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A Game Theoretic Approach on an Optimal Investment-Consumption-Insurance Strategy

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Abstract

We find the possible risk minimizing portfolio strategies in a two dimensional market consisting of a risk asset and risk-less asset. The investor in the market is subjected to consumption, purchasing of life insurance and stochastic income with inflation risk. The problem is formulated as zero sum game problem between the market and the investor. The strategies are determined for the different generations of the life of an investor, that is before the investor dies and after the investor dies. We used the concept of convex risk measures and monetary utility maximizing problem-concept studied before finding the risk minimizing portfolios which was solved using the game theoretic approach to obtain the strategies explicitly given in the propositions in the study.

Keywords

Zero Sum Games, Inflation Risk, Convex Risk Measures, Stochastic Optimization

1. Introduction

Portfolio theory was introduced by Markowitz [1]. The study was further extended by different scholars; Gábor and Kondor [2] did portfolios with nonlinear constraints and spin glasses, Schroder and Skiadas [3] looked into optimal consumption and portfolio selection with stochastic differential utility, Liu *et al.* [4] studied optimal investment problem under non-extensive statistical mechanics. Recently, Campani *et al.* [5] analysed optimal portfolio strategies with the presence of regimes in asset returns, Diaz and Esparcia [6] built optimal dynamic portfolios under time-changing risk aversion and optimal portfolio deleveraging under market impact and margin restrictions was done by Edirisinghe *et al.* [7]. Other scholars Holger *et al.* [8] studied consumption-portfolio optimization with recursive utility in incomplete markets of Epstein-Zin type by application of dynamic programming and given a proof of their verification theorem result to prove their problem. Richard [9] generalized Merton's [10] model to study optimal consumption, portfolio and life insurance rules for an investor with arbitrary but known distribution of lifetime in a continuous time model. We apply game theory in formulation of our problem. Game theoretic concept has always been applied to different fields, Thakor [11] says that game theory was widely adopted in 1970s by economics after the study by Von Neumann and Morgenstern [12] in 1944. The domination of game theory in economics was also appreciated by Gibbons [13] on the applicability of game theory to model implications of rationality, equilibrium in market interaction and non-market interactions.

Von Neumann and Morgenstern [12] contributed significantly in applying game theory in various fields, economics, statistical physics and social sciences. With optimal portfolio not an exception, the theory of differential games has also been viewed as a game to minimize the risk for high expected utility versus minimization of the maximal expected utility of the opponent. Øksendal and Sulem [14] studied a game theoretic approach to martingale measures in incomplete markets and showed that the optimal strategy for the market was to choose an equivalent martingale measure in considering a stochastic differential game in a financial jump diffusion market. The continued study is also seen in game theory as theory of differential games, the zero sum differential and nonzero sum differential games. A zero sum game is a competitive situation where one cannot win unless the other loses and the expected value of the game is zero. Many researchers have been applying the zero sum differential game to their problems. Moon and Basar [15] studied zero sum game differential game on the Wasserstein space by considering two player zero sum differential games with state processes depending on random initial condition and state's process's distribution.

Moon and Basar [15] show that the set of probability measures and the set of random variables both with finite second moments are equivalent and satisfies the dynamic programming principle and the value functions are unique viscosity solutions to their defined Hamilton-Jacobi-Isaacs equations. Bell and Cover [16] studied game theoretic optimal portfolios by showing that for various payoff functions the expected log optimal portfolio is also game theoretically optimal in a single or multiple plays of the stock market. Their findings also show that by maximizing the conditional expected log return you obtain a good short-term and long run performance. Laraki and Solan [17] studied the value of zero-sum stopping games in continuous time with no assumptions of conditions on the relations between the payoff processes. They proved that the value in randomized stopping times exists as soon as the payoff processes are right continuous.

Mataramvura and Øksendal [18] studied risk minimizing portfolios and HJBI equations for stochastic differential games that minimizes the convex risk measure of the terminal wealth in a jump diffusion market. They extended the approach to solving a zero-sum stochastic differential game between an agent and a market as a min-max problem obtained in the process by proving the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. The continued application of game theory in financial modelling is also shown by the study on non-zero sum games as Xiong *et al.* [19] as they studied a class of partially observed non-zero stochastic differential game basing their study on forward and backward stochastic differential equation (FBSDEs). They established a maximum principle as a must have condition and derived the verification theorem as their sufficient condition by applications of stochastic filtering theory and obtained the explicit investment strategy of a partial information financial problem. Zhou *et al.* [20] studied an interesting topic on non-zero games between two insurers on non-zero sum reinsurance-investment game with delay and asymmetric information in which both insurers can buy proportional reinsurance and have investments in the market with a risky and risk-free asset. They considered the asymmetric information effect with an assumption that both insurers have access to different levels of information in the market. Each one's aim is to maximize the expected utility relative to its competitor. Zhou *et al.* [20] applied the dynamic programming principle to derive the Hamilton-Jacobi Bellman's equation (HJB) and obtained the results that the insurer with less information completely ignores its own risk aversion factor and imitates the investment strategy of its competitor who has more information on the market.

Savku and Weber [21] applied game theory to study stochastic differential games for optimal investment problems in a Markov regime-switching jump-diffusion market by applying the dynamic programming principle in two optimal investment problems by using zero sum and nonzero sum approaches of stochastic games in continuous time Markov regime switching within the finance frame work. The zero sum game is between the investor and the market and the nonzero sum game is applied as the sensitivity of two investors terminal wealth/gain. They further obtain the explicit portfolio strategies with Feynman-Kac representation of value functions. Karoui *et al.* [22] studied Backward stochastic differential equations (BSDEs) and risk sensitive control, zero sum and nonzero sum game problems of stochastic functional differential equations by the use of BSDEs to show the existence of an optimal control, saddle point and equilibrium point for both games. Bensoussan *et al.* [23] did a research on a class of nonzero sum stochastic differential investment and reinsurance games by applying the dynamic programming principle to solve the resulting nonzero sum game problem. In their study the nonzero sum game is between two insurance companies where each company's surplus process comprises of a proportional reinsurance protection and investment in both risky and risk-free asset.

In the present problem the study is based on minimizing risk in an Itô Lévy setting in an investment where consumption and purchasing of insurance is made. The strategy consists of taking in consideration, inflation risk and stochastic income. The problem is formulated as a zero sum differential game where two players including the investor and the market. It is the min-max problem we get when the investor chooses a portfolio that minimizes the risk of the terminal wealth while the market aims to minimize the maximal payoff of an investor. We consider two player stochastic differential game and employ the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation and prove it.

The article is structured as follows. In Section 2, the model framework is presented in detail with the characteristics of the positions of investment. Section 3 has the solutions to the different generations of the portfolio problem in obtaining the optimal strategies. Section 4 concludes followed by the references.

2. Statement of the Problem

We suppose that our model is built in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ which follows the characteristics of the Brownian motion $B_t = \{B_t^0, B_t^1, B_t^3, B_t^4, 0 \le t \le T\}$ in a finite time horizon, *i.e.* $0 \le t \le T$. to be considered. The Brownian motions $\langle B_t^0, B_t^1, B_t^3, B_t^4 \rangle$ are all correlated in a correlation matrix such that $\mathrm{d}B_t^i \cdot \mathrm{d}B_t^j = \rho_{i,j}\mathrm{d}t$ for $i, j \in \{0,1,3,4\}$, such that, $i \ne j$, are correlated with the correlation coefficient $-1 \le \rho_{\cdot,\cdot} \le 1$. For $\rho_{\cdot,\cdot} = 1$ then the Brownian motions are strongly correlated and for $\rho_{\cdot,\cdot} = -1$ then they are negatively correlated.

We take into account an important economic inflation factor, since the value for money in a long term investment has to be considered as inflation can lead to reduction in the value of investment returns of an investor. This was also discussed by Zhang and Zheng [24] that inflation could lead to diminution of financial wealth of the insurer when studying optimal investment Reinsurance policy with stochastic interest and inflation rates. We can measure inflation by the inflation rate, by considering the consumer price index (CPI) which includes the consumer price and retail price indices for any change in the index.

The investor inherits in three assets namely, a money market account, an inflation-linked index bond and stock at time $t \in [0,T]$. The wealth of an insurance policy holder is indicated by a portfolio distributing the wealth accordingly with π_1 being the fraction of the wealth held in stock, π_2 being held in the inflation-linked index bond which shall be defined in depth along the descriptions and $(1-\pi_1-\pi_2)$ denotes the fraction held in the money market account. We note that all the fractions held must add up to 1. We denote the market value of the money market account R_t at a given time $t \in [0,T]$ as

$$\mathrm{d}R_t = rR_t\mathrm{d}t, \tag{1}$$

where $r > 0$ denote the interest rate to be compounded over *t*.

The dynamics of the stock follows an Itô Lêvy process;

$$\mathrm{d}S_t = S_{t-}\left[\alpha\mathrm{d}t + \sigma\mathrm{d}B_t^0 + \int_{\mathbb{R}} \gamma(t,\eta)\tilde{J}(\mathrm{d}\eta,\mathrm{d}t)\right] \tag{2}$$

where $\alpha > 0$, $\sigma > 0$ are the stock's drift and volatility rates respectively, $\gamma > -1$ and B_t^0 is the Brownian motion driving the stock. We assume the inequality $\alpha - r > 0$ is satisfied for economic equilibrium purposes. $J(\mathrm{d}t, \mathrm{d}\eta)$ is a compensated random measure given by;

$$\tilde{J}(\mathrm{d}t, \mathrm{d}\eta) \equiv J(\mathrm{d}t, \mathrm{d}\eta) - \nu(\mathrm{d}\eta)\mathrm{d}t, \tag{3}$$

with η being the generic jump size and ν is the Lêvy measure of a given Lêvy process. $\gamma(t, \eta)$ is a predictable process satisfying the following;

$$\int_0^T \int_{\mathbb{R}} |\gamma(t, \eta)|^2 \nu(\mathrm{d}\eta)\mathrm{d}t < \infty. \tag{4}$$

We consider the consumer good[1] an investor purchases in their time of life in the economy over a time interval satisfying a stochastic process:

$$\mathrm{d}Q(t) = Q(t)\left[Z(t)\mathrm{d}t + \varrho(t)\mathrm{d}B^1(t)\right], \; Q(0) = Q_0, \tag{5}$$

for all $t \in [0, T]$ with $B^1(t)$ being a Brownian motion driving the consumption good, $\rho(t)$ is the volatility price index rate of the commodity good and $Z(t)$ is the stochastic drift rate representing the expected inflation rate over time governed by the time dependent Ornstein-Uhlenbeck (OU) process also applied by Chaiyapo and Phewchean [25] and is given by:

$$\mathrm{d}Z(t) = \varsigma(t)\left[\beta(t) - Z(t)\right]\mathrm{d}t + \bar{\varrho}(t)\mathrm{d}B^1(t), \tag{6}$$

where $\beta(t)$ is the long-run mean of the inflation rate, $\bar{\varrho}(t)$ is the volatility rate and $\varsigma(t)$ is the rate of mean reversion of the inflation rate. The quantity functions $\varrho(t), \beta(t), \varsigma(t)$ and $\bar{\varrho}(t)$ are continuous and deterministic with $t \in [0, T]$. We have the index bond $M(t)$ which is linked to inflation and has the price level process,

$$\frac{\mathrm{d}M(t)}{M(t)} = k(t)\mathrm{d}t + \frac{\mathrm{d}Q(t)}{Q(t)} = \left(k(t) + Z(t)\right)\mathrm{d}t + \varrho(t)\mathrm{d}B^1(t), \tag{7}$$

where $k(t)$ is the interest rate at time t.

The insured have an incoming source of funds which comes in at different time intervals therefore we consider it to be random over a given period of time. The income rate at time t is given by $\psi(Y_t, t)$ where Y_t is referred to as the state variable which is an Itô process such that,

$$\mathrm{d}Y_t = \beta_1(Y_t, t)\mathrm{d}t + \sigma_1(Y_t, t)\mathrm{d}B_t^4, \tag{8}$$

and follows the properties of Itô processes and B_t^4 is the Brownian motion driving the process. With the assumption that $\beta_1(Y_t, t)$ and $\sigma_1(Y_t, t)$ satisfies Lipschitz and growth conditions in Y_t and are continuous so we obtain a unique solution.

The policy holder purchases an insurance policy that pays premiums $P_t > 0$ over a unit time interval, (supposedly on monthly basis) which is an $\mathcal{F}_t$ adapted and measurable process with,

$$\int_0^T P(s)\mathrm{d}s < \infty. \tag{9}$$

[1]Items that individuals and households buy, (they include packaged goods, clothing, beverages, automobiles, and electronics) for their own use and enjoyment.

Life insurance can be purchased for different purposes, covering mortgages, personal loans and also leaving your family with a non taxable amount when one dies. Bayraktar *et al.* [26] looked into buying life insurance to reach bequest goal by separating it from the wealth of an investor. We consider the lifetime of the policy holder at $t > 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The insurance is determined by the retirement and the time of death of the policy holder τ which is finite.

Interesting studies on how to develop optimal retirement plan using stochastic programming approach and multi-stage stochastic programming have been considered. Bonsoo Koo *et al.* [27] developed a model to improve the standard of living in retirement and Owadally *et al.* [28] developed an optimal investment for a retirement plan with deferred annuities assumed to be bought over the continuous working lifetime of an investor. In our case, death benefit is paid to the beneficiaries at the time of death of the policy holder. The continuation of the policy can be renegotiated by the family members when the holder dies based on the predetermined terms and conditions of the insurance policy. We consider the probability mass function $f(t_l)$ to be the lifetime of the holder, then the cumulative mass function of t is given by,

$$F(t_l) = P(t < t_l) = \int_0^{t_l} f(s)\,\mathrm{d}s,$$

then the probability that $t > t_l$ is the survival function given by,

$$G(t_l) = P(t > t_l) = 1 - F(t_l).$$

The instantaneous force of mortality $\mu(t_l)$ for a policy holder to be alive at time t_l is given by,

$$\begin{aligned}\mu(t_l) &= \lim_{\Delta t_l \to 0} \frac{P(t_l \le t < t_l + \Delta t_l \mid t \ge t_l)}{\Delta t_l} \\ &= \lim_{\Delta t_l \to 0} \frac{P(t_l \le t < t_l + \Delta t_l)}{\Delta t_l P(t \ge t_l)}\end{aligned}$$

$$\frac{f(t_l)}{1 - F(t_l)} = -\frac{\mathrm{d}}{\mathrm{d}t_l}\left(\ln\left(1 - F(t_l)\right)\right),$$

which is the hazard rate function. The conditional probability of survival of the holder is defined as,

$$\bar{F}(t_l) = 1 - F(t_l) = P(t > t_l \mid \mathcal{F}_t) = \exp\left(-\int_0^{t_l} \mu(s)\,\mathrm{d}s\right). \tag{10}$$

and the conditional survival probability density of the death of the holder is given by,

$$f(t_l) = \mu(t_l)\exp\left(-\int_0^{t_l} \mu(s)\,\mathrm{d}s\right).$$

With the conditions above, the wealth process is given below where;

S_t is the price of the stock, R_t being the price of the bond, W_t is the wealth of the investor at time $t \in [0, T]$ and $\pi_1, \pi_2, (1 - \pi_1 - \pi_2)$ are as defined previously.

The total wealth process at time $t \in [0,T]$, where T is the time of retirement is given by:

1) **Pre-death case**

For $t < \tau \wedge T$ that is $t \in [0, \tau \wedge T]$ we have,

$$\begin{aligned} \mathrm{d}W_t = {} & \pi_1 W_t \left(\alpha \mathrm{d}t + \sigma \mathrm{d}B_t^0 + \int_{\mathbb{R}} \gamma(t,\eta) \tilde{J}(\mathrm{d}\eta, \mathrm{d}t) \right) - C(t)\mathrm{d}t \\ & - \mu(t)\left(P(t) - W_t\right)\mathrm{d}t + \pi_2 W(t)\left[\left(k(t) + Z(t)\right)\mathrm{d}t + \varrho(t)\mathrm{d}B^1(t)\right] \\ & + (1 - \pi_1 - \pi_2) r W_t \mathrm{d}t + \psi(Y_t, t)\mathrm{d}t \end{aligned} \tag{11}$$

$$\begin{aligned} = {} & \Big[W_t \left(\pi_1(\alpha - r) + \pi_2\left(k(t) + Z(t) - r\right) + r + \mu(t) \right) + \psi(Y_t, t) \\ & - \mu(t)P(t) - C(t) \Big]\mathrm{d}t + W_t\left[\pi_1 \sigma \mathrm{d}B_t^0 - \pi_2 \varrho(t)\mathrm{d}B_t^1\right] \\ & + \pi_1 W_t \int_{\mathbb{R}} \gamma(t,\eta)\tilde{J}(\mathrm{d}\eta, \mathrm{d}t). \end{aligned} \tag{12}$$

2) **Post-death case**

We have;

$$\begin{aligned} \mathrm{d}W_t = {} & \pi_1 W_t \left(\alpha \mathrm{d}t + \sigma \mathrm{d}B_t^0 + \int_{\mathbb{R}} \gamma(t,\eta)\tilde{J}(\mathrm{d}t, \mathrm{d}\eta) \right) + (1 - \pi_1 - \pi_2) r W_t \mathrm{d}t \\ & + \pi_2 W(t)\left[\left(k(t) + Z(t)\right)\mathrm{d}t + \varrho(t)\mathrm{d}B^1(t)\right] - C(t)\mathrm{d}t + \psi(Y_t, t)\mathrm{d}t \end{aligned} \tag{13}$$

$$\begin{aligned} = {} & \left(W_t\left[\pi_1(\alpha - r) + \pi_2\left(k(t) + Z(t) - r\right) + r\right] - C(t) + \psi(Y_t, t) \right)\mathrm{d}t \\ & + W_t\left[\sigma \pi_1 \mathrm{d}B_t^0 - \pi_2 \varrho(t)\mathrm{d}B_t^1\right] + \pi_1 W_t \int_{\mathbb{R}} \gamma(t,\eta)\tilde{J}(\mathrm{d}t, \mathrm{d}\eta) \end{aligned} \tag{14}$$

where $\tau \wedge T = \min\{\tau, T\}$, and $\mu(t)\left(P(t) - W_t\right)\mathrm{d}t$ corresponds to the risk premium rate to pay for the life insurance at time *t*, $\psi(Y_t, t)$ is the stochastic income rate and the consumption rate $C(t)$ which are non negative and continuous.

3. Minimizing Portfolios

Our main interest is to find the portfolio $\pi_1(t), \pi_2(t), c(t), p(t)$ that minimizes the risk of the wealth of an investor. We will use the concept of convex risk measure defined by Mataramvura and Øksendal [18] generalized from the coherent risk measure definition considered by Foellmer and Schied [29].

We refer the reader to [18] [29] for the definitions of the quantitative risk measure and utility monetary function.

Our risk minimizing portfolios given the convex risk measure ρ_r involves finding π_1, π_2, c, p that minimizes the risk of the terminal wealth,

$$\rho_r = \sup_{Q \in Q^1} \left\{ E_Q\left[-W^{(\pi_1, \pi_2, c, p)}(T)\right] - \mathcal{K}(Q) \right\}. \tag{15}$$

where Q^1 is a family of measures Q on a set Ω and $\mathcal{K}(\cdot)$ is a certain "penalty function" on *P*. Note that $\mathcal{K} : Q^1 \to \mathbb{R}$. The choice of penalty function $\mathcal{K}(Q)$ is the relative entropy of Q with respect to *P*. See [18]. E_Q being the expectation with respect to *Q*.

With the risk minimizing portfolio established in Equation (15), this will be equivalent to the monetary utility maximizing problem given below;

$$\Theta = \sup_{\pi_1,\pi_2,c,p}\left(\inf_{Q\in Q^1}\left\{E_Q\left[W^{(\pi_1,\pi_2,c,p)}(T)\right]+\mathcal{K}(Q)\right\}\right). \tag{16}$$

We find the optimal portfolios π_1^*,π_2^*,c^*,p^* which are the controls of player 1 which is the investor and the measure Q^* is the control of player 2 which is the market and the problem will be solved in a game theoretic concept, such that:

$$\Theta = E_{Q^*}\left[W^{(\pi_1^*,\pi_2^*,c^*,p^*)}(T)\right]+\mathcal{K}(Q^*). \tag{17}$$

We suppose that the measure Q^* is the control of player 2 which is the market and π_1,π_2,c,p are the controls of player 1 who is the investor in this case.

Given the functions $\tilde{f}:\mathbb{R}_k\times k\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ with k being a given subset of $\mathbb{R}_p$. We let $\tilde{f}$ be the profit rate and g be a bequest function. Suppose that given a family of admissible controls we have:

$$E\left[\int_0^{\tau_s}\left|\tilde{f}\left(Y_1^*(t)\right)\right|\mathrm{d}t+\left|g\left(Y_1^*(\tau_s)\right)\right|\right]<\infty, \forall y\in\mathcal{S}. \tag{18}$$

$\mathcal{S}\subset\mathbb{R}^k$ is the given solvency region and τ_s is the exit time defined as:

$$\tau_s=\inf\left\{t>0;Y_1^*(t)\notin\mathcal{S}\right\}. \tag{19}$$

We define the performance function $J(y)$ as:

$$J(y)=E\left[\int_0^{\tau_s}\left|\tilde{f}\left(Y_1^*(t)\right)\right|\mathrm{d}t+\left|g\left(Y_1^*(\tau_s)\right)\right|\right]. \tag{20}$$

Note that $g(Y(\tau_s))=0$ if $\tau_s=\infty$. We present the value function $V(t)$ defined by:

$$V(t)=\sup_{(\pi_1,\pi_2,C,P)\in\chi}\left(\inf_{\theta_0,\theta_1\in\theta}J^{(\pi_1,\pi_2,C,P,\theta_0,\theta_1)}(t)\right)=J^{(\pi_1^*,\pi_2^*,C^*,P^*,\theta_0^*,\theta_1^*)}(t) \tag{21}$$

and $\pi_1^*,\pi_2^*,C^*,P^*,\theta_0^*,\theta_1^*$ as the optimal controls. We present the BSDE with a diffusion jump as:

$$f(t,H(t),\Upsilon(t))\mathrm{d}t=\int_{\mathbb{R}}\Upsilon(t,\eta)\tilde{J}(\mathrm{d}\eta,\mathrm{d}t)-\mathrm{d}H(t). \tag{22}$$

where the following conditions are considered:

1) The terminal condition $\xi\in\mathbb{L}^2(\Omega,\{\mathcal{F}_t\},\mathbb{P},\mathbb{R})$, where $H(T)=\xi$.

2) A mapping f(generator) $f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{L}^2(\mathbb{R})\mapsto\mathbb{R}$ is predictable.

3) $E\left[\int_0^T|f(t,0,0)|^2\,\mathrm{d}t\right]<\infty$; with

$$|f(t,y,v)-f(t,y,v')|^2\le h\left(|y-y'|^2+\int_{\mathbb{R}}|v(\eta)-v'(\eta)|^2\nu(\mathrm{d}\eta)\right).$$

The parameter Υ is a control process that controls the process H so that the first condition above is satisfied.

4. Main Results

4.1. Pre-Death Case

Let $\mathrm{d}V=(\mathrm{d}V_0(t),\mathrm{d}V_1(t),\mathrm{d}V_2(t))$ where we have,

$$\mathrm{d}V_0(t)=\mathrm{d}t, \tag{23}$$

$$\begin{aligned}\mathrm{d}V_1(t)=&\Big[V_1\big(\pi_1(\alpha-r)+\pi_2(k(t)+Z(t)-r)+r+\mu(t)\big)+\psi(Y_t,t)\\&-\mu(t)P(t)-C(t)\Big]\mathrm{d}t+V_1\left[\pi_1\sigma\mathrm{d}B_t^0-\pi_2\varrho(t)\mathrm{d}B_t^1\right]\\&+\pi_1V_1\int_{\mathbb{R}}\gamma(t,\eta)\tilde{J}(\mathrm{d}\eta,\mathrm{d}t)\end{aligned} \tag{24}$$

$$\mathrm{d}V_2(t)=\theta_1V_2(t)\mathrm{d}B_t^3+V_2\int_{\mathbb{R}}\theta_2(t,\eta)\tilde{J}(\mathrm{d}\eta,\mathrm{d}t). \tag{25}$$

In consideration with the conditions above, we are certain that there is a unique solution (H,Υ) to (22). Let $\varphi:\mathbb{R}^k\to\mathbb{R}$ where $\varphi\in C_0^2(\mathbb{R}^k)$ be a twice differentiable function. Suppose its partial derivatives exists, then the generator of $V(t)$ will be given by:

$$\begin{aligned}&\mathcal{L}^{\pi_1,\pi_2,c,p,\theta_1,\theta_2}\left[\varphi(t,v_1v_2,z,y,h)\right]\\&=\varphi_t+\varphi_{v_1}\left[v_1(r+\mu)+\psi(y_t,t)-\mu P-C\right]+\varphi_y\beta_1(y_t,t)+\frac{1}{2}\varphi_{yy}\sigma_1^2(y_t,t)\\&\quad+\varphi_z\varsigma[\beta-Z]+\frac{1}{2}\varphi_{zz}\bar{\varrho}^2-\varphi_{v_1y}\rho_{0,4}\varrho\sigma_1(y_t,t)v_1(\pi_1\sigma-\pi_2\varrho)\\&\quad-\varphi_{v_1z}\bar{\varrho}\varrho v_1(\pi_1\sigma-\pi_2\varrho)\rho_{0,1}+\varphi_{yz}\rho_{4,1}\sigma_1(Y_t,t)\bar{\varrho}+\frac{1}{2}\varphi_{v_1v_1}\sigma^2\pi_1^2v_1^2\\&\quad+\varphi_{v_1}v_1\pi_1(\alpha-r)+\frac{1}{2}\theta_1^2v_2^2\varphi_{v_2v_2}+\frac{1}{2}\varphi_{v_1v_1}\varrho^2\pi_2^2v_1^2+\varphi_{v_1}v_1\pi_2(k+z-r)\\&+\varphi_{v_1v_2}\theta_1\rho_{0,3}v_1v_2(\pi_1\sigma-\pi_2\varrho)-\varphi_hf(t,H(t),\Upsilon(t))+U(C)+U(P)\\&+\int_{\mathbb{R}}\left[\varphi(t,v_1v_2+\theta_2,z,y,h)-\varphi(t,v_1v_2,z,y,h)-v_1v_2\theta_2\varphi_{v_2}\right]\nu(\mathrm{d}\eta)\\&+\int_{\mathbb{R}}\left[\varphi(t,v_1v_2(1+\pi_1\gamma),z,y,h)-\varphi(t,v_1v_2,z,y,h)-\pi_1v_1v_2\gamma\varphi_{v_1}\right]\nu(\mathrm{d}\eta)\\&+\int_{\mathbb{R}}\left[\varphi(t,v_1v_2,z,y,h+\Upsilon)-\varphi(t,v_1v_2,z,y,h)-\Upsilon\varphi_{v_1}\right]\nu(\mathrm{d}\eta).\end{aligned} \tag{26}$$

Differentiating Equation (26) with respect to θ_1 and we obtain:

$$\theta_1^*=-\frac{v_1}{v_2}\frac{\varphi_{v_1v_2}}{\varphi_{v_2v_2}}\rho_{0,3}(\pi_1\sigma-\pi_2\varrho). \tag{27}$$

Differentiating Equation (26) with respect to θ_2 we obtain the following, θ_2^* as the solution of:

$$\begin{aligned}&\int_{\mathbb{R}}\Bigg[\frac{\partial}{\partial\theta_2}\Big(\varphi(t,v_1v_2+\theta_2(t,\eta),z,y,h)-\varphi(t,v_1v_2,z,y,h)\\&-v_1v_2\theta_2(t,\eta)\varphi_{v_2}\Big)\Bigg]\nu(\mathrm{d}\eta)=0.\end{aligned} \tag{28}$$

Substituting Equation (27) into Equation (26) and differentiating to find the controls π_1 and π_2 we obtain:

π_1^* as the solution of the equation below;

$$\begin{aligned}0=&\varphi_{v_1v_1}\sigma^2\pi_1v_1^2+\varphi_{v_1}v_1(\alpha-r)+\varphi_{v_1y}\sigma\sigma_1v_1\rho_{0,4}\varrho+\varphi_{v_1z}\bar{\varrho}\varrho v_1\sigma\rho_{0,1}\\&-v_1^2\frac{\varphi_{v_1v_2}^2}{\varphi_{v_2v_2}}\sigma\rho_{0,3}(\pi_1\sigma-\pi_2\varrho)+\int_{\mathbb{R}}\Big[\varphi_{\pi_1}(t,v_1+\pi_1v_1\gamma,z,y,H)\\&-v_1\gamma(t,\eta)\varphi_{v_1}\Big]\nu(\mathrm{d}\eta).\end{aligned} \tag{29}$$

The fraction invested in the inflation linked bond is given by;

$$\pi_2^* = \frac{1}{v_1 \varrho^2} \frac{\varphi_{v_1}\varphi_{v_2 v_2}\left(k+z-r\right)+2\rho_{0,3}\varphi_{v_1 v_2}^2 v_1 \pi_1 \sigma \varrho}{2\rho_{0,3}\varphi_{v_1 v_2}^2 - \varphi_{v_1 v_1}\varphi_{v_2 v_2}} \tag{30}$$

for

$$2\rho_{0,3}\varphi_{v_1 v_2}^2 - \varphi_{v_1 v_1}\varphi_{v_2 v_2} \neq 0 \tag{31}$$

respectively.

We have the following general differentials of the optimal controls for consumption and premium payment:

$$U'(c) = \varphi_{v_1} \tag{32}$$

and

$$U'(p) = \mu \varphi_{v_1} \tag{33}$$

respectively.

4.2. Constant Relative Risk Aversion (CRRA)

We choose the value function as,

$$\varphi(v_1, v_2) = \frac{(v_1 v_2)^\lambda}{\lambda} \mathrm{e}^{-(\lambda-1)H(t)}, \ \lambda \in \mathbb{R} \setminus \{0\}. \tag{34}$$

Therefore we have the controls given by:

$$\theta_1^* = \frac{\lambda \rho_{0,3}\left(\pi_1 \sigma \varrho (\lambda-1) + \rho_{0,3}(k+z-r)\right)}{\varrho\left(2\lambda^2 - (\lambda-1)^2\right)}. \tag{35}$$

We have θ_2^* as the solution of the equation below;

$$\int_{\mathbb{R}} \mathrm{e}^{-(\lambda-1)H(t)}\left[(v_1 v_2 + \theta_2)^{\lambda-1} - v_1 (v_1 v_2)^\lambda\right] \nu(\mathrm{d}\eta) = 0 \tag{36}$$

from (36) $\mathrm{e}^{-(\lambda-1)H(t)} \neq 0$ therefore we have;

$$(v_1 v_2 + \theta_2)^{\lambda-1} - v_1 (v_1 v_2)^\lambda = 0. \tag{37}$$

We obtain θ_2 as;

$$\theta_2 = v_1^{\frac{\lambda+1}{\lambda-1}} v_2^{\frac{\lambda}{\lambda-1}} - v_1 v_2. \tag{38}$$

We have the proportion invested in the stock being the solution of the equation below;

$$\begin{aligned} &-(v_1 v_2)^\lambda \mathrm{e}^{-(\lambda-1)H(t)}(\alpha - r) \\ &= (v_1 v_2)^\lambda \rho_{0,3} \mathrm{e}^{-(\lambda-1)H(t)}\left[\sigma^2 \pi_1 - \sigma \lambda^2 (\pi_1 \sigma - \pi_2 \varrho)\right] \\ &\quad + \int_{\mathbb{R}}\left[\gamma (v_1 v_2)^\lambda \mathrm{e}^{-(\lambda-1)H(t)}\left((1+\pi_1\gamma)^{(\lambda-1)} - 1\right)\right]\nu(\mathrm{d}\eta). \end{aligned} \tag{39}$$

The proportion invested in the inflation linked bond is given by;

$$\pi_2^* = \frac{2\lambda^2 \rho_{0,3}\pi_1 \sigma \varrho + (k+z-r)(\lambda-1)\rho_{0,3}}{\varrho^2\left(2\lambda^2 - (\lambda-1)^2\right)}. \tag{40}$$

We present the controls below for consumption and premiums to be paid given by;

$$C^* = v_1 v_2^{\frac{\lambda}{\lambda-1}} \mathrm{e}^{-H(t)}, \tag{41}$$

$$P^* = v_1 \left(\mu v_2^{\lambda}\right)^{\frac{1}{\lambda-1}} \mathrm{e}^{-H(t)}, \tag{42}$$

respectively.

We obtain the functions $f(t,H,\Upsilon)$ and $H(t)$ by substituting the considered differentials and the controls π_1, π_2, C^*, P^* above into Equation (26). Note that $(v_1 v_2)^{\lambda} \mathrm{e}^{-(\lambda-1)H} \neq 0$ therefore diving both sides with the term we obtain;

$$\begin{aligned}
f(t,H,\Upsilon) = \frac{\lambda}{\lambda-1}\Bigg\{ &(r+\mu) + \frac{\psi(Y_t,t)}{v_1} + \frac{1}{2}\sigma^2\left(\pi_1^*\right)^2(\lambda-1) \\
&+ \pi_1^*(\alpha - r) + \frac{\rho_{0,3}(\lambda-1)\left(2\lambda^2\pi_1^*\sigma\varrho + (k+z-r)(\lambda-1)\right)}{2\left(2\lambda^2 - (\lambda-1)^2\right)} \\
&+ \frac{\rho_{0,3}(k+z-r)\left(2\lambda^2\pi_1^*\sigma\varrho + (k+z-r)(\lambda-1)\right)}{\varrho^2\left(2\lambda^2 - (\lambda-1)^2\right)} \\
&+ \frac{\rho_{0,3}\lambda^2(\lambda-1)\left((k+z-r)^2 - \left(\pi_1^*\sigma\varrho(\lambda-1)\right)^2\right)}{\varrho^2\left(2\lambda^2 - (\lambda-1)^2\right)} \\
&+ \frac{\rho_{0,3}\lambda^2(\lambda-1)\left((k+z-r) + \pi_1^*\sigma\varrho(\lambda-1)\right)^2}{2\varrho^2\left(2\lambda^2 - (\lambda-1)^2\right)^2} \\
&+ \frac{1}{\lambda}\Bigg[\int_{\mathbb{R}}\left(\left(1+\pi_1^*\gamma\right)^{\lambda} - 1 - \pi_1^*\gamma\lambda\right)\nu(\mathrm{d}\eta) \\
&+ \int_{\mathbb{R}}\left(\mathrm{e}^{-(\lambda-1)\gamma} - 1 - \frac{\gamma\lambda}{v_1}\right)\nu(\mathrm{d}\eta) \\
&+ \int_{\mathbb{R}}\left[\left(1 + \frac{\theta_2^*}{v_1 v_2}\right)^{\lambda} - 1 - \frac{v_1\theta_2^*}{\lambda}\right]\nu(\mathrm{d}\eta)\Bigg] \\
&- \left(\left(\mu v_2\right)^{\frac{\lambda}{\lambda-1}} + v_2^{\frac{\lambda}{\lambda-1}} - \frac{v_2^{\frac{\lambda}{\lambda-1}} - v_2^{\frac{\lambda}{\lambda-1}}\mu^{\frac{\lambda}{\lambda-1}}}{\lambda}\right)\mathrm{e}^{-H}\Bigg\},
\end{aligned} \tag{43}$$

the function $H(t)$ is given by;

$$\begin{aligned}
H(t) = -\ln\Bigg\{ &\frac{1}{\left(\mu v_2\right)^{\frac{\lambda}{\lambda-1}} + v_2^{\frac{\lambda}{\lambda-1}} - \frac{v_2^{\frac{\lambda}{\lambda-1}} - v_2^{\frac{\lambda}{\lambda-1}}\mu^{\frac{\lambda}{\lambda-1}}}{\lambda}} \\
&\times\Bigg((r+\mu) + \frac{\psi(Y_t,t)}{v_1} + \frac{1}{2}\sigma^2\left(\pi_1^*\right)^2(\lambda-1) + \pi_1^*(\alpha - r)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\rho_{0,3}(\lambda-1)\left(2\lambda^2\pi_1^*\sigma\varrho+(k+z-r)(\lambda-1)\right)}{2\left(2\lambda^2-(\lambda-1)^2\right)}\\
&+\frac{\rho_{0,3}(k+z-r)\left(2\lambda^2\pi_1^*\sigma\varrho+(k+z-r)(\lambda-1)\right)}{\varrho^2\left(2\lambda^2-(\lambda-1)^2\right)}\\
&+\frac{\rho_{0,3}\lambda^2(\lambda-1)\left((k+z-r)^2-\left(\pi_1^*\sigma\varrho(\lambda-1)\right)^2\right)}{\varrho^2\left(2\lambda^2-(\lambda-1)^2\right)}\\
&+\frac{\rho_{0,3}\lambda^2(\lambda-1)\left((k+z-r)+\pi_1^*\sigma\varrho(\lambda-1)\right)^2}{2\varrho^2\left(2\lambda^2-(\lambda-1)^2\right)^2}\\
&+\frac{1}{\lambda}\left(\int_{\mathbb{R}}\left((1+\pi_1^*\gamma)^{\lambda}-1-\pi_1^*\gamma\lambda\right)\nu(\mathrm{d}\eta)\right.\\
&+\int_{\mathbb{R}}\left(\mathrm{e}^{-(\lambda-1)\gamma}-\frac{v_1-\gamma\lambda}{v_1}\right)\nu(\mathrm{d}\eta)\\
&\left.\left.+\int_{\mathbb{R}}\left[\left(1+\frac{\theta_2^*}{v_1v_2}\right)^{\lambda}-1-\frac{v_1\theta_2^*}{\lambda}\right]\nu(\mathrm{d}\eta)\right)\right\}.
\end{aligned}
\tag{44}
$$

With the procedure followed we obtain the following proposition using the CRRA utility function defined in (4.2) with different value functions:

Proposition 1 *Given the optimal investment-consumption-insurance problem with inflation risk and stochastic income in an Itô Lévy setting, and the value function*

$$
\varphi(t,v_1v_2,z,y,h)=\frac{(v_1v_2)^{\lambda}}{\lambda}\mathrm{e}^{-(\lambda-1)H(t)}
$$

then,

$$
\theta_1^*=\frac{\rho_{0,3}\lambda\left(\pi_1^*\sigma\varrho(\lambda-1)+\rho_{0,3}(k+z-r)\right)}{\varrho\left(2\lambda^2-(\lambda-1)^2\right)},\ C^*(t)=V_1V_2^{\frac{\lambda}{\lambda-1}}\mathrm{e}^{-H(t)}
$$

$$
\pi_2^*(t)=\frac{2\rho_{0,3}\lambda^2\pi_1^*\varrho\sigma+\rho_{0,3}(k+Z-r)(\lambda-1)}{\varrho^2\left(2\lambda^2-(\lambda-1)^2\right)},\ \theta_2=v_1^{\frac{\lambda+1}{\lambda-1}}v_2^{\frac{\lambda}{\lambda-1}}-v_1v_2
$$

$$
P^*(t)=V_1\left(\mu V_2^{\lambda}\right)^{\frac{1}{\lambda-1}}\mathrm{e}^{-H(t)}\ \text{ and }\ \pi_1^*(t)\ \text{ solves;}
$$

$$
\begin{aligned}
&-(v_1v_2)^{\lambda}\mathrm{e}^{-(\lambda-1)H(t)}(\alpha-r)\\
&=(v_1v_2)^{\lambda}\rho_{0,3}\mathrm{e}^{-(\lambda-1)H(t)}\left[\sigma^2\pi_1-\sigma\lambda^2(\pi_1\sigma-\pi_2\varrho)\right]\\
&+\int_{\mathbb{R}}\left[\gamma(v_1v_2)^{\lambda}\mathrm{e}^{-(\lambda-1)H(t)}\left((1+\pi_1\gamma)^{(\lambda-1)}-1\right)\right]\nu(\mathrm{d}\eta)
\end{aligned}
$$

where $(v_1v_2)^{\lambda}\mathrm{e}^{-(\lambda-1)H(t)}\neq 0$.

4.3. Constant Absolute Risk-Aversion (CARA)

Under CARA we choose the following value function;

$$\varphi(v_1, v_2) = \frac{1 - e^{H - \alpha_1 v_1 v_2}}{\alpha_1}, \ \alpha_1 \neq 0 \tag{45}$$

Following the same procedure we obtain the following strategies;

θ_1^* is given by;

$$\theta_1^* = \frac{\rho_{0,3}(1 - \alpha_1 v_1 v_2)\left[(k + z - r) - \varrho \pi_1^* \sigma \alpha_1 v_1 v_2\right]}{2\varrho(1 - \alpha_1 v_1 v_2)^2 - \varrho \alpha_1^2 v_1^2 v_2^2}. \tag{46}$$

θ_2^* solves the following equation;

$$\int_{\mathbb{R}} \left[e^{H - \alpha_1 v_1 v_2} \left(e^{-\alpha_1 \theta_2} - v_1^2 v_2 \right) \right] \nu(d\eta) = 0. \tag{47}$$

Since $e^{H - \alpha_1 v_1 v_2} \neq 0$ we have $e^{-\alpha_1 \theta_2} - v_1^2 v_2 = 0$ thus,

$$\theta_2 = \frac{-1}{\alpha_1} \ln\left[v_1^2 v_2\right]. \tag{48}$$

The proportion invested in stock is the solution of the following equation;

$$\begin{aligned} 0 &= v_1 v_2(\alpha - r) - v_1^2 v_2^2 \alpha_1 \pi_1^* \sigma^2 \rho_{0,3} + \frac{\sigma \rho_{0,3}\left(\pi_1^* \sigma - \pi_2^* \varrho\right)(1 - \alpha_1 v_1 v_2)^2}{\alpha_1} \\ &\quad + \int_{\mathbb{R}} \left[v_1 \gamma e^{-\alpha_1 \pi_1^* \gamma v_1} - \gamma v_1 v_2 \right] \nu d\eta, \end{aligned} \tag{49}$$

for $e^{H - \alpha_1 v_1 v_2} \neq 0$.

π_2^* is given by the following;

$$\pi_2^* = \frac{1}{\varrho^2} \frac{v_1^2 \rho_{0,3}(k + z - r) + 2\rho_{0,3} \pi_1^* \sigma \varrho (1 - \alpha_1 v_1 v_2)^2}{2(1 - 2\alpha_1 v_1 v_2) + \alpha_1^2 v_1^2 v_2^2}. \tag{50}$$

We have the following strategies for the consumption and premiums to be paid as follows;

$$C^*(t) = \frac{-(H - \alpha_1 v_1 v_2)\ln(v_2)}{\alpha_1}, \tag{51}$$

$$P^*(t) = \frac{-(H - \alpha_1 v_1 v_2)\ln(\mu v_2)}{\alpha_1}, \tag{52}$$

respectively. We have the function $f(t, H, \Upsilon)$ given by;

$$\begin{aligned} f(t, H, \Upsilon) &= -\alpha_1 v_1 v_2 (r + \mu) - \alpha_1 v_2 \psi(Y_t, t) - \mu v_2 \ln(\mu v_2)(H - \alpha_1 v_1 v_2) \\ &\quad - (H - \alpha_1 v_1 v_2)\ln(v_2) + \frac{1}{2}\left(\sigma \pi_1^* \alpha_1 v_1 v_2\right)^2 - \pi_1^* \alpha \alpha_1 v_1 v_2 \\ &\quad + \pi_1^* r \alpha_1 v_1 v_2 - \frac{\alpha_1^2 v_1^2 v_2^2 \rho_{0,3}^2}{\varrho^2} \left(\frac{2\pi_1^* \sigma \varrho (1 - \alpha_1 v_1 v_2)^2 - \alpha_1 v_1 v_2 (k + z - r)}{2(1 - \alpha_1 v_1 v_2)^2 - \varrho \alpha_1^2 v_1^2 v_2^2} \right)^2 \\ &\quad - \frac{v_1 v_2 \alpha_1 \rho_{0,3}(k + z - r)}{\varrho^2} \frac{2\pi_1^* \sigma \varrho (1 - \alpha_1 v_1 v_2)^2 - \alpha_1 v_1 v_2 (k + z - r)}{2(1 - \alpha_1 v_1 v_2)^2 - \varrho \alpha_1^2 v_1^2 v_2^2} \\ &\quad - \frac{\rho_{0,3}\left[(k + z - r) - \pi_1^* \sigma \varrho \alpha_1 v_1 v_2\right]\left[\alpha_1 v_1 v_2 (k + z - r) - \alpha_1^2 v_1^2 v_2^2 \pi_1^* \sigma \varrho\right]}{2(1 - \alpha_1 v_1 v_2)^2 - \varrho \alpha_1^2 v_1^2 v_2^2} \end{aligned}$$

$$\begin{aligned}
&\times\frac{v_1v_2\alpha_1(1-\alpha_1v_1v_2)^2}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}-\int_{\mathbb{R}}\left[1-e^{-\alpha_1\theta_2^*}-v_1v_2\theta_2^*\alpha_1\right]\nu(d\eta)\\
&+\frac{\rho_{0,3}v_1^2v_2^2\alpha_1^2}{2}\frac{(1-\alpha_1v_1v_2)\left[(k+z-r)-\pi_1^*\sigma\varrho\alpha_1v_1v_2\right]}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}\\
&-\int_{\mathbb{R}}\left[1-e^{-\alpha_1v_1v_2\pi_1^*\gamma}-\alpha_1v_1v_2^2\pi_1^*\gamma\right]\nu(d\eta)-\int_{\mathbb{R}}\left[1-e^{\gamma}-\alpha_1v_2\gamma\right]\nu(d\eta)\\
&+v_2+\mu v_2-2e^{\alpha_1v_1v_2-H}.
\end{aligned}\tag{53}$$

following above;

$$\begin{aligned}
&2e^{\alpha_1v_1v_2-H}+H\left(\mu v_2\ln[\mu v_2]-\ln(v_2)\right)\\
&=-\alpha_1v_1v_2(r+\mu)-\alpha_1v_2\psi(Y_t,t)+\frac{1}{2}\left(\sigma\pi_1^*\alpha_1v_1v_2\right)^2-\pi_1^*(\alpha-r)\alpha_1v_1v_2\\
&\quad-\frac{\rho_{0,3}^2\alpha_1^2v_1^2v_2^2}{\varrho^2}\left(\frac{2\pi_1^*\sigma\varrho(1-\alpha_1v_1v_2)^2-\alpha_1v_1v_2(k+z-r)}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}\right)^2\\
&\quad-\frac{\rho_{0,3}v_1v_2\alpha_1(k+z-r)}{\varrho^2}\frac{2\pi_1^*\sigma\varrho(1-\alpha_1v_1v_2)^2-\alpha_1v_1v_2(k+z-r)}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}\\
&\quad-\frac{\rho_{0,3}\left[(k+z-r)-\pi_1^*\sigma\varrho\alpha_1v_1v_2\right]\left[\alpha_1v_1v_2(k+z-r)-\alpha_1^2v_1^2v_2^2\pi_1^*\sigma\varrho\right]}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}\\
&\quad\times\frac{v_1v_2\alpha_1(1-\alpha_1v_1v_2)^2}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}-\int_{\mathbb{R}}\left[1-e^{-\alpha_1\theta_2^*}-v_1v_2\theta_2^*\alpha_1\right]\nu(d\eta)\\
&\quad+\frac{\rho_{0,3}v_1^2v_2^2\alpha_1^2}{2}\frac{(1-\alpha_1v_1v_2)\left[(k+z-r)-\pi_1^*\sigma\varrho\alpha_1v_1v_2\right]}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}\\
&\quad-\int_{\mathbb{R}}\left[1-e^{-\alpha_1v_1v_2\pi_1^*\gamma}-\alpha_1v_1v_2^2\pi_1^*\gamma\right]\nu(d\eta)-\int_{\mathbb{R}}\left[1-e^{\gamma}-\alpha_1v_2\gamma\right]\nu(d\eta)\\
&\quad+\mu v_2^2\alpha_1v_1\ln[\mu v_2]+v_2\alpha_1v_1\ln[v_2].
\end{aligned}\tag{54}$$

Let;

$$\begin{aligned}
K^*&=-\alpha_1v_1v_2(r+\mu)-\alpha_1v_2\psi(Y_t,t)+\frac{1}{2}\left(\sigma\pi_1^*\alpha_1v_1v_2\right)^2-\pi_1^*(\alpha-r)\alpha_1v_1v_2\\
&\quad-\frac{\rho_{0,3}\alpha_1^2v_1^2v_2^2}{\varrho^2}\left(\frac{2\pi_1^*\sigma\varrho(1-\alpha_1v_1v_2)^2-\alpha_1v_1v_2(k+z-r)}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}\right)^2\\
&\quad-\frac{\rho_{0,3}v_1v_2\alpha_1(k+z-r)}{\varrho^2}\frac{2\pi_1^*\sigma\varrho(1-\alpha_1v_1v_2)^2-\alpha_1v_1v_2(k+z-r)}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}\\
&\quad-\frac{\rho_{0,3}\left[(k+z-r)-\pi_1^*\sigma\varrho\alpha_1v_1v_2\right]\left[\alpha_1v_1v_2(k+z-r)-\alpha_1^2v_1^2v_2^2\pi_1^*\sigma\varrho\right]}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}\\
&\quad\times\frac{v_1v_2\alpha_1(1-\alpha_1v_1v_2)^2}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}-\int_{\mathbb{R}}\left[1-e^{-\alpha_1\theta_2^*}-v_1v_2\theta_2^*\alpha_1\right]\nu(d\eta)\\
&\quad+\frac{\rho_{0,3}v_1^2v_2^2\alpha_1^2}{2}\frac{(1-\alpha_1v_1v_2)\left[(k+z-r)-\pi_1^*\sigma\rho\alpha_1v_1v_2\right]}{2(1-\alpha_1v_1v_2)^2-\varrho\alpha_1^2v_1^2v_2^2}
\end{aligned}$$

$$-\int_{\mathbb{R}}\left[1-\mathrm{e}^{-\alpha_1 v_1 v_2 \pi_1^* \gamma}-\alpha_1 v_1 v_2^2 \pi_1^* \gamma\right] \nu(\mathrm{d}\eta)-\int_{\mathbb{R}}\left[1-\mathrm{e}^{\gamma}-\alpha_1 v_2 \gamma\right] \nu(\mathrm{d}\eta) \\ +\mu v_2^2 \alpha_1 v_1 \ln\left[\mu v_2\right]+v_2 \alpha_1 v_1 \ln\left[v_2\right], \tag{55}$$

$$K_1^* = \mu v_2 \ln\left[\mu v_2\right]-\ln\left(v_2\right) \tag{56}$$

and

$$K_2^* = \alpha_1 v_1 v_2 \tag{57}$$

therefore we have;

$$K^* + H K_1^* - 2\mathrm{e}^{K_2^* - H} = 0. \tag{58}$$

Using the Lambert W(.) function/omega function in finding the solution of $H(t)$ we obtain

$$H(t) = W\left(\frac{-2}{K_1^* \mathrm{e}^{\frac{K^* - K_1^* K_2^*}{K_1^*}}}\right). \tag{59}$$

for the constants K^*, K_1^*, K_2^* defined above.

We obtain the following proposition;

Proposition 2 *Given the optimal investment-consumption-insurance problem with inflation risk and stochastic income in an Itô Lévy setting, and the value function*

$$\varphi(t, v_1 v_2, z, y, h) = \frac{1-\mathrm{e}^{H-\alpha_1 v_1 v_2}}{\alpha_1}$$

then,

$$\theta_1^* = \frac{\rho_{0,3}\left(1-\alpha_1 v_1 v_2\right)\left[(k+z-r)-\varrho \pi_1^* \sigma \alpha_1 v_1 v_2\right]}{2\varrho\left(1-\alpha_1 v_1 v_2\right)^2-\varrho \alpha_1^2 v_1^2 v_2^2},\ \theta_2^* = \frac{-1}{\alpha_1}\ln\left[v_1^2 v_2\right] \tag{60}$$

$$C^*(t) = \frac{-\left(H-\alpha_1 v_1 v_2\right)\ln\left(v_2\right)}{\alpha_1}, P^*(t) = \frac{-\left(H-\alpha_1 v_1 v_2\right)\ln\left(\mu v_2\right)}{\alpha_1}, \tag{61}$$

$$\pi_2^* = \frac{1}{\varrho^2} \frac{\rho_{0,3} v_1^2 (k+z-r)+2\rho_{0,3}\pi_1^* \sigma \varrho\left(1-\alpha_1 v_1 v_2\right)^2}{2\left(1-2\alpha_1 v_1 v_2\right)+\alpha_1^2 v_1^2 v_2^2}, \tag{62}$$

The proportion invested in stock is the solution of the following equation;

$$0 = v_1 v_2(\alpha-r)-v_1^2 v_2^2 \alpha_1 \pi_1^* \sigma^2 \rho_{0,3}+\frac{\sigma \rho_{0,3}\left(\pi_1^* \sigma-\pi_2^* \varrho\right)\left(1-\alpha_1 v_1 v_2\right)^2}{\alpha_1} \\ +\int_{\mathbb{R}}\left[v_1 \gamma \mathrm{e}^{-\alpha_1 \pi_1^* \gamma v_1}-\gamma v_1 v_2\right] \nu(\mathrm{d}\eta),\ \text{ for } \mathrm{e}^{H-\alpha_1 v_1 v_2} \neq 0. \tag{63}$$

Following the same procedure in the post-death case with the Constant absolute risk aversion utility we obtain the same strategies beneficiaries will hold when the investor dies.

4.4. Post-Death Case

When the investor dies at time τ after the retirement time *T*, that is $\tau > T$. The investor does not pay premiums thus $p = 0$. We have the wealth process as given by Equation (14) and the beneficiaries obtain a share as given by the following;

$$W(\tau)+\frac{p(\tau)}{\tilde{\pi}(\tau)}, \tag{64}$$

where $\tilde{\pi}(\tau)$ is the premium insurance ratio, $\frac{p(\tau)}{\tilde{\pi}(\tau)}$ is the insurance benefit and $W(\tau)$ is the wealth of the investor in their time of life to their time of death τ. We have the investment-consumption strategy for the beneficiaries given by

$$\chi := (\overline{\pi}_1, \overline{\pi}_2, c_a) \tag{65}$$

where $\overline{\pi}_1, \overline{\pi}_2$ are the proportions of wealth to be obtained by the beneficiaries from stock and the inflation linked bond respectively and c_a is the consumption rate after the investor dies. It is defined by;

$$\int_0^T c_a(s)\,\mathrm{d}s < \infty. \tag{66}$$

The performance function is given below as:

$$\begin{aligned} &J\left(t, W(t), \overline{\pi}_1, \overline{\pi}_2, c_a\right) \\ &:= \sup_{(\overline{\pi}_1, \overline{\pi}_2, c_a)\in\chi} \mathbb{E}_{t,w}\left[\int_t^T \mathrm{e}^{-\int_0^{\tau\wedge T}(\Phi(u)+\mu(u))\mathrm{d}u}\left(U_1\left(c_a(s)\right)\right)\mathrm{d}s\right. \\ &\quad \left. + \mathrm{e}^{-\int_t^T(\Phi(u)+\mu(u))\mathrm{d}u}\left(U_2\left(W(T)\right)\right)\right]. \end{aligned} \tag{67}$$

where U_1 and U_2 are the utility functions for the consumption of the beneficiaries and the terminal wealth of an investor respectively. The set $(\overline{\pi}_1, \overline{\pi}_2, c_a)$ is $\mathcal{F}_t$ adapted.

Let;

$$\mathrm{d}\mathcal{V} = \left(\mathrm{d}V_0(t), \mathrm{d}V_1(t), \mathrm{d}V_2(t)\right) \tag{68}$$

where,

$$\mathrm{d}V_0(t) = \mathrm{d}t \tag{69}$$

$$\begin{aligned} \mathrm{d}V_1(t) = &\left(V_1\left[\pi_1(\alpha - r) + \pi_2\left(k(t) + Z(t) - r\right) + r\right] - C(t) + \psi(Y_t, t)\right)\mathrm{d}t \\ &+ V_1\left[\sigma\pi_1\mathrm{d}B_t^0 - \pi_2\varrho(t)\mathrm{d}B_t^1\right] + \pi_1 V_1\int_{\mathbb{R}}\gamma(t,\eta)\tilde{J}(\mathrm{d}t, \mathrm{d}\eta) \end{aligned} \tag{70}$$

$$\mathrm{d}V_2(t) = \theta_1 V_2(t)\mathrm{d}B_t^3 + V_2\int_{\mathbb{R}}\theta_2(t,\eta)\tilde{J}(\mathrm{d}t, \mathrm{d}\eta). \tag{71}$$

The generator is given by;

$$\begin{aligned} &\mathcal{L}^{\overline{\pi}_1, \overline{\pi}_2, c_a, \theta_0}\left[\varphi(t, v_1 v_2, z, y, h)\right] \\ &= \varphi_t + \varphi_{v_1}\left[v_1(r+\mu) + \psi(y_t, t) - c_a\right] + \varphi_y\beta_1(y_t, t) + \frac{1}{2}\varphi_{yy}\sigma_1^2(y_t, t) \\ &\quad + \varphi_z\varsigma[\beta - Z] + \frac{1}{2}\varphi_{zz}\overline{\varrho}^2 - \varphi_{v_1 y}\rho_{0,4}\varrho\sigma_1(y_t, t)v_1(\overline{\pi}_1\sigma - \overline{\pi}_2\varrho) \\ &\quad - \varphi_{v_1 z}\overline{\varrho}\varrho v_1(\overline{\pi}_1\sigma - \overline{\pi}_2\varrho)\rho_{0,1} + \varphi_{yz}\rho_{4,1}\sigma_1(y_t, t)\overline{\varrho} + \frac{1}{2}\varphi_{v_1 v_1}\sigma^2\overline{\pi}_1^2 v_1^2 \\ &\quad + \varphi_{v_1}v_1\overline{\pi}_1(\alpha - r) + \frac{1}{2}\theta_1^2 v_2^2\varphi_{v_2 v_2} + \frac{1}{2}\varphi_{v_1 v_1}\varrho^2\overline{\pi}_2^2 v_1^2 + \varphi_{v_1}v_1\overline{\pi}_2(k + z - r) \end{aligned}$$

$$\begin{aligned}&+\varphi_{v_1v_2}\theta_1\rho_{0,3}v_1v_2\left(\bar{\pi}_1\sigma-\bar{\pi}_2\varrho\right)-\varphi_h f\left(t,H(t),\Upsilon(t)\right)+U\left(c_a\right)\\&+\int_{\mathbb{R}}\left[\varphi\left(t,v_1v_2+\theta_2,z,y,h\right)-\varphi\left(t,v_1v_2,z,y,h\right)-v_1v_2\theta_2\varphi_{v_2}\right]\nu(\mathrm{d}\eta)\\&+\int_{\mathbb{R}}\left[\varphi\left(t,v_1v_2\left(1+\bar{\pi}_1\gamma\right),z,y,h\right)-\varphi\left(t,v_1v_2,z,y,h\right)-\bar{\pi}_1v_1v_2\gamma\varphi_{v_1}\right]\nu(\mathrm{d}\eta)\\&+\int_{\mathbb{R}}\left[\varphi\left(t,v_1v_2,z,y,h+\Upsilon\right)-\varphi\left(t,v_1v_2,z,y,h\right)-\Upsilon\varphi_{v_1}\right]\nu(\mathrm{d}\eta).\end{aligned}\tag{72}$$

We obtain the optimal control θ_1^* given by;

$$\theta_1^*=-\frac{v_1}{v_2}\frac{\varphi_{v_1v_2}}{\varphi_{v_2v_2}}\rho_{0,3}\left(\bar{\pi}_1\sigma-\bar{\pi}_2\varrho\right),\tag{73}$$

θ_2^* is the solution of the equation below;

$$\begin{aligned}&\frac{\partial}{\partial\theta_2}\Big(\int_{\mathbb{R}}\big[\varphi\left(t,v_1v_2+\theta_2(t,\eta),z,y,h\right)-\varphi\left(t,v_1v_2,z,y,h\right)\\&-v_1v_2\theta_2(t,\eta)\varphi_{v_2}\big]\nu(\mathrm{d}\eta)\Big)=0.\end{aligned}\tag{74}$$

Substituting Equation (73) into the generator above we obtain the following general controls;

$\bar{\pi}_1^*$ is the solution of the equation below;

$$\begin{aligned}0=&\varphi_{v_1v_1}\sigma^2\bar{\pi}_1v_1^2+\varphi_{v_1}v_1(\alpha-r)+\varphi_{v_1y}\sigma\sigma_1v_1\rho_{0,4}\varrho\\&+\varphi_{v_1z}\bar{\varrho}\varrho v_1\sigma\rho_{0,1}-v_1^2\frac{\varphi_{v_1v_2}^2}{\varphi_{v_2v_2}}\sigma\rho_{0,3}\left(\bar{\pi}_1\sigma-\bar{\pi}_2\varrho\right)\\&+\int_{\mathbb{R}}\left[\varphi_{\bar{\pi}_1}\left(t,v_1+\bar{\pi}_1v_1\gamma,z,y,H\right)-v_1\gamma(t,\eta)\varphi_{v_1}\right]\nu(\mathrm{d}\eta).\end{aligned}\tag{75}$$

The fraction held in the inflation linked bond is given by;

$$\bar{\pi}_2^*=\frac{1}{v_1\varrho^2}\frac{\varphi_{v_1}\varphi_{v_2v_2}(k+z-r)+2\rho_{0,3}\varphi_{v_1v_2}^2v_1\bar{\pi}_1\sigma\varrho}{2\rho_{0,3}\varphi_{v_1v_2}^2-\varphi_{v_1v_1}\varphi_{v_2v_2}},\ \text{for}\tag{76}$$

$$2\rho_{0,3}\varphi_{v_1v_2}^2-\varphi_{v_1v_1}\varphi_{v_2v_2}\neq0.$$

The optimal control for consumption is given by:

$$U'\left(c_a\right)=\varphi_{v_1}.\tag{77}$$

Following the same procedure as in the pre-death case we obtain the following theorem;

Proposition 3 *Given the optimal investment-consumption-insurance problem with inflation risk and stochastic income in an Itô Lévy setting, and the value function* $\varphi\left(t,v_1v_2,z,y,h\right)=\frac{\left(v_1v_2\right)^{\lambda}}{\lambda}\mathrm{e}^{-(\lambda-1)H(t)}$, *then,*

$$\theta_1^*=\frac{\rho_{0,3}\lambda\left(\bar{\pi}_1\sigma\varrho(\lambda-1)+\rho_{0,3}(k+z-r)\right)}{\varrho\left(2\lambda^2-(\lambda-1)^2\right)},\ c_a^*(t)=V_1V_2^{\frac{\lambda}{\lambda-1}}\mathrm{e}^{-H(t)}$$

$$\bar{\pi}_2^*(t)=\frac{2\lambda^2\rho_{0,3}\bar{\pi}_1\sigma\varrho+(k+z-r)(\lambda-1)\rho_{0,3}}{\varrho^2\left(2\lambda^2-(\lambda-1)^2\right)},\ \theta_2^*=v_1^{\frac{\lambda+1}{\lambda-1}}v_2^{\frac{\lambda}{\lambda-1}}-v_1v_2.$$

$\bar{\pi}_1^*(t)$ solves;

$$-(v_1v_2)^{\lambda}\,\mathrm{e}^{-(\lambda-1)H(t)}(\alpha-r)$$
$$=(v_1v_2)^{\lambda}\,\rho_{0,3}\mathrm{e}^{-(\lambda-1)H(t)}\left[\sigma^2\bar{\pi}_1-\sigma\lambda^2\left(\bar{\pi}_1\sigma-\bar{\pi}_2\varrho\right)\right]$$
$$+\int_{\mathbb{R}}\left[\gamma(v_1v_2)^{\lambda}\,\mathrm{e}^{-(\lambda-1)H(t)}\left((1+\bar{\pi}_1\gamma)^{(\lambda-1)}-1\right)\right]\nu(\mathrm{d}\eta)$$

where $(v_1v_2)^{\lambda}\,\mathrm{e}^{-(\lambda-1)H(t)}\neq 0$.

Using the proposition above we obtain the function $f(t,H,\Upsilon)$ as follows;

$$\begin{aligned}
f(t,H,\Upsilon)=\frac{1}{(\lambda-1)}\Bigg[&(r+\mu)+\frac{\psi(Y_t,t)}{v_1}+\frac{1}{2}\sigma^2\left(\bar{\pi}_1^*\right)^2(\lambda-1)+\bar{\pi}_1^*(\alpha-r)\\
&+\frac{(\lambda-1)(1-\varrho)}{\varrho^2}\left(\frac{2\rho_{0,3}\lambda^2\bar{\pi}_1^*\sigma\varrho+\rho_{0,3}(k+z-r)(\lambda-1)}{\left(2\lambda^2-(\lambda-1)^2\right)}\right)^2\\
&+\frac{\rho_{0,3}(k+z-r)}{\varrho^2}\left(\frac{2\lambda^2\bar{\pi}_1^*\sigma\varrho+(k+z-r)(\lambda-1)}{\left(2\lambda^2-(\lambda-1)^2\right)}\right)\\
&+\frac{1}{\lambda}\Bigg[\int_{\mathbb{R}}\left(\left(1+\bar{\pi}_1^*\gamma\right)^{\lambda}-1-\bar{\pi}_1^*\gamma\lambda v_2\right)\nu(\mathrm{d}\eta)\\
&+\int_{\mathbb{R}}\left(\mathrm{e}^{-(\lambda-1)\gamma}-1-\frac{\gamma\lambda}{v_1}\right)\nu(\mathrm{d}\eta)\\
&+\int_{\mathbb{R}}\left[\left(1+\frac{\theta_2^*}{v_1v_2}\right)^{\lambda}-1-\frac{v_1\theta_2^*}{\lambda}\right]\nu(\mathrm{d}\eta)\Bigg]-\frac{\mathrm{e}^{-H(t)}\left(\lambda v_2^{\frac{\lambda}{\lambda-1}}-v_2^{\frac{\lambda}{\lambda-1}}\right)}{\lambda}\Bigg],
\end{aligned}\tag{78}$$

with $H(t)$ be given by;

$$\begin{aligned}
H(t)=-\ln\Bigg\{&\frac{1}{(\mu v_2)^{\frac{\lambda}{\lambda-1}}+v_2^{\frac{\lambda}{\lambda-1}}}\Bigg((r+\mu)+\frac{\psi(Y_t,t)}{v_1}+\frac{1}{2}\sigma^2\left(\bar{\pi}_1^*\right)^2(\lambda-1)\\
&+\bar{\pi}_1^*(\alpha-r)+\frac{(\lambda-1)\left(2\rho_{0,3}\lambda^2\bar{\pi}_1^*\sigma\varrho+\rho_{0,3}(k+z-r)(\lambda-1)\right)}{2\left(2\lambda^2-(\lambda-1)^2\right)}\\
&+\frac{(k+z-r)\left(2\lambda^2\rho_{0,3}\bar{\pi}_1^*\sigma\varrho+(k+z-r)(\lambda-1)\rho_{0,3}\right)}{\varrho^2\left(2\lambda^2-(\lambda-1)^2\right)}\\
&+\frac{\lambda^2\rho_{0,3}(\lambda-1)\left((k+z-r)^2-\left(\bar{\pi}_1^*\sigma\varrho(\lambda-1)\right)^2\right)}{\varrho^2\left(2\lambda^2-(\lambda-1)^2\right)}\\
&+\frac{\lambda^2(\lambda-1)\rho_{0,3}\left((k+z-r)+\bar{\pi}_1^*\sigma\varrho(\lambda-1)\right)^2}{2\varrho^2\left(2\lambda^2-(\lambda-1)^2\right)^2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\lambda}\Bigg(\int_{\mathbb{R}}\Big(\big(1+\bar{\pi}_1^*\gamma\big)^{\lambda}-1-\bar{\pi}_1^*\gamma\lambda\Big)\nu(\mathrm{d}\eta)\\
&+\int_{\mathbb{R}}\left(\mathrm{e}^{-(\lambda-1)\gamma}-1-\frac{\gamma\lambda}{v_1}\right)\nu(\mathrm{d}\eta)\\
&+\int_{\mathbb{R}}\left[\left(1+\frac{\theta_2^*}{v_1v_2}\right)^{\lambda}-1-\frac{v_1\theta_2^*}{\lambda}\right]\nu(\mathrm{d}\eta)\Bigg)\Bigg\}.
\end{aligned}
\tag{79}
$$

5. Conclusions and Suggestions

In the study, a game theoretic problem is formulated with incorporation to the risk minimizing portfolio strategies when investment is made on the two models, pre-death and post-death of the investor. The investor and the market are in competition that the investor wants to minimize the risk of his/her terminal wealth to maximize the monetary returns while the market is minimizing the chances of the investor maximizing from the investment. The strategies for each case are presented in the propositions (1) and (3) using the different utility functions, the constant relative risk aversion (CRRA) and the constant absolute risk aversion (CARA) functions. In both cases, the defined function $H(t)$ is also obtained from the solutions of the strategies obtained in the study.

The study can be extended by using other approaches or techniques in finding the optimal strategies. A nonzero sum game approach can be applied to compare with the solutions obtained when using this approach. The portfolio of an investor can further be expanded and consider a specific type of insurance to buy.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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