

A New Parallel Difference Method for Solving Time Fractional Black-Scholes Model

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Abstract

A modified Black-Scholes (B-S) model with time fractional derivative is studied when the price change of the underlying is considered as a fractal transmission system. It is very practical in application to study the numerical computation of this time fractional B-S model (TFBSM) governing European options. This paper constructs mixed alternative segment Crank-Nicolson (MASC-N) parallel difference scheme, which accuracy is spatially 2 order and temporally $2-\alpha$ order. In addition, we have provided a theoretical proof for the stability and convergence of the MASC-N scheme, and demonstrated the accuracy and effectiveness of the proposed method through numerical experiments. The numerical results show that the MASC-N scheme is easy to implement when applied to time fractional B-S model.

Keywords

Time Fractional B-S Model, MASC-N Scheme, Stability, Convergence, Numerical Experiments

1. Introduction

Option is one of the most widely used financial derivatives, and it has shown very important significance in both theory and practice to study the option pricing. The Black-Scholes (B-S) model, which was proposed by Black F., Scholes M. and Merton R. (1973) [1] [2], led to a boom in options trading and scientifically legitimized the activities of the Chicago board options exchange and other option markets around the world. The classical B-S model is of seven basic assumptions, sometimes not meet the actual financial market applications. Therefore, the researchers improved the classical B-S model and obtained some new models such as B-S model with transaction cost [3] [4], jump-diffusion model [5], stochastic

volatility model [6] and fractional B-S model [7] [8].

Through observation and research on the stock market, it is found that the most fundamental characteristics and basic state of the capital market are random fluctuations, and some complex dynamic systems are usually fractional, such as fractional Brownian motion, continuous time walk, etc. Based on the fractional stochastic differential equation driven by fractional Brownian motion, many scholars have made some progress in the research of fractional B-S equation in recent years. W. Wyss (2000) [9] deduced a time fractional B-S equation governing European call option. Cartea A. and Del-Castillo-Negrete D. (2007) [10] obtained several fractional diffusion models for pricing option in market with jumps. By considering the fractional dynamics driven, Jumarie G. (2008, 2010) [11] [12] derived time fractional B-S model and time-and-space fractional B-S model. Soon after that, Zhang J.R. *et al.* (2010) [13] assumed that the underlying price change follows a fractional Itô process and the change in option price with time is a fractal transmission system, obtained a bi-fractional Black-Scholes-Merton model. Chen W. *et al.* (2015) [14] proposed a simplified version of Liang *et al.*'s model, they assumed that the underlying price change still follows the classical Brownian motion, but considered fractal transmission system. As a result, they obtained double barrier options pricing based on time fractional B-S model.

The analytical solution of the fractional B-S models is usually via integral transform methods [9] [12] [13] [14], homotopy perturbation methods [15], wavelet based hybrid methods [16], or via Lie symmetry transformations [17] and so on. The solutions usually are difficult to calculate, because they usually contain the form of convolution of some special functions or infinite series with an integral. Therefore, studying numerical approximate solutions of fractional B-S models is a very practical and important research project. The existing numerical algorithms for solving the fractional order differential equations mainly include: finite difference method, finite element method, spectral method, moving mesh method, series approximation method, matrix transformation method and so on [18] [19] [20]. Roughly speaking, finite difference method has been well established.

For solving the time fractional B-S equation, a few achievements on the numerical methods are as follow. Song L.N. and Wang W.G. (2013) [21] employed implicit finite difference method to solve the time fractional B-S equation together with the conditions satisfied by the standard put option. Kumar S. and Singh D. (2014) [22] presented a numerical algorithm for the time fractional B-S equation with boundary condition by homotopy perturbation method and homotopy analysis method. Bhowmik S.K. (2014) [23] proposed an explicit-implicit numerical scheme with a low order of convergence for a partial integro-differential equation that arises in option pricing theory by using a finite difference technique. Yang X.Z. *et al.* (2015) [24] deduced an Explicit-Implicit and Implicit-Explicit difference scheme for time fractional B-S equation. Zhang H. *et al.* (2016) [25] derived an implicit difference scheme for the time fractional B-S equation governing European options. Phaochoo P. *et al.* (2016) [26] pro-

posed a meshless local Petrov-Galerkin method, which involves not only a meshless interpolation for the trial functions, but also a meshless integration of the weak-form. Nuugulu S.M. *et al.* (2021) [27] propose a corresponding robust numerical method which is based on the extension of a Crank-Nicolson (C-N) finite difference method. For solving time-fractional B-S equation, An X. *et al.* (2021) [28] proposed space-time spectral method employs the Jacobi polynomials for the temporal discretisation and Fourier-like basis functions for the spatial discretisation. Based on the alternating segment technology, Yan R.F. *et al.* (2021) [29] applied C-N format, and four kinds of Saul'yev asymmetric format to construct the alternating segmented C-N parallel scheme. Sarboland M. and Aminataei A. (2022) [30] applied the multiquadric quasi-interpolation scheme and the integrated radial basis function networks scheme to provide a numerical method to approximate the solution of the time fractional Black-Scholes equation.

With the rapid development of multi-core and cluster technology, it is of great theoretical significance and practical value to construct the parallel difference scheme with good stability for solving time fractional B-S model (TFBSM). In combination with the call option, we construct the mixed alternative segment C-N (MASC-N) scheme for TFBSM. The stability and convergence of the MASC-N scheme are analyzed. Finally, numerical examples demonstrate the effectiveness and accuracy of the MASC-N scheme for solving the TFBSM.

2. MASC-N Parallel Difference Method

2.1. Time Fractional Black-Scholes Model

In the present work, the option price $P(S, t)$ is suggested to be subject to the time fractional Black-Scholes equation with the following form [25] [31]:

$$\frac{\partial^\alpha P(S, t)}{\partial t^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P(S, t)}{\partial S^2} + rS \frac{\partial P(S, t)}{\partial S} - rP(S, t) = 0. \quad (1)$$

where $t > 0$, $0 < \alpha \leq 1$, P is the price of European call option; S and t are asset price and time; r is risk free interest rate; σ represents volatility of underlying asset. The fractional derivative $P_t^{(\alpha)}(S, t)$ is Riemann-Liouville time fractional derivative with respect to t ; P_S and P_{SS} refer to first derivative and second derivative with respect to S .

The value of European call option is taken as a solution of (1) with the following terminal and boundary conditions [1]:

- 1) $P(S, T) = \max\{S - K, 0\}$. The profit and loss when the option expires is its price. Here K is the exercise price;
- 2) $S \rightarrow \infty$, $P(S, t) \sim S - Ke^{-r(T-t)}$. When S is sufficiently large, the option price is close to $S - Ke^{-r(T-t)}$, T is the due date of the options;
- 3) $P(0, t) = 0$, which implies that the option price is approaching to zero when S is zero.

The European call option pricing model is to solve the follow time fractional B-S equation:

$$\begin{cases} \frac{\partial^\alpha P(S,t)}{\partial t^\alpha} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P(S,t)}{\partial S^2} + rS \frac{\partial P(S,t)}{\partial S} - rP(S,t) = 0, \\ P(S,T) = \max\{S - K, 0\}. \end{cases} \tag{2}$$

The boundary conditions $P(0,t) = 0$, $\lim_{S \rightarrow \infty} P(S,t) = S - Ke^{-r(T-t)}$, and the solution region is $\Sigma = \{0 \leq S \leq \infty, 0 \leq t \leq T\}$.

Equation (2) is an anti-variable coefficient parabolic equation. With the change of variables [2] [25]: $S = e^x, t = T - \tau, V(x, \tau) = P(e^x, T - \tau)$.

Equation (2) can be transformed into the parabolic equations as follows:

$$\begin{cases} V_\tau^{(\alpha)}(x, \tau) + \left(r - \frac{1}{2}\sigma^2\right)V_x(x, \tau) - \frac{1}{2}\sigma^2 V_{xx}(x, \tau) - rV(x, \tau) = 0, \\ V(x, 0) = \max\{e^x - K, 0\}. \end{cases} \tag{3}$$

The solution region converses into: $\Sigma_0 = \{-\infty \leq x \leq +\infty, 0 \leq \tau \leq T\}$.

To solve the TFBSM numerically it is necessary to truncate the original unbounded domain into a finite interval. Here we truncate the range of variable x in problem (1) to a finite interval $[M^-, M^+]$. Then the solution region we consider is of the following finite domain: $\Sigma_1 = \{M^- \leq x \leq M^+, 0 \leq \tau \leq T\}$.

Meanwhile, boundary conditions are transformed into the form:

$$V(M^+, \tau) = e^{M^+} - Ke^{-r\tau}, V(M^-, \tau) = 0.$$

2.2. Construction of the MASC-N Scheme

Let $h = (M^+ - M^-)/M$ and $k = T/N$, $(M, N \in \mathbb{Z}_+)$ be space and time steps. The computation domain Σ_1 is discretized by uniform grid (x_i, τ_n) :

$$\begin{cases} x_i = M^- + (i-1)h, & i = 1, 2, \dots, M+1, \\ \tau_n = (n-1)k, & n = 1, 2, \dots, N+1. \end{cases}$$

$V_i^n = V(x_i, \tau_n)$ denotes an approximate solution (exact solution of MASC-N scheme) of (3) in x_i and τ_n . The corresponding initial and boundary conditions are:

$$V_i^1 = f(x_i) = \max\{e^{x_i} - K, 0\}, V_{M+1}^n = f_2(\tau_n) = e^{M^+} - Ke^{-r(n-1)k}.$$

The discrete format of time fractional derivative $P_t^{(\alpha)}(S, t)$ in the point (x_i, τ_{n+1}) using L1 interpolation approximation of the modified Riemann-Liouville fractional derivative is as follows [32] [33]:

$$\begin{aligned} & \frac{\partial^\alpha V(x_i, \tau_{n+1})}{\partial \tau^\alpha} \\ &= \frac{k^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n [V(x_i, \tau_{n+2-j}) - V(x_i, \tau_{n+1-j})] [j^{1-\alpha} - (j-1)^{1-\alpha}] + O(k^{2-\alpha}). \end{aligned}$$

Letting $l_j = j^{1-\alpha} - (j-1)^{1-\alpha}$, we can define the operator $L_t^\alpha V(x_i, \tau_{n+1})$ as:

$$L_t^\alpha V(x_i, \tau_{n+1}) = \frac{k^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n l_j [V(x_i, \tau_{n+2-j}) - V(x_i, \tau_{n+1-j})]. \tag{4}$$

Moreover, the discrete formats of the space derivatives are:

$$\begin{aligned} \frac{\partial V(x_i, \tau_n)}{\partial x} &= \frac{V_{i+1}^n - V_{i-1}^n}{2h} + O(h^2), \\ \frac{\partial^2 V(x_i, \tau_n)}{\partial x^2} &= \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{h^2} + O(h^2). \end{aligned} \tag{5}$$

Substituting (4) and (5) into (3), we can derive the classical explicit scheme, implicit scheme and C-N scheme of time fractional B-S Equation (3):

The classical explicit scheme of time fractional B-S Equation (3) is

$$\begin{aligned} &L_\tau^\alpha V(x_i, \tau_{n+1}) \\ &= \frac{1}{2}\sigma^2 \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{h^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{V_{i+1}^n - V_{i-1}^n}{2h} - rV_i^n + O(k^{2-\alpha} + h^2). \end{aligned}$$

After combining similar terms and omitting the truncation errors, it can be rewritten as

$$V_i^{n+1} = aV_{i-1}^n - 2bV_i^n + cV_{i+1}^n + \sum_{j=1}^{n-1} w_j V_i^{n+1-j} + l_n V_i^1. \tag{6}$$

The classical implicit scheme of time fractional B-S Equation (3) gives

$$\begin{aligned} L_\tau^\alpha V(x_i, \tau_{n+1}) &= \frac{1}{2}\sigma^2 \frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{h^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2h} \\ &\quad - rV_i^{n+1} + O(k^{2-\alpha} + h^2). \end{aligned}$$

It can be rewritten as

$$-aV_{i-1}^{n+1} + (1+2b)V_i^{n+1} - cV_{i+1}^{n+1} = \sum_{j=1}^{n-1} w_j V_i^{n+1-j} + l_n V_i^1. \tag{7}$$

The classical C-N scheme of time fractional B-S Equation (3) gives

$$\begin{aligned} &-\frac{1}{2}aV_{i-1}^{n+1} + (1+b)V_i^{n+1} - \frac{1}{2}cV_{i+1}^{n+1} \\ &= \frac{1}{2}aV_{i-1}^n - bV_i^n + \frac{1}{2}cV_{i+1}^n + \sum_{j=1}^{n-1} w_j V_i^{n+1-j} + l_n V_i^1. \end{aligned} \tag{8}$$

where

$$\begin{aligned} a &= \frac{m\sigma^2}{2h^2} - \frac{m(r-0.5\sigma^2)}{2h}, b = \frac{m\sigma^2}{2h^2} + \frac{mr}{2}, c = \frac{m\sigma^2}{2h^2} + \frac{m(r-0.5\sigma^2)}{2h}, \\ m &= k^\alpha \Gamma(2-\alpha), w_j = 2j^{1-\alpha} - (j+1)^{1-\alpha} - (j-1)^{1-\alpha}. \end{aligned}$$

Simulating the method of alternating explicit and implicit scheme [34], the design of MASC-N scheme we construct is as follows. Let $M-1 = (2s+1)l = ql$, here q and l are positive integers $q, l \geq 3$, we can divide the points on each time level into q sections which are sequentially recorded as S_1, S_2, \dots, S_q .

On the odd level, we apply explicit scheme (6) to calculate points $x_i (i = i_1'', i_2'', i_3'', i_4'', \dots, i_{2s-1}'', i_{2s}'')$, at points $x_i (i = i_2'', i_3'', i_4'', i_5'', \dots, i_{2s}'', i_{2s+1}'')$, we apply explicit scheme (7) to calculate. At the remaining points, we apply C-N scheme (8). When it turns to the even level, we apply explicit scheme (6) to calculate points $x_i (i = i_2'', i_3'', i_4'', i_5'', \dots, i_{2s}'', i_{2s+1}'')$, at points $x_i (i = i_1'', i_2'', i_3'', i_4'', \dots, i_{2s-1}'', i_{2s}'')$, we apply explicit scheme (7) to calculate. At the remaining points, we apply C-N scheme (8). The schematic design of the MASC-N scheme is shown in **Figure 1**.

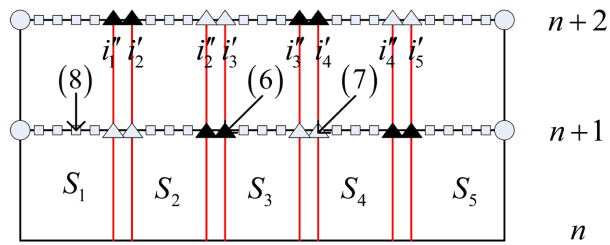


Figure 1. Schematic design with segment nodes of the MASC-N scheme.

Then the approximate discrete (MASC-N) scheme of Equation (1) can be expressed in the matrix form:

$$\begin{cases} (I + A_1 G)V^{n+1} = -A_2 G V^n + w_1 V^n + \dots + w_{n-1} V^2 + I_n V^1 + b^n, \\ (I + A_2 G)V^{n+2} = -A_1 G V^{n+1} + w_1 V^{n+1} + \dots + w_n V^2 + I_{n+1} V^1 + b^{n+1}. \end{cases} \quad (9)$$

where $n = 1, 3, \dots$, $V^n = (V_2^n \ V_3^n \ \dots \ V_{M-1}^n \ V_M^n)^T$,

$$b^n = \left(\frac{a}{2}(V_1^n + V_1^{n+1}) \ 0 \ \dots \ 0 \ \frac{c}{2}(V_{M+1}^n + V_{M+1}^{n+1}) \right)^T.$$

A_1 and A_2 are $(M-1)$ -order diagonal matrices and they meet $A_1 + A_2 = I$, $A_1 = \text{diag}(\theta_1, \theta_2, \dots, \theta_{M-2}, \theta_{M-1})$.

$$\theta_i = \begin{cases} 0, i = i''_{2l-1}, i'_{2l}, 0 < l \leq s \\ 1, i = i''_{2l}, i'_{2l+1}, 0 < l \leq s \\ \frac{1}{2}, \text{elsewhere} \end{cases}, \quad G = \begin{pmatrix} 2b & -c & & & \\ -a & 2b & -c & & \\ & \ddots & \ddots & \ddots & \\ & & & -a & 2b & -c \\ & & & & -a & 2b \end{pmatrix}_{M-1}.$$

Here I is an $(M - 1)$ order unit matrix.

3. Theoretical Analysis of MASC-N Scheme

3.1. Existence and Uniqueness of the MASC-N Scheme Solution

Lemma 1. The matrices defined by the MASC-N scheme (9) are the non-negative real matrices.

Proof.

$$G + G^T = \begin{pmatrix} 4b & -(a+c) & & & \\ -(a+c) & 4b & -(a+c) & & \\ & \ddots & \ddots & \ddots & \\ & & -(a+c) & 4b & -(a+c) \\ & & & -(a+c) & 4b \end{pmatrix},$$

where $2b = \frac{m\sigma^2}{h^2} + mr$, $-(a+c) = -\frac{m\sigma^2}{h^2}$, $m = k^\alpha \Gamma(2-\alpha)$, we have $G + G'$

as a strictly diagonally dominant three diagonal matrix, since the main diagonal elements are positive real numbers, $G + G'$ is a positive definite matrix, $G + G'$ is nonnegative definite matrix, G is nonnegative definite matrix.

Combined with lemma 1, $(I + A_1G)^{-1}$ and $(I + A_2G)^{-1}$ exist, and then we have following theorem:

Theorem 1. Solution of the MASC-N scheme (9) of time fractional Black-Scholes Equation (1) is existing and unique.

3.2. Stability of the MASC-N Scheme

Lemma 2 (Kellogg [35] [36]). If the matrix C is a nonnegative real matrix, that is, $C + C^T$ meet the nonnegative, then for any parameter $\rho \geq 0$, there are estimators $\|(\rho I + C)^{-1}\| \leq \rho^{-1}$.

Lemma 3 ([37]). The coefficients $w_j = l_j - l_{j+1}$ satisfy:

$$1 > w_1 > w_2 > \dots > w_n > 0, \quad \sum_{j=1}^{n-1} w_j = 1 - l_n, \quad l_1 = 1.$$

Let \tilde{V}_i^n be the approximate solution of Equation (1) at mesh point (x_i, t_n) . Defining $\varepsilon_i^n = \tilde{V}_i^n - V_i^n$, $E^n = (\varepsilon_2^n, \varepsilon_3^n, \dots, \varepsilon_M^n)$, $1 \leq n \leq N + 1$, and substituting $V_i^n = \tilde{V}_i^n - \varepsilon_i^n$ into MASC-N scheme, we have E^n satisfying:

$$\begin{cases} (I + A_1G)E^{n+1} = -A_2GE^n + w_1E^n + \dots + w_{n-1}E^2 + l_nE^1, \\ (I + A_2G)E^{n+2} = -A_1GE^{n+1} + w_1E^{n+1} + \dots + w_nE^2 + l_{n+1}E^1, \end{cases} \quad n = 1, 3, \dots \quad (10)$$

When $n \geq 3$, we have

$$\begin{aligned} E^{n+2} &= (I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1} (w_1I - A_2G)E^n \\ &\quad + (I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1} (w_2E^{n-1} + \dots + w_{n-1}E^2 + l_nE^1) \\ &\quad + (I + A_2G)^{-1} (w_2E^n + \dots + w_nE^2 + l_{n+1}E^1). \end{aligned} \quad (11)$$

Take the norm of both sides of Equation (11)

$$\begin{aligned} \|E^{n+2}\| &= \left\| (I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1} (w_1I - A_2G) \right\| \|E^n\| \\ &\quad + \left\| (I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1} \right\| \left\| (w_2E^{n-1} + \dots + w_{n-1}E^2 + l_nE^1) \right\| \\ &\quad + \left\| (I + A_2G)^{-1} \right\| \left\| (w_2E^n + \dots + w_nE^2 + l_{n+1}E^1) \right\|. \end{aligned}$$

Due to the growth matrix $T = (I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1} (w_1I - A_2G)$ and we define $\tilde{T} = (I + A_2G)T(I + A_2G)^{-1}$, assume that the eigenvalue of G is r , then

$$\begin{aligned} \|T\| = \|\tilde{T}\| &= \left\| (w_1I - A_1G)(I + A_1G)^{-1} (w_1I - A_2G)(I + A_2G)^{-1} \right\| \\ &= \max \left\{ \left(\frac{w_1 - \theta r}{1 + \theta r} \right)^2 \right\} \leq 1. \end{aligned}$$

Then we use mathematical induction to prove $\|E^n\| \leq \|E^1\|$.

When $n = 1$, $(I + A_1G)E^2 = (I - A_2G)E^1$

$$\|E^2\| = \left\| (I + A_1G)^{-1} (I - A_2G)E^1 \right\| \leq \|E^1\|.$$

When $\max\{w_1, \theta r\} = w_1$ is established,

$$\begin{aligned} \|E^3\| &\leq \|(I + A_2G)^{-1}(w_1I - A_1G)(I + A_1G)^{-1}(I - A_2G)\| \|E^1\| + \|(I + A_2G)^{-1}\| \|L_2E^1\| \\ &\leq \max\left\{\frac{w_1 - \theta r}{1 + \theta r}\right\} \|E^1\| + \max\left\{\frac{l_2}{1 + \theta r}\right\} \|E^1\| \\ &\leq \max\left\{\frac{1 - \theta r}{1 + \theta r}\right\} \|E^1\| \\ &\leq \|E^1\|. \end{aligned}$$

When $\max\{w_1, \theta r\} = \theta r$ is established,

$$\begin{aligned} \|E^3\| &\leq \|(I + A_2G)^{-1}(w_1I - A_1G)(I + A_1G)^{-1}(I - A_2G)\| \|E^1\| + \|(I + A_2G)^{-1}\| \|L_2E^1\| \\ &\leq \max\left\{\frac{w_1 - \theta r}{1 + \theta r}\right\} \|E^1\| + \max\left\{\frac{l_2}{1 + \theta r}\right\} \|E^1\| \\ &\leq \max\left\{\frac{\theta r + l_2 - w_1}{1 + \theta r}\right\} \|E^1\| \\ &\leq \|E^1\|. \end{aligned}$$

Assuming that $n \leq k + 1$, we have $\|E^n\| \leq \|E^1\|$. When $n = k + 2$, we prove it in two cases:

Case 1: $\max\{w_1 - \theta r\} \leq \max\{w_1, \theta r\} \leq w_1$

$$\begin{aligned} \|E^{n+2}\| &= \|(I + A_2G)^{-1}(w_1I - A_1G)(I + A_1G)^{-1}(w_1I - A_2G)\| \|E^n\| \\ &\quad + \|(I + A_2G)^{-1}(w_1I - A_1G)(I + A_1G)^{-1}\| \|(w_2E^{n-1} + \dots + w_{n-1}E^2 + l_nE^1)\| \\ &\quad + \|(I + A_2G)^{-1}\| \|(w_2E^n + \dots + w_nE^2 + l_{n+1}E^1)\| \\ &\leq \left(\frac{w_1}{1 + \theta r}\right)^2 \|E^1\| + \frac{w_1(1 - w_1)}{(1 + \theta r)^2} \|E^1\| + \frac{1 - w_1}{1 + \theta r} \|E^1\| \\ &\leq w_1 \|E^1\| + (1 - w_1) \|E^1\| \\ &\leq \|E^1\|. \end{aligned}$$

Case 2: $\max\{w_1 - \theta r\} \leq \max\{w_1, \theta r\} \leq \theta r$

$$\begin{aligned} \|E^{n+2}\| &= \|(I + A_2G)^{-1}(w_1I - A_1G)(I + A_1G)^{-1}(w_1I - A_2G)\| \|E^n\| \\ &\quad + \|(I + A_2G)^{-1}(w_1I - A_1G)(I + A_1G)^{-1}\| \|(w_2E^{n-1} + \dots + w_{n-1}E^2 + l_nE^1)\| \\ &\quad + \|(I + A_2G)^{-1}\| \|(w_2E^n + \dots + w_nE^2 + l_{n+1}E^1)\| \\ &\leq \left(\frac{\theta r}{1 + \theta r}\right)^2 \|E^1\| + \frac{\theta r(1 - w_1)}{(1 + \theta r)^2} \|E^1\| + \frac{1 - w_1}{1 + \theta r} \|E^1\| \\ &\leq \frac{\theta r}{1 + \theta r} \left(\frac{\theta r}{1 + \theta r} + \frac{1 - w_1}{1 + \theta r}\right) \|E^1\| + \frac{1 - w_1}{1 + \theta r} \|E^1\| \\ &\leq \frac{\theta r}{1 + \theta r} \|E^1\| + \frac{1 - w_1}{1 + \theta r} \|E^1\| \\ &\leq \|E^1\|. \end{aligned}$$

To sum up, we obtain the following theorem.

Theorem 2. The MASC-N scheme (9) of time-fractional Black-Scholes Equation (1) is unconditional stability.

3.3. Convergence of the MASC-N Scheme

Let $v_i^n = v(x_i, t_n) (i = 1, 2, \dots, M + 1; n = 1, 2, \dots, N + 1)$ be the exact solution of Equation (1) at mesh point (x_i, t_n) . Defining $e_i^n = v_i^n - V_i^n$ and $e^n = (e_2^n, e_3^n, \dots, e_{M-1}^n, e_M^n)^T$, and substituting $e_i^n = v_i^n - V_i^n$ into MASC-N scheme, we have e^n satisfying:

$$\begin{cases} (I + A_1G)e^{n+1} = -A_2Ge^n + w_1e^n + \dots + w_{n-1}e^2 + l_n e^1 + R^n, \\ (I + A_2G)e^{n+2} = -A_1Ge^{n+1} + w_1e^{n+1} + \dots + w_n e^2 + l_{n+1}e^1 + R^{n+1}, \end{cases} \quad n = 1, 3, \dots \quad (12)$$

and $e^1 = 0, R^n = k^\alpha O(k^{2-\alpha} + h^2)$, which means there exists a positive constant C such that $\|R^n\| \leq Ck^\alpha (k^{2-\alpha} + h^2)$.

Imitate the proof of stability, when $n \geq 3$, we have

$$\begin{aligned} e^{n+2} &= (I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1} (w_1I - A_2G)e^n \\ &\quad + (I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1} (w_2e^{n-1} + \dots + w_{n-1}e^2 + R^n) \\ &\quad + (I + A_2G)^{-1} (w_2e^n + \dots + w_n e^2 + R^{n+1}). \end{aligned}$$

When $n = 1$, it is straightforward to see that:

$$\|e^2\| \leq \|(I + A_1G)^{-1}\| \|R^1\| \leq l_1^{-1} k^\alpha C (k^{2-\alpha} + h^2).$$

When $\max\{w_1, \theta r\} = w_1$ is established,

$$\begin{aligned} \|e^3\| &\leq \|(I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1}\| \|R^1\| + \|(I + A_2G)^{-1}\| \|R^2\| \\ &\leq \left| \frac{w_1 - \theta r}{(1 + \theta r)^2} \right| Ck^\alpha (k^{2-\alpha} + h^2) + \frac{1}{1 + \theta r} Ck^\alpha (k^{2-\alpha} + h^2) \\ &\leq \frac{w_1 + 1}{(1 + \theta r)^2} Ck^\alpha (k^{2-\alpha} + h^2) \\ &\leq l_2^{-1} Ck^\alpha (k^{2-\alpha} + h^2). \end{aligned}$$

When $\max\{w_1, \theta r\} = \theta r$ is established,

$$\begin{aligned} \|e^3\| &\leq \|(I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1}\| \|R^1\| + \|(I + A_2G)^{-1}\| \|R^2\| \\ &\leq \left| \frac{\theta r}{1 + \theta r} \right| Ck^\alpha (k^{2-\alpha} + h^2) + \left| \frac{1}{1 + \theta r} \right| Ck^\alpha (k^{2-\alpha} + h^2) \leq Ck^\alpha (k^{2-\alpha} + h^2) \\ &\leq l_2^{-1} Ck^\alpha (k^{2-\alpha} + h^2). \end{aligned}$$

Assuming that $n \leq k + 1$, we have $\|e^n\| \leq l_{n-1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2)$. When $n = k + 2$, we prove it in two cases:

Case 1: $\max\{w_1 - \theta r\} \leq \max\{w_1, \theta r\} \leq w_1$

$$\begin{aligned} \|e^{n+2}\| &\leq \|(I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1} (w_1I - A_2G)\| \|e^n\| \\ &\quad + \|(I + A_2G)^{-1} (w_1I - A_1G)(I + A_1G)^{-1}\| \|(w_2e^{n-1} + \dots + w_{n-1}e^2) + R^n\| \end{aligned}$$

$$\begin{aligned}
 & + \left\| (I + A_2G)^{-1} \right\| \left\| (w_2e^n + \dots + w_n e^2) + R^{n+1} \right\| \\
 & \leq \left(\frac{w_1}{1 + \theta r} \right)^2 l_{n-1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) + \left(\frac{w_1 - \theta r}{(1 + \theta r)^2} + \frac{1}{1 + \theta r} \right) \\
 & \quad \times \left(\sum_{i=2}^n l_{n+1}^{-1} w_i + 1 \right) Ck^\alpha (k^{2-\alpha} + h^2) \\
 & \leq \left(\frac{w_1}{1 + \theta r} \right)^2 l_{n-1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) + \frac{w_1 + 1}{(1 + \theta r)^2} \left(\sum_{i=2}^n w_i + l_{n+1} \right) l_{n+1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) \\
 & \leq w_1^2 l_{n-1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) + (1 + w_1)(1 - w_1) l_{n+1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) \\
 & \leq l_{n+1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2).
 \end{aligned}$$

Case 2: $\max \{w_1 - \theta r\} \leq \max \{w_1, \theta r\} \leq \theta r$

$$\begin{aligned}
 \|e^{n+2}\| & = \left\| (I + A_2G)^{-1} (w_1 I - A_1G) (I + A_1G)^{-1} (w_1 I - A_2G) \right\| \|e^n\| \\
 & \quad + \left\| (I + A_2G)^{-1} (w_1 I - A_1G) (I + A_1G)^{-1} \right\| \left\| (w_2 e^{n-1} + \dots + w_{n-1} e^2) + R^n \right\| \\
 & \quad + \left\| (I + A_2G)^{-1} \right\| \left\| (w_2 e^n + \dots + w_n e^2) + R^{n+1} \right\| \\
 & \leq \left(\frac{\theta r}{1 + \theta r} \right)^2 l_{n-1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) + \left(\frac{\theta r - w_1}{(1 + \theta r)^2} + \frac{1}{1 + \theta r} \right) \\
 & \quad \times \left(\sum_{i=2}^n l_{n+1}^{-1} w_i + 1 \right) Ck^\alpha (k^{2-\alpha} + h^2) \\
 & \leq \left(\frac{\theta r}{1 + \theta r} \right)^2 l_{n-1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) + \frac{2\theta r + 1}{(1 + \theta r)^2} \left(\sum_{i=2}^n l_{n+1}^{-1} w_i + 1 \right) Ck^\alpha (k^{2-\alpha} + h^2) \\
 & \leq \left(\frac{\theta r}{1 + \theta r} \right)^2 l_{n-1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) + \frac{2\theta r + 1}{(1 + \theta r)^2} \left(\sum_{i=2}^n w_i + l_{n+1} \right) l_{n+1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) \\
 & \leq \left(\frac{\theta r}{1 + \theta r} \right)^2 l_{n-1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) + \frac{2\theta r + 1}{(1 + \theta r)^2} l_{n+1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2) \\
 & \leq l_{n+1}^{-1} Ck^\alpha (k^{2-\alpha} + h^2).
 \end{aligned}$$

Due to

$$\lim_{n \rightarrow \infty} \frac{l_n^{-1}}{n^\alpha} = \lim_{n \rightarrow \infty} \frac{n^{-\alpha}}{n^{1-\alpha} - (n-1)^{1-\alpha}} = \lim_{n \rightarrow \infty} \frac{n^{-1}}{1 - \left(1 - \frac{1}{n}\right)^{1-\alpha}} = \frac{1}{1 - \alpha}.$$

There is a positive constant C_1 , such that

$$\|e^n\| \leq n^\alpha C_1 (k^2 + k^\alpha h^2) = (nk)^\alpha C_1 (k^{2-\alpha} + h^2), n = 1, 2, \dots, N.$$

Here $C_1 = \frac{C}{1 - \alpha}$, it is clear that $nk < T$, then we can know that

$$|v(x_i, \tau_n) - V_i^n| \leq C_2 (k^{2-\alpha} + h^2) \text{ and } C_2 = T^\alpha C_1.$$

To sum up, we obtain the following theorem.

Theorem 3. The MASC-N scheme (9) of time-fractional Black-Scholes Equation (1) is convergent, and $|v(x_i, \tau_n) - V_i^n| \leq C(k^{2-\alpha} + h^2)$, where C is a positive

constant.

4. Numerical Experiments

Numerical experiments will be done in Matlab 2013b, based on the Intel Core i3-2330 CPU@2.20GHz. The analytic solution of TFBSM is very complex to be obtained [13]. We use the MASC-N scheme (9), implicit numerical method (INM) [25] to calculate European call option prices governed by the time fractional B-S model. To show the accuracy and effectiveness of the schemes numerically, we firstly take example 1 with the given terminal and boundary condition as an example.

Example 1 [25] [31]. Consider the following time fractional B-S model with homogeneous boundary conditions

$$\begin{cases} {}_0D_t^\alpha U(x, \tau) = a \frac{\partial^2 U(x, \tau)}{\partial x^2} + b \frac{\partial U(x, \tau)}{\partial x} - cU(x, \tau) + g(x, \tau), \\ U(0, \tau) = 0, U(1, \tau) = 0, \\ U(x, 0) = x^2(1-x). \end{cases}$$

where the source term

$$g = \left(\frac{2\tau^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2\tau^{1-\alpha}}{\Gamma(2-\alpha)} \right) x^2(1-x) - (t+1)^2 [a(2-6x) + b(2x-3x^2) - cx^2(1-x)]$$

is chosen so that the exact solution of example 1 is $U = (t+1)^2 x^2(1-x)$. Here we take the parameters values as

$$r = 0.05, \sigma = 0.25, \alpha = 0.7, a = \frac{1}{2}\sigma^2, b = r - a, c = r, T = 1.$$

Next, we verify the calculation precision and convergence order of MASC-N parallel difference schemes for solving time fractional B-S equation. Convergence order of spatial layer (COS) and convergence order of time horizon (COT) are defined respectively [38] [39]. V_i^n is the exact solution and \bar{V}_i^n is the solution of MASC-N.

$$E_{2,\Delta t}^m = \|V_j^m - \bar{V}_j^m\|_2 = \left\{ \sum_{j=1}^n (V_j^m - \bar{V}_j^m)^2 \Delta t \right\}^{\frac{1}{2}},$$

$$E_{\infty,\Delta t}^m = \|V_j^m - \bar{V}_j^m\|_\infty = \max_{1 \leq j \leq n} |V_j^m - \bar{V}_j^m|,$$

$$\text{COT} \approx \log_2 \left(E_{\Delta x, \Delta t}^m / E_{\Delta x, \Delta t/2}^m \right), l = 2, \infty,$$

$$E_{2,\Delta x}^n = \|V_i^n - \bar{V}_i^n\|_2 = \left\{ \sum_{i=1}^m (V_i^n - \bar{V}_i^n)^2 \Delta x \right\}^{\frac{1}{2}},$$

$$E_{\infty,\Delta x}^n = \|V_i^n - \bar{V}_i^n\|_\infty = \max_{1 \leq i \leq m} |V_i^n - \bar{V}_i^n|,$$

$$\text{COS} \approx \log_2 \left(E_{\Delta x, \Delta t}^n / E_{\Delta x/2, \Delta t/4}^n \right), l = 2, \infty.$$

Table 1 and **Table 2** list the numerical errors in 2-norm and the maximum-norm and their corresponding convergence rates. As can be seen from the table, the numerical solutions obtained by the MASC-N scheme are in excellent agreement with exact solutions. We can see that for fixed space step $h = 1/100$ and some different time steps $k = 1/40, 1/80, 1/160$ and $1/320$, the corresponding orders of convergence for the MASC-N scheme are close to 1.3 when $\alpha = 0.7$. The space convergence orders of the MASC-N scheme approach to 2. It shows good agreement with the conclusion of Theorem 3.

Example 2 [25]. Assuming that the current price of the underlying stock is 97 dollars, the strike price of option is 50 dollars, the risk-free interest rate is 0.05 per year, the deadline of the option is 12 months, the volatility is 0.25 per year.

First, let $M = 101, N = 100, l = 200, q = 5$. The parameters are $S = 97, K = 50, T = 12, r = 0.05, \sigma = 0.25, M^- = \ln 0.1, M^+ = \ln 100$.

We give the plots of the option price curves $P(S, T)$ under the INM, MASC-N scheme.

The parameter values are selected according to paper [25]. From **Figure 2**, we can see that the characteristics of **Figure 2** are completely consistent with **Figure 2** in paper [25] and the numerical solutions of MASC-N scheme are very close to that of implicit scheme.

Second, in order to verify the stability and the accuracy of MASC-N scheme, we regard the solution V_i^n of INM as the control solution, and the solution \bar{V}_i^n of MASC-N is used as the perturbation solution, then define the sum of relative error for every time level (SRET) and difference total energy (DTE):

$$\text{SRET}(n) = \sum_{i=1}^M \frac{|V_i^n - \bar{V}_i^n|}{V_i^n}, \quad \text{DTE}(i) = \frac{1}{2} \sum_{n=1}^N (V_i^n - \bar{V}_i^n)^2.$$

Table 1. Numerical errors and orders of convergence of MASC-N scheme for the example 1 when $M = 101$.

Δt	$\ \cdot\ _2$	COT	$\ \cdot\ _\infty$	COT
1/40	0.00549407537		0.00642653898	
1/80	0.00163256675	1.25073472300	0.00262767491	1.29025517900
1/160	0.00048949099	1.23778776807	0.00108954783	1.27005725963
1/320	0.00015143065	1.19262497528	0.000468875620	1.21645237116

Table 2. Numerical errors and orders of convergence of MASC-N scheme for the example 1 when $N = 100$.

Δt	Δx	$\ \cdot\ _2$	COS	$\ \cdot\ _\infty$	COS
1/10	1/15	0.0455637460106		0.0206037934344	
1/40	1/30	0.0136296111988	1.74114195458324	0.0043408515398	2.246859993971
1/160	1/60	0.0042681481212	1.67506225886374	0.0009528489608	2.187658631184
1/640	1/120	0.0013767764693	1.63231589868382	0.0002160284100	2.141026491177

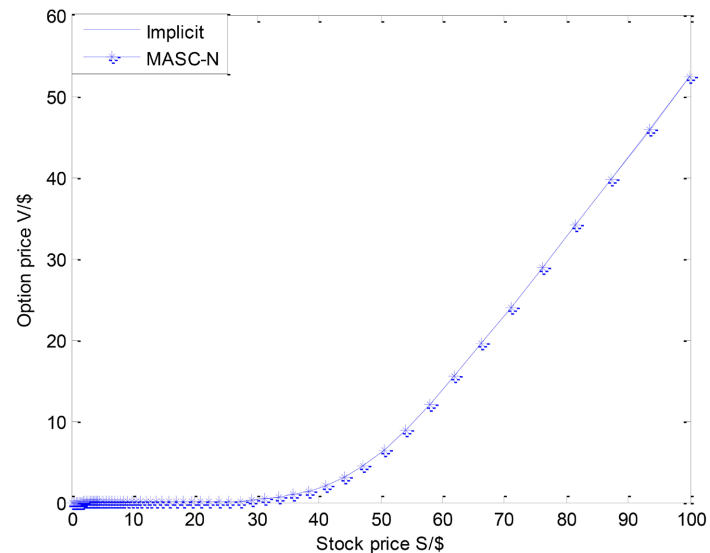


Figure 2. Call option prices obtained by MASC-N scheme and INM for $\alpha = 0.7$.

Let $M = 101$, $N = 100$, other parameters are the same as above. The results of numerical experiments are as follows:

From **Figure 3**, we can see that the SRET curves of the MASC-N scheme are larger at the beginning and decrease along with the movement of time step and gradually keep a constant level. This shows that the MASC-N scheme has better stability.

Figure 4 suggests that the DTE of numerical solutions of the MASC-N scheme has the same order of magnitude compared with that of INM, showing that the accuracy of MASC-N scheme is close to that of INM. On the other hand, maximum value of DTE doesn't exceed 0.014, indicating that the MASC-N scheme to solve the time fractional B-S equation have good accuracy.

From the viewpoint of computational efficiency, INM is the serial format and needs to use the catch-up method to solve three diagonal equations. The advantage of the MASC-N scheme of this paper is that they can divide the large numerical problems $AX = b$ into five small problems $A_i X = b, i = 1, 2, \dots, 5$. Using "parfor" to implement parallel computing means that the cycle is divided into independent parts and the various parts can carry out parallel. It leads to the improvement of the efficiency of program execution and the effective reduction of computation time [40] [41].

Finally, in order to better reflect the superiority of parallel difference schemes, we choose INM, MASC-N scheme (9) to perform numerical tests. The number of time layers is 1000 ($N = 1000$), and the spatial grid points gradually increase from 201 to 1001. Next, the number of space grid points is 1001 ($M = 1001$), and the time layers gradually increase from 200 to 1000. And results are shown in **Figure 5** and **Table 3**.

Obviously, MASC-N parallel scheme has superiority in saving the computing

time with the increase of time layer as shown in **Figure 5**. Compared with the INM, the computing time of the MASC-N scheme can save nearly 90% when the space grid is the same (see in **Table 3**). With the encryption of the spatial grid, the computational efficiency of the MASC-N scheme is significantly higher than that of the INM. Especially when the number of spatial points is large enough, the MASC-N method has obvious localization characteristics in computing and communication.

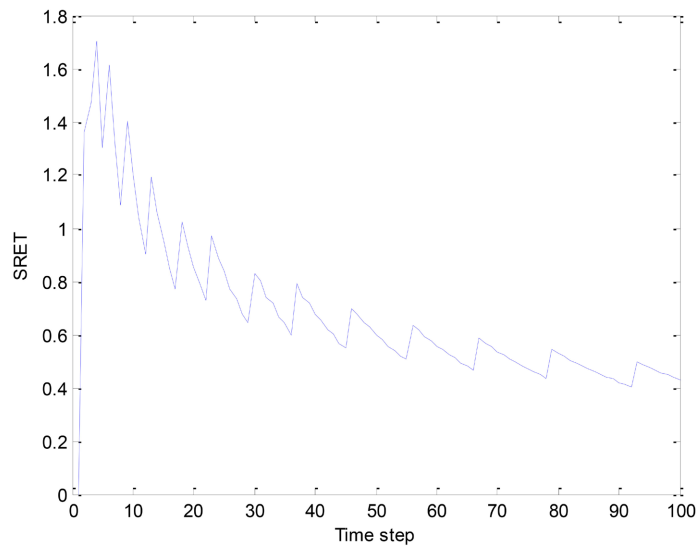


Figure 3. The curves of SRET at time layer.

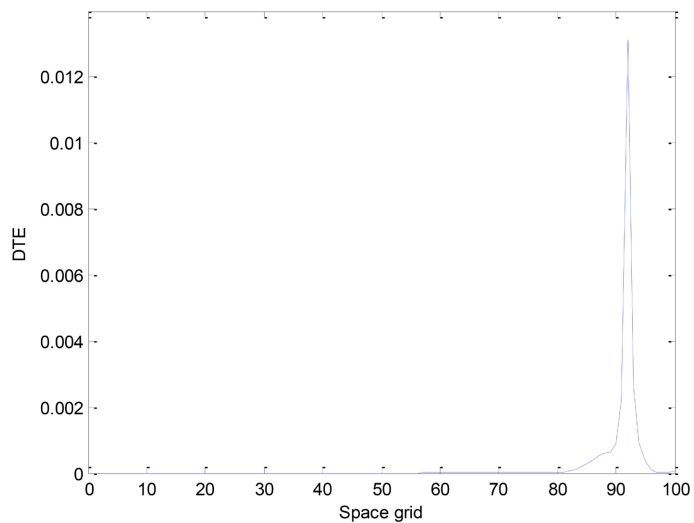


Figure 4. The curve of DTE at space layer.

Table 3. Comparison of two schemes' computing time at $M = 1001$.

Space grid (N)	200	400	600	800	1000
INM [25]	25.7785534	52.3843632	80.9568081	109.7452905	140.4224646
MASC-N	1.312375357	3.716396959	7.247458624	12.07522917	18.10542618

Example 3. Similarly, we consider the case that the underlying asset is a call option. Here we take the parameters values as $S = 97$, $K = 50$, $r = 0.05$, $\sigma = 0.25$ and $T = 6$. Using INM, MASC-N difference scheme to compute the price of European call options with different values of α and the relative error when $\alpha = 0.7$, the results are calculated as follows:

In general, option price obtained by the classical B-S equation ($\alpha = 1$) is lower than real financial market price whose deadline is 6 months [42]. From **Figure 6** and **Table 4**, the option price is a decreasing function of α , the option price calculated by time fractional B-S equation is higher than that of traditional B-S equation. We conclude that the TFBSM can more correctly capture the characteristics of a jump or large movement, compared with the classical B-S process. It is visible enough that we can get closer to the real financial market by properly selecting the value of α .

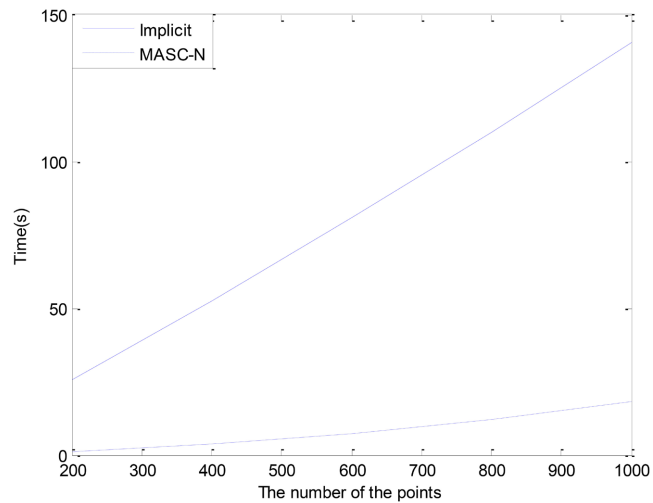


Figure 5. Comparison of the two schemes' computing time.

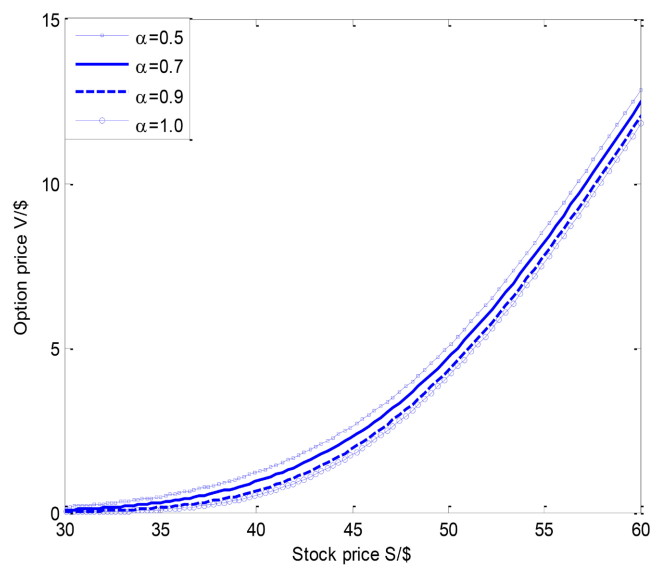


Figure 6. Call option prices using MASC-N scheme for different values of α .

Table 4. Comparison of numerical results of several schemes for some values of α .

α	0.5	0.7	0.9	1.0	Time/(s)	Relativeerror
INM [25]	48.39938155	48.38462401	48.37244188	48.36813716	138.02153018	---
MASC-N	48.40010322	48.38539992	48.37338163	48.36918903	17.133438792	1.60362453 e-05

5. Conclusions

For the time fractional B-S equation with boundary conditions satisfied by standard European call options, this paper constructs the MASC-N parallel difference scheme. This scheme has been shown to be uniquely solved, unconditionally stable and convergent. And convergence rates are temporally $2-\alpha$ order and spatially 2-order.

Moreover, compared with the implicit difference scheme, the MASC-N difference scheme is easy to realize parallel computing. So this scheme can greatly improve the computational efficiency, and it has obvious advantages in solving the multi-asset option pricing problems. All the results imply that MASC-N difference scheme is feasible to solve the time fractional B-S equation.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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