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Dynamic Reinsurance Strategy

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Abstract

In this paper, I consider insurers' reinsurance strategies to find an optimal reinsurance cover ratio for underwritten insurance exposure. First, I describe the one-period model and the continuous time dynamic model by stochastic differential equation in the same structure. Second, I translate the one-period model solution, where VaR is used as a risk measure (a target function to minimize), into the kinked CRRA utility dynamic model for a reinsurance strategy. Numerical simulations are also performed. I show that the reinsurance premium buffer divided by the variance of underwritten risk and divided by the insurer's risk averseness indicates the optimal ratio of how much risk should be mitigated by reinsurance.

Keywords

Reinsurance, Stochastic Process, Optimal Dynamic Strategy

1. Introduction

The question of how to design a dynamic optimal reinsurance coverage for underwritten insurance is an essential management topic for insurers. Many researchers have used one-period model or dynamic model approaches of the stochastic differential equation to examine this (as introduced in Section 2). Some researchers used VaR or CVaR as risk measures (a target function to be minimized) because the cost of capital tends to be proportional to VaR or CVaR. This study makes use of the stochastic differential equation, Hamilton Jacobi Bellman (HJB) model, in an investment asset allocation area. I show that the expansion of one-period research is a dynamic model approach of stochastic differential equation research and that, based on the one-period solution with VaR constraint, the kinked utility model in [1] can be utilized for a dynamic reinsurance strategy.

Regarding the appropriate use of reinsurance, when insurers manage the risk they underwrite, if modeled as minimizing VaR, they use reinsurance which takes the risk of loss above a certain level and below a certain upper level. From the insurer's perspective, it takes the risk of loss below a certain lower level and above a certain upper level. When the reinsurance situation, which is taking the risk of loss above a certain level and below a certain upper level, is applied to investment management in terms of how many risky assets should be invested, it is a strategy of investing in so-called risk assets with buying put option (protective put) and selling a call option (covered call), to avoid a loss greater than a certain amount, while surrendering a certain level of profit. This study shows that both of the VaR-constrained models of one-period and dynamic stochastic differential equations for optimal reinsurance strategy are essentially related to the kinked utility approach of stochastic differential equations for optimal reinsurance strategy.

The remainder of the paper is structured as follows. Section 2 introduces typical one-period models and their optimal solutions. Section 3 explains the typical dynamic approach of stochastic differential equations for dynamic reinsurance strategy. Section 4 describes the investment model of the stochastic differential equations (HJB) approach. Section 5 shows how the basic one-period model, and the basic approach of the stochastic differential equation model can be set identically. Section 6 presents related stochastic investment solutions, where essentially identical characteristics of the dynamic reinsurance strategies are pointed out, following their simulation results in Section 7. Section 8 is dedicated to the practical implementation of the strategy. And last, Section 9 concludes and describes future issues that need to be addressed.

2. One-Period Models

2.1. Notation and Implication for the Type of Insurance and Reinsurance

In the context of decision-making for insurance, I use the following notations, with the explanations based on [2].

X_i or X: losses to the individual i

x: $x \in X$

d: a lower limit of loss

l: an upper limit of loss

$f(X_i)$ or $f(X)$: loss coverage to the individual i by the insurer

Pfi: insurance premium

$X_i - f(X_i)$ or $X - f(X)$: total retained loss for individual i

$Y = \sum_{i=1}^{n} f(X_i)$: total losses for the insurer

PI or P: reinsurance premium determined by some premium principle

$I(Y)$: insurer's ceded losses

$C = Y - I(Y) + PI$: insurer's total loss

$R = I(Y) - PI$: reinsurer's total loss

w_i or w: individual's wealth

w_1: insurer's wealth

w_2: reinsurer's wealth

$u(\ )$: insurer's utility

$v(\ )$: reinsurer's utility

The risk is described by the loss X and its distribution is not specified. The insurer underwrites the insurance contract to cover the loss X. Depicting x as an outcome of X, in case $f(x) = kx$ ($0 < k < 1$), the type of the reinsurance is proportional reinsurance with the ratio k. If $f(x) = (x - d)_+$, the type of the reinsurance is reinsurance with the deductible amount "d". If $f(x) = x \wedge l$, the type of the reinsurance is reinsurance with the excess loss amount, that is larger than "l".

2.2. Individuals Who Buy the Insurance

I follow [3] concept of maximizing utility, namely individuals are risk-averse with the utility function U:

$$Max\ E\left[U\left(w - X + f(X) - Pfi\right)\right] \tag{1}$$

The results show that:

If $Pfi = E\left[f(X)\right]$, then $f(x) = x$ for all x, and that (2)

If $Pfi = (1+\theta)E\left[f(X)\right]$, where $\theta > 0$ is a risk loading (insurance premium buffer), then $f(x) = (x-d)_+$. (3)

Equation (3) shows that optimal reinsurance payoff has a deductible amount "d".

2.3. Insurers

Researches of [4] and [5] minimized value-at-risk:

$VaR_\alpha(Y) = F^{-1}(Y(\alpha)) = inf\{y : F(\alpha) \geq \alpha\}$.

$$Min\ VaR_\alpha\left[\left(w - Y + I(Y) - P\right)\right] \text{ with } P = (1+\theta)E\left[I(Y)\right] \tag{4}$$

Roughly,

$$I^*(y) = \left(y - S_Y^{-1}(\theta^*)\right)_+ \wedge l \text{, where } \theta^* = 1/(1+\theta). \tag{5}$$

The illustrative $I(y)$ is in **Figure 1**, as quoted from [2] page 17's Figure 3. In general, also refer to [6]. Equation (5) means that optimal reinsurance payoff has both a deductible amount d and loss cover limit (max) at level l ($d = S_Y^{-1}(\theta^*)$).

2.4. Insurers and Reinsurers

2.4.1. Pareto Optimal

The research [7] studied Pareto optimality using the following problem of $J(I, P)$.

For $k \geq 0$,

$$Max\ J(I,P) = Max\left\{E_1\left[u\left(w_1 - Y + I(Y) - P\right)\right] + kE_2\left[v\left(w_2 - I(Y) + P\right)\right]\right\} \tag{6}$$

Subject to $E_1\left[u\left(w_1 - Y + I(Y) - P\right)\right] \geq E_1\left[u\left(w_1 - Y\right)\right]$, and to (7)

$$E_2\left[v\left(w_2 - I(Y) + P\right)\right] \geq v(w_2) \tag{8}$$

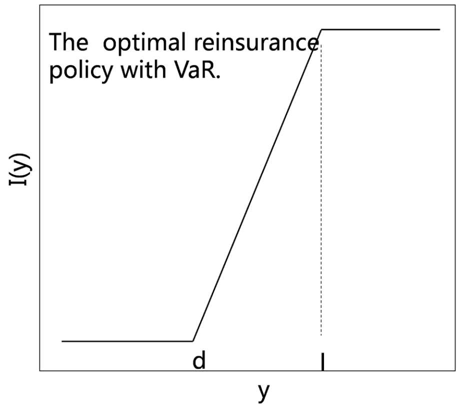

Figure 1. VaR Optimal $I(Y)$. The vertical axis is $I(Y)$, and the horizontal axis is Y. Quoted from [2], page 17, Figure 3.

By changing the value of k, one gets a set of Pareto optimal policies (efficiency frontier). For other generalizations, see [8].

2.4.2. Nash Equilibrium

To obtain the best policy, one can apply the classical game-theoretical equilibrium introduced by [9], where the reinsurance policy solves:

$$Max_{P\in[0,min\{w_1,P\}]}\left\{E_1\left[u\left(w_1-Y+I(Y)-P\right)\right]-E_1\left[u\left(w_1-Y\right)\right]\right\} \times\left\{E_2\left[v\left(w_2-I(Y)+P\right)\right]-v\left(w_2\right)\right\} \quad (9)$$

For other generalizations, please see [10].

3. Dynamic Model Approach of the Stochastic Differential Equation

To obtain the best policy, one can apply the classical game-theoretical equilibrium. I follow [11] and assume that $(\Omega, F, (F_t), P)$ denotes a complete filtered probability space that satisfies the general condition with a reference filtration $F_t \geq 0$. P is a martingale probability measure equivalent to the real-world probability, and $T > 0$ is a time horizon.

I suppose that the surplus of the insurance company follows Brownian motion with drift. For a better understanding of the model's formulation, I introduce the Cramér-Lundberg model as follows:

$$R(t)=x_0+ct-Z(t), t>0 \quad (10)$$

$R(t)$ denotes the insurer's capital at time t, x_0 is the initial capital, and $c > 0$ and $Z(t)$ denotes the premium income rate and claims, respectively. The basis for calculating the premium rate of an insurer is based on the expected value principle. Therefore, the respective mathematical expressions are:

$$Z(t)=\sum_{t=1}^{N(t)} Y_i \text{ and } c=(1+\theta)\lambda\mu \quad (11)$$

where $N(t)$ is the number of claims up to time t and follows a Poisson process with intensity $\lambda > 0$. In addition, $E[Y_i] = \mu$. Moreover, $\theta > 0$ is the safety loading interpreted as a risk premium buffer, and Y_i denotes the *i-th* claim, and they are independent and identically distributed random variables. I assume $c > \lambda\mu$ as the necessary condition to instantly avoid the insurer's bankruptcy. Let the claim process $Z(t)$ follow the Brownian motion with drift based on [12] [13] and [14]:

$$Z(t) = \mu\lambda \mathrm{d}t - \sigma \mathrm{d}W(t). \tag{12}$$

Therefore, the surplus process for insurance becomes:

$$R(t) = c\mathrm{d}t - \mathrm{d}Z(t) = \lambda\mu\theta \mathrm{d}t + \sigma \mathrm{d}W. \tag{13}$$

Thus, the risk process is perturbed by Brownian motion, and the insurer can choose proportional reinsurance over other types of reinsurance.

Furthermore, I denote risk exposure by $\alpha = \alpha(t) = [0,1]$, the proportional reinsurance level by $(1 - \alpha)$, and the premium buffer for reinsurance by $c = (1+\phi)\lambda\mu$. Therefore, the insurer diverts a portion of the premium to the reinsurer at the rate with buffer $(1+\phi)\lambda\mu(1-\alpha)$, where $\phi > 0$ is the safety loading of the reinsurer and $\phi > 0$. Thus, the surplus process $R(t)$ without investment satisfies the stochastic differential equation:

$$\mathrm{d}R(t) = \mu\lambda\left[-(\phi-\theta) + \alpha(t)\phi\right]\mathrm{d}t + \alpha(t)\sigma \mathrm{d}W(t). \tag{14}$$

See [14] and [15] for other generalizations.

With the comparison of the description in 2.1., the risk is counted by loss X's deviation from expected value of X, and its motion is expressed by Brownian motion. The insurer underwrites the insurance contract to cover the loss X and the expected value of X is covered by part of the insurance premium for sure.

4. Investment Model

4.1. Model Basics

I use the investment model and follow [1]. The objective is set to maximize the expected utility (denoted by U) of end-of-period investor wealth by w_T allocating wealth w_t between two assets, a risky security (risky asset) and a riskless security (risk-free asset), over some investment horizon [0, T], which is called a strategy and expressed by the risky asset weight ϕ_t. In Section 3, the character ϕ is used for another parameter but here I use [1] notation. Section 4's ϕ_t is Section 3's $\alpha(t)$. w_t does not become 0 or negative. Other assumptions are as follows.

1) A strategy process X_t manages the portfolio, and it consists of investing in both risky and risk-free assets.

2) The asset amount w_t consists of portfolio assets and derivatives (options), if any.

3) The risky asset's characteristic is set as price S under geometric Brownian motion with drift and volatility.

Brownian motion B_t is on a complete filtered probability space (Ω, F, (F_t), P) with an initial value of 0, almost certain. Filtration F_t is all available time t information for the pension fund. In Section 3, Brownian motion is written as $W(t)$ and I use another description B_t when I discuss an investment model. Setting a finite time T, $(F_t)_{0\le t\le T}$ satisfies the usual conditions, and the augmented sigmafield generated by B_t up to time t. In general, X is a controlled state process valued in R and satisfies the following equation:

$$\mathrm{d}X_s = b(s, X_s, \phi_s)\mathrm{d}s + \sigma(s, X_s, \phi_s)\mathrm{d}B_s . \tag{15}$$

The decision of the risky asset weight ϕ_s is the control. Generally, the control is set as $\phi = (\phi_s)_{0\le s\le T}$, and it is a progressively measurable process valued in the control set A, a subset of R. The Borelian functions b, σ on $[0,T]\times R\times A$ satisfy the usual conditions to ensure the existence of a strong solution to the above stochastic process. This is typically satisfied when b and σ satisfy a Lipschitz condition on (s, x) uniformly in A, and α satisfies a square integrability condition.

In this study, the risky asset's characteristics, which are set as price S is under geometric Brownian motion with drift μ^S and volatility σ^S, and the risk-free asset's interest rate is assumed fixed at r^f.

$$\mathrm{d}S_t = S_t\mu^f\mathrm{d}t + S_t\sigma^S\mathrm{d}B_t,\ \ \mu^S,\ \ \sigma^S,\ \text{and}\ \ r^f = const. \tag{16}$$

$$\mathrm{d}X_t = X_t\left[\phi_t\left(\mu^S - r^f\right) + r^f\right]\mathrm{d}t + \phi_t X_t\sigma^S\mathrm{d}B_t . \tag{17}$$

Regarding the risky asset, the P measure of $\mathrm{d}S_t$ and its equivalent martingale Q measure are assumed to exist. I analyze the following stochastic process:

$$\mathrm{d}S_t = S_t r^f\mathrm{d}t + S_t\sigma^S\mathrm{d}\hat{B} . \tag{18}$$

where $\hat{B}$ is defined by $\theta = \dfrac{\mu^S - r^f}{\sigma^S}$ and $\mathrm{d}\hat{B} = \mathrm{d}B + \theta\mathrm{d}t$.

Again, in Section 3, I use ϕ as the reinsurance premium buffer and in this section, character ϕ_t is used as the ratio of risky assets. I use another description when I discuss an investment model.

4.2. Merton Model

The utility functions treated in this study are shown in their mathematical form below. The CRRA utility maximization problem is set as follows:

$$\underset{\phi_t}{Sup}\, E\left[U^{STD}(w_T)\right] \tag{19}$$

where

$$U^{STD}(x) = \frac{w_t^{1-\gamma} - 1}{1-\gamma}\ \ \text{(risk averseness}\ \gamma\ \text{is set as a constant)} \tag{20}$$

Subject to:

$$V(w_0, 0) = \mathrm{e}^{-r^f T}E_Q\left[w_T\right] \tag{21}$$

where P is the market measure, and Q is the risk-neutral measure. (See [1])

4.3. Kinked Utility Model

The kinked utility functions are shown in a mathematical form below. Setting the CRRA utility maximization problem, I denote two utility function features: kinks at the minimum level (M) and the target level (L) of asset wealth. M is for modeling a minimum solvency level, and L is for liability, which should be constant. I describe the minimum and target levels in one equation and aim to do this simultaneously. In Yamashita [1], they are not treated simultaneously. The mathematical expression is as follows:

$$\underset{\phi_t}{Sup}\, E\left[U^{M,L}\left(w_T\right)\right] \tag{22}$$

$$U^{M,L}\left(x\right)=\begin{cases}-\infty & \text{if } 0<x\le M\\ \dfrac{w_t^{1-\gamma}-1}{1-\gamma} & \text{if } M<x<L\\ U^{M,L}\left(L\right) & \text{if } L\le x\end{cases} \tag{23}$$

Risk averseness γ is set as a constant.

Subject to:

$$V\left(w_0,0\right)=\mathrm{e}^{-r^f T}E_Q\left[w_T\right] \tag{24}$$

$$E_P\left[U^{M,L}\left(w_T\right)\right]>U^{M,L}\left(M\right) \tag{25}$$

$$E_P\left[U^{M,L}\left(w_T\right)\right]<U^{M,L}\left(L\right) \tag{26}$$

where P is the market measure, and Q is the risk-neutral measure.

5. The Model

5.1. One-Period Model

As a one-period model, the following applies for the [7] type problem setting and is rewritten as follows:

W^i: insurer wealth with $t = 0$, $W0^i$;

W^r: insurer wealth with $t = 0$, $W0^r$;

X: losses;

$I(X)$: losses covered by reinsurance;

θ: insurance premium buffer, meaning premium with risk loading = $\left(1+\theta\right)E\left[X\right]$;

ϕ: reinsurance premium buffer, meaning premium with risk loading = $\left(1+\phi\right)E\left[I\left(X\right)\right]$;

α: risk retention, 1 − α is receded risk portion (in this section, I use α differently from the way in Section 4).

The wealth and wealth changes of the insurer and reinsurer in the case of no reinsurance and in the case of reinsurance (ratio α), are presented as follows.

$$W^i \text{ no reinsurance}=W0^i+E\left[X\right]\left(1+\theta\right)-\left\{E\left[X\right]+\left(X-E\left[X\right]\right)\right\} \tag{27}$$

$$\mathrm{d}W^i \text{ no reinsurance}=\theta E\left[X\right]-\left(X-E\left[X\right]\right) \tag{28}$$

$$W^r \text{ reinsurance}=W0^r+E\left[I\left(X\right)\right]\left(1+\phi\right)-\left\{E\left[I\left(X\right)\right]+\left(I\left(X\right)-E\left[I\left(X\right)\right]\right)\right\} \tag{29}$$

$$\mathrm{d}W^i \text{ reinsurance} = \phi E\left[I(X)\right] - \left(I(X) - E\left[I(X)\right]\right) \tag{30}$$

$$\begin{aligned}
&\mathrm{d}W^i \text{ with reinsurance} \\
&= \left\{\theta E[X] - \left(X - E[X]\right)\right\} - (1-\alpha)\left[\phi E\left[I(X)\right] - \left\{I(X) - E\left[I(X)\right]\right\}\right] \\
&= \left[-\left\{\phi E\left[I(X)\right] - \theta E[X]\right\} + \alpha\phi E\left[I(X)\right]\right] \\
&\quad - \left[\left\{\left(X - E[X]\right) - \left(I(X) - E\left[I(X)\right]\right)\right\} + \alpha\left(I(X) - E\left[I(X)\right]\right)\right]
\end{aligned} \tag{31}$$

$$\mathrm{d}W^r \text{ reinsurance} = \phi E\left[I(X)\right] - \left\{I(X) - E\left[I(X)\right]\right\} \tag{32}$$

5.2. Stochastic Differential Equation Model

I describe the stochastic differential equation based on the previous section.

B: (geometric) Brownian motion:

$$\mathrm{d}W^i \text{ no reinsurance} = (1+\theta)\lambda\mu\mathrm{d}t - \left(\lambda\mu\mathrm{d}t - \sigma\mathrm{d}B\right) = \theta\lambda\mu\mathrm{d}t + \sigma\mathrm{d}B \tag{33}$$

$$\mathrm{d}W^i \text{ reinsurance} = (1+\phi)\lambda\mu\mathrm{d}t - \left(\lambda\mu\mathrm{d}t - \sigma\mathrm{d}B\right) = \phi\lambda\mu\mathrm{d}t + \sigma\mathrm{d}B \tag{34}$$

$\alpha = \alpha(t)$: risk retention

$1-\alpha$: receded risk portion

$$\begin{aligned}
\mathrm{d}W^i \text{ with reinsurance} &= \left\{\theta\lambda\mu\mathrm{d}t + \sigma\mathrm{d}B\right\} - (1-\alpha)\left\{\phi\lambda\mu\mathrm{d}t + \sigma\mathrm{d}B\right\} \\
&= \left\{-(\phi-\theta) + \alpha\phi\right\}\mathrm{d}t + \alpha\sigma\mathrm{d}B
\end{aligned} \tag{35}$$

$$\mathrm{d}W^r \text{ reinsurance} = \phi\lambda\mu\mathrm{d}t + \sigma\mathrm{d}B \tag{36}$$

The one-period model (Equations (31) and (32)) and the stochastic differential equation model (Equations (35) and (36)) are essentially identical. Furthermore, Equation (35) is also essentially identical to the investment model depicted in Equation (37):

$$\mathrm{d}X_t = X_t\left[\phi_t\left(\mu^S - r^f\right) + r^f\right]\mathrm{d}t + \phi_t X_t \sigma^S \mathrm{d}B_t \tag{37}$$

where r^f is $-(\phi-\theta)$ (and usually negative) and $\mu^S - r^f$ is ϕ, and $\mathrm{d}X_t/X_t$ is $\mathrm{d}W^i$, and ϕ_t is $\alpha = \alpha(t)$ (in this sense, I use arithmetic Brownian motion instead of a geometric one).

5.3. VaR as Risk Measure

In Section 4, I use CRRA utility, and even in this case, it is related to using variance as a risk measure. Here I use the Taylor series of the difference w_t and w_{t+dt} as follows as used in [16]:

$$w_{t+dt} - w_t = \left[\phi_t\left(\mu_t^S\mathrm{d}t + \sigma_t^S b_t\sqrt{\mathrm{d}t}\right) + (1-\phi_t)r_t^f\right]w_t \tag{38}$$

$$U(w_{t+dt}) \approx U(w_t) + U'(w_t)(w_{t+dt} - w_t) + \frac{1}{2}U''(w_t)(w_{t+dt} - w_t)^2 \tag{39}$$

$$\begin{aligned}
E_{dt}\left(U(w_{t+dt})\right) &\approx U(w_t) + U'(w_t)w_t\mathrm{d}t\left[\phi_t\mu_t^S + (1-\phi_t)r_t^f\right] \\
&\quad + \frac{1}{2}U''(w_t)w_t^2\left(\sigma_t^S\right)^2\mathrm{d}t
\end{aligned} \tag{40}$$

$$\approx U\left(w_t\right)+w_t \mathrm{d}t U'\left(w_t\right)\left[\left[r_t^f+\frac{1}{2}\frac{\left(\frac{\mu_t^S-r_t^f}{\sigma_t^S}\right)^2}{\left[-w_t\frac{U''\left(w_t\right)}{U'\left(w_t\right)}\right]}\right]-\frac{1}{2}\left[-w_t\frac{U''\left(w_t\right)}{U'\left(w_t\right)}\right]\left(\sigma_t^S\right)^2\left[\phi_t-\frac{\left(\frac{\mu_t^S-r_t^f}{\sigma_t^S}\right)}{\left[-w_t\frac{U''\left(w_t\right)}{U'\left(w_t\right)}\right]\sigma_t^S}\right]^2\right] \tag{41}$$

In case $U=U^{STD}\left(w_t\right)$,

$$\approx U\left(w_t\right)+w_t \mathrm{d}t U'\left(w_t\right)\left[\left[r^f+\frac{1}{2}\frac{\left(\frac{\mu^S-r^f}{\sigma^S}\right)^2}{\gamma}\right]-\frac{1}{2}\gamma\left(\sigma^S\right)^2\left[\phi_t-\frac{\left(\frac{\mu^S-r^f}{\sigma^S}\right)}{\gamma\sigma^S}\right]^2\right] \tag{42}$$

The risk is related to variance (later part of [ ] in the second term's coefficient of [ ]). Note that ϕ_t with maximizes $E_{dt}\left(U\left(w_{t+dt}\right)\right)$ is $\phi_t=\frac{\left(\frac{\mu^S-r^f}{\sigma^S}\right)}{\gamma\sigma^S}$ (ϕ_t is time independent and constant). Brownian motion means that VaR is proportionate to the standard deviation, the square root of variance; so here, using the CRRA utility and using VaR as a risk measure is essentially the same. If the VaR-constrained solution is an option type (a buy put option and a sell call option), kinked utility using CRRA utility matches for dynamic strategy optimization using a stochastic differential equation. Using VaR as a major part of the utility, as in the case of [17] (**Appendix A**), does not reflect its use in the one-period model, where VaR is used as a risk constraint. Moreover, from the perspective of the Arrow-Pratt measure, the variance of w_t can be treated as a risk (see **Appendix B**).

6. Solutions

6.1. Merton Model Solution

I solve the problem below. Equation (47) indicates that the optimal dynamic reinsurance strategy is to keep the reinsurance cover ratio constant (time independent). The ratio can be calculated from the following parameters.

$$\underset{\phi_t}{Sup}\, E\left[U^{STD}\left(w_T\right)\right] \tag{43}$$

$$\text{Subject to: } V^{STD}\left(w_T,T\right)=U^{STD}\left(w_T\right) \tag{44}$$

$$V^{STD}\left(w_t,t\right)=\underset{\phi_t}{Sup}\, E\left[U^{STD}\left(w_t\right)\right] \tag{45}$$

$$V_t^{STD}+V_w^{STD}w_t\left[\phi_t\left(\mu^S-r^f\right)+r^f\right]+V_{ww}^{STD}\left(\sigma^s\right)^2 w_t^2\phi_t^2/2=0 \tag{46}$$

$$\phi_t = \frac{\left(\mu^S - r^f\right) / \left(\sigma^S\right)^2}{-V_{ww}^{STD} / \left(w_t V_w^{STD}\right)} = \frac{\left(\mu^S - r^f\right) / \left(\sigma^S\right)^2}{\gamma} \quad \text{(Constant, independent from } t\text{)} \tag{47}$$

(See **Appendix A**).

6.2. Kinked Utility Model Solution

The illustrative solution can be described as follows. The utility of the reinsurer should not be below or equal to minimum *M*—it would be like buying a put option and not necessarily above *L*, the upper limit, or selling a call option. This is related (similar) to **Figure 1**'s *I*(*y*) payoff; moreover, see the next section's simulation results for buying a put option and selling a call option. The upper limit *I* is *L*, and the lower limit *d* is *M*.

Equation (5), **Figure 2**, and **Figure 3** show that the optimal reinsurance strategy has the lower and upper limit of risk taking. In a dynamic model context of an investment model, the solution of the case of both constraints (lower and upper bounds) is described below, following [1] calculated the optimal solution w_t, which sets w_T^{***} at $t = T$, w_t^{***}:

$$w_t^{***} = \zeta_t X_t^{STD} N\left(d_1\right) + L e^{-r^f(T-t)} N\left(-d_2\right) - \zeta_t X_t^{STD} N\left(-d_1\right) + L e^{-r^f(T-t)} N\left(d_2\right) \tag{48}$$

ζ_t = scalar, varies if time varies. This is decided by self-financing constraints.

$$d_1 = \frac{\ln\left(\frac{X_t^{STD}}{\frac{L e^{-r^f(T-t)}}{\zeta_t}}\right) + \frac{1}{2}\left(\sigma^S\right)^2}{\sigma^S}, \quad d_2 = \frac{\ln\left(\frac{X_t^{STD}}{\frac{L e^{-r^f(T-t)}}{\zeta_t}}\right) - \frac{1}{2}\left(\sigma^S\right)^2}{\sigma^S}, \tag{49}$$

where the utility function is:

$$U^{M,L}(x) = \begin{cases} -\infty & \text{if } 0 < x \le M \\ \frac{w_t^{1-\gamma} - 1}{1-\gamma} & \text{if } M < x < L \\ U^{M,L}(L) & \text{if } L \le x \end{cases} \tag{50}$$

7. Simulations

I performed a Monte Carlo simulation, using the solution of the case of the kinked utility at both an upper limit and a lower limit. The solution is for its reinsurer. The solution strategy means that the reinsurer's wealth is generated from the dynamic payoff between insurance underwriting and reinsurance strategy results. In the simulation, 10,000 return patterns are generated for underwritten insurance's risk using arithmetic Brownian motion. The details of the parameters are as follows. Each period means one year, and the total number of years is 20. I denote the Merton model solution as a benchmark as "standard strategy (STD)". The solution of our strategy is a "conventional reinsurance cover strategy (CC)". The target "L" and "M" are exogenously given as 150 and

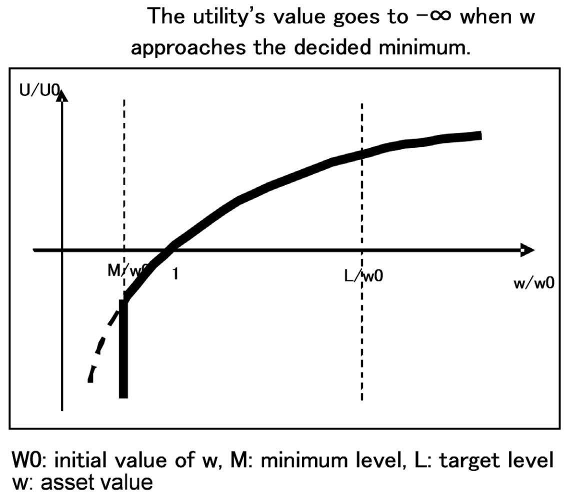

Figure 2. An illustration of the utility function $U(w)$ with the kink at $w = M$. Quoted from [1], page 57, Figure 2. (U is the utility function, and w is the asset value. The utility's value goes to $-\infty$ when w approaches the minimum asset value level M)

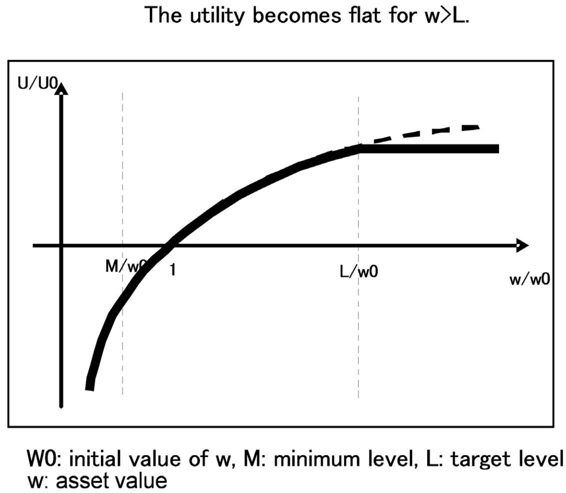

Figure 3. An illustration of the utility function $U(w)$ with the kink at $w = L$. Quoted from [1], page 57, Figure 3. (U is the utility function, and w is the asset value. L is the target level of the asset value, and the utility becomes flat for $w > L$).

50, respectively, for all instances as an upper limit and a lower limit, with wealth starting at 100. CC is an optimal strategy under the kinked utility and STD is an optimal strategy under the "normal" utility, which means the reinsurance strategy is a constant ratio of reinsurance to buy and insurance to be underwritten. Utility is $\gamma = 1.411$ as in the research of Yamashita [1].

In the equations of:

$$\mathrm{d}R(t) = \left[-(\phi - \theta)\mu\lambda + \phi\mu\lambda \times \alpha(t)\right]\mathrm{d}t + \alpha(t) \times \sigma \mathrm{d}W(t) \text{, and} \tag{51}$$

$$\mathrm{d}X_t / X_t = \left[r^f + \left(\mu^S - r^f \right) \times \phi_t \right] \mathrm{d}t + \phi_t \times \sigma^S \mathrm{d}B_t , \tag{52}$$

I set the following **Table 1**'s cases and parameters. Equations (51) and (52) are identical to Equations (35) and (37), respectively, and only the order of the parameters have changed. In Equation (52), Equation (51)'s $\alpha(t)$ is treated as ϕ_t, Equation (51)'s $\phi\mu\lambda$ is treated as ($\mu^S - r^f$), and Equation (51)'s $-(\phi-\theta)\mu\lambda$ is treated as r^f. The table of (a) shows the parameters of Equation (51) and the table of (b) shows the parameters of Equation (52). **Figure 4** shows Case 1. The vertical axis shows the CC strategy's wealth return at the end of $t = 20$, and the horizontal axis shows the STD strategy's wealth at the end of $t = 20$. The red line (45-degree line) shows that CC and STD ultimately have the same as wealth performance. **Table 2**'s WIN is the percentage of the samples in which CC is superior to STD from the end wealth growth viewpoint. The return is calculated by an annualized return of the change in wealth from $t = 0$ to $t = 20$. Some of the characteristics are explained below.

The data assumptions used in μ, λ, ϕ, θ are as follows. Regarding μ and λ, I use typical risk of 0.50 and 0.20. Regarding ϕ and θ, taking typical insurance premium buffer rates into account, I decided 0.20 and 0.10 respectively.

1) How reinsurance premium buffer is larger than insurance premium buffer is naturally matters as $-(\phi-\theta)\mu\lambda$ of Equation (51) is always reduce the wealth.

2) STD keeps the reinsurance ratio at 0.89 whatever the value of the wealth is. The number 0.89 means the insurer usually (neither VaR constraints nor risk measure burden) reinsures 11% of the underwritten and keeps 89% of the risk.

3) The condition for CC is, from a dynamic strategy point of view, avoiding more than the value of 50 (translated as upper VaR limit = (150 − 100)/100/20% = 2.5 (2.5σ)) and giving up more than the additional value of 50 units. So the reinsurance ratio is not always 0.89.

4) The distribution of returns shows upper and lower limit characteristics. This is because the reinsurer's insurance result is limited bad results, which is that loss is larger than premium inflow and limited good results, which is that loss is smaller than premium inflow. This simulation is discrete time base and that is the reason the plots are bounded to some extent.

5) In Case 1, WIN is 21.5%, which means that because of VaR constraints or the risk measure burden, for the reinsurer, the CC result (final term's wealth) is inferior to the STD case with a 78.5% chance. Instead, the insurer gets the merit.

The results of Cases 1 to 4 are shown in **Figure 5** and **Table 2**. Comparing Cases 1 and 4 (or/and Cases 2 and 3) reveals the reinsurance costs—Case 1 costs more than Case 4, and Case 2 costs more than Case 3. Comparing Cases 1 and 2 (or/and Cases 3 and 4) reveals risk volatility. Case 1 shows greater volatility than Case 2, and Case 3 greater volatility than Case 3.

Figure 5 shows the four cases: Case 1 (upper right): ($\mu^S - r^f$) = 3%, $r^f = -2\%$, $\sigma = 20\%$, Merton model solution 0.89; Case 2 (upper left): ($\mu^S - r^f$) = 3%, $r^f = -2\%$, $\sigma = 15\%$, Merton model solution 1.57; Case 3 (lower left): ($\mu^S - r^f$) = 2%, $r^f = -1\%$, $\sigma = 15\%$, Merton model solution 0.95); Case 4 (lower right): ($\mu^S - r^f$) = 2%, $r^f = -1\%$, $\sigma = 20\%$, Merton model solution 0.53).

Table 1. Parameter settings of Equations (51) (left) and (52) (right).

(a)

	Case 2		Case 1
$\mu =$	50%	$\mu =$	50%
$\lambda =$	20%	$\lambda =$	20%
$\varphi =$	30%	$\varphi =$	30%
$\theta =$	10%	$\theta =$	10%
	Case 3		Case 4
$\mu =$	50%	$\mu =$	50%
$\lambda =$	20%	$\lambda =$	20%
$\varphi =$	20%	$\varphi =$	20%
$\theta =$	10%	$\theta =$	10%

(b)

	Case 2		Case 1
$\mu^S - r^f =$	3%	$\mu^S - r^f =$	3%
$r^f =$	−2%	$r^f =$	−2%
$\sigma =$	15%	$\sigma =$	20%
	Case 3		Case 4
$\mu^S - r^f =$	2%	$\mu^S - r^f =$	2%
$r^f =$	−1%	$r^f =$	−1%
$\sigma =$	15%	$\sigma =$	20%

Table 2. How much CC is better than STD?

	Case 2		Case 1
Upper	2.0%	Upper	2.0%
Lower	−3.4%	Lower	−3.4%
STD	1.57	STD	0.89
WIN	16.7%	WIN	21.5%
	Case 3		Case 4
Upper	2.0%	Upper	2.0%
Lower	−3.4%	Lower	−3.4%
STD	0.95	STD	0.53
WIN	26.2%	WIN	27.3%

The four cases tell us the following.

1) In each case, as seen in the previous description regarding Case 1, the distribution of returns shows upper and lower limit characteristics. This is because the reinsurer's insurance result is limited bad results, which is that loss is larger than premium inflow and limited good results, which is that loss is smaller than premium inflow.

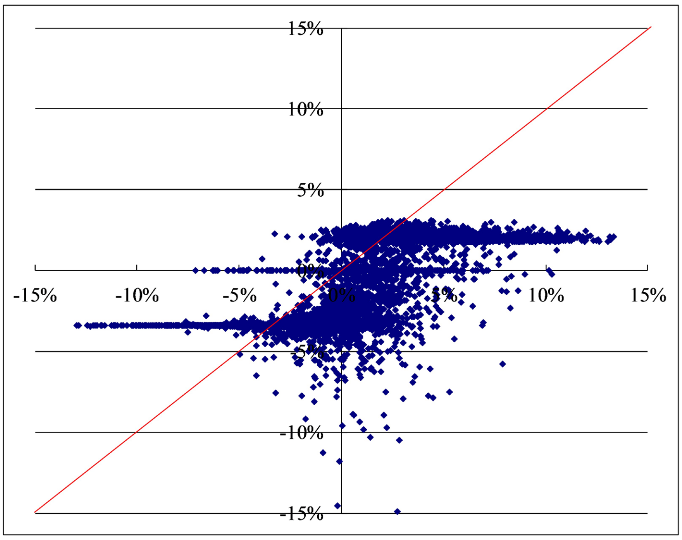

Figure 4. An illustration of the results of the reinsurance strategy (Case 1), where $(\mu^S - r^f) = 3\%$, $r^f = -2\%$, $\sigma = 20\%$, and the Merton model solution is 0.71. The vertical axis shows the CC strategy's wealth return at the end of $t = 20$ and the horizontal axis shows the STD strategy' wealth at the end of $t = 20$. The red line (45-degree line) shows that CC and STD ultimately have the same as wealth performance.

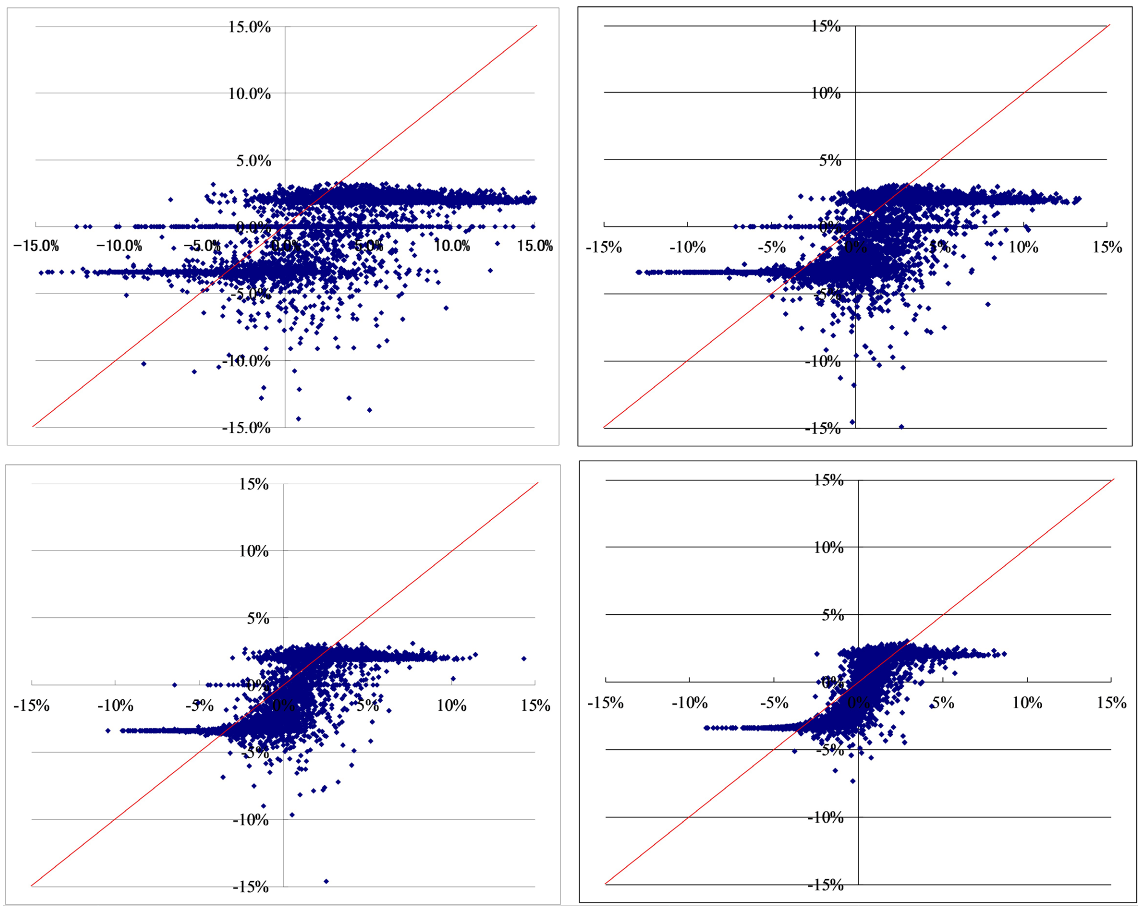

Figure 5. An illustration of the result of the reinsurance strategy (CC), Cases 1 to 4.

2) The Merton model solution shows that the value of insurers' risk-taking matters regarding the wide variety of returns, which means a smaller WIN rate (VaR constraint cost is larger.) The order of Case 1 to Case 4 is as follows:

Case 4 < Case 3 ≈ Case 1 < Case 2.

Merton model solution (reinsurance ratio) is calculated by $\phi\mu\lambda/\sigma^2$ divided by risk averseness if described by Equation (51) parameters.

3) Suppose the Merton model solution figure is almost the same; volatility matters. Larger volatility (Case 1) has a wider variety of returns (smaller WIN rate, larger VaR constraint cost), comparing Case 3.

Last, the limitation of the proposed strategy is that the model assumes the insurance premium consists of the expected loss value and the buffer which is proportion to the expected loss value. The simpleness of Equation (58) comes from the premise.

8. Practical Use of the Strategy

8.1. Reinsurance Strategy of the Insurance Industry

The reinsurance strategy depends on many factors such as if the insurance is non-life or life. However, typically, insurers use reinsurance from the following reasons:

- The risk is so huge that it is natural to share the risk among several insurers.
- The insurer needs to reduce risk exposure because of insufficient economic capital, because of risk management requirement, or because of improvement of cash flow and return of equity.
- The insurer needs to diversify underwriting risks.

Regarding the type of reinsurance, proportional type is the industry's traditional and non-proportional type like excess loss cover, is increasing. From contract type point of view, there are facultative reinsurance and treaty reinsurance. The prior is that the insurance contract decision is made case by case and the latter is that the reinsurance contract lasts usually for a long term and its conditions don't change often.

The dynamic reinsurance strategy solution in this paper considers non-proportional type of reinsurance and, in addition, the strategy dynamically changes reinsurance exposure. It is not a traditional way and whether facultative reinsurance and treaty reinsurance, it needs some tactics to fit for a dynamic strategy.

8.2. Implication of the Solution and Simulation Results

The implications and potential impact of the results on the insurance industry are that, in the case risk deviation is larger, the reinsurer suffers more, even though the reinsurer takes only the middle range of risk, because the dynamic strategy costs the reinsurer which comes from VaR constraint. There is a lucky opportunity to save the loss below expected value. The reinsurer welcomes a larger reinsurance premium buffer but the merit is limited.

There are some external factors that affect the results. One of those is that the model supposes the insurance premium buffer and reinsurance buffer is time independently constant to expected loss value. In addition, current regulatory economic capital requirement amount is basically proportion to VaR and that is the reason I treated VaR constrained problem. The regulation change might change the needed problem setting.

9. Conclusions

I show that one-period model research and the dynamic model approach of the stochastic differential equation can be described as having the same structure. I made use of the investment model for the dynamic model approach, which gives us a wider variety of reinsurer strategies. I also noted that the optimal VaR minimization measure model under a certain condition relates to the kinked CRRA utility dynamic investment strategy.

Generally, and naturally, higher reinsurance cost (reinsurance premium burden) makes using reinsurance less attractive. From an optimal dynamic strategy point of view, the reinsurance premium burden divided by the risk's volatility and divided by the insurer's risk averseness affects how much insurance exposure should be covered by reinsurance.

There are additional challenges in this area that should be addressed in future studies. One of them is to test a stochastic volatility case.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A

The following description is following [16] [17] (henceforth, H-V) analyzed the utility maximization problem below. At first sight, it differs from the Merton model.

$$\underset{\phi(w_t)}{Sup}\left(E\left[w_T\right]-\alpha Var\left[w_T\right]\right) \tag{A.1}$$

Subject to: $w_t = w_0$, $w_t \geq 0$, $t = 1,2,\cdots,T$.

Using $\mathrm{d}S_t = S_t\mu^S\mathrm{d}t + S_t\sigma^S\mathrm{d}B_t$ and solving the Hamilton-Jacobi-Bellman equation, they derive the equation below.

$$\phi(w_t)=\frac{\mu^S-r^f}{\sigma_S^2}\cdot\left(\frac{\mathrm{e}^{-r^f(T-t)+\left(\frac{\mu^S-r^f}{\sigma_S}\right)^2}}{2\alpha w_t}+\frac{1}{w_t\Big/\left(w_0\mathrm{e}^{r^f}\right)}-1\right) \tag{A.2}$$

Just in case, Merton Model is solved in the following way. The equation below is solved by Lagrangian L:

$$\underset{\phi_t}{Sup}\, E_P\left[U^{STD}(w)\,|\,t=T\right] \tag{A.3}$$

$$\text{Subject to: } V(w_0,0)=\mathrm{e}^{-r^fT}E_Q\left[w_T\right] \tag{A.4}$$

$$L=E_P\left[U^{STD}(w_T)\right]-\lambda\left(\mathrm{e}^{-r^fT}E_Q\left[w_T\right]-V(w_0,0)\right) \tag{A.5}$$

Radon-Nikodym derivative is set as g_t:

$$w\,|\,\underset{w}{Sup}\left(\frac{\partial}{\partial w}U^{STD}(w)\,|\,t=T\right)=\lambda\cdot g_t\,|\,t=T$$

The optimal solution w_T^* is shown below.

$$w_T^*=\left[\frac{\partial}{\partial w}U^{STD}\right]^{-1}(\lambda g_t)\,|\,t=T=(\lambda g_t)^{-\frac{1}{\gamma}}\,|\,t=T \tag{A.6}$$

$$\lambda=\left[w_0\mathrm{e}^{r^fT}\right]^{-\gamma}E_P\left[(g_t)^{\frac{1-\gamma}{\gamma}}\right]^{\gamma}\,|\,t=T \tag{A.7}$$

The Black-Sholes premise produces the equation below.

$$g(s)=\left[\mathrm{e}^{-\frac{1}{2}\left(\frac{\mu^S-r^f}{\sigma^S}\right)^2T+\frac{\mu^S-r^f}{\left(\sigma^S\right)^2}\left(\mu^S-\frac{1}{2}\left(\sigma^S\right)^2\right)T}\right]\left(\frac{S_0}{s}\right)^{\frac{\mu^S-r^f}{\left(\sigma^S\right)^2}},\quad S_0=S_{t=0}=constant \tag{A.8}$$

$$w_T^*=\mathrm{e}^{[1-\phi]\cdot\left[r^f+\frac{1}{2}\phi\left(\sigma^S\right)^2\right]}w_0\left(\frac{S_T}{S_0}\right)^{\phi},\quad \phi=\frac{\frac{\mu-r^f}{\left(\sigma^S\right)^2}}{\gamma} \tag{A.9}$$

Next, I discuss the difference between the two models. I set $U^{STD}(x)=\log x$ with Brownian motion for the price of the risky asset. For the risky asset,

$$\mathrm{d}S_t=S_t\mu\mathrm{d}t+S_t\sigma\mathrm{d}B_t \tag{A.10}$$

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma B_t} \tag{A.11}$$

For the strategy portfolio,

$$X_t = f(t, B_t) = f(t, x) \tag{A.12}$$

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} dt \tag{A.13}$$

$$dX_t = X_t\left[\phi_t(\mu - r) + r\right]dt + \phi_t X_t \sigma dB_t \tag{A.14}$$

$$\frac{\partial f}{\partial x} = \phi_t \sigma f \quad \text{leads} \quad f(t, x) = f(t, 0) e^{\phi_t \sigma x} \tag{A.15}$$

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \left[\phi_t(\mu - r) + r\right] f \tag{A.16}$$

$$\left(\frac{\partial \phi_t}{\partial t}\sigma x + 1\right) e^{\phi_t \sigma x} \frac{\partial f(t,0)}{\partial t} + \frac{1}{2}\sigma^2 \phi_t^2 f(t,0) e^{\phi_t \sigma x} = \left[\phi_t(\mu - r) + r\right] f(t,0) e^{\phi_t \sigma x} \tag{A.17}$$

$$\frac{\partial \log f(t,0)}{\partial t} = \frac{\phi_t(\mu - r) + r - \frac{1}{2}\sigma^2 \phi_t^2}{\frac{\partial \phi_t}{\partial t}\sigma x + 1} \tag{A.18}$$

$$X_t = f(t, B_t) = X_0 e^{\int \frac{\phi_t(\mu - r) + r - \frac{1}{2}\sigma^2 \phi_t^2}{\frac{\partial \phi_t}{\partial t}\sigma x + 1} dt + \sigma B_t} \tag{A.19}$$

Regarding the Merton model:

$$\underset{\phi_t}{Sup}\, E\left[U^{STD}(w_T)\right] = \underset{\phi}{Sup}\, w_0 E\left[\int \frac{\phi_t(\mu - r) + r - \frac{1}{2}\sigma^2 \phi_t^2}{\frac{\partial \phi_t}{\partial t}\sigma x + 1} dt + \sigma \phi B_T\right] \tag{A.20}$$

Regarding the H-V model:

$$\underset{\phi_t(w_t)}{Sup}\, w_0 \left(E\left[e^{\int_0^T \frac{\phi_t(\mu - r) + r - \frac{1}{2}\sigma^2 \phi_t^2}{\frac{\partial \phi_t}{\partial t}\sigma x + 1} dt + \sigma B_T} \right] - \alpha Var\left[e^{\int_0^T \frac{\phi_t(\mu - r) + r - \frac{1}{2}\sigma^2 \phi_t^2}{\frac{\partial \phi_t}{\partial t}\sigma x + 1} dt + \sigma B_T} \right] \right) \tag{A.21}$$

If $\frac{\partial \phi_t}{\partial t} = 0$, ϕ_t is ϕ (independent from *t*). In addition,

$$\frac{\partial f(t,0)}{\partial t} = \left[\phi(\mu - r) + r - \frac{1}{2}\sigma^2 \phi^2\right] f(t,0) \tag{A.22}$$

$$X_t = f(t, B_t) = X_0 e^{\left[\phi(\mu - r) + r - \frac{1}{2}\sigma^2 \phi^2\right] t + \sigma \phi B_t}. \tag{A.23}$$

The Merton model with $U^{STD}(x) = \log x$:

$$\underset{\phi_t}{Sup}\, E\left[U^{STD}(w_T)\right] = \underset{\phi}{Sup}\, w_0 E\left[\left[\phi(\mu - r) + r - \frac{1}{2}\sigma^2 \phi^2\right] T + \sigma \phi B_T\right] \tag{A.24}$$

$\phi = \frac{\mu - r}{\sigma^2}$ maximize the *U*.

For the case H-V, $\phi_t = \phi$ means:

$$\underset{\phi}{Sup}\, E\left[w_0 e^{\left[\phi(\mu-r)+r-\frac{1}{2}\sigma^2\phi^2\right]T+\sigma\phi B_T} \right] - \alpha Var\left[w_0 e^{\left[\phi(\mu-r)+r-\frac{1}{2}\sigma^2\phi^2\right]T+\sigma\phi B_T} \right] \quad (A.25)$$

$$= \underset{\phi}{Sup}\left[w_0 e^{\left[\phi(\mu-r)+r-\frac{1}{2}\sigma^2\phi^2\right]T} - \alpha w_0^2 e^{2\left[\phi(\mu-r)+r-\frac{1}{2}\sigma^2\phi^2\right]T}\left(e^{\sigma^2 T}-1\right) \right] \quad (A.26)$$

Arg ϕ indicates:

$$\left(\phi - \frac{\mu - r}{\sigma^2}\right)\left(1 - 2\alpha w_0\left(e^{\sigma^2 T}-1\right)\right) = 0 \quad (A.27)$$

$$\phi = \frac{\mu - r}{\sigma^2} \quad (A.28)$$

$\underset{\phi_t(w_t)}{Sup}\left(E[w_T] - \alpha Var[w_T]\right)$ leads,

$$\phi_t(w_t) = \frac{\mu - r}{\sigma^2}\cdot\left(\frac{e^{-r(T-t)+\left(\frac{\mu-r}{\sigma}\right)^2 T}}{2\alpha w_t} + \frac{1}{w_t/\left(w_0 e^r\right)} - 1 \right) \quad (A.29)$$

The Merton model solution is the case of $\gamma = 1$, and above ( ) = 1. Therefore, they are different.

Appendix B

The explanation of the common risk variance by Arrow-Platt measure is described below. See [18] and [19].

$$\begin{aligned} E_t\left(U(w_t)\right) &= E_t\left(U\left(E(w_t) + w_t - E(w_t)\right)\right) \\ &\approx E_t\left(U\left(E(w_t)\right)\right) + \left(w_t - E(w_t)\right)U'\left(E(w_t)\right) + \frac{1}{2}U''\left(E(w_t)\right)\left(w_t - E(w_t)\right)^2 \\ &= E_t\left(U\left(E(w_t)\right)\right) + \frac{1}{2}E_t\left(U''\left(E(w_t)\right)\right)Var_t(w_t) \\ &= E_t\left(U\left(E(w_t)\right)\right) - \frac{1}{2}E_t\left(w_t U'\left(E(w_t)\right)\left[-w_t\frac{U''\left(E(w_t)\right)}{U'\left(E(w_t)\right)}\right]\Big/ w_t^2 \right)Var_t(w_t) \end{aligned} \quad (B.1)$$

The risk is approximated as:

$$risk \approx \frac{1}{2}\left[-w_t\frac{U''\left(E(w_t)\right)}{U'\left(E(w_t)\right)}\right]Var_t(w_t)/w_t^2 \quad (B.2)$$