tag{18}$$

$dw_{e,x}$: Work done for infinitesimal displacement of electron in infinitesimal velocity in SEF of proton.

By the law of energy conservation, work done to the electron in such manner is gained by the electron as self energy increment,

$$dE_{e,x} = dw_{e,x} \quad \rightarrow \quad \frac{dE_{e,x}}{d\rho} = E_{e,\boldsymbol{i}} \frac{\beta_e^{1/2}}{\rho^2}. \tag{19}$$

$E_{e,x}$: Self energy of electron at rest in SEF of proton.

It has been shown that static Planck constant in centripetal force field is field invariant [4] and

$$E_{s,0,f} = \beta_f E_{s,\boldsymbol{i}}, \; \beta_f \equiv \frac{c_{s,0,f}}{c_{\boldsymbol{i}}}. \tag{20}$$

$E_{s,0,f}$: Self energy of a particle at rest in static centripetal force field. $c_{s,0,f}$: SLV as measured at rest at location of the particle in the field. $E_{s,i}$: Self energy of the particle in RS.

SEF is centripetal force field, therefore, with Equation (19),

$$\frac{d\beta_e}{d\rho} = \frac{\beta_e^{1/2}}{\rho^2} \quad \rightarrow \quad \beta_e^{1/2} = 1 - \frac{1}{2\rho} \quad \rightarrow \quad \begin{array}{l} \rho_{\text{ESS}} \equiv \rho\big|_{\beta_e=0} = \dfrac{1}{2}. \\ \rho_{\text{ESS}} \le \rho < \infty \end{array} \tag{21}$$

ρ_{ESS} : Radius of ESS in reduced unit.

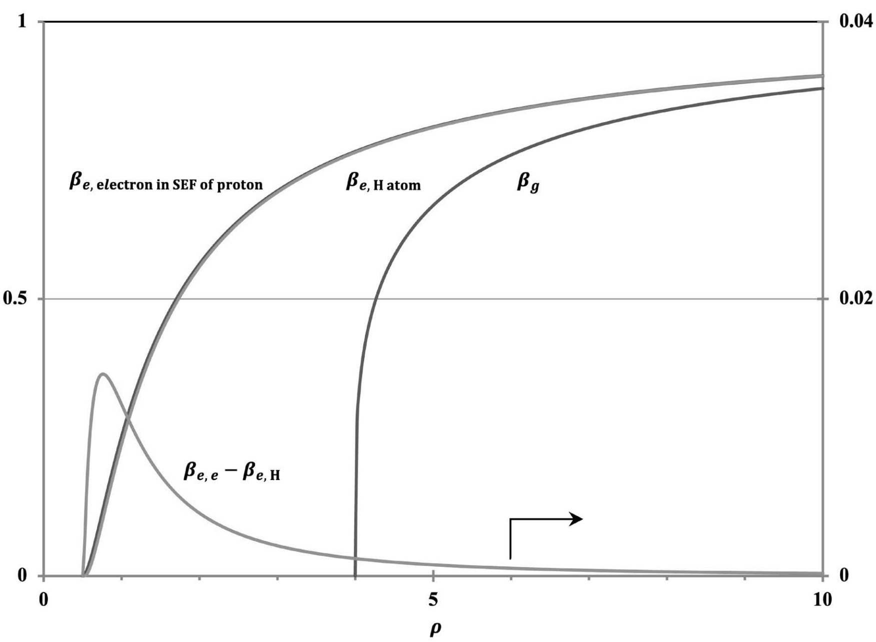

Figure 1. Self energy of particle in field, in reduced unit.

Figure 1 plots out self energy of electron in SEF of proton, in reduced unit. Note that left bound of the domain is at radius of ESS. Left side of the left bound is forbidden zone since, by definition, $\beta_e^{1/2}$ must be a real entity.

For $\mathbf{S}^3$,

$$\beta_e^{1/2} = 1 - \frac{\cot\varphi}{2\mathcal{R}_e},\ \varphi \equiv \frac{\rho}{\mathcal{R}_e} \ \rightarrow\ \varphi_{\text{ESS}} = \arctan\frac{1}{2\mathcal{R}_e} < \varphi \le \frac{\pi}{2}. \tag{22}$$

$\mathcal{R}_e$: External radius of $\mathbf{S}^3$ in unit of CLF.

No point in finite space is free of field of any kind. Therefore, reference state for the field is chosen at internal radius of the space, whereat, strength of the field centered at internal origin is minimal hence an approximation to true RS.

From Equation (17), under ISA,

$$\begin{gathered}\boldsymbol{\mathcal{F}}_e \equiv \frac{r_e}{E_{e,\boldsymbol{i}}}\boldsymbol{F}_e \ \rightarrow\ \boldsymbol{\mathcal{F}}_{e,x} = -\left(1-\frac{1}{2\rho}\right)\frac{\hat{\boldsymbol{\rho}}}{\rho^2},\ \frac{1}{2} \le \rho < \infty \ \rightarrow \\ \mathcal{F}_{e,x,\max} = \frac{16}{27},\ \rho_{\mathcal{F}_{e,x},\max} = \frac{3}{4},\ \mathcal{F} \equiv \|\boldsymbol{\mathcal{F}}\| \end{gathered}. \tag{23}$$

$\boldsymbol{\mathcal{F}}_{e,x}$: Electrostatic force experienced by an electron at rest in SEF of proton, in reduced unit.

That is, strength of electrostatic attraction between electron and proton is not a monotonic function of the distance between the parties but shall reach maxima at 3/4 of the distance unit and then approach zero towards ESS and become so at ESS. Therefore, if electron is sufficiently close to proton then electrostatic interaction between the parties shall become diminishing. On the other hand, reduced charges of the parties are SIT hence shall remain intact.

From Equation (20), under ISA,

$$E_{e,s,0,e} = E_{e,x} = \beta_e E_{e,\boldsymbol{i}} = \left(1-\frac{1}{2\rho}\right)^2 E_{e,\boldsymbol{i}} \ \rightarrow\ E_{e,x}\big|_{\rho_{\text{ESS}} \le \rho < \infty} < E_{e,\boldsymbol{i}}. \tag{24}$$

$E_{e,s,0,e}$: Self energy of electron at rest in SEF of proton. $E_{e,x}$: Rest energy of electron at rest in SEF of proton.

That is, self energy of an electron shall approach zero towards ESS and become so at ESS. Therefore, if an electron is at rest in vicinity of ESS then self energy of the electron hence difference thereof shall be approaching zero. Further, at ESS, electron can neither emit nor absorb photon of any kind. For same reason, if an electron is in motion in close proximity of ESS then emission/absorption of photon shall not be an effective channel for the electron to lose/gain energy. Therefore, close proximity of ESS is zone of optical silence.

By definition of Coulomb Factor, Expression (20),

$$c_{e,x} = \beta_e c_{\boldsymbol{i}} \ \rightarrow\ c_{e,x}\big|_{\rho_{\text{ESS}} \le \rho < \infty} < c_{\boldsymbol{i}}. \tag{25}$$

That is, SLV in SEF shall approach zero towards ESS and become so at ESS. Therefore, if an electron is at rest in vicinity of ESS then SLV as measured by the electron shall be slower than that as measured away from ESS. Therefore, if an electron is in motion in vicinity of ESS then velocity of the electron shall be slower than that in region away from ESS even if it were approaching local SLV in the field. Further, if an electron were to land at ESS then magnitude of its touchdown velocity cannot exceed local SLV thereat, which is zero.

By definition of SLV [10], with state invariance of unit of length and Planck constant in SEF,

$$\mathbb{U}_{t,x} = \frac{\mathbb{U}_{t,\boldsymbol{i}}}{\beta_e} \quad \rightarrow \quad \mathbb{U}_{t,x}\Big|_{\rho_{\mathrm{ESS}} \le \rho < \infty} > \mathbb{U}_{t,\boldsymbol{i}}. \tag{26}$$

$\mathbb{U}_{t,x}$: Unit of self time of electron at rest in SEF of proton. $\mathbb{U}_{t,i}$: Unit of Rest Time [2].

That is, time dilation of unit of self time of an electron in SEF shall occur regardless of choice/definition of the self time. If the electron is to approach ESS then the clock defining self time of the electron shall cease to work. On the other hand, lacking of particular set of recurring events, by itself, shall have no effect to any other event of the electron, *i.e.*, such by itself shall not prevent other event of the electron from happening, since events happen or not happen regardless of status of assigned set of recurring events [2].

By LME, from Equation (20),

$$m_{e,s,0,e} = m_{e,x} = \frac{m_{e,\boldsymbol{i}}}{\beta_e} \quad \rightarrow \quad m_{e,s,0,e}\Big|_{\rho_{\mathrm{ESS}} \le \rho < \infty} > m_{e,\boldsymbol{i}}. \tag{27}$$

$m_{e,s,0,e}$: Self mass of electron at rest in SEF of proton. $m_{e,x}$: Rest mass of electron at rest in SEF of proton. $m_{e,i}$: Self/rest mass of electron in RS.

That is, rest mass of an electron at rest in SEF of proton shall approach infinity towards ESS and become so at ESS. Therefore, if an electron is at rest in vicinity of ESS then rest mass of the electron as measured by itself and/or at rest at location of the electron in the field shall be heavier than otherwise. Therefore, according to the law of gravitation of Newton, gravitation interaction between the parties shall become non-negligible. Therefore, the electron shall be gravitationally attracted towards or repelled from the proton pending on type of rest mass of electron with respect to that of proton.

5. Motion of Electron in Static Electric Field of Proton

In particle model ignoring spin of electron, self field interaction and magnetic field induced by motion of electron, other types of forces, etc., equation of motion of an electron in SEF of proton is, under **Assumption 3** and ISA, from Equation (17),

$$\frac{d\boldsymbol{P}_{e,f,u,e}}{dt_{e,s,u,e}} = \boldsymbol{F}_{e,f,0,e} = -\frac{m_{e,\boldsymbol{i}} c_{\boldsymbol{i}}^2}{r_e} \frac{\beta_e^{1/2}}{\rho^2} \hat{\boldsymbol{\rho}},\ \beta_e^{1/2} = 1 - \frac{1}{2\rho},\ \frac{1}{2} \le \rho < \infty. \tag{28}$$

$\boldsymbol{P}_{e,f,u,e}$: Field momentum of an electron in motion in SEF of proton at origin, as perceived at rest at location of the electron in the field. $t_{e,s,u,e}$: Self time of the electron in motion in the field [4].

According to the second law of Newtonian mechanics, alteration of momentum of a particle is caused by and equal to net force experienced by the particle. However, momentum is an attribute of the object it is in association with that is perspective dependent, and there are two but only two perspectives in particle-field system, *i.e.*, perspective of particle and that of field. From perspective of a particle, momentum of the particle is and is always zero regardless of the force it is experienced since a particle is and is always at rest with respect to itself, regardless. Therefore, the momentum referred to by the law has to be the momentum perceived from perspective of field, *i.e.*, field momentum. There also have two but only two types of time available in particle-field system, *i.e.*, self time of particle and local time in field. However, due to time dilation in centripetal force field, Equation (26), there can be no such thing, in general, as locally defined common time of the field. In other words, local time in such field is generally non differentiable. Therefore, the time differential referred to by the law has to be self time of particle.

By definition of field momentum, under LME,

$$\boldsymbol{P}_{e,f,u,e} \equiv m_{e,f,u,e}\boldsymbol{v}_{e,f,u,e} = \frac{m_{e,f,0,e}}{\beta_u}c_{e,f,u,e}\boldsymbol{u} = \frac{m_{e,f,0,e}}{\beta_u}c_{e,f,0,e}\boldsymbol{u} = m_{e,\boldsymbol{i}}c_{\boldsymbol{i}}\frac{\boldsymbol{u}}{\beta_u}. \quad (29)$$

$m_{e,f,u,e}$: Mass of an electron in motion in SEF of proton as perceived at rest at location of the electron in the field. $\boldsymbol{v}_{e,f,u,e}$: Velocity of an electron in motion in SEF of proton as perceived at rest at location of the electron in the field. β_u: Lorentz Factor of electron in absolute motion in finite space/bound state [2]. $\boldsymbol{u}$: Reduced velocity of electron in motion.

By definition of self time, under LME,

$$dt_{e,s,u,e} = \frac{dt_{\boldsymbol{i}}}{\beta_u\beta_e} \rightarrow \beta_u\frac{d}{d\tau}\frac{\boldsymbol{u}}{\beta_u} = -\frac{1}{\beta_e^{1/2}}\frac{\hat{\boldsymbol{\rho}}}{\rho^2},\ d\tau \equiv \frac{c_{\boldsymbol{i}}}{r_e}dt_{\boldsymbol{i}}. \quad (30)$$

t_i: Rest Time [2] [4].

Thus, in reduced units, equation of motion of an electron in SEF of proton is

$$\boldsymbol{a} - f_u\boldsymbol{u} + f_e\hat{\boldsymbol{\rho}} = \boldsymbol{0},\ \boldsymbol{a} \equiv \frac{d\boldsymbol{u}}{d\tau},\ f_u \equiv \frac{d\ln\beta_u}{d\tau},\ \beta_u \equiv \sqrt{1-u^2},\ f_e \equiv \frac{1}{\beta_e^{1/2}\rho^2}. \quad (31)$$

Therefore, motion of charge particle in SEF is identical in form to that of mass particle in static gravitation field (SGF) [4].

With cylindrical coordination system, Equation (31) is decomposed to

$$\begin{aligned}
&\begin{matrix} 0 = \rho\psi + 2\omega q - f_u\rho\omega \\ 0 = \chi - \rho\omega^2 - f_u q + f_e \end{matrix},\quad \begin{matrix} q \equiv \rho' & \omega \equiv \theta' \\ \chi \equiv q' & \psi \equiv \omega' \end{matrix},\quad ' \equiv \frac{d}{d\tau} \rightarrow \\
&\begin{matrix} \psi = q\omega\left(f_e - \dfrac{2}{\rho}\right) \\ \chi = \rho\omega^2 - \left(1-q^2\right)f_e \end{matrix} \rightarrow \begin{matrix} d\ln\dfrac{\beta_e}{\beta_u} = 0 \\ d\ln\dfrac{\rho^2\omega}{\beta_u} = 0 \end{matrix} \rightarrow \begin{matrix} \dfrac{\beta_e}{\beta_u} = \epsilon \\ \dfrac{\rho^2\omega}{\beta_u} = \gamma \end{matrix}.
\end{aligned} \quad (32)$$

ϵ: Integration constant, total energy of an electron in SEF of proton in reduced unit. γ: Integration constant, angular momentum of an electron in SEF of proton in reduced unit.

Thus,

$$\begin{aligned}
&\omega^2 = \frac{\gamma^2\beta_e^2}{\epsilon^2\rho^4},\ q^2 = 1 - \frac{\beta_e^2}{\epsilon^2}\left(1 + \frac{\gamma^2}{\rho^2}\right),\ \epsilon \neq 0 \rightarrow \\
&\lim_{\beta_e\to 0}\beta_u = 0,\ \lim_{\beta_u\to 0}\rho\omega = 0 \rightarrow \lim_{\beta_e\to 0}u^2 = 1,\ \lim_{\beta_u\to 0}q^2 = 1
\end{aligned}. \quad (33)$$

That is, if an electron were to rendezvous onto ESS of proton then, at landing site, transverse velocity of the electron shall approach zero and impact velocity approach local SLV, which is zero along surface normal at landing site, while total energy and angular momentum of the electron shall remain intact during such rendezvous.

For an electron in orbital motion outside ESS of SEF of proton, at apsis distances,

$$q\big|_{\rho=a,b}=0,\ \frac{1}{2}<b\le a\quad\rightarrow\quad\begin{array}{l}\epsilon^2=\dfrac{\left(a^2-b^2\right)\beta_{e,a}^2\beta_{e,b}^2}{a^2\beta_{e,b}^2-b^2\beta_{e,a}^2}\\ \gamma^2=\dfrac{a^2b^2\left(\beta_{e,a}^2-\beta_{e,b}^2\right)}{a^2\beta_{e,b}^2-b^2\beta_{e,a}^2}\end{array}.\tag{34}$$

a and b: Maximum and minimum distance between orbiting electron and SEF center in reduced unit.

Therefore, orbital energy of electron in SEF of proton is completely determined by orbital apsides, and circular orbit is of relatively lower energy. Further, exists minima in orbital energy of electron in circular motion in SEF of proton, referred to herein as ground state,

$$\lim_{b\to a}\epsilon_{\mathrm{o}}=\frac{(2a-1)^{5/2}}{4a^2\sqrt{2a-3}}\quad\rightarrow\quad a>\frac{3}{2},\ \epsilon_{\mathrm{o,min}}=\epsilon_{\mathrm{o}}\Big|_{a=3}=\frac{1}{4}\left(\frac{5}{3}\right)^{5/2},\ u_{\mathrm{o,min}}=\left(\frac{2}{5}\right)^{1/2}.\tag{35}$$

ϵ_{o}: Total energy of an electron in circular orbit in SEF of proton, in reduced unit.

If an electron is orbiting in vicinity of ESS of proton closer than that of the ground state then orbital energy of the electron shall become higher for such maneuvering, which shall approach infinity at orbital radius $\rho_{\mathrm{ESS}}+1$, *i.e.*, one r_e away from the ESS. Therefore, even under tremendous electrostatic attraction in vicinity of ESS, an electron in circular orbital motion cannot maneuver onto the ESS. On the other hand, even in the ground state, total energy of an electron is marginally lower than that in RS, *i.e.*, $\sim 0.987E_{e,i}$.

By definition of apsidal precession, from Equation (33),

$$\begin{aligned}T_{AP}&=\frac{\pi(a+b)^{3/2}}{\sqrt{2}}\left(1+\frac{2}{a+b}+\cdots\right),\\ \theta_{AP}&=\frac{5\pi}{4}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{95\pi}{64}\left(\frac{1}{a}+\frac{1}{b}\right)^2+\cdots.\end{aligned}\tag{36}$$

T_{AP}: Apsidal orbital period, duration from a moment of an apside configuration to next of same, in reduced Rest Time. θ_{AP}: Apsidal precession of an electron orbiting in SEF of proton per apsidal orbital period.

That is, electron orbiting in SEF of proton shall have apsidal precession, which is always positive, *i.e.*, beyond 2π per apsidal orbital period.

6. Energy State of Electron in Static Electric Field of Proton

In contrast to particle model, object in wave mechanics (WM) [11] is perceived as extended object (EO) or in association with such. Due to time dilation in centripetal force field, self time of an EO in such field is generally indefinable hence the concept meaningless. On the other hand, total energy of an electron in such field is invariant to location, time, state of motion of the electron, etc., ignoring spin, gravitation, interaction of self field and magnetic field associated with motion of the electron, and assuming stativity of the field. Such invariance of such attribute can be utilized to decipher energy state of such EO in such field by WM. From Expression (33), under **Assumption 3** and ISA,

$$\epsilon_e \equiv \frac{E_{e,f,u,e}}{E_{e,\boldsymbol{i}}} = \frac{\beta_e}{\beta_u},\ \epsilon_e \neq 0 \ \rightarrow\ \frac{\epsilon_e^2}{\beta_e^2} = \frac{1}{1-u^2},\ \beta_e \neq 0. \tag{37}$$

ϵ_e : Total energy of an electron in motion in SEF as measured at rest in the field, in reduced unit. $E_{e,f,u,e}$: Total energy of an electron in motion in SEF as measured at rest in the field. $E_{e,i}$: Self energy of electron in RS.

From Expression (29),

$$\begin{gathered}\boldsymbol{P}_{e,f,u,e} = m_{e,\boldsymbol{i}} c_{\boldsymbol{i}} \frac{\boldsymbol{u}}{\beta_u} \ \rightarrow\ \boldsymbol{p}_e \equiv \frac{\boldsymbol{P}_{e,f,u,e}}{m_{e,\boldsymbol{i}} c_{\boldsymbol{i}}} = \frac{\boldsymbol{u}}{\beta_u} \ \rightarrow \\ 1 + p_e^2 = \frac{1}{1-u^2} \ \rightarrow\ -p_e^2 + \frac{\epsilon_e^2}{\beta_e^2} = 1.\end{gathered} \tag{38}$$

$\boldsymbol{P}_{e,f,u,e}$: Field momentum of an electron in motion in SEF as perceived at rest in the field. $\boldsymbol{p}_e$: Field momentum of an electron in motion in SEF as perceived at rest in the field, in reduced unit.

According to WM [6], exists correspondence relationship between field momentum of an object and spacial gradient of wave function (WF) of same, expressed as

$$\boldsymbol{P}_e \sim i\hbar\nabla_r,\ i \equiv \sqrt{-1} \ \rightarrow\ \boldsymbol{p}_e \sim \frac{1}{m_{e,\boldsymbol{i}} c_{\boldsymbol{i}}} \frac{i\hbar_{\boldsymbol{i}}}{r_e} \nabla_\rho = \frac{i}{\alpha_e} \nabla_\rho. \tag{39}$$

∇_r : Gradient operator in unit of length of space. ∇_ρ : Gradient operator in unit of CLF. α_e : Fine structure constant.

Thus, under **Assumption 3**,

$$\begin{gathered}\frac{\Delta_\rho}{\alpha_e^2} + \frac{\epsilon_e^2}{\beta_e^2} = \hat{\mathrm{I}},\ \beta_e^{1/2} = 1 - \frac{1}{2\rho},\ \frac{1}{2} < \rho < \infty \ \rightarrow \\ \left(\frac{\Delta_\rho}{\alpha_e^2} + \frac{\epsilon_e^2}{\beta_e^2} - 1\right)\Psi = 0,\ \ \begin{matrix}\lim\limits_{\rho\to 1/2} \Psi = 0 \\ \lim\limits_{\rho\to\infty} \Psi = 0\end{matrix}.\end{gathered} \tag{40}$$

Δ_ρ : Laplace operator in unit of CLF. $\hat{\mathrm{I}}$: Identity operator. Ψ: WF of EO associated with an electron in motion in SEF of proton.

This is the WME complying with LME and finite space/bound state for electron in SEF of proton under the approximations aforementioned. In such, attributes are expressed in reduced form, *i.e.*, in units of the corresponding ones measured/defined in RS, e.g., energy in unit of $m_{e,\boldsymbol{i}} c_{\boldsymbol{i}}^2$, momentum in $m_{e,\boldsymbol{i}} c_{\boldsymbol{i}}$, length in r_e, etc. Note that it is the domain boundaries of the Coulomb Factor that determines the boundaries of ρ in the WME, even though the equation itself may appear not being bounded by such. The boundary conditions of Ψ are to ensure convergence of the WF in entire domain that is valid physically.

In far field,

$$\rho \gg \frac{1}{2} \quad \to \quad \frac{1}{\beta_e^2} \simeq 1 + \frac{2}{\rho} \quad \to \quad \left(\frac{\Delta_\rho}{2\alpha_e^2} + \frac{1}{\rho}\right)\Psi \simeq -\varepsilon_e \Psi, \quad \begin{matrix} \epsilon_e \equiv 1 - \varepsilon_e \\ 0 \le \varepsilon_e \ll 1 \end{matrix}. \tag{41}$$

Therefore, the classical equation is but an approximation of WME (40) under the far field condition, whiles form of the approximated WME may lead to over interpretation of the corresponding WF in near field.

Note also that any WF has or should have unit in association since WF is an attribute of the object it is in association with, and attribute must have unit by definition of attribute [2]. **Assuming** unit associated with WF is field invariant, then such unit can be eliminated from WME due to form of WME. In addition, if a WF is a solution of a WME then any WF in constant proportion to the WF is also a solution of the WME, due to form of WME. Therefore, standardization of WF is necessary, via, e.g., normalization.

Formal solution of WME (40) involves D-finite function, which is not transparent for analysis. To solve the WME for analysis, take the first order far field approximation while retaining the domain boundaries of ρ,

$$\begin{gathered} \rho \gg \frac{1}{2} \quad \to \quad \frac{1}{\beta_e^2} \simeq 1 + \frac{2}{\rho - 1/2} \equiv f_1, \ \frac{1}{\rho^2} \simeq \frac{1}{(\rho - 1/2)^2} \quad \to \\ \frac{1}{X}\frac{d^2X}{dx^2} - \frac{l(l+1)}{x^2} - \frac{1}{4} + \frac{\delta}{x} \simeq 0, \quad \begin{matrix} \Psi \equiv R Y_l^m \\ X \equiv \rho R \end{matrix}, \\ x \equiv 2\nu(\rho - 1/2), \ \nu \equiv \alpha_e \sqrt{1 - \epsilon_e^2}, \ \delta \equiv \frac{\alpha_e \epsilon_e^2}{\sqrt{1 - \epsilon_e^2}} \end{gathered} \tag{42}$$

Y_l^m: Normalized spherical harmonic function.

Solution of the approximated WME is

$$\Psi = \frac{1}{\rho}\left(c_1 M_\delta^{l+1/2}\left[2\nu\left(\rho - \frac{1}{2}\right)\right] + c_2 W_\delta^{l+1/2}\left[2\nu\left(\rho - \frac{1}{2}\right)\right]\right) Y_l^m. \tag{43}$$

$c_{1\&2}$: Linear combination coefficients. M_a^b: Whittaker M function. W_a^b: Whittaker W function.

The solution above is generally divergent at the domain boundaries pending on values of δ and l. However, solution of WME represents physical aspect of the object in association with and physical entity cannot have aspect, attribute, or property being/becoming divergent, even at domain boundary. Therefore, Expression (43) cannot be the WF representing the object. On the other hand, the expression is a general solution of Equation (42) hence *the* solution of *the* WME according to the uniqueness theorem of solution of differential equation [12].

To obtain WF for WME (40), let

$$\delta = n, \ n = 1, 2, \cdots \quad \to \quad \epsilon_n^2 = \frac{1}{2}\frac{n^2}{\alpha_e^2}\left(\sqrt{1 + 4\frac{\alpha_e^2}{n^2}} - 1\right) = 1 - \frac{\alpha_e^2}{n^2} + 2\frac{\alpha_e^4}{n^4} + \cdots. \tag{44}$$

Define

$$\begin{gathered}\Phi_{n,l,m} \equiv R_n^l Y_l^m, \; R_n^l \equiv \frac{\alpha_e \epsilon_n \rho^{-1}}{n\sqrt{(n-l-1)!(n+l)!}} W_n^{l+\frac{1}{2}}\left[\frac{2\alpha_e^2\epsilon_n^2}{n}\left(\rho-\frac{1}{2}\right)\right] \to \\ n > l \geq |m| \geq 0 \\ \lim_{\rho \to 1/2} \Phi_{n,l,m} = 0, \; \lim_{\rho \to \infty} \Phi_{n,l,m} = 0\end{gathered} \quad (45)$$

Φ: Base function for WME (40).

All the base functions above are real, finite, continuous, differentiable and integrable in the domain meeting the boundary conditions that form a base function set that is complete and irreducible. Any real, finite, and continuous function in real domain with specified boundary conditions can be expressed as a linear combination of members of complete and irreducible set of real, finite, continuous, and differentiable functions in same domain under same boundary conditions, if the function is decomposable with respect to such set [13]. WF is real, finite, and continuous function meeting boundary conditions specified, by definition of WF. Therefore, WF of WME (40) can be expressed as

$$\begin{gathered}\Psi = \sum_{n,l,m} c_{n,l,m}\Phi_{n,l,m}, \; \sum_{n,l,m} c_{n,l,m}^2 \neq 0 \to \\ \sum_{n,l,m} c_{n,l,m}\left(\Delta_\rho + \frac{\alpha_e^2\epsilon_e^2}{\beta_e^2} - \alpha_e^2\right)\Phi_{n,l,m} = 0\end{gathered} . \quad (46)$$

$c_{n,l,m}$: Linear combination coefficient, real numeral constant determined by function decomposition procedure.

Therefore,

$$\sum_{n,l,m} c_{n,l,m}\left(\frac{\epsilon_e^2}{\beta_e^2} - f_1\epsilon_n^2 - \frac{l(l+1)}{\alpha_e^2}\left(\frac{1}{\rho^2} - \frac{1}{(\rho-1/2)^2}\right)\right)\left|R_n^l Y_l^m\right\rangle = 0 . \quad (47)$$

$|\ \rangle$: Dirac notation, right bracket [14].

For $l \neq 0$,

$$\sum_{n,n'>l>0} c_{n,l}\left\langle R_{n'}^l\right|\frac{\epsilon_e^2}{\beta_e^2} - f_1\epsilon_n^2 - \frac{l(l+1)}{\alpha_e^2}\left(\frac{1}{\rho^2} - \frac{1}{(\rho-1/2)^2}\right)\left|R_n^l\right\rangle = 0 . \quad (48)$$

$\langle\ |$: Dirac notation, left bracket [14].

The equations are linearly independent with respect to each other and organized per the variable l.

In matrix form,

$$\begin{gathered}\left(\boldsymbol{\eta}\epsilon_e^2 - \boldsymbol{\nu} - l(l+1)\boldsymbol{\mu}\right)\boldsymbol{c} = \mathbf{0} \\ \boldsymbol{\eta}_{j,k} \equiv \left\langle R_j^l\right|\frac{1}{\beta_e^2}\left|R_k^l\right\rangle, \; \boldsymbol{\mu}_{j,k} \equiv \frac{1}{\alpha_e^2}\left\langle R_j^l\right|\frac{1}{\rho^2} - \frac{1}{(\rho-1/2)^2}\left|R_k^l\right\rangle . \\ \boldsymbol{\nu}_{j,k} \equiv \left\langle R_j^l\right| f_1 \left|R_k^l\right\rangle\epsilon_k^2, \; j,k \in \{n > l > 0\}.\end{gathered} \quad (49)$$

$\boldsymbol{c}$: Real coefficient vector to be determined. $\mathbf{0}$: Column vector of zeros.

Since the base function set is complete and irreducible, matrix $\boldsymbol{\eta}$ is invertible. Therefore,

$$\left(\boldsymbol{I}\epsilon_e^2 - \boldsymbol{\lambda}\right)\boldsymbol{c} = \mathbf{0}, \; \boldsymbol{\lambda} \equiv \boldsymbol{\eta}^{-1}\left(\boldsymbol{\nu} + l(l+1)\boldsymbol{\mu}\right) \to \left|\boldsymbol{I}\epsilon_e^2 - \boldsymbol{\lambda}\right|\left|\boldsymbol{c}\right| = \left|\mathbf{0}\right| = 0 . \quad (50)$$

I: Identity matrix. $\left|\ \right|$: For vector, square root of dot product of the vector; for square matrix, determinant of the matrix.

Since the linear coefficient vector c is nonzero, determinant of the matrix above, which is but a polynomial of ϵ_e^2, has to be zero,

$$\left|\boldsymbol{c}\right| \neq 0 \ \rightarrow \ \left|\boldsymbol{I}\epsilon_e^2 - \boldsymbol{\lambda}\right| = 0 \ \rightarrow \ \epsilon_{e,n,l}^2 = \lambda_{n,l} \ \rightarrow \ \epsilon_{e,n,l} = \sqrt{\lambda_{n,l}}\,. \tag{51}$$

$\lambda_{n,l}$: The nth root of the polynomial, *i.e.*, eigenvalue of matrix $\boldsymbol{\lambda}$, which is l-dependent.

That is, total energy of an electron in SEF of proton is discrete by WM. In comparison with that in the classical case, the energy levels are l-dependent.

For $l = 0$, element of matrix $\boldsymbol{\eta}$ is divergent due to insufficient rank of zero point of the base functions at left boundary of ρ. Therefore, take far field approximation of β_e^{-2} to the second order,

$$\begin{gathered}\rho \gg \frac{1}{2} \ \rightarrow \ \frac{1}{\beta_e^2} \simeq 1 + \frac{2}{\rho - 1/2} + \frac{3}{\left(\rho - 1/2\right)^2} \equiv f_2 \ \rightarrow \\ \sum_{n,n'>0} c_{n,0}\left\langle R_{n'}^0 \middle| f_2\epsilon_e^2 - f_1\epsilon_n^2 \middle| R_n^0\right\rangle \simeq 0.\end{gathered} \tag{52}$$

In matrix form,

$$\left(\boldsymbol{I}\epsilon_e^2 - \boldsymbol{\lambda}\right)\boldsymbol{c} \simeq \boldsymbol{0},\ \boldsymbol{\lambda} \equiv \boldsymbol{\eta}^{-1}\boldsymbol{\nu},\ \begin{aligned}\boldsymbol{\eta}_{j,k} &\equiv \left\langle R_j^0 \middle| f_2 \middle| R_k^0\right\rangle \\ \boldsymbol{\nu}_{j,k} &\equiv \left\langle R_j^0 \middle| f_1 \middle| R_k^0\right\rangle\epsilon_k^2\end{aligned} \ \rightarrow \ \epsilon_{e,n,0} \simeq \sqrt{\lambda_{n,0}}\,. \tag{53}$$

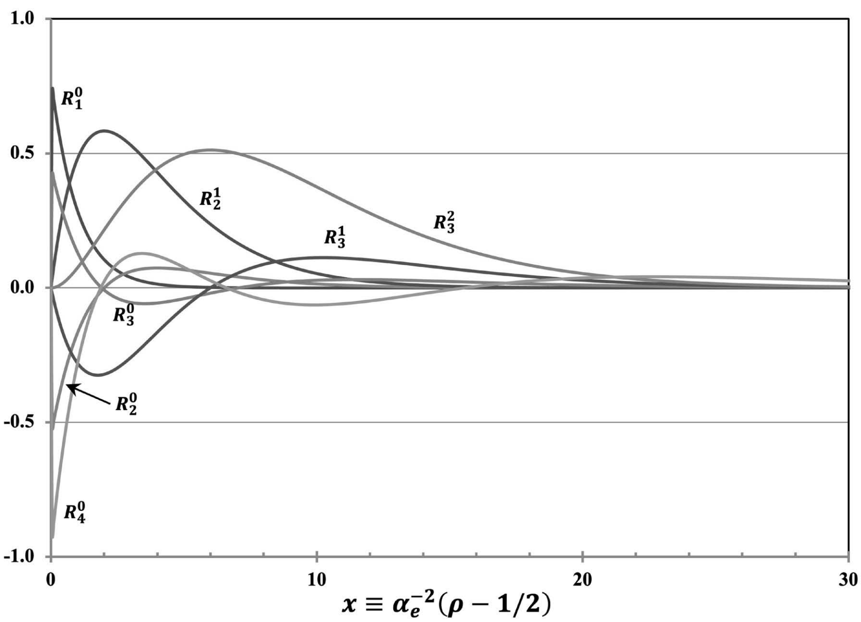

Figure 2. Some base functions for an electron in SEF of proton.

Due to the far field approximation taken, the result above is not exact with respect to WME (40). On the other hand, retention of the boundary condition of ρ assures correctness of qualitative aspect of profiles of the WFs in near field, especially that of the ground state, which is distinctly different from that in the classical case. Figure 2 plots out some of the base functions for an electron in SEF of proton, wherein, abscissa of the plot is in unit of r_e/α_e^2. It can be seen from the plot that profiles of the base functions are similar to that in the classical case except the one corresponding to the ground state, which is zero at left bound of the domain hence different from that in the classical case. In general, base functions are not eigenfunctions of WME and linear combinations of the base functions are, that is also different with respect to that in the classical case.

7. Hydrogen Atom

Regard hydrogen atom as a binary particle system (BPS) at rest in RF. From Equation (8), under ISA,

$$\begin{gathered} \boldsymbol{F}_{e,x} = -\frac{E_{e,\boldsymbol{i}}\beta_{e,x}^{1/2}\beta_{p,x}^{1/2}}{r_e}\frac{\boldsymbol{\rho}_e - \boldsymbol{\rho}_p}{\left\|\boldsymbol{\rho}_e - \boldsymbol{\rho}_p\right\|^3},\ \boldsymbol{F}_{p,x} = -\frac{E_{e,\boldsymbol{i}}\beta_{p,x}^{1/2}\beta_{e,x}^{1/2}}{r_e}\frac{\boldsymbol{\rho}_p - \boldsymbol{\rho}_e}{\left\|\boldsymbol{\rho}_p - \boldsymbol{\rho}_e\right\|^3} \rightarrow \\ \left.\begin{aligned} \frac{dE_{e,x}}{E_{e,\boldsymbol{i}}} &= d\beta_{e,x} = -\frac{r_e\boldsymbol{F}_{e,x}\cdot d\boldsymbol{\rho}_e}{E_{e,\boldsymbol{i}}} = \frac{\beta_{e,x}^{1/2}\beta_{p,x}^{1/2}}{\left\|\boldsymbol{\rho}_e - \boldsymbol{\rho}_p\right\|^3}\left(\boldsymbol{\rho}_e - \boldsymbol{\rho}_p\right)\cdot d\boldsymbol{\rho}_e \\ \frac{dE_{p,x}}{E_{p,\boldsymbol{i}}} &= d\beta_{p,x} = -\frac{r_e\boldsymbol{F}_{p,x}\cdot d\boldsymbol{\rho}_p}{E_{p,\boldsymbol{i}}} = \frac{\mu\beta_{p,x}^{1/2}\beta_{e,x}^{1/2}}{\left\|\boldsymbol{\rho}_p - \boldsymbol{\rho}_e\right\|^3}\left(\boldsymbol{\rho}_p - \boldsymbol{\rho}_e\right)\cdot d\boldsymbol{\rho}_p \end{aligned}\right., \quad \begin{aligned} x &\equiv s,0,e \\ \mu &\equiv \frac{m_{e,\boldsymbol{i}}}{m_{p,\boldsymbol{i}}} \end{aligned} \end{gathered} \cdot \tag{54}$$

$\boldsymbol{F}_{e\|p}$: Electrostatic force experienced by electron or proton in field of the other. $E_{e\|p,\boldsymbol{i}}$: Self/rest energy of electron or proton in RS. r_e : CLF, classical electron radius. $\beta_{e\|p}$: Coulomb Factor of electron or proton. $\boldsymbol{\rho}_{e\|p}$: Location vector of electron or proton, in reduced unit. $E_{e\|p,x}$: Self/rest energy of electron or proton at rest in field of the other. $m_{e\|p,\boldsymbol{i}}$: Self/rest mass of electron or proton in RS. X : State indicator.

Set origin of RF to RS rest mass center of the BPS and approximate that

$$\begin{gathered} \left.\begin{aligned} \boldsymbol{s} &\equiv \boldsymbol{\rho}_e - \boldsymbol{\rho}_p \\ \boldsymbol{\rho}_e &\simeq +\frac{1}{1+\mu}\boldsymbol{s} \\ \boldsymbol{\rho}_p &\simeq -\frac{\mu}{1+\mu}\boldsymbol{s} \end{aligned}\right. \rightarrow \left.\begin{aligned} \frac{d\beta_{e,x}}{ds} &= \frac{\beta_{e,x}^{1/2}\beta_{p,x}^{1/2}}{(1+\mu)s^2} \\ \frac{d\beta_{p,x}}{ds} &= \frac{\mu^2\beta_{p,x}^{1/2}\beta_{e,x}^{1/2}}{(1+\mu)s^2} \end{aligned}\right. \rightarrow \frac{d}{ds}\left(\mu^2\beta_{e,x} - \beta_{p,x}\right) = 0 \rightarrow \\ \beta_{p,x} = 1 - \mu^2\left(1-\beta_{e,x}\right) \rightarrow \frac{d\beta_{e,x}^{1/2}}{ds} = \frac{\sqrt{1-\mu^2\left(1-\beta_{e,x}\right)}}{2(1+\mu)s^2} \end{gathered} . \tag{55}$$

$\boldsymbol{s}$: Distance vector between particles of the BPS in reduced unit.

Therefore,

$$\begin{gathered} \beta_{e,x}^{1/2} = \cosh\left[\frac{\mu}{1+\mu}\frac{1}{2s}\right] - \frac{1}{\mu}\sinh\left[\frac{\mu}{1+\mu}\frac{1}{2s}\right] \rightarrow \\ s_{\mathrm{ESS}} = \frac{\mu}{2}\frac{\operatorname{arccoth}\mu^{-1}}{1+\mu} \equiv s_0 \end{gathered} . \tag{56}$$

s_{ESS} : Electric Schwarzschild distance between particles of the BPS, in reduced unit.

That is, minimal distance between parties of the BPS is a bit shorter of ½ of the

CLF. Although expression for self energy of the electron appears different in comparison to that of Expression (21), the profile is actually quite similar to that of electron in SEF of proton, as can be seen in **Figure 1**.

From Equation (54), with Equation (55),

$$
\begin{gathered}
\frac{d\boldsymbol{P}_{e,f,u,e}}{dt_{e,s,u,e}} = -\frac{E_{e,i}}{r_e}\beta_{e,x}^{1/2}\beta_{p,x}^{1/2}\frac{\hat{\boldsymbol{s}}}{s^2} \quad \rightarrow \\
\boldsymbol{a}_e - f_u\boldsymbol{u}_e + f_e\hat{\boldsymbol{s}} = \boldsymbol{0}, \quad f_u \equiv \frac{d\ln\beta_{u,e}}{d\tau}, \quad f_e \equiv \frac{\beta_{p,x}^{1/2}}{\beta_{e,x}^{1/2}s^2} \quad \rightarrow \\
d\ln\frac{\beta_{e,x}}{\beta_{e,u}} = 0 \quad \rightarrow \quad \frac{\beta_{e,x}}{\beta_{e,u}} = \epsilon_e.
\end{gathered}
\tag{57}
$$

ϵ_e : Integration constant, total energy of electron in hydrogen atom in RS.

Therefore,

$$
\left(\frac{\Delta_s}{\alpha^2} + \frac{\epsilon_e^2}{\beta_{e,x}^2} - 1\right)\Psi = 0, \ \alpha \equiv \frac{\alpha_e}{1+\mu}, \ \begin{matrix} \lim\limits_{s\to s_0}\Psi = 0 \\ \lim\limits_{s\to\infty}\Psi = 0 \end{matrix}.
\tag{58}
$$

Δ_s : Laplace operator with respect to s.

Expand the Coulomb Factor of the electron,

$$
s \gg s_0 \quad \rightarrow \quad \begin{gathered} \frac{1}{\beta_{e,x}^2} \simeq 1 + \frac{2}{(1+\mu)(s-s_0)} + \frac{\gamma}{(1+\mu)(s-s_0)^2} \equiv f_2 \\ \gamma \equiv \frac{5 - 2\mu/\operatorname{arctanh}\mu - \mu^2}{2(1+\mu)}. \end{gathered}
\tag{59}
$$

Define the base function set as

$$
\Phi_{n,l,m} \equiv R_n^l Y_l^m, \ R_n^l \propto \frac{1}{s} W_n^{\iota_{n,l}}\left[2\nu_n\left(s-s_0\right)\right], \ \begin{matrix} n > l \ge |m| \ge 0 \\ n,l,m \in \text{Integers} \end{matrix}.
\tag{60}
$$

Φ: Base function for WME (58).

Wherein,

$$
\begin{gathered}
\nu_n \equiv \alpha\sqrt{1-\epsilon_n^2}, \ \iota_{n,l} \equiv \sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{\alpha^2\gamma\epsilon_n^2}{1+\mu}}, \\
\epsilon_n^2 = \frac{(1+\mu)^2 n^2}{2\alpha^2}\left(\sqrt{1+\frac{4\alpha^2}{(1+\mu)^2 n^2}} - 1\right)
\end{gathered}.
\tag{61}
$$

WME (58) becomes

$$
\begin{gathered}
\sum_{n,l,m} c_{n,l,m}\left(\frac{\epsilon_e^2}{\beta_{e,x}^2} - f_2\epsilon_n^2 - \frac{l(l+1)}{\alpha^2}\left(\frac{1}{s^2} - \frac{1}{(s-s_0)^2}\right)\right)\left|R_n^l Y_l^m\right\rangle = 0 \quad \rightarrow \\
\sum_{n,n'>l} c_{n,l}\left\langle R_{n'}^l\right|\frac{\epsilon_e^2}{\beta_{e,x}^2} - f_2\epsilon_n^2 - \frac{l(l+1)}{\alpha^2}\left(\frac{1}{s^2} - \frac{1}{(s-s_0)^2}\right)\left|R_n^l\right\rangle = 0
\end{gathered}.
\tag{62}
$$

In matrix form, for $l \neq 0$,

$$
\begin{aligned}
&\left(\boldsymbol{\eta}\epsilon_e^2-\boldsymbol{\nu}-l(l+1)\boldsymbol{\mu}\right)\boldsymbol{c}=\mathbf{0}\\
&\boldsymbol{\eta}_{j,k}\equiv\left\langle R_j^l\left|\frac{1}{\beta_{e,x}^2}\right|R_k^l\right\rangle,\ \boldsymbol{\mu}_{j,k}\equiv\frac{1}{\alpha^2}\left\langle R_j^l\left|\frac{1}{s^2}-\frac{1}{(s-s_0)^2}\right|R_k^l\right\rangle\\
&\boldsymbol{\nu}_{j,k}\equiv\left\langle R_j^l\left|f_2\right|R_k^l\right\rangle\epsilon_k^2,\ \ j,k\in\{n>l>0\}.
\end{aligned}
\tag{63}
$$

Therefore,

$$
\left(\boldsymbol{I}\epsilon_e^2-\boldsymbol{\lambda}\right)\boldsymbol{c}=\mathbf{0},\ \boldsymbol{\lambda}\equiv\boldsymbol{\eta}^{-1}\left(\boldsymbol{\nu}+l(l+1)\boldsymbol{\mu}\right)\ \rightarrow\ \epsilon_{e,n,l}=\sqrt{\lambda_{n,l}}\,. \tag{64}
$$

For $l=0$, with far field approximation of the second order,

$$
s\gg s_0\ \ \rightarrow\ \ \boldsymbol{\eta}\boldsymbol{\epsilon}\boldsymbol{c}=\mathbf{0},\ \begin{aligned}&\boldsymbol{\eta}_{j,k}\equiv\left\langle R_j^l\left|f_2\right|R_k^l\right\rangle\\&\boldsymbol{\epsilon}_{j,k}\equiv\delta_{j,k}\left(\epsilon_e^2-\epsilon_k^2\right)\end{aligned}\ \rightarrow\ \epsilon_{e,n,0}=\epsilon_n\,. \tag{65}
$$

$\delta_{j,k}$: Kronecker Delta function.

The result is not exact due to the far field approximations as well as the assumption on stativity of the rest mass center of the BPS.

8. Energy State of Mass Object in Static Gravitation Field

Similar WME for energy state of a mass particle in SGF is obtainable if the particle is perceived as an EO or in association with such. Total energy of a mass particle in SGF is [4]

$$
\epsilon_g\equiv\frac{E_{f,u,g}}{E_i}=\frac{\beta_g}{\beta_u}=\frac{1}{\sqrt{1-u^2}}\left(1-\frac{4}{\rho}\right)^{1/4}\ \rightarrow\ u^2+\frac{\beta_g^2}{\epsilon_g^2}=1\ \rightarrow\ -p_g^2+\frac{\epsilon_g^2}{\beta_g^2}=1\,. \tag{66}
$$

$E_{f,u,g}$: Total energy of a mass particle in motion in SGF as measured at rest at location of the particle in the field. E_i: Self energy of the mass particle in RS. ϵ_g: Total energy of a mass particle in motion in SGF as measured at rest at location of the particle in the field in reduced unit. β_g: Schwarzschild Factor for mass particle in SGF. p_g: Field momentum of a mass particle in reduced unit.

The equation is for total energy of a mass particle in motion in SGF under ISA, ignoring spin and self field interaction associated with motion of the particle and assuming stativity of the field. Self energy of mass particle in SGF is also shown in **Figure 1**. Note that left bound of the domain is at radius of Schwarzschild Sphere (SS) of the field, which is four length units in terms of r_g as defined below.

According to WM,

$$
\begin{aligned}
&\boldsymbol{P}_g\sim i\hbar\nabla_r\ \ \rightarrow\ \ \boldsymbol{p}_g\sim\frac{1}{m_ic_i}\frac{i\hbar}{r_g}\nabla_\rho,\ r_g\equiv\frac{G_iM_i}{c_i^2}\ \ \rightarrow\\
&\frac{\Delta_\rho}{\alpha_g^2}+\frac{\epsilon_g^2}{\beta_g^2}=\hat{\mathrm{I}},\ \alpha_g\equiv\frac{G_iM_im_i}{\hbar c_i}.
\end{aligned}
\tag{67}
$$

$\boldsymbol{P}_g$: Field momentum of mass particle in SGF. r_g: CLF of SGF. ∇_ρ: Gradient operator in unit of CLF of SGF. M_i: Rest mass of the field causing object in RS. m_i: Rest mass of mass particle in RS.

Thus,

$$
\left(\frac{\Delta_\rho}{\alpha_g^2}+\frac{\epsilon_g^2}{\beta_g^2}-1\right)\Psi=0,\ \begin{aligned}&\rho_{\mathrm{SS}}=4\\&\rho_{\mathrm{SS}}\le\rho<\infty\end{aligned},\ \begin{aligned}&\lim_{\rho\to4}\Psi=0\\&\lim_{\rho\to\infty}\Psi=0\end{aligned}. \tag{68}
$$

Ψ: WF of mass object in SGF.

This is the WME complying with LME and finite space/bound state for mass object in SGF under the approximations aforementioned. In such, attributes are expressed in reduced form, *i.e.*, in unit of the corresponding ones measured/defined in RS, e.g., energy in unit of $m_i c_i^2$, momentum in $m_i c_i$, length in r_g, etc. Note that it is the domain boundary of Schwarzschild Factor that sets the lower bound of ρ in the WME.

Define the base function set as

$$\Phi_{n,l,m} \equiv R_n^l Y_l^m,\ R_n^l \equiv \frac{1}{\rho}\frac{\sqrt{2\nu_n}W_n^{l+1/2}\left[2\nu_n(\rho-4)\right]}{\sqrt{2n(n-l+1)!(n-l)!}},\quad \begin{matrix} n>l\ge|m|\ge 0 \\ n,l,m\in\text{Integers}\end{matrix}. \tag{69}$$

Φ: Base function for WME (68).

Wherein,

$$\nu_n \equiv \alpha_g\sqrt{1-\epsilon_n^2},\ \epsilon_n^2 = \frac{1}{2}\frac{n^2}{\alpha_g^2}\left(\sqrt{1+\frac{4\alpha_g^2}{n^2}}-1\right) = \frac{n}{\alpha_g}\left(1-\frac{1}{2}\frac{n}{\alpha_g}+\frac{1}{8}\frac{n^2}{\alpha_g^2}+\cdots\right). \tag{70}$$

WME (68) becomes

$$\begin{gathered}\sum_{n,l,m} c_{n,l,m}\left(\frac{\epsilon_g^2}{\beta_g^2}-f_1\epsilon_n^2-\frac{l(l+1)}{\alpha_g^2}\left(\frac{1}{\rho^2}-\frac{1}{(\rho-4)^2}\right)\right)\left|\Phi_{n,l,m}\right\rangle = 0,\\ f_1 \equiv 1+\frac{2}{\rho-4} \quad \rightarrow \\ \sum_{n,n'>l} c_{n,l}\left\langle R_{n'}^l\right|\frac{\epsilon_g^2}{\beta_g^2}-f_1\epsilon_n^2-\frac{l(l+1)}{\alpha_g^2}\left(\frac{1}{\rho^2}-\frac{1}{(\rho-4)^2}\right)\left|R_n^l\right\rangle = 0\end{gathered}. \tag{71}$$

In matrix form,

$$\begin{gathered}\left(\alpha_g^{1/2}\epsilon_g^2\boldsymbol{\eta}-\boldsymbol{\nu}-l(l+1)\boldsymbol{\mu}\right)\boldsymbol{c}=\boldsymbol{0}\\ \boldsymbol{\eta}_{j,k} \equiv \frac{1}{\alpha_g^{1/2}}\left\langle R_j^l\right|\frac{1}{\beta_g^2}\left|R_k^l\right\rangle,\quad \boldsymbol{\mu}_{j,k} \equiv \frac{1}{\alpha_g^2}\left\langle R_j^l\right|\frac{1}{\rho^2}-\frac{1}{(\rho-4)^2}\left|R_k^l\right\rangle\\ \boldsymbol{\nu}_{j,k} \equiv \epsilon_k^2\left\langle R_j^l\right|f_1\left|R_k^l\right\rangle, \qquad j,k\in\{n>l\ge 0\}.\end{gathered} \tag{72}$$

Therefore,

$$\left(\boldsymbol{\lambda}-\frac{1}{\alpha_g^{1/2}\epsilon_g^2}\boldsymbol{I}\right)\boldsymbol{c}=\boldsymbol{0},\ \boldsymbol{\lambda}\equiv\left(\boldsymbol{\nu}+l(l+1)\boldsymbol{\mu}\right)^{-1}\boldsymbol{\eta} \ \rightarrow\ \epsilon_{g,n,l}=\frac{1}{\alpha_g^{1/4}\sqrt{\lambda_{n,l}}}. \tag{73}$$

Table 1 lists some values of $1/\sqrt{\lambda_{n,l}}$ for the states near ground state ($n \ll \alpha_g$). In general, the parameter α_g is a large number. For instance, for neutron in gravitation field of ten solar masses, $\alpha_{g,n} \approx 7.030\,897\,757\times 10^{19}$; for neutrino in same, $\alpha_{g,nu} \approx 8.98\times 10^9$. Therefore, if a neutron were to transit between, e.g., 1s and 2s states, the energy quanta involved in such transition would be ~2.186 keV, which is in the range of gamma ray if transfer of the energy is in form of electromagnetic radiation. If neutrino were to transit between 1s and 2s states then the energy quanta involved would be ~83 μeV or ~20 GHz if the energy is transferred in form of electromagnetic radiation. Conversely, if a mass particle in gravitation field is identifiable and energy spectrum of state transitions of the particle in the field is detectable, then the rest mass causing the field shall become knowable/measurable.

Table 1. $\lambda_{n,l}^{-1/2}$ for mass object in static gravitation field.

n \ l	0	1	2	3	4	5	6	7	8	9
10	1.873	1.725	1.553	1.421	1.338	1.293	1.270	1.259	1.255	1.255
9	1.824	1.666	1.490	1.362	1.287	1.248	1.230	1.223	1.223	
8	1.771	1.601	1.421	1.299	1.233	1.201	1.189	1.187		
7	1.713	1.529	1.347	1.233	1.176	1.153	1.148			
6	1.647	1.448	1.265	1.162	1.118	1.106				
5	1.573	1.354	1.174	1.087	1.059					
4	1.487	1.242	1.072	1.008						
3	1.382	1.106	0.959							
2	1.244	0.932								
1	1.031									

Figure 3 plots out some of the base functions for a mass particle in SGF. It can be seen from the plot that profiles of the base functions are qualitatively similar to those for electron in SEF, except significant difference in length scale, that major portion of the profiles is confined at length scale of a nucleon. Accordingly, scale of the profiles is on the order of $\alpha_g^{1/2}$. It is also indicative that eigenfunctions of the WME shall not be dominated by a single base function. In other words, convergence of the finite matrix shall be slow relative to that in the case of, e.g., electron in SEF of proton.

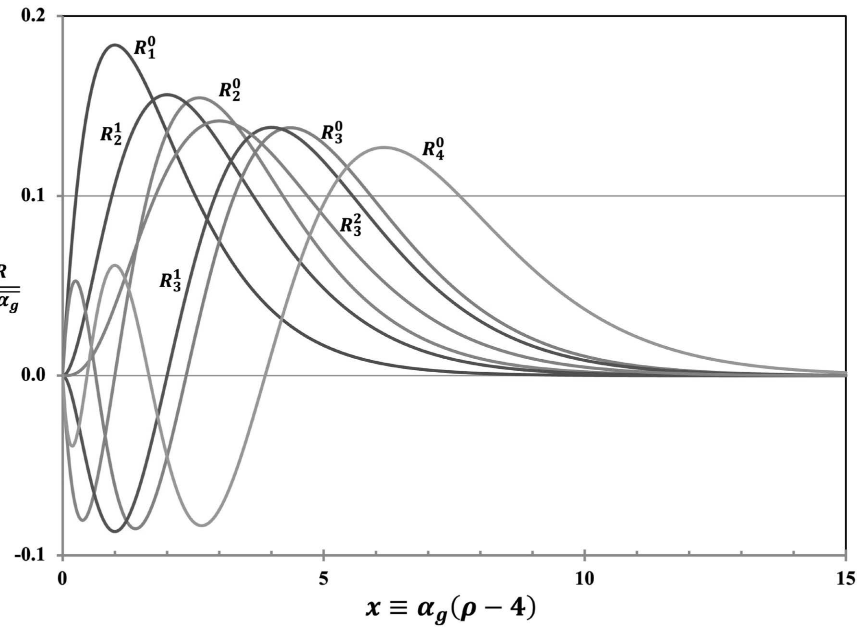

Figure 3. Some base functions for a mass particle in SGF.

From Expression (69), mean radii of the base functions are

$$\bar{\rho}_{n,l} \equiv \left\langle R_n^l \middle| \rho \middle| R_n^l \right\rangle = 4 + \delta\rho, \quad \delta\rho \equiv \frac{3n^2 - l(l+1)}{2n\alpha_g\sqrt{1-\epsilon_n^2}} \rightarrow \\ \delta r = \frac{3n^2 - l(l+1)}{2n\sqrt{1-\epsilon_n^2}} \frac{\hbar}{m_i c_i} . \quad (74)$$

Therefore, for neutron in ground state, it would be, on average, ~0.32 fm away from SS, regardless of strength of the gravitation field. For neutrino, it would be ~2.5 μm, regardless of strength of the gravitation field. If a mass object in near ground states were to have internal transition, then gravitation redshift of the photon from such process is

$$\bar{z}_G = \left(1 - \frac{4}{\bar{\rho}}\right)^{-1/4} - 1 \sim \left(\frac{8n\alpha_g}{3n^2 - l(l+1)}\right)^{1/4} \sim \alpha_g^{1/4} . \quad (75)$$

$\bar{z}_G$: Mean redshift of photon caused by gravitation field at emission site.

Therefore, if a neutron in ground state in the gravitation field were to emit a photon via radioactive beta decay and energy of the photon would have been 100 keV if the photon were from a neutron in RS, then the photon would be observed as an infrared radiation of ~1.45 μm.

For mass particle in gravitation field, $\beta_g = 0$ brings the particle to a distinctly different state of being, $\rho = \rho_{\mathrm{SS}}$, *i.e.*, the particle shall be confined at the SS, and the latter is but a finite subspace. It has been shown that, for particle in motion in finite space [2],

$$\epsilon_u \equiv \frac{E_{\mathrm{SS},u}}{E_{\mathrm{SS},0}} = \frac{1}{\sqrt{1-u^2}} \neq 0 \rightarrow u^2 + \frac{1}{\epsilon_u^2} = 1 \rightarrow -p_u^2 + \epsilon_u^2 = 1 . \quad (76)$$

$E_{\mathrm{SS},u}$: Total energy of a mass particle in motion at SS as measured at rest at location of the particle at the SS. $E_{\mathrm{SS},0}$: Self energy of a mass particle at rest at SS. ϵ_u and p_u : Total energy and momentum of a mass particle in motion at SS as measured at rest at location of the particle at SS, in reduced unit.

Therefore,

$$\frac{\hat{\mathrm{L}}}{\alpha_g^2\rho_{\mathrm{SS}}^2} + \epsilon_u^2 = \hat{\mathrm{I}} \rightarrow \Psi_{\mathrm{SS},l} = Y_l^m, \ \epsilon_{u,l}^2 = 1 + \frac{l(l+1)}{16\alpha_g^2} \geq 1 \rightarrow \\ E_{\mathrm{SS},u} = E_{\mathrm{SS},0}\sqrt{1 + \frac{l(l+1)}{16\alpha_g^2}}. \quad (77)$$

$\hat{\mathrm{L}}$: Harmonic operator.

That is, spherical harmonic functions are eigenfunctions of the WME. However, there is no overlap between these functions and WFs outside the SS even for WFs of identical l. Therefore, mass entity cannot transit between such states, unless certain mechanism is built-in *a prior* into the corresponding WMEs, e.g., retardation of self field of gravitation [4]. On the other hand, $\beta_g \rightarrow 0$ also means rest mass of the mass particle in association would approach infinity. Therefore, stativity assumption of the gravitation field hence that of SS can no longer hold. Therefore, dynamics of the process is beyond the coverage of the WMEs. Note also that, if most of the rest masses of a black hole are enclosed by its SS then $E_{\mathrm{SS},0} \to 0$; if most of the masses are at the SS then $E_{\mathrm{SS},0} \approx 83\%$ of the rest mass of the mass particle in RS [15]. Therefore, without aid of foreign energy, a mass object in near ground states outside the SS may not be able to transit to the SS even if such transition is allowable by mechanism/circumstance.

9. Energy State of Mass Particle in Free Motion in Finite Space

For mass particle in free motion in finite space, ignoring spin, retardation effect of self field associated with motion of the particle, and potential influence from its counterpart, total energy of the particle is [2]

$$\epsilon_u \equiv \frac{E_u}{E_i} = \frac{1}{\sqrt{1-u^2}} \neq 0 \quad \to \quad u^2 = 1 - \frac{1}{\epsilon_u^2}. \tag{78}$$

E_u: Total energy of a mass particle in free motion in finite space as measured at rest in RF. E_i: Self/rest energy of mass particle in RS. ϵ_u: Total energy of the particle in free motion in finite space as measured at rest in RF in reduced unit.

By definition,

$$\begin{aligned} &\boldsymbol{P}_u \equiv m_u \boldsymbol{v}_u = m_i c_i \frac{\boldsymbol{u}}{\beta_u} \quad \to \quad \boldsymbol{p}_u \equiv \frac{\boldsymbol{P}_u}{m_i c_i} = \frac{\boldsymbol{u}}{\beta_u} \quad \to \\ &u^2 = \frac{p_u^2}{1+p_u^2} \quad \to \quad -p_u^2 + \epsilon_u^2 = 1 \end{aligned}. \tag{79}$$

$\boldsymbol{P}_u$, m_u, and $\boldsymbol{v}_u$: Momentum, mass, and velocity of a mass particle in free motion in finite space as measured at rest in RF. m_i: Self/rest mass of particle in RS.

According to WM,

$$\boldsymbol{P}_u \sim i\hbar \nabla_r \quad \to \quad \boldsymbol{p}_u \sim \frac{i\hbar}{m_i c_i} \nabla_r \quad \to \quad \frac{\hbar^2 c_i^2}{E_i^2} \Delta_r + \epsilon_u^2 = \hat{\mathrm{I}}. \tag{80}$$

Denote

$$\alpha_u \equiv \frac{E_i R}{\hbar c_i} \quad \to \quad \frac{\Delta_R}{\alpha_u^2} + \epsilon_u^2 = \hat{\mathrm{I}} \quad \to \quad (\Delta_R + \delta)\Psi = 0,\ \delta \equiv \alpha_u^2 \left(\epsilon_u^2 - 1\right) \geq 0. \tag{81}$$

R: External radius of finite space in unit of length of the space. Δ_R: Laplace operator in unit of external radius of finite space.

For $\mathbf{S}^3$, with spherical coordinate system, define

$$\Psi \equiv Z Y_l^m \quad \to \quad \frac{1}{Z \sin^2\varphi} \frac{d}{d\varphi}\left(\sin^2\varphi \frac{dZ}{d\varphi}\right) - \frac{l(l+1)}{\sin^2\varphi} + \delta = 0. \tag{82}$$

The solution is

$$Z_P \propto \frac{P_{\sqrt{1+\delta}-1/2}^{l+1/2}[\cos\varphi]}{\sqrt{\sin\varphi}},\ Z_Q \propto \frac{Q_{\sqrt{1+\delta}-1/2}^{l+1/2}[\cos\varphi]}{\sqrt{\sin\varphi}}. \tag{83}$$

P_a^b: Associated Legendre polynomial. Q_a^b: Associated Legendre polynomial of the second kind.

Z_P is divergent at internal origin of the space for any value of δ, and Z_Q is finite at the origin and the antipode thereof but only if $\sqrt{1+\delta}$ is positive integer. However, origin and antipode of finite space are ordinary spacial points assigned artificially/arbitrarily. Therefore, the general solutions of Expression (83) are not qualified as WF of the WME.

Construct the base function set by

$$\begin{gathered} \begin{matrix} n=\sqrt{1+\delta} \\ n=1,2,\cdots \end{matrix} \rightarrow \epsilon_n^2 = 1+\frac{n^2-1}{\alpha_u^2} \rightarrow \\ \Phi_{nlm} \equiv \frac{2}{\pi}\sqrt{\frac{(n+1)(n-l)!}{(n+l+1)!}}\frac{Q_{n-1/2}^{l-1/2}[\cos\varphi]Y_l^m}{\sqrt{\sin\varphi}} \end{gathered} . \tag{84}$$

Φ: Base function for WME (81).

Accordingly,

$$\begin{gathered} \Psi = \sum_{n,l,m} c_{nlm}\Phi_{nlm},\ \sum_{n,l,m} c_{nlm}^2 \neq 0 \rightarrow \sum_{n,l,m} c_{nlm}\left(\epsilon_u^2-\epsilon_n^2\right)\Phi_{nlm}=0 \rightarrow \\ \epsilon_{u,n} = \sqrt{1+\frac{n^2-1}{\alpha_u^2}} \end{gathered} . \tag{85}$$

Wherein,

$$\left(\Delta_R + \alpha_u^2\left(\epsilon_n^2-1\right)\right)\Phi_{nlm}=0,\ \left\langle \Phi_{n'l'm'} \middle| \Phi_{nlm}\right\rangle = \delta_{n'n}\delta_{l'l}\delta_{m'm}. \tag{86}$$

Therefore, the base functions of Expression (84) are eigenfunctions of the WME that correspond to discrete energy states. Figure 4 plots out radial portion of some of the eigenfunctions of the WME. Note that abscissa of the plot is in unit of external radius of the finite space. As can be seen from the plot, the eigenfunctions are but three dimensional spherical harmonic functions.

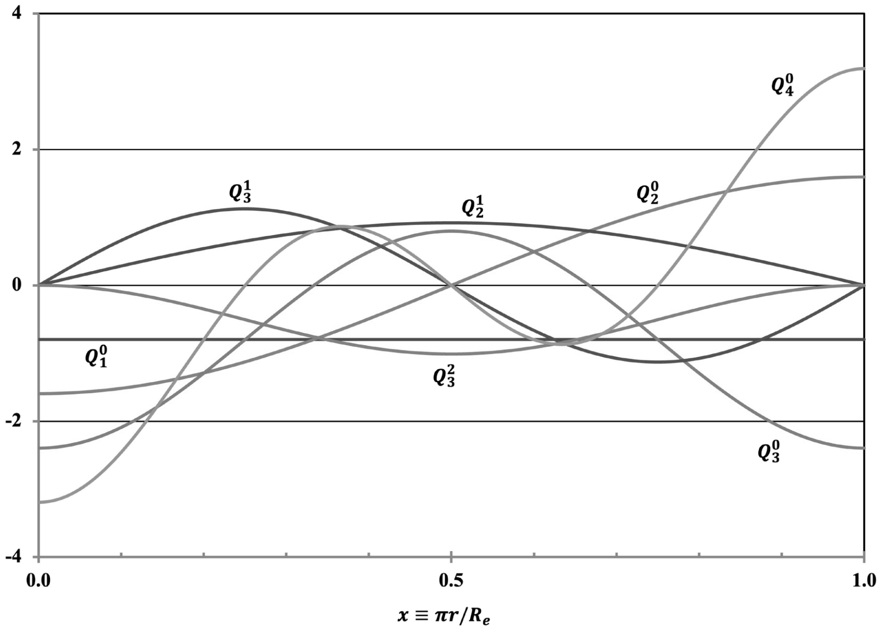

Figure 4. Radial portion of some of the eigenfunctions for mass particle in S^3.

Therefore, as a consequence of finiteness of the space, total energy of a mass entity in free motion in such space is discrete by WM. This discreteness is particularly referring to kinetic energy of the entity, defined as

$$\epsilon_{K,n} \equiv \epsilon_{u,n} - 1 = \sqrt{1 + \frac{n^2 - 1}{\alpha_u^2}} - 1 \quad \rightarrow$$
$$\lim_{n \to \infty} \delta\epsilon_{K,n} = \frac{1}{\alpha_u}, \quad \delta\epsilon_{K,1} \simeq \frac{3}{2\alpha_u^2}, \quad \delta\epsilon_{K,n} \equiv \epsilon_{K,n+1} - \epsilon_{K,n}. \tag{87}$$

ϵ_K : Kinetic energy of mass entity in free motion in finite space in reduced unit.

That is, kinetic energy difference between consecutive kinetic states of a mass entity in finite space/subspace is a nonlinear function of the kinetic energy that shall approach constant if the energy is sufficiently high. From Expression (81), with internal radius of physical space measured at 1.0 billion light years [16], for, e.g., hydrogen atom,

$$\alpha_{u,\mathrm{H}} = \frac{4m_{\mathrm{H},i} c_i R_i}{h} \approx 2.9 \times 10^{40}. \tag{88}$$

$m_{\mathrm{H},i}$: Rest mass of hydrogen atom in RS. R_i : Internal radius of physical space.

Therefore, kinetic energy difference between neighboring eigenstates of a mass entity is extremely small, hence discreteness of motion of the entity shall be minuscule as if the entity in motion in finite space is continuous.

10. Fine Structure Constant

Fine structure constant is a dimensionless entity. However, by itself, dimensionlessness of an entity does not guarantee state invariance of the entity. By definition, Expression (3), and from Maxwell electrodynamics [17],

$$\alpha_e = \frac{\boldsymbol{e}^2}{4\pi\varepsilon_0 \hbar c}, \ \varepsilon_0 \mu_0 = \frac{1}{c^2} \quad \rightarrow \quad \alpha_e = \frac{\boldsymbol{e}^2 \mu_0 c}{4\pi\hbar} \quad \rightarrow \quad \alpha_{e,x} = \frac{\boldsymbol{e}_x^2 \mu_{0,x} c_x}{4\pi\hbar_x}. \tag{89}$$

α_e : Fine structure constant. $\boldsymbol{e}$: Elementary charge. ε_0 : Electric constant. μ_0 : Magnetic constant. $\hbar$: Planck constant. c : SLV.

By definition [10],

$$\begin{matrix} \boldsymbol{e} \equiv \mathcal{N}_e \mathbb{U}_\mathrm{C} & & \boldsymbol{e}_x = \mathcal{N}_e \mathbb{U}_{\mathrm{C},x} \\ \mu_0 \equiv n_\mu \dfrac{\mathbb{U}_F \mathbb{U}_t^2}{\mathbb{U}_\mathrm{C}^2} & \rightarrow & \mu_{0,x} = n_{\mu,x} \dfrac{\mathbb{U}_{F,x} \mathbb{U}_{t,x}^2}{\mathbb{U}_{\mathrm{C},x}^2} = n_{\mu,x} \dfrac{\mathbb{U}_{E,x} \mathbb{U}_{t,x}^2}{\mathbb{U}_{L,x} \mathbb{U}_{\mathrm{C},x}^2}. \end{matrix} \tag{90}$$

$\mathbb{U}_\mathrm{C}$: Unit of charge, Coulomb in SI. $\mathbb{U}_F$: Unit of force, Newton in SI. $\mathbb{U}_t$: Unit of time, second in SI. $\mathbb{U}_E$: Unit of energy, Joule in SI. $\mathbb{U}_L$: Unit of length, meter in SI.

Thus,

$$\alpha_{e,x} = \frac{\left(\mathcal{N}_e \mathbb{U}_{\mathrm{C},x}\right)^2}{2\left(n_{h,x} \mathbb{U}_{E,x} \mathbb{U}_{t,x}\right)} \left(\frac{n_{\mu,x} \mathbb{U}_{E,x} \mathbb{U}_{t,x}^2}{\mathbb{U}_{L,x} \mathbb{U}_{\mathrm{C},x}^2}\right) \left(\frac{n_{c,x} \mathbb{U}_{L,x}}{\mathbb{U}_{t,x}}\right) = \frac{\mathcal{N}_e^2}{2} \frac{n_{\mu,x} n_{c,x}}{n_{h,x}}$$
$$\mathcal{N}_e \equiv 1.602\ 176\ 634 \times 10^{-19}, \ n_{\mu,x} \approx 4\pi \times 10^{-7}. \tag{91}$$

n_c : Numeral aspect of SLV. n_h : Numeral aspect of Planck constant.

That is, α_e is unit-independent if and only if all the units involved therein are in one and the same state.

By the rule of metrology [4],

$$\begin{matrix} n_{\mu,x} = n_\mu \\ n_{c,x} = n_c \\ n_{h,x} = n_h \end{matrix} \quad \rightarrow \quad \alpha_{e,x} = \frac{\mathcal{N}_e^2}{2} \frac{n_\mu n_c}{n_h} \subset \text{SIT} \quad \rightarrow \quad \alpha_{e,x} = \alpha_e \,. \tag{92}$$

That is, fine structure constant is SIT metrologically, regardless of choice/definition of the entities involved therein. Therefore, **Assumption 3** is indeed true and valid.

Fine structure constant is generally believed calculable, e.g., via Standard Model [18]. Regardless of specifics of models, state invariance of α_e, n_μ, n_c, and n_h has always been assumed *a prior* in any such calculations, although implicitly. With the definition of SLV and the new definition of Planck constant [10],

$$\begin{matrix} n_h \equiv \mathcal{N}_h \\ n_c \equiv \mathcal{N}_c \end{matrix} \quad \rightarrow \quad \alpha_e = n_\mu \frac{\mathcal{N}_e^2 \mathcal{N}_c}{2\mathcal{N}_h} \,. \tag{93}$$

Therefore, calculation of fine structure constant is equivalent to calculation of magnetic constant, and the latter is used to be an assigned entity, $n_\mu \equiv 4\pi \times 10^7$. Therefore, it might be suffice to show why there should or should not be a factor of 4π in n_μ, since all other elements of α_e are due to artificial construct.

11. Discussion

For electron in SEF of proton, $\beta_e \rightarrow 0$ is indicative of some distinctive states that distance between the electron and center of the field is fixed at one half classical electron radius while electrostatic force experienced by the electron is none according to Equation (17). Such states are therefore unlikely physical. On the other hand, at such length scale, other forces between electron and proton are non-negligible except gravitation. In comparison, for mass particle in SGF of mass entity of same type, $\beta_g \rightarrow 0$ represents unique states that motion of the mass particle is confined at SS of the field, which is comprehensible in terms of infinite gravitational attraction and/or zero SLV along surface normal of the SS. Therefore, motion of mass particle at SS of such SGF and that in a finite space of same dimension are physically indistinguishable if the finite space is not mass/charge balanced. However, for observer confined at SS, some physical parameters, such as c_i, G_i, etc., are imperceptible/inaccessible but only c_{SS}, G_{SS}, etc., instead.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Einstein, A. (1905) Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig? *Annalen der Physik*, **323**, 639-641. https://doi.org/10.1002/andp.19053231314

[2] Wang, Y. (2024) Physical Space Is Finite. *Journal of Modern Physics*, **15**, 550-595. https://doi.org/10.4236/jmp.2024.155027

[3] Planck, M. (1900) Zur Theorie des Gesetzes der Energieverteilung im Normalspec-trum. *Verhandlungen der Deutschen Physikalischen Gesellschaft* (*Leipzig*), **17**, 237-245.

[4] Wang, Y. (2024) Refinement of Newton Gravitation Law. *Journal of Modern Physics*, **15**, 720-763. https://doi.org/10.4236/jmp.2024.155033

[5] Giacomo, P. (1984) News from the BIPM. *Metrologia*, **20**, 25-30. https://doi.org/10.1088/0026-1394/20/1/005

[6] Schrödinger, E. (1926) An Undulatory Theory of the Mechanics of Atoms and Molecules. *Physical Review*, **28**, 1049-1070. https://doi.org/10.1103/physrev.28.1049

[7] Coulomb, C.A. (1785) Construction et usage d'une balance électrique, fondée sur la propriété qu'ont les fils de métal, d'avoir une force de réaction de torsion proportionnelle à l'angle de torsion. *Histoire de l'Académie Royale des sciences avec les mémoires de mathématiques et de physique*, **1785**, 569-577.

[8] Sommerfeld, A. (1916) Zur Quantentheorie der Spektrallinien. *Annalen der Physik*, **356**, 1-94. https://doi.org/10.1002/andp.19163561702

[9] CODATA (2018) Fundamental Physical Constants. http://physics.nist.gov/constants

[10] Newell, D.B. and Tiesinga, E. (Eds.) (2019) The International System of Units (SI), NIST Special Publication 330. https://doi.org/10.6028/NIST.SP.330-2019

[11] Feynman, R.P., Leighton, R.B. and Sands, M. (1966) The Feynman Lectures on Physics. Addison-Wesley.

[12] Boyce, W.E. and Diprima, R.C. (1986) Elementary Differential Equations and Boundary Value Problems. 4th Edition, John Wiley & Sons.

[13] Rudin, W. (1987) Real and Complex Analysis. McGraw-Hill.

[14] Dirac, P.A.M. (1930) The Principles of Quantum Mechanics. Oxford University Press.

[15] Wang, Y. (2024) Exists Gravitation Inverse Matter. *Journal of Modern Physics*, **15**, 1001-1009. https://doi.org/10.4236/jmp.2024.157042

[16] Wang, Y. (2024) Physical Space Was Not Expanding. *Journal of Modern Physics*, **15**, 634-673. https://doi.org/10.4236/jmp.2024.155030

[17] Maxwell, J.C. (1865) A Dynamical Theory of the Electromagnetic Field. *Philosophical Transactions of the Royal Society of London Serries I*, **155**, 459-512. https://doi.org/10.1098/rstl.1865.0008

[18] Aoyama, T., Hayakawa, M., Kinoshita, T. and Nio, M. (2012) Tenth-Order QED Contribution to the Electron g-2 and an Improved Value of the Fine Structure Constant. *Physical Review Letters*, **109**, Article ID: 111807. https://doi.org/10.1103/physrevlett.109.111807