Journal of Mathematical Finance, 2022, 12, 607-628
https://www.scirp.org/journal/jmf
ISSN Online: 2162-2442
ISSN Print: 2162-2434

Optimal Investment and Consumption Problem with Stochastic Environments

Stanley Jere*, Elias Rabson Offen, Othusitse Basmanebothe

Department of Mathematics, University of Botswana, Gaborone, Botswana
Email: *sjere@mu.edu.zm, elias.offen@gmail.com, basimanebotlheos@UB.AC.BW

How to cite this paper: Jere, S., Offen, E.R. and Basmanebothe, O. (2022) Optimal Investment and Consumption Problem with Stochastic Environments. *Journal of Mathematical Finance*, **12**, 607-628.
https://doi.org/10.4236/jmf.2022.124032

Received: February 2, 2022
Accepted: October 18, 2022
Published: October 21, 2022

Abstract

Optimal investment and consumption problem for a CRRA investor or agent is solved in this study. An agent invests in the financial market with one risk-free security and one risky security. The stochastic interest rate dynamics of risk-free security follow a Ho-Lee model and the risky security is modeled as Heston's model with its volatility parameter dynamics following a Cox-Ingersoll-Ross (CIR) model. Interest rates and volatility rates, in reality, are stochastic due to uncertain events such as the Coronavirus disease 2019 (COVID19) pandemic, climate change, etc. Our main goal is to allocate initial wealth x_0 between risk-free security and risky security in order to maximize the discounted expected utility of consumption and terminal wealth over a finite horizon. Applying the Dynamic Programming Principle (DPP), the HJB PDE for the value function is established. The power utility function which belongs to the Constant Relative Risk Aversion (CRRA) class is employed for our analysis to obtain the value function and optimal policies. Finally, numerical examples and simulations are provided and discussed.

Keywords

Investment-Consumption Problem, Ho-Lee Model, Heston's Model, Cox-Ingersoll-Ross (CIR) Model, HJB PDE, Value Function, Optimal Policies

1. Introduction

One key area of mathematical finance is the problem of an investor who seeks to maximize the expected utility of consumption and terminal wealth. This research work builds on the celebrated work of Merton in [1] [2] who originally studied continuous time investment and consumption problems when stock price follows a geometric Brownian motion. For Merton's work, both the interest rate and volatility rate are constants. In real life, Interest rates and volatility rates are not constant. For example, the US 2007-2008 global financial crisis (housing bubble) made the US central banks adjust interest rates considerably. Interest rates and volatility rates, in reality, are stochastic due to uncertain events such as the Coronavirus disease 2019 (Covid19) pandemic, climate change, wars, inflation, natural disasters, fiscal policy and financial policy adjustments. In this study, a stochastic control problem of a single investor with stochastic interest rate from a bond modeled as a Ho-Lee process and a stock modeled as a Heston's process with its volatility dynamics following a Cox-Ingersoll-Ross (CIR) process is investigated. These models are industry standard for option pricing and maximization problems. Thus, it is worth considering in solving an optimal investment and consumption problem in this mixed structure. Our main goal is to allocate initial wealth x_0 between risk-free security and risky security in order to maximize the discounted expected utility of consumption and terminal wealth over a finite horizon. Bellman's optimality principle, introduced by Bellman [3] called the Dynamic Programming Principle (DPP) will be applied in order to determine the Hamilton-Jacobi-Bellman Partial Differential equation (HJB PDE). The investor preferences are modeled as a Constant Relative Risk Aversion (CRRA) function. Our major contribution is that we have extended Merton's work in [1] [2] problems with a unique mixture of consumption, stochastic interest rate and stochastic volatility rate simultaneously. So far, many researchers have studied such a control problem by considering constant interest and constant volatility. Some have considered one stochastic parameter in their analysis. However, such assumptions are unrealistic and not practical in the real financial world. In addition, we have also linked probability theory to PDE mathematics.

2. Links to the Literature

The problem of optimal investment and consumption has attracted a number of extensions. For instance, a paper by Benth [4] analyzed Merton's portfolio optimization problem with stochastic volatility of Ornstein Uhlenbeck type. Yi and Guan [5] treated consumption and investment problem with volatility being constant. Zariphopoulou [6] explored consumption and investment problem with an interest rate, mean rate of return, and dispersion coefficient being constant. A paper by Sandjo *et al.* [7] considered constant expected return and stochastic volatility. Wang *et al.* [8] researched on optimal portfolio and consumption rule with a short interest rate driven by the CIR model under HARA utility function. Harrison and Kreps [9] and Harrison and Pliska [10] applied a different approach called martingale methods to solve an optimization problem. Cox and Huang [11] discussed optimal consumption and portfolio policy when asset prices follow a diffusion process. Jinzhu and Rong [12] considered a Cox-Ingersoll-Ross (CIR) model to describe the stochastic interest rate and stochastic volatility of the stock. Noh and Kim [13] Studied optimal portfolio model with stochastic volatility and stochastic interest rate with an assumption that the risky asset prices follow geometric Brownian motion. Pang [14] investigated the interest rate which varies according to a Markov diffusion process and that risky asset price obeyed a logarithmic Brownian motion. Korn and Kraft [15] obtained optimal policy for the case of a Ho-Lee model and Vasicek model for interest rates. Kraft [16] examined optimal portfolios and considered only Heston's stochastic volatility model. Liu [17] considered the stock portfolio selection problem when stock return volatility is stochastic via Heston's model. Zariphopoulou [18] studied optimization models in a market where assets are modeled as a diffusion process with coefficients changing with time according to correlated diffusion factors. Zariphopoulou [19] considered an optimization problem with bond price deterministic and the stock price modeled as a diffusion process such that coefficients of the stock price diffusion are arbitrary nonlinear functions of the underlying process. The paper by Fleming [20] investigated on consumption model with stochastic volatility and constant interest rate. Fouque *et al.* [21] considered a portfolio optimization problem with stochastic volatility and constant interest rate. In recent years jump-diffusion models, as well as Levy process models, have become popular in financial research. This is due to the shortcomings of the simple Brownian motion model developed in Black and Scholes [22].

In this study, Merton [1] [2] are extended in a unique way by studying the stochastic control problem for an agent who faces consumption, stochastic interest rates and stochastic volatility rates simultaneously. So far, many researchers have studied such models by considering either stochastic interest or stochastic volatility rates separately. However, such an assumption is unrealistic and not practical in the real financial world. Therefore, introducing stochastic interest and stochastic volatility rates simultaneously makes our model more realistic and practical although such stochastic control problems led to complex or sophisticated HJB PDE.

The outline of this paper is as follows: Section 1 Introduction. Section 2 Literature review. Section 3 Description of the financial market model. In section 4, the wealth model is determined. Section 5 Optimization criterion description. In section 6, the HJB PDE for the value function is derived. Section 7, we investigate the value function, optimal investment and consumption policies. In Section 8, numerical examples and simulations are provided. Here, the effect of market parameters on the optimal investment and consumption policies are illustrated. In Section 9, the conclusion and suggested possible future research work are stated.

3. Financial Market Model

Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered complete probability space with filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions such as $(\mathcal{F}_t)_{0\le t\le T}$ being right continuous complete filtration and $\mathbb{P}$-complete. Let all stochastic processes be well defined and adapted in the filtered complete probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$.

Consider a financial market of a single investor with a portfolio consisting of one risk-free security (e.g. a money market account or bond) $B(t)$ and one risky security (e.g. a stock or stock index) $S(t)$.

Let the price dynamics of the risk-free security $B(t)$ evolve as follows:

$$\begin{cases} \mathrm{d}B(t) = r(t)B(t)\mathrm{d}t, \\ B(0) = 1, \end{cases} \tag{1}$$

with stochastic interest rate $r(t)$ following a Ho-Lee model given by:

$$\begin{cases} \mathrm{d}r(t) = \theta_0(t)\mathrm{d}t + \sigma_0 \mathrm{d}W^r(t), \\ r(0) = r_0 > 0, \end{cases} \tag{2}$$

where $\theta_0(t)$ is the expected instantaneous change in the interest rate, $\sigma_0 > 0$ is a constant volatility factor and $W^r(t)$ is a one-dimensional wiener process on a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. Assumed that $\theta_0(t)$ can be written as $\theta_0(t) = \gamma[\beta - r(t)]$, where γ and β are constants.

Let the price dynamics of the risky security a stock (or share) $S(t)$, follow a Heston's model given by:

$$\begin{cases} \mathrm{d}S(t) = S(t)[r(t) + k\eta(t)]\mathrm{d}t + \sigma_1\sqrt{\eta(t)}S(t)\mathrm{d}W^S(t), \\ S(0) = s_0 > 0, \end{cases} \tag{3}$$

where $k\eta(t)$ is the appreciation factor, $\sigma_1\sqrt{\eta(t)}$ is the volatility of the risky price and $W^S(t)$ is a Wiener process on a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. Note that $S(t)$ is risky stock price, $r(t)$ is risk-free interest rate, $k > 0$ is the expected returns parameter of risky asset and σ_1 is the volatility of the volatility $\sqrt{\eta(t)}$ of risky asset.

In addition, let $\eta(t)$ follow a Cox-Ingersoll-Ross (CIR) model given by:

$$\begin{cases} \mathrm{d}\eta(t) = [\theta_2 - b\eta(t)]\mathrm{d}t + \sigma_2\sqrt{\eta(t)}\mathrm{d}W^\eta(t), \\ \eta(0) = \eta_0 > 0, \end{cases} \tag{4}$$

where $\theta_2 > 0$, $b > 0$, and $\sigma_2 > 0$ are constants. Also note that $\eta(t) > 0$ for all $t \geq 0$. W^η is a wiener process on a filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$.

4. The Wealth Model

Consider an investor with an initial amount of money $x_0 > 0$ and a time horizon of interest *T*. Over the time interval $[0, T]$, the investor changes his portfolio dynamically. Let $\mathcal{C}(t)$ denote the rate of continuous consumption. Let $\pi(t)$ denote the wealth to be invested in the risky asset *S*. Then the amount invested in the risk-free security is given by $\mathcal{X}(t) - \pi(t)$. Note that the pair $(\mathcal{C}(t), \pi(t))$ is an investment and consumption strategy.

Lemma 1 *The net wealth for an investor who faces intermediate consumption, stochastic interest rate and stochastic volatility rate evolve as follows*:

$$\begin{cases} d\mathcal{X}(t) = [\mathcal{X}(t)r(t) - \mathcal{C}(t) + \pi(t)k\eta(t)]\mathrm{d}t + \pi(t)\sigma_1\sqrt{\eta(t)}\mathrm{d}W^S(t), \\ \mathcal{X}(0) = x_0 > 0. \end{cases} \tag{5}$$

Proof. In lemma 1, we prove the net wealth model for our financial market. Note that net wealth with intermediate consumption is defined by:

$$\begin{cases} \mathrm{d}\mathcal{X}(t) = \left[\left(\mathcal{X}(t) - \pi(t)\right) \dfrac{\mathrm{d}B(t)}{B(t)} + \pi(t) \dfrac{\mathrm{d}S(t)}{S(t)} \right] - \mathcal{C}(t) d(t), \\ \mathcal{X}(0) = x_0 > 0. \end{cases} \tag{6}$$

Substituting 1 and 3 into 6 gives:

$$\begin{aligned} \mathrm{d}\mathcal{X}(t) = & \left[\mathcal{X}(t) - \pi(t)\right] r(t) \mathrm{d}t + \pi(t) \Big[\left(r(t) + k\eta(t) \right) \mathrm{d}t \\ & + \sigma_1 \sqrt{\eta(t)} \mathrm{d}W^S(t) \Big] - \mathcal{C}(t) d(t). \end{aligned} \tag{7}$$

Rearranging 7 gives us that:

$$\begin{cases} \mathrm{d}\mathcal{X}(t) = \left[\mathcal{X}(t) r(t) - \mathcal{C}(t) + \pi(t) k\eta(t) \right] \mathrm{d}t + \pi(t) \sigma_1 \sqrt{\eta(t)} \mathrm{d}W^S(t), \\ \mathcal{X}(0) = x_0 > 0. \end{cases} \tag{8}$$

5. The Optimization Criterion

Suppose the set of all admissible strategies is denoted by $\mathcal{A}$.

Definition 5.1 *An investment and consumption strategy pair* $\left(\pi(t), \mathcal{C}(t)\right) \in \mathcal{A}$ *is said to be admissible if the following conditions are satisfied.*

1) The pair $\left(\pi(t), \mathcal{C}(t)\right)$ is progressively $\mathcal{F}_t$ *-measurable* and $\int_0^T \pi(t)^2 \mathrm{d}t < \infty$, $\int_0^T \mathcal{C}(t) \mathrm{d}t < \infty$, for all $T > 0$.

2) $\mathbb{E}\left[\int_0^T \left(\pi(t) \sigma_1 \sqrt{\eta(t)} \right)^2 \mathrm{d}t \right] < \infty$.

3) For all admissible pair $\left(\pi(t), \mathcal{C}(t)\right)$, the wealth process 5 with $\mathcal{X}(0) = x_0 > 0$ has a path wise unique solution.

Remark 1 *The investor's objective is to maximize the net expected discounted utility of consumption plus the expected discounted utility of terminal wealth.*

In this study, the power utility function which belongs to the CRRA class is used. The investor's objective is to maximize the expected discounted utility of consumption plus the expected discounted utility of terminal wealth formulated mathematically as follows:

$$\max_{(\pi(t), \mathcal{C}(t)) \in \mathcal{A}} \mathbb{E}\left[\int_0^T \phi \exp^{-\lambda t} U_1\left[\mathcal{C}(t)\right] \mathrm{d}t + (1 - \phi) \exp^{-\lambda T} U_2\left[\mathcal{X}(T)\right] \right]. \tag{9}$$

subject to the budget constraint

$$\begin{cases} \mathrm{d}\mathcal{X}(t) = \left[\mathcal{X}(t) r(t) - \mathcal{C}(t) + \pi(t) k\eta(t) \right] \mathrm{d}t + \pi(t) \sigma_1 \sqrt{\eta(t)} \mathrm{d}W^S(t), \\ \mathcal{X}(0) = x_0 > 0, \end{cases} \tag{10}$$

$$\begin{cases} \mathrm{d}r(t) = \theta_0(t) \mathrm{d}t + \sigma_0 \mathrm{d}W^r(t), \\ r(0) = r_0 > 0, \end{cases} \tag{11}$$

and

$$\begin{cases} \mathrm{d}\eta(t) = \left[\theta_2 - b\eta(t)\right]\mathrm{d}t + \sigma_2\sqrt{\eta(t)}\mathrm{d}W^{\eta}(t), \\ \eta(0) = \eta_0 > 0. \end{cases} \tag{12}$$

Definition 5.2 *The value function is defined as*

$$V(t,r,\eta,x) = \sup_{(\pi(t),\mathcal{C}(t))\in\mathcal{A}} \mathbb{E}\left[\int_0^T \phi\exp^{-\lambda t} U_1\left[\mathcal{C}(t)\right]\mathrm{d}t + (1-\phi)\exp^{-\lambda T} U_2\left[\mathcal{X}(T)\right]\right], \tag{13}$$

with boundary conditions

$$V(T,r,\eta,x) = (1-\phi)\exp^{-\lambda T} U_2\left[\mathcal{X}(T)\right]. \tag{14}$$

where $\mathcal{X}(t) \geq 0$ for all t, with T being the date of death, $\mathcal{X}(T)$ is the value at time T of a trading strategy. The parameter λ is the subjective discount rate and ϕ determines the relative importance of the intermediate consumption. $\mathbb{E}$ denotes the conditional expectation operator. $U_1\left[\mathcal{C}(t)\right]$ and $U_2\left[\mathcal{X}(T)\right]$ are consumption and bequest functions respectively.

Remark 2 *Note that* $U_1\left[\mathcal{C}(t)\right]$ *and* $U_2\left[\mathcal{X}(T)\right]$ *are such that* $U(.)$ *is twice differentiable with* $U'(.) > 0$ *and* $U''(.) < 0$.

Remark 3 *When* $\phi = 0$, *the expected utility only depends on the terminal wealth and the problem is reduced to an investment problem without intermediate consumption.*

Definition 5.3 *Let* $\mathcal{X}(0) > 0$ *be the initial wealth, the investor's optimal investment and consumption problem is to maximize the expected discounted utility over the set of all admissible strategies* $(\pi(t),\mathcal{C}(t))$ *such that:*

$$V\left(\pi^{\star}(t)\right) = \sup_{(\pi(t),\mathcal{C}(t))\in\mathcal{A}} V(\pi) \tag{15}$$

and

$$V\left(\mathcal{C}^{\star}\right) = \sup_{(\pi(t),\mathcal{C}(t))\in\mathcal{A}} V(\mathcal{C}), \tag{16}$$

for all $\left(\pi^{\star}(t),\mathcal{C}^{\star}(t)\right) \in \mathcal{A}$, $t \in [0,T]$.

6. The Hamilton-Jacobi-Bellman PDE

Bellman's optimality principle, introduced by Bellman [3] called the Dynamic Programming Principle (DPP) will be applied in order to determine the Hamilton-Jacobi-Bellman Partial Differential equation (HJB PDE). By applying DPP, the fully HJB PDE associated with the stochastic control problem 13 is the nonlinear second order PDE given as follows:

$$\begin{aligned} V_t + \sup_{(\pi,\mathcal{C})\in\mathcal{A}} \Bigg[&(rx - \mathcal{C} + \pi k\eta)V_x + \frac{1}{2}\pi^2\sigma_1^2\eta V_{xx} + \theta_0 V_r + \frac{1}{2}\sigma_0^2 V_{rr} \\ &+ (\theta_2 - b\eta)V_\eta + \frac{1}{2}\sigma_2^2\eta V_{\eta\eta} + \pi\sigma_1\sigma_2\eta\rho V_{x\eta} + \phi\exp^{-\lambda t} U_1(\mathcal{C}) \Bigg] = 0. \end{aligned} \tag{17}$$

where V_t, V_x, V_{xx}, V_r, V_{rr}, V_η, $V_{\eta\eta}$ and $V_{x\eta}$ denotes partial derivatives.

Remark 4 *For the sake of closed form solutions from the HJB PDE* 17, *we assume the following:*

1) The correlation coefficient $\rho \in \{-1,1\}$ of $\mathrm{d}W^r\mathrm{d}W^S$, $\mathrm{d}W^\eta\mathrm{d}W^S$ and $\mathrm{d}W^r\mathrm{d}W^\eta$.

2) Interest rate for risk-free security and risky security are equal.

Remark 5 *The reduction of the initial nonlinear HJB PDE to a linear PDE is useful for obtaining both the value function and the optimal policies.*

Definition 6.1 *Applying the first-order maximizing conditions to* 17, *we obtain the following candidate optimizers*:

$$\pi^*(t) = -\frac{kV_x}{\sigma_1^2 V_{xx}} - \frac{\sigma_2 \rho V_{x\eta}}{\sigma_1 V_{xx}} \tag{18}$$

and

$$U_1'\left(\mathcal{C}^*\right) = \frac{V_x}{\phi \exp^{-\lambda t}}. \tag{19}$$

Substituting the candidate optimizers 18 and 19 into the PDE 17 we get the following after simplification:

$$\begin{aligned} &V_t + rxV_x - \mathcal{C}^*(t)V_x - \frac{k^2\eta V_x^2}{2\sigma_1^2 V_{xx}} - \frac{\sigma_2^2\rho^2\eta V_{x\eta}^2}{2V_{xx}} + \theta_0 V_r + \frac{\sigma_0^2 V_{rr}}{2} + (\theta_2 - b\eta)V_\eta \\ &+ \frac{\sigma_2^2 \eta V_{\eta\eta}}{2} - \frac{k\sigma_2\rho\eta}{\sigma_1}\frac{V_x V_{x\eta}}{V_{xx}} + \phi\exp^{-\lambda t} U_1\left(\mathcal{C}^*\right) = 0. \end{aligned} \tag{20}$$

At this stage, we can apply power transformation and change of variable techniques to reduce PDE 20 to a linear PDE with well-defined solutions.

7. The Value Function and Optimal Policies

Solving PDE 20 by applying power transformation and change of variable techniques results in a linear PDE with well-defined solutions and thus, the value function and optimal policies can be established.

Taking a trial solution for PDE 20 take the form

$$V(t,r,\eta,x) = \exp^{-\lambda t}\frac{x^\delta}{\delta} g(t,r,\eta),\ g(T,r,\eta) = 1-\phi. \tag{21}$$

The partial derivatives for 21 are given by:

$$\begin{cases} V_t = -\lambda\exp^{-\lambda t}\dfrac{x^\delta}{\delta} g + \exp^{-\lambda t}\dfrac{x^\delta}{\delta} g_t, \\ V_x = \exp^{-\lambda t} x^{\delta-1} g, \\ V_{xx} = \exp^{-\lambda t}(\delta-1)x^{\delta-2} g, \\ V_r = \exp^{-\lambda t}\dfrac{x^\delta}{\delta} g_r, \\ V_{rr} = \exp^{-\lambda t}\dfrac{x^\delta}{\delta} g_{rr}, \\ V_\eta = \exp^{-\lambda t}\dfrac{x^\delta}{\delta} g_\eta, \\ V_{\eta\eta} = \exp^{-\lambda t}\dfrac{x^\delta}{\delta} g_{\eta\eta}, \\ V_{x\eta} = \exp^{-\lambda t} x^{\delta-1} g_\eta. \end{cases} \tag{22}$$

Note that the optimal consumption strategy becomes

$$C^{*}(t)=\phi^{\frac{-1}{\delta-1}}xg^{\frac{1}{\delta-1}}. \tag{23}$$

Substituting 22 and 23 into 20 gives the following after simplification:

$$\begin{aligned}&\exp^{-\lambda t}\frac{x^{\delta}}{\delta}\Bigg[g_t+\left(-\lambda+r\delta-\frac{1}{2}\frac{k^2\eta}{\sigma_1^2}\frac{\delta}{\delta-1}\right)g-\frac{1}{2}\frac{\sigma_2^2\rho^2\eta\delta}{\delta-1}\frac{g_\eta g_\eta}{g}+\theta_0 g_r\\&+\frac{1}{2}\sigma_0^2 g_{rr}+\frac{1}{2}\sigma_2^2\eta g_{\eta\eta}+\left(\theta_2-b\eta-\frac{k\eta\sigma_2\rho}{\sigma_1}\frac{\delta}{\delta-1}\right)g_\eta+(1-\delta)\phi^{\frac{-1}{\delta-1}}g^{\frac{\delta}{\delta-1}}=0\Bigg].\end{aligned} \tag{24}$$

Eliminating the dependence on x gives a PDE of the form:

$$\begin{aligned}&g_t+\left(-\lambda+r\delta-\frac{k^2\eta\delta}{2\sigma_1^2(\delta-1)}\right)g-\frac{\sigma_2^2\rho^2\eta\delta g_\eta^2}{2(\delta-1)g}+\theta_0 g_r+\frac{\sigma_0^2}{2}g_{rr}+\frac{\sigma_2^2\eta}{2}g_{\eta\eta}\\&+\left(\theta_2-b\eta-\frac{k\eta\sigma_2\rho\delta}{\sigma_1(\delta-1)}\right)g_\eta+(1-\delta)\phi^{\frac{-1}{\delta-1}}g^{\frac{\delta}{\delta-1}}=0.\end{aligned} \tag{25}$$

Again we assume PDE 25 is of the form:

$$g(t,r,\eta)=f(t,r,\eta)^{1-\delta},\ f(T,r,\eta)=(1-\phi)^{\frac{1}{\delta-1}}. \tag{26}$$

The partial derivatives for 26 are given by:

$$\begin{cases}g_t=(1-\delta)f^{-\delta}f_t,\\g_r=(1-\delta)f^{-\delta}f_r,\\g_{rr}=(1-\delta)(-\delta)f^{-\delta-1}f_r^2+(1-\delta)f^{-\delta}f_{rr},\\g_\eta=(1-\delta)f^{-\delta}f_\eta,\\g_{\eta\eta}=(1-\delta)(-\delta)f^{-\delta-1}f_\eta^2+(1-\delta)f^{-\delta}f_{\eta\eta}.\end{cases} \tag{27}$$

Substituting 27 into PDE 25, we obtain:

$$\begin{aligned}&(1-\delta)f^{-\delta}\Bigg[f_t+\left(\frac{-\lambda}{1-\delta}+\frac{r\delta}{1-\delta}-\frac{1}{2}\frac{k^2\eta}{\sigma_1^2}\frac{\delta}{(\delta-1)(1-\delta)}\right)f\\&+\frac{1}{2}\sigma_2^2\rho^2\eta\delta f^{-1}f_\eta^2+\theta_0 f_r+\frac{1}{2}\sigma_0^2\left(f^{-1}f_r^2+f_{rr}\right)\\&-\frac{1}{2}\sigma_2^2\eta\left(\delta f^{-1}f_\eta^2+f_{\eta\eta}\right)+\left(\theta_2-b\eta-\frac{k\eta\sigma_2\rho}{\sigma_1}\frac{\delta}{\delta-1}\right)f_\eta+\phi^{\frac{1}{1-\delta}}=0\Bigg].\end{aligned} \tag{28}$$

Eliminating $(1-\delta)f^{-\delta}$ and simplifying further, 28 result to a nonlinear second-order PDE in f given by:

$$\begin{aligned}&f_t+\left(\frac{r\delta-\lambda}{1-\delta}+\frac{1}{2}\frac{k^2\eta}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}\right)f+\frac{1}{2}\sigma_2^2\eta(\delta)\left[\rho^2-1\right]f^{-1}f_\eta^2+\theta_0 f_r+\frac{1}{2}\sigma_0^2 f_{rr}\\&-\frac{1}{2}\sigma_0^2\delta\frac{f_r^2}{f}+\frac{1}{2}\sigma_2^2\eta f_{\eta\eta}+\left(\theta_2-b\eta-\frac{k\eta\sigma_2\rho}{\sigma_1}\frac{\delta}{\delta-1}\right)f_\eta+\phi^{\frac{1}{1-\delta}}=0.\end{aligned} \tag{29}$$

Remark 6 *Note that PDE* 29 *is still complex and cannot be solved directly since there exists the term* $\phi^{\frac{1}{1-\delta}}$.

Inspired by the paper of Liu [17], we further assume a solution to 29 as stated below:

Lemma 2 *Assume* f *given by*

$$f(t,r,\eta)=\phi^{\frac{1}{1-\delta}}\int_t^T \hat{f}(u,r,\eta)\mathrm{d}u+(1-\phi)^{\frac{1}{\delta-1}}\hat{f}(t,r,\eta), \tag{30}$$

is the solution to 29. Then we can prove that $\hat{f}(t,r,\eta)$ can be written as:

$$\begin{aligned}&\hat{f}_t+\left(\frac{r\delta-\lambda}{1-\delta}+\frac{1}{2}\frac{k^2\eta}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}\right)\hat{f}+\frac{1}{2}\sigma_2^2\eta(\delta)\left[\rho^2-1\right]\hat{f}^{-1}\hat{f}_\eta^2+\theta_0\hat{f}_r\\&+\frac{1}{2}\sigma_0^2\hat{f}_{rr}-\frac{1}{2}\sigma_0^2\delta\frac{\hat{f}_r^2}{\hat{f}}+\frac{1}{2}\sigma_2^2\eta\hat{f}_{\eta\eta}+\left(\theta_2-b\eta-\frac{k\eta\sigma_2\rho}{\sigma_1}\frac{\delta}{\delta-1}\right)\hat{f}_\eta=0,\end{aligned} \tag{31}$$

with the boundary condition $\hat{f}(T,r,\eta)=1$.

Proof. In lemma 2, we seek to convert PDE 29 to PDE 31. Define the differential operator Δ on any function $f(t,r,\eta)$ as

$$\begin{aligned}\Delta f=&\left(\frac{r\delta-\lambda}{1-\delta}+\frac{1}{2}\frac{k^2\eta}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}\right)f+\frac{1}{2}\sigma_2^2\eta(\delta)\left[\rho^2-1\right]f^{-1}f_\eta^2+\theta_0 f_r\\&+\frac{1}{2}\sigma_0^2 f_{rr}-\frac{1}{2}\sigma_0^2\delta\frac{f_r^2}{f}+\frac{1}{2}\sigma_2^2\eta f_{\eta\eta}+\left(\theta_2-b\eta-\frac{k\eta\sigma_2\rho}{\sigma_1}\frac{\delta}{\delta-1}\right)f_\eta\\=&\,0.\end{aligned} \tag{32}$$

Then equation 29 can also be written as

$$\frac{\partial f}{\partial t}+\nabla f+\phi^{\frac{1}{1-\delta}}=0, \tag{33}$$

where

$$f(T,r,\eta)=(1-\phi)^{\frac{1}{\delta-1}}. \tag{34}$$

Notice that, on the other hand, we find

$$\begin{aligned}\frac{\partial f}{\partial t}+\nabla f=&\frac{\partial}{\partial t}\left(\phi^{\frac{1}{1-\delta}}\int_t^T\hat{f}(u,r,\eta)\right)+\nabla\left(\phi^{\frac{1}{1-\delta}}\int_t^T\hat{f}(u,r,\eta)\mathrm{d}u\right)\\&+(1-\phi)^{\frac{1}{\delta-1}}\left(\frac{\partial}{\partial t}\hat{f}(t,r,\eta)+\nabla\hat{f}(t,r,\eta)\right)\\=&-\phi^{\frac{1}{1-\delta}}\hat{f}(t,r,\eta)+\phi^{\frac{1}{1-\delta}}\int_t^T\nabla\hat{f}(u,r,\eta)\mathrm{d}u\\&+(1-\phi)^{\frac{1}{\delta-1}}\left(\frac{\partial}{\partial t}\hat{f}(t,r,\eta)+\nabla\hat{f}(t,r,\eta)\right)\\=&-\phi^{\frac{1}{1-\delta}}.\end{aligned} \tag{35}$$

Note also that

$$\begin{cases}\hat{f}(t,r,\eta)+\int_t^T\nabla\hat{f}(u,r,\eta)\mathrm{d}u=1\\ \frac{\partial}{\partial t}\hat{f}(t,r,\eta)+\nabla\hat{f}(t,r,\eta)=0\end{cases} \tag{36}$$

Therefore,

$$\frac{\partial \hat{f}}{\partial t}+\nabla \hat{f}=0, \tag{37}$$

where

$$\hat{f}(T,r,\eta)=1. \tag{38}$$

Implying 29 can be converted to the following:

$$\begin{aligned}&\hat{f}_t+\left(\frac{r\delta-\lambda}{1-\delta}+\frac{1}{2}\frac{k^2\eta}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}\right)\hat{f}+\frac{1}{2}\sigma_2^2\eta(\delta)\left[\rho^2-1\right]\hat{f}^{-1}\hat{f}_\eta^2+\theta_0\hat{f}_r\\&+\frac{1}{2}\sigma_0^2\hat{f}_{rr}-\frac{1}{2}\sigma_0^2\delta\frac{\hat{f}_r^2}{\hat{f}}+\frac{1}{2}\sigma_2^2\eta\hat{f}_{\eta\eta}+\left(\theta_2-b\eta-\frac{k\eta\sigma_2\rho}{\sigma_1}\frac{\delta}{\delta-1}\right)\hat{f}_\eta=0,\end{aligned} \tag{39}$$

with the boundary condition $\hat{f}(T,r,\eta)=1$.

PDE 29 has been converted to PDE 31. PDE 31 has well defined solutions.

Thus, solving 20 or 29 is equivalent to solving 31.

Theorem 1 *Suppose* $V(t,r,\eta,x)$ *is continuously differentiable and twice continuously differentiable for all* $t\in[0,T]$ *and* $(r,x,\eta)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, *then the solution of the HJB PDE* 20 *is given by*

$$\begin{aligned}V(t,r,\eta,x)=\exp^{-\lambda t}\frac{x^\delta}{\delta}\Bigg[&\phi^{\frac{1}{1-\delta}}\int_t^T\exp\{\mathcal{H}(u)\eta+\mathcal{L}(u)r+\mathcal{M}(u)\}\mathrm{d}u\\&+(1-\phi)^{\frac{1}{1-\delta}}\exp\{\mathcal{H}(t)\eta+\mathcal{L}(t)r+\mathcal{M}(t)\}\Bigg]^{1-\delta},\end{aligned} \tag{40}$$

with $\mathcal{H}(t)$, $\mathcal{L}(t)$ and $\mathcal{M}(t)$ given by:

$$\mathcal{H}(t)=\frac{\xi_1\xi_2\left[1-\exp\left[-\frac{\sigma_2^2\sigma_1}{2}\left(\delta\rho^2+1-\delta\right)(\xi_1-\xi_2)(T-t)\right]\right]}{(\xi_1-\xi_2)\exp\left[-\frac{\sigma_2^2\sigma_1}{2}\left(\delta\rho^2+1-\delta\right)(\xi_1-\xi_2)(T-t)\right]}, \tag{41}$$

$$\mathcal{L}(t)=\frac{\delta}{\delta-1}(T-t) \tag{42}$$

and

$$\mathcal{M}(t)=\int_t^T\left[\frac{1}{2}\sigma_0^2(1-\delta)\mathcal{L}^2(s)-\theta_0\mathcal{L}(s)-\theta_2\mathcal{H}(s)+\frac{\lambda}{1-\delta}\right]\mathrm{d}s. \tag{43}$$

In addition, the pair $(\pi^*,\mathcal{C}^*)\in\mathcal{A}$ given by

$$\pi^*(t)=\frac{k}{\sigma_1^2(1-\delta)}\mathcal{X}(t)+\frac{\sigma_2\rho f_\eta}{\sigma_1 f}\mathcal{X}(t) \tag{44}$$

and

$$\mathcal{C}^*(t)=\phi^{\frac{1}{1-\delta}}\mathcal{X}(t)f^{-1} \tag{45}$$

are the optimal investment and consumption policies when interest rates of a risk-free security follow a Ho-Lee model and stock price dynamics evolve as a Heston's model.

Proof. In theorem 1, assume we can fit a solution $\hat{f}(t,r,\eta)$ given by:

$$\hat{f}(t,r,\eta)=\exp\{\mathcal{H}(t)\eta+\mathcal{L}(t)r+\mathcal{M}(t)\}, \tag{46}$$

with the boundary condition $\mathcal{H}(T)=\mathcal{L}(T)=\mathcal{M}(T)=0$.

From equation 46, we have the following partial derivatives:

$$\begin{cases}\hat{f}_t=\left[\mathcal{H}'(t)\eta+\mathcal{L}'(t)r+\mathcal{M}'(t)\right]\hat{f}(t,r,\eta),\\ \hat{f}_r=\mathcal{L}(t)\hat{f}(t,r,\eta),\\ \hat{f}_{rr}=\mathcal{L}^2(t)\hat{f}(t,r,\eta),\\ \hat{f}_\eta=\mathcal{H}(t)\hat{f}(t,r,\eta),\\ \hat{f}_{\eta\eta}=\mathcal{H}^2(t)\hat{f}(t,r,\eta).\end{cases} \tag{47}$$

Substituting 47 into 31 gives:

$$\begin{aligned}&\left(\mathcal{H}'(t)\eta+\mathcal{L}'(t)r+\mathcal{M}'(t)\right)\hat{f}(t,r,\eta)+\left(\frac{\delta r-\lambda}{1-\delta}+\frac{1}{2}\frac{k^2\eta}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}\right)\hat{f}(t,r,\eta)\\&+\frac{1}{2}\sigma_2^2\eta(\delta)\left(\rho^2-1\right)\mathcal{H}^2\hat{f}(t,r,\eta)+\theta_0\mathcal{L}(t)\hat{f}(t,r,\eta)+\frac{1}{2}\sigma_0^2\mathcal{L}^2(t)\hat{f}(t,r,\eta)\\&-\frac{1}{2}\sigma_0^2\delta\mathcal{L}^2(t)\hat{f}(t,r,\eta)+\frac{1}{2}\sigma_2^2\eta\mathcal{H}^2(t)\hat{f}(t,r,\eta)\\&+\left(\theta_2-\eta-\frac{k\sigma_2\rho\eta}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t)\hat{f}(t,r,\eta)=0.\end{aligned} \tag{48}$$

Canceling the term $\hat{f}(t,r,\eta)$ on both sides of 48 gives:

$$\begin{aligned}&\left(\mathcal{H}'(t)\eta+\mathcal{L}'(t)r+\mathcal{M}'(t)\right)+\left(\frac{\delta r-\lambda}{1-\delta}+\frac{1}{2}\frac{k^2\eta}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}\right)\\&+\frac{1}{2}\sigma_2^2\eta(\delta)\left(\rho^2-1\right)\mathcal{H}^2+\theta_0\mathcal{L}(t)+\frac{1}{2}\sigma_0^2\mathcal{L}^2(t)-\frac{1}{2}\sigma_0^2\delta\mathcal{L}^2(t)\\&+\frac{1}{2}\sigma_2^2\eta\mathcal{H}^2(t)+\left(\theta_2-b\eta-\frac{k\sigma_2\rho\eta}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t)=0.\end{aligned} \tag{49}$$

Rewriting equation 49 to collect like terms in r and η gives:

$$\begin{aligned}&\eta\left[\mathcal{H}'(t)+\frac{1}{2}\sigma_2^2\left(\rho^2-1+\frac{1}{\delta}\right)\mathcal{H}^2(t)-\left(b+\frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t)+\frac{1}{2}\frac{k^2}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}\right]\\&+r\left[\mathcal{L}'(t)+\frac{\delta}{1-\delta}\right]+\left[\mathcal{M}'(t)-\frac{1}{2}\sigma_0^2(1-\delta)\mathcal{L}^2(t)+\theta_0\mathcal{L}(t)+\theta_2\mathcal{H}(t)-\frac{\lambda}{1-\delta}\right]=0.\end{aligned} \tag{50}$$

After eliminating η and r, we can split Equation (50) into three ODE's as follows:

$$\begin{cases}\mathcal{H}'(t)+\frac{1}{2}\sigma_2^2\left(\rho^2-1+\frac{1}{\delta}\right)\mathcal{H}^2(t)-\left(b+\frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t)+\frac{1}{2}\frac{k^2}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}=0,\\ \mathcal{H}(T)=0.\end{cases} \tag{51}$$

$$\begin{cases}\mathcal{L}'(t)+\frac{\delta}{1-\delta}=0,\\ \mathcal{L}(T)=0.\end{cases} \tag{52}$$

and

$$\begin{cases} \mathcal{M}'(t) - \frac{1}{2}\sigma_0^2(1-\delta)\mathcal{L}^2(t) + \theta_0\mathcal{L}(t) + \theta_2\mathcal{H}(t) - \frac{\lambda}{1-\delta} = 0, \\ \mathcal{M}(T) = 0. \end{cases} \tag{53}$$

Rewriting Equations (51)-(53), we get

$$\begin{cases} \mathcal{H}'(t) = -\frac{1}{2}\sigma_2^2\left(\rho^2 - 1 + \frac{1}{\delta}\right)\mathcal{H}^2(t) + \left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t) - \frac{1}{2}\frac{k^2}{\sigma_1^2}\frac{\delta}{(\delta-1)^2}, \\ \mathcal{H}(T) = 0. \end{cases} \tag{54}$$

$$\begin{cases} \mathcal{L}'(t) = -\frac{\delta}{1-\delta}, \\ \mathcal{L}(T) = 0. \end{cases} \tag{55}$$

and

$$\begin{cases} \mathcal{M}'(t) = \frac{1}{2}\sigma_0^2(1-\delta)\mathcal{L}^2(t) - \theta_0\mathcal{L}(t) - \theta_2\mathcal{H}(t) + \frac{\lambda}{1-\delta}, \\ \mathcal{M}(T) = 0. \end{cases} \tag{56}$$

Rewriting Equation alone (54), we obtain:

$$\begin{aligned} \mathcal{H}'(t) = -\frac{1}{2}\sigma_2^2\left(\delta\rho^2 + 1 - \delta\right)\Bigg[\mathcal{H}^2(t) - \frac{2}{\sigma_2^2\left(\delta\rho^2+1-\delta\right)}\left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t) \\ + \frac{k^2}{\sigma_1^2}\frac{\delta}{(\delta-1)^2\sigma_2^2\left(\delta\rho^2+1-\delta\right)}\Bigg]. \end{aligned} \tag{57}$$

Let $\Delta_{\mathcal{H}}$ denote the discriminant of the quadratic equation given by:

$$\mathcal{H}^2(t) - \frac{2}{\sigma_2^2\left(\delta\rho^2+1-\delta\right)}\left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)\mathcal{H}(t) + \frac{k^2}{\sigma_1^2}\frac{\delta}{(\delta-1)^2\sigma_2^2\left(\delta\rho^2+1-\delta\right)}. \tag{58}$$

Implying

$$\begin{aligned} \Delta_{\mathcal{H}} &= \frac{4}{\sigma_2^4\sigma_1^2\left(\delta\rho^2+1-\delta\right)^2}\left(b + \frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right)^2 - \frac{4k^2}{\sigma_1^2}\frac{\delta}{(\delta-1)^2\sigma_2^2\left(\delta\rho^2+1-\delta\right)} \\ &= \frac{4}{\sigma_2^4\sigma_1^2\left(\delta\rho^2+1-\delta\right)^2}\left[\frac{-k^2}{\delta-1} + \frac{\delta}{\delta-1}\left[(k\rho+k\rho)^2 + b^2\left(1-\rho^2\right)\right]\right]. \end{aligned} \tag{59}$$

Let the discriminant $\Delta_{\mathcal{H}}$ have distinct real solutions, that is $\Delta_{\mathcal{H}} > 0$, then we obtain the following condition for δ necessary for numerical analysis:

$$\delta < \frac{k^2}{(k\rho+k\rho)^2 + b^2\left(1-\rho^2\right)} < 1. \tag{60}$$

Considering condition 60, if we integrate both sides of 57 with respect to *t*, we obtain:

$$\frac{1}{\xi_1-\xi_2}\int_t^T\left[\frac{1}{\mathcal{H}-\xi_1} - \frac{1}{\mathcal{H}-\xi_2}\right]\mathrm{d}\mathcal{H}(t) = \frac{-1}{2}\sigma_2^2\sigma_1\left(\delta\rho^2+1-\delta\right)(T-t), \tag{61}$$

where ξ_1 and ξ_2 are two distinct real solutions for 57 given by:

$$\begin{aligned}\xi_{1,2} = & \frac{1}{\sigma_2^2\sigma_1\left(\delta\rho^2+1-\delta\right)^2}\left[b+\frac{k\rho\sigma_2}{\sigma_1}\frac{\delta}{\delta-1}\right] \\ & \pm\sqrt{\frac{1}{\sigma_2^4\sigma_1^2\left(\delta\rho^2+1-\delta\right)^2}\left[\frac{-k^2}{\delta-1}+\frac{\delta}{\delta-1}\left[\left(k\rho+k\rho\right)^2+b^2\left(1-\rho^2\right)\right]\right]}.\end{aligned} \tag{62}$$

Solving 61 with terminal conditions $\mathcal{H}(T)=0$, we obtain:

$$\mathcal{H}(t)=\frac{\xi_1\xi_2\left[1-\exp\left[-\frac{\sigma_2^2\sigma_1}{2}\left(\delta\rho^2+1-\delta\right)\left(\xi_1-\xi_2\right)\left(T-t\right)\right]\right]}{\left(\xi_1-\xi_2\right)\exp\left[-\frac{\sigma_2^2\sigma_1}{2}\left(\delta\rho^2+1-\delta\right)\left(\xi_1-\xi_2\right)\left(T-t\right)\right]}. \tag{63}$$

The solutions to the Equations (55) and (56) are obtained directly as follows:

$$\mathcal{L}(t)=\frac{\delta}{\delta-1}(T-t) \tag{64}$$

and

$$\mathcal{M}(t)=\int_t^T\left[\frac{1}{2}\sigma_0^2(1-\delta)\mathcal{L}^2(s)-\theta_0\mathcal{L}(s)-\theta_2\mathcal{H}(s)+\frac{\lambda}{1-\delta}\right]\mathrm{d}s. \tag{65}$$

Therefore, the value function is represented as follows:

$$\begin{aligned}V(t,r,\eta,x)=\exp^{-\lambda t}\frac{x^\delta}{\delta}\Bigg[&\phi^{\frac{1}{1-\delta}}\int_t^T\exp\left\{\mathcal{H}(u)\eta+\mathcal{L}(u)r+\mathcal{M}(u)\right\}\mathrm{d}u \\ &+(1-\phi)^{\frac{1}{1-\delta}}\exp\left\{\mathcal{H}(t)\eta+\mathcal{L}(t)r+\mathcal{M}(t)\right\}\Bigg]^{1-\delta}.\end{aligned} \tag{66}$$

where $\mathcal{H}(t)$, $\mathcal{L}(t)$ and $\mathcal{M}(t)$ are given in 63, 64 and 65 respectively.

In addition, the optimal feedback portfolio functions are given as follows:

$$\pi^*(t)=\left(\frac{k}{\sigma_1^2(1-\delta)}+\frac{\sigma_2\rho}{\sigma_1}\cdot\frac{f_\eta}{f}\right)(\mathcal{X}) \tag{67}$$

and

$$\mathcal{C}^*(t)=\phi^{\frac{1}{1-\delta}}(\mathcal{X})f^{-1}, \tag{68}$$

where

$$\begin{aligned}f(t,r,\eta)=&\phi^{\frac{1}{1-\delta}}\int_t^T\exp\left\{\mathcal{H}(u)\eta+\mathcal{L}(u)r+\mathcal{M}(u)\right\}\mathrm{d}u \\ &+(1-\phi)^{\frac{1}{1-\delta}}\exp\left\{\mathcal{H}(t)\eta+\mathcal{L}(t)r+\mathcal{K}(t)\right\},\end{aligned} \tag{69}$$

with $\mathcal{H}(t)$, $\mathcal{L}(t)$ and $\mathcal{M}(t)$ determined in 63, 64 and 65.

8. Numerical Examples and Simulations

In this section, we determine how parameters affect investment $\pi^*(t)$ and consumption $\mathcal{C}^*(t)$ controls.

When $\theta_0(t)=\sigma_0=\sigma_2=0$, then problem becomes an investment and consumption problem with the following optimal policies:

$$\pi^*(t)=\frac{k}{\sigma_1^2}\cdot\frac{1}{1-\delta}\mathcal{X}(t) \tag{70}$$

and

$$\mathcal{C}^*(t)=\phi^{\frac{1}{1-\delta}}\mathcal{X}(t)f^{-1}. \tag{71}$$

When $\rho=1$, then the problem becomes an investment and consumption problem giving the following optimal policies:

$$\pi^*(t)=\frac{k}{\sigma_1^2}\cdot\frac{1}{1-\delta}\mathcal{X}(t)+\frac{\sigma_2}{\sigma_1}\cdot\frac{f_\eta}{f}\mathcal{X}(t) \tag{72}$$

and

$$\mathcal{C}^*(t)=\phi^{\frac{1}{1-\delta}}\mathcal{X}(t)f^{-1}. \tag{73}$$

When $\rho=-1$, then the problem becomes an investment and consumption problem having the following optimal policies:

$$\pi^*(t)=\frac{k}{\sigma_1^2}\cdot\frac{1}{1-\delta}\mathcal{X}(t)-\frac{\sigma_2}{\sigma_1}\cdot\frac{f_\eta}{f}\mathcal{X}(t) \tag{74}$$

and

$$\mathcal{C}^*(t)=\phi^{\frac{1}{1-\delta}}\mathcal{X}(t)f^{-1}. \tag{75}$$

When $\phi=0$, the problem becomes an asset allocation problem without consumption. In such a case, optimal policies can be investigated further in another study.

8.1. Effects of Wealth $\mathcal{X}$ on Optimal Investment $\pi^*(t)$ and Consumption $C^*(t)$

Here, we assess the effects of Wealth on investment and consumption. Note that

$$\frac{\partial\pi^*}{\partial\mathcal{X}}=\frac{k}{\sigma_1^2}\cdot\frac{1}{1-\delta}+\frac{\sigma_2\rho}{\sigma_1}\cdot\frac{f_\eta}{f}>0. \tag{76}$$

and

$$\frac{\partial\mathcal{C}^*}{\partial\mathcal{X}}=\phi^{\frac{1}{1-\delta}}f^{-1}>0. \tag{77}$$

Figure 1 and Figure 2 show the effect of the wealth $\mathcal{X}(t)$ on the investment $\pi^*(t)$ and consumption $\mathcal{C}^*(t)$. The curve results analysis indicates that wealth $\mathcal{X}(t)$ affects investment and consumption rates in a positive way when $\rho=-1$ and $\rho=1$. In summary, we can confidently state that optimal investment $\pi^*(t)$ and consumption $\mathcal{C}^*(t)$ increase with the accumulation of the wealth $\mathcal{X}(t)$. This agrees with practical investments and our intuition.

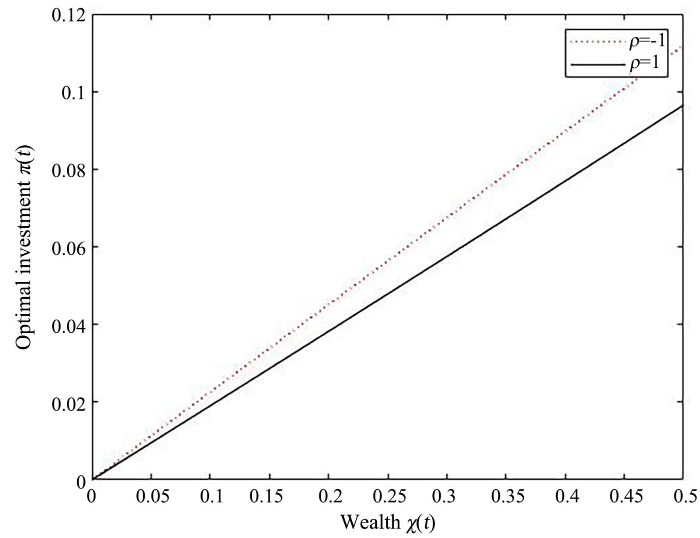

Figure 1. The effects of wealth $\mathcal{X}(t)$ on optimal investment $\pi^*(t)$ when $t = 0:0.01:0.5$; $k = 0.6$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\delta = -1$; $\sigma_1 = 1.2$; $\eta = 0.6$; $\lambda = 0.5$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\phi = 1$ and $\sigma_2 = 1.1$.

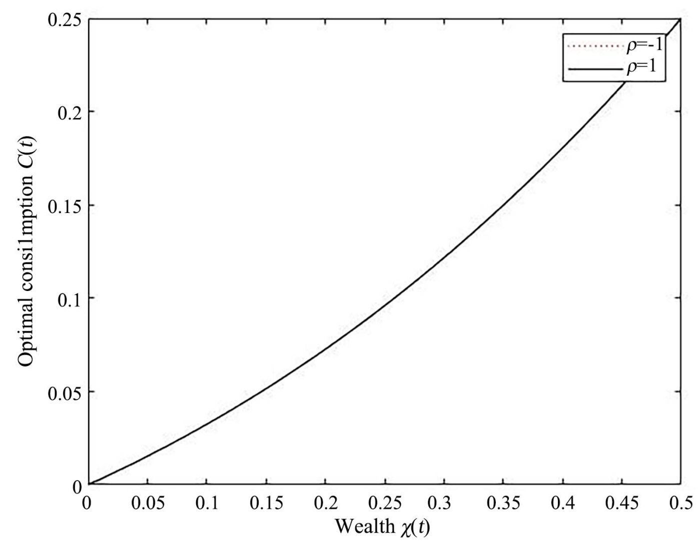

Figure 2. The effects of wealth $\mathcal{X}(t)$ on consumption $\mathcal{C}^*(t)$ when $t = 0:0.01:0.5$; $k = 0.6$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\delta = -1$; $\sigma_1 = 1.2$; $\eta = 0.6$; $\lambda = 0.5$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\phi = 1$ and $\sigma_2 = 1.1$.

8.2. Effects of the Expected Returns Parameter of Risky Asset *k* on Optimal Investment $\pi^*(t)$ and Consumption $C^*(t)$

In **Figure 3** and **Figure 4**, the optimal investment $\pi^*(t)$ and optimal consumption $\mathcal{C}^*(t)$ increases with respect to the increase in expected returns of risky asset k when $\rho = -1$ and $\rho = 1$. Note that in 3, the product $k\eta(t)$ is considered as the appreciation rate of the stock implying the more the investor wishes to invest in the stock for more wealth and consumption. This agrees with practical investments and our intuition.

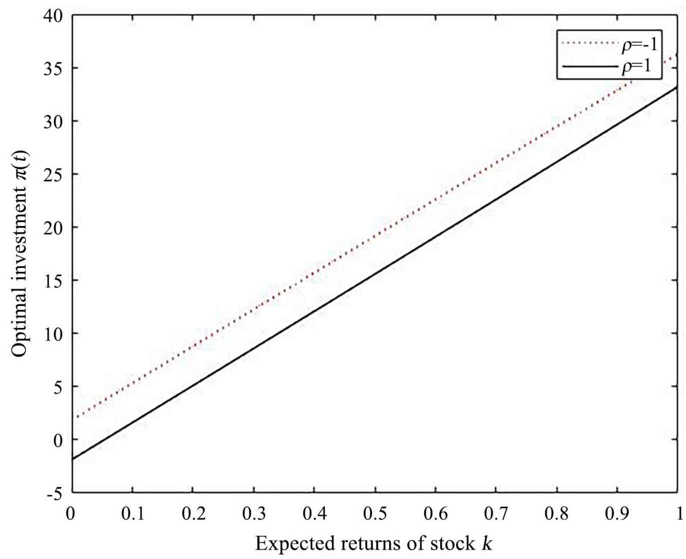

Figure 3. The effects of the expected returns parameter of risky asset k on optimal investment $\pi^*(t)$ when $t = 0:0.01:1$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\sigma_1 = 1.2$; $\eta = 0.6$; $\delta = -1$; $\lambda = 0.5$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\phi = 1$ and $\sigma_2 = 1.1$.

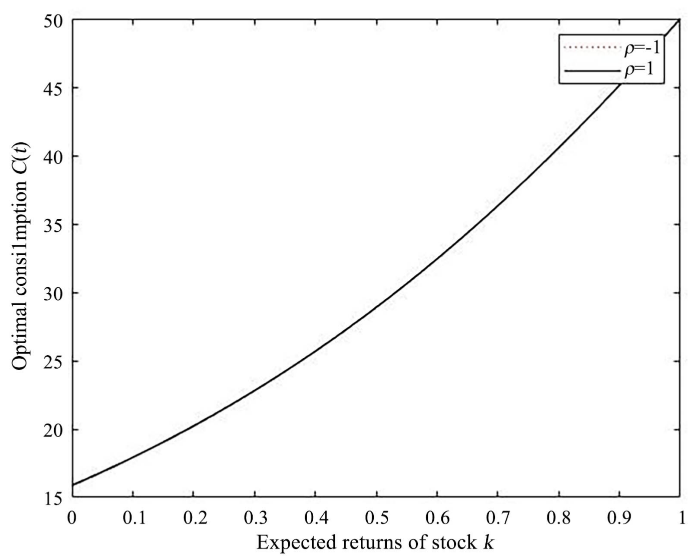

Figure 4. The effects of the expected returns parameter of risky asset k on consumption $C^*(t)$ when $t = 0:0.01:1$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\sigma_1 = 1.2$; $\eta = 0.6$; $\delta = -1$; $\lambda = 0.5$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\phi = 1$ and $\sigma_2 = 1.1$.

8.3. Effects of Risk Aversion Factor δ on Optimal Investment $\pi^*(t)$ and Consumption $C^*(t)$

In Figure 5 and Figure 6, the optimal investment $\pi^*(t)$ and consumption $C^*(t)$ increase with larger values of risk aversion factor δ as this lead to smaller relative risk aversion $1-\delta$ for the investor. The investor becomes vigorous in investing in the stock resulting in more wealth and thus more consumption.

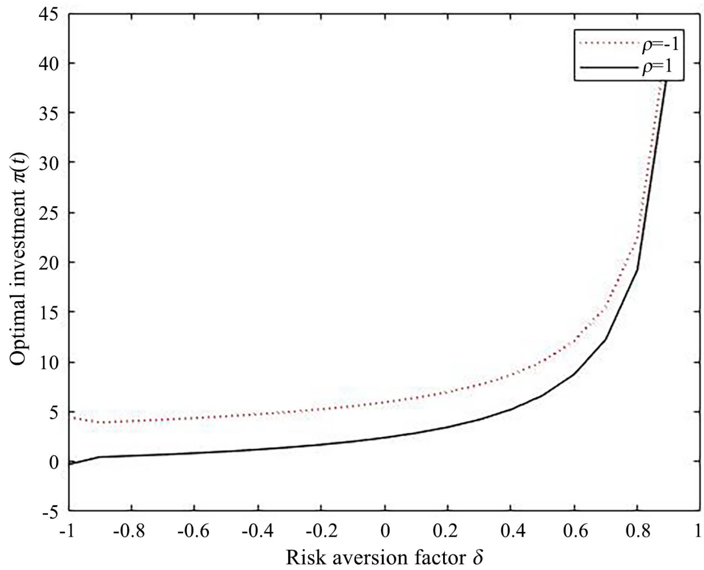

Figure 5. The effects of risk aversion factor δ on optimal investment $\pi^*(t)$ when $t=-1:0.1:1$; $k=0.6$; $\sigma_0=0.2$; $a=0.8$; $b=0.9$; $\sigma_1=1.2$; $\eta=0.6$; $\lambda=0.5$; $\theta_0=0.25$; $\theta_2=0.26$; $\phi=1$ and $\sigma_2=1.1$.

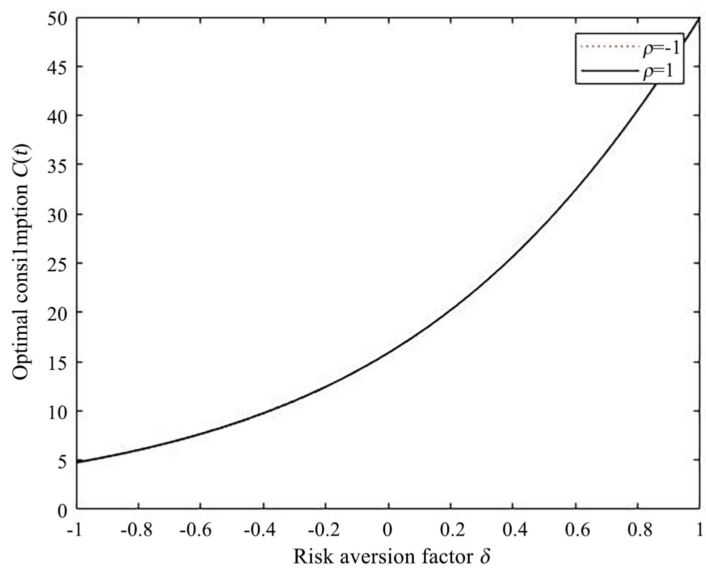

Figure 6. The effects of risk aversion factor δ on optimal consumption $C^*(t)$ when $t=-1:0.1:1$; $k=0.6$; $\sigma_0=0.2$; $a=0.8$; $b=0.9$; $\sigma_1=1.2$; $\eta=0.6$; $\lambda=0.5$; $\theta_0=0.25$; $\theta_2=0.26$; $\phi=1$ and $\sigma_2=1.1$.

8.4. Effects of Weight for Intermediate Consumption ϕ on Optimal Consumption $C(t)$

In **Figure 7**, $C^*(t)$ is increasing as weight for intermediate consumption ϕ increase. When ϕ gets larger, optimal consumption amount will also increase for $\rho=-1$ and $\rho=1$. In conclusion, the amount to consume increases for larger values of ϕ.

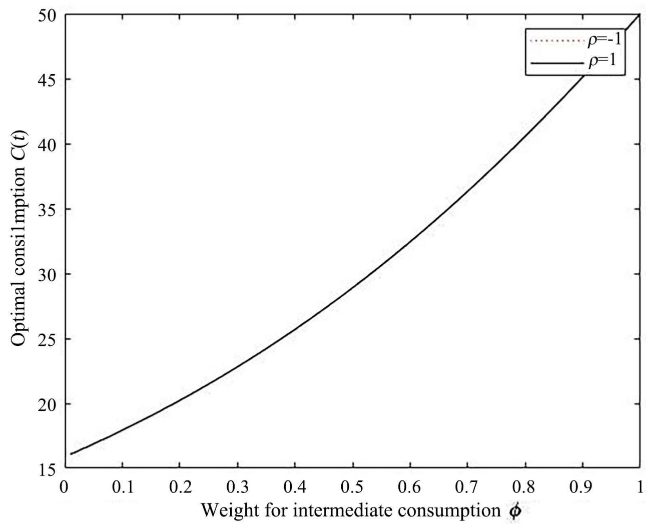

Figure 7. The effects of weight for intermediate consumption ϕ on optimal consumption $C^{*}(t)$ when $t=0:0.01:1$; $k=0.6$; $\sigma_0=0.2$; $a=0.8$; $b=0.9$; $\delta=-1$; $\eta=0.6$; $\lambda=0.5$; $\theta_0=0.25$; $\theta_2=0.26$ and $\sigma_1=1.2$.

8.5. Effects of Risk-Free Interest Rate *r* on Optimal Investment $\pi^{*}(t)$ and Consumption $C^{*}(t)$

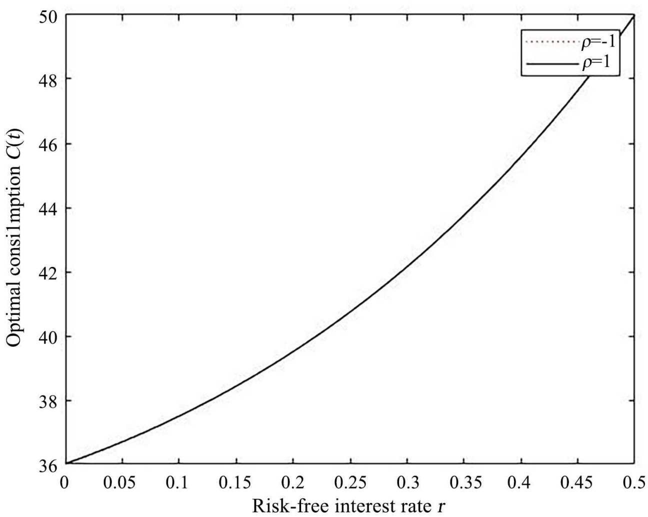

Figure 8. The effects of risk-free interest rate r on optimal investment $\pi^{*}(t)$ when $t=0:0.01:0.5$; $k=0.6$; $\sigma_0=0.2$; $a=0.8$; $b=0.9$; $\sigma_1=1.2$; $\eta=0.6$; $\lambda=0.5$; $\theta_0=0.25$; $\theta_2=0.26$; $\phi=1$ and $\sigma_2=1.1$.

8.6. Effects of Risk-Free Interest Rate *r* on Optimal Consumption $C^{*}(t)$

In **Figure 8**, the optimal investment $\pi^{*}(t)$ decreases as interest rate r increases when $\rho=1$ and vise verse for $\rho=-1$. In this case, the investor will reduce the investment amount in the stock in order to avoid the risks and invest more in risk-free assets since income is increasing in this asset. In **Figure 9**, consumption

$\mathcal{C}^*(t)$ increases as interest rate r increases when $\rho \in \{-1,1\}$. Again the investor will reduce the investment amount in the stock in order to avoid the risks and invest more in risk-free assets since income is increasing in these assets. Net wealth still increases resulting in more consumption.

8.7. Effects of Volatility of Risky Security σ_2 on Optimal Investment $\pi^*(t)$ and Consumption $C^*(t)$

In **Figure 10**, the optimal investment policy $\pi^*(t)$ increases as the volatility of risky security η increases when $\rho=-1$ but decreases when $\rho=1$. Implying the value of correlation is key when making an investment decision in this financial market setup. In **Figure 11**, the consumption policy $\mathcal{C}^*(t)$ increases as

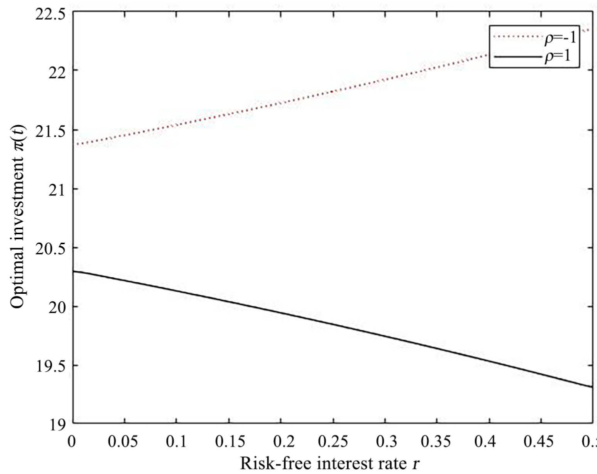

Figure 9. The effects of risk-free interest rate r on optimal consumption $\mathcal{C}^*(t)$ when $t=0:0.01:0.5$; $k=0.6$; $\sigma_0=0.2$; $a=0.8$; $b=0.9$; $\sigma_1=1.2$; $\eta=0.6$; $\lambda=0.5$; $\theta_0=0.25$; $\theta_2=0.26$; $\phi=1$ and $\sigma_2=1.1$.

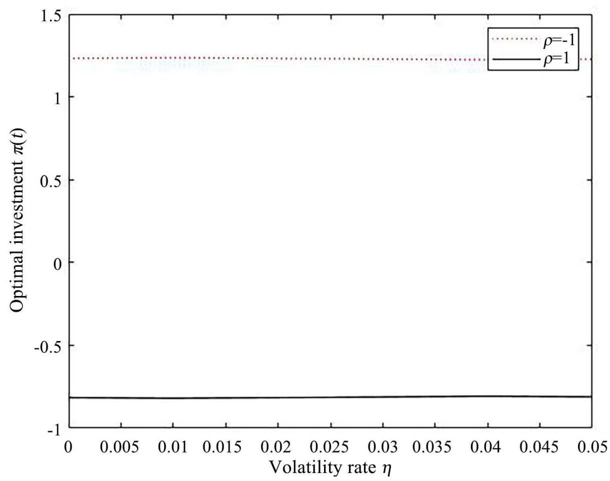

Figure 10. The effects of volatility of risky security η on optimal investment $\pi^*(t)$ when $t=0:0.001:0.5$; $k=0.6$; $\sigma_0=0.2$; $a=0.8$; $b=0.9$; $\delta=-1$; $r=0.6$; $\lambda=0.5$; $\theta_0=0.25$; $\theta_2=0.26$; $\phi=1$ and $\sigma_1=1.2$.

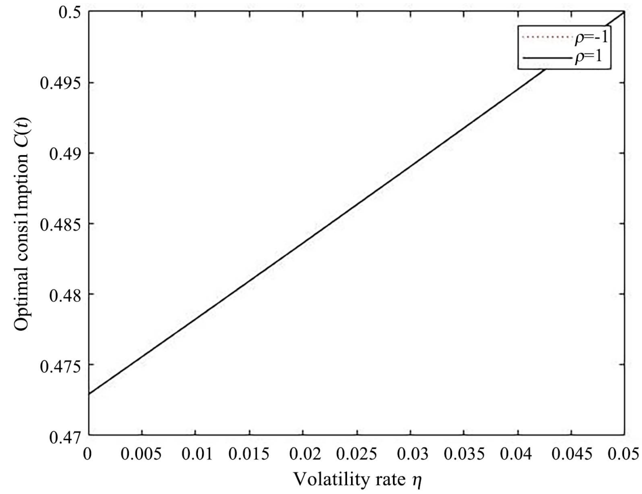

Figure 11. The effects of volatility of risky security η on optimal consumption $\mathcal{C}^*(t)$ when $t = 0:0.001:0.5$; $k = 0.6$; $\sigma_0 = 0.2$; $a = 0.8$; $b = 0.9$; $\delta = -1$; $r = 0.6$; $\lambda = 0.5$; $\theta_0 = 0.25$; $\theta_2 = 0.26$; $\phi = 1$ and $\sigma_1 = 1.2$.

the volatility of risky security η increases. Higher risky investments yield higher returns implying more consumption. This agrees with practical investments and our intuition.

9. Conclusion

This research work builds on the celebrated work of Merton [1] [2] who originally studied continuous-time investment and consumption problems. We investigate an optimal investment and consumption problem for a single investor with a portfolio consisting of one risk-free security (e.g. a money market account or bond) $B(t)$ and one risky security (e.g. a stock or stock index) $S(t)$. The interest rate dynamics of risk-free security follow a Ho-Lee model. In addition, the risky asset price follows Heston's model with its volatility evolving as the CIR model. Our main goal is to allocate initial wealth x_0 between risk-free security and risky security to maximize the discounted expected utility of consumption and terminal wealth over a finite horizon. By applying the Dynamic Programming Principle (DPP), we obtain the HJB PDE. Upon solving the HJB PDE, we derive the closed-form solutions of optimal investment and consumption strategies for the power utility case. The impact and economic implications of market parameters on optimal investment and consumption strategies showed that the wealth $\mathcal{X}$, the weight for intermediate consumption ϕ, the risk aversion factor δ and the expected returns parameter of risky asset k affect the optimal investment $\pi^*(t)$ and optimal consumption $\mathcal{C}^*(t)$ is a positive way regardless of the value of the correlation coefficient $\rho \in \{-1,1\}$. In addition, an increase in risk-free interest rate r and volatility of risky security η led to an increase in net wealth resulting in more consumption. Also the value of $\rho \in \{-1,1\}$ is key for optimal investment in this financial market setup. The future research will focus on extending our study to include other utility functions. We will also introduce multiple risky securities resulting in more sophisticated nonlinear second-order partial differential equations.

Acknowledgements

Special thanks go to the anonymous reviewers for their helpful comments on our earlier submitted version of this paper. This work was fully supported financially by Mulungushi University.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Merton, R.C. (1969) Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *The Review of Economics and Statistics*, **51**, 247-257. https://doi.org/10.2307/1926560

[2] Merton, R.C. (1971) Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economics Theory*, **3**, 373-413. https://doi.org/10.1016/0022-0531(71)90038-X

[3] Bellman, R. (1957) Dynamic Programming. Princeton University Press, Princeton.

[4] Benth, F.E., Karlsen, K.H. and Reikvam, K. (2003) Merton's Portfolio Optimization Problem in a Black and Scholes Market with Non-Gaussian Stochastic Volatility of Ornstein-Uhlenbeck Type. *Mathematical Finance*, **13**, 215-244. https://doi.org/10.1111/1467-9965.00015

[5] Ma, Q., Yi, F. and Guan, C. (2018) Consumption and Investment Problem with Constraints on Minimum and Maximum Consumption Rates. *Journal of Computational and Applied Mathematics*, **338**, 185-198. https://doi.org/10.1016/j.cam.2018.02.006

[6] Zariphopoulou, T. (1994) Consumption and Investment Models with Constraints. *SIAM Journal on Control and Optimization*, **32**, 59-85. https://doi.org/10.1137/S0363012991218827

[7] Sandjo, A., Colin, F. and Moutari, S. (2017) An Explicit Solution for a Portfolio Selection Problem with Stochastic Volatility. *Journal of Mathematical Finance*, **7**, 199-218. https://doi.org/10.4236/jmf.2017.71011

[8] Wang, C., Chang, H. and Fang, Z. (2018) Optimal Portfolio and Consumption Rule with a CIR Model under HARA Utility. *Journal of the Operations Research Society of China*, **6**, 107-137. https://doi.org/10.1007/s40305-017-0189-8

[9] Harrison, M. and Kreps, D. (1979) Martingales and Arbitrage in Multi Period Securities Markets. *Journal of Economic Theory*, **20**, 381-408. https://doi.org/10.1016/0022-0531(79)90043-7

[10] Harrison, M. and Pliska, S. (1981) Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and Their Applications*, **11**, 215-260. https://doi.org/10.1016/0304-4149(81)90026-0

[11] Cox, J. and Huang, C.F. (1989) Optimal Consumption and Portfolio Policy When Asset Prices Follow a Diffusion Process. *Journal of Economic Theory*, **49**, 33-83. https://doi.org/10.1016/0022-0531(89)90067-7

[12] Jinzhu L. and Rong, W. (2009) Optimal Investment Problem with Stochastic Interest Rate and Stochastic Volatility: Maximizing a Power Utility. *Applied Stochastic Models in Business and Industry*, **25**, 407-420. https://doi.org/10.1002/asmb.759

[13] Noh, E. and Kim, J. (2011) Optimal Portfolio Model with Stochastic Volatility and Stochastic Interest Rate. *Journal of Mathematical Analysis and Applications*, **375**, 510-522. https://doi.org/10.1016/j.jmaa.2010.09.055

[14] Pang, T. (2006) Stochastic Portfolio Optimization with Log Utility. *Journal of Mathematical Analysis and Applications*, **9**, 869-880. https://doi.org/10.1142/S0219024906003858

[15] Korn, R. and Kraft, H. (2002) A Stochastic Control Approach to Portfolio Problems with Stochastic Interest Rates. *SIAM Journal on Control and Optimization*, **40**, 1250-1269. https://doi.org/10.1137/S0363012900377791

[16] Kraft, H. (2005) Optimal Portfolios and Heston's Stochastic Volatility Model: An Explicit Solution for Power Utility. *Quantitative Finance*, **5**, 303-313. https://doi.org/10.1080/14697680500149503

[17] Liu, J. (2007) Portfolio Selection in Stochastic Environments. *Review of Financial Studies*, **20**, 1-39. https://doi.org/10.1093/rfs/hhl001

[18] Zariphopoulou, T. (2001) A Solution Approach to Valuation with Unhedgeable risks. *Finance and Stochastics*, **5**, 61-82. https://doi.org/10.1007/PL00000040

[19] Zariphopoulou, T. (1999) Optimal Investment and Consumption Models with Non-Linear Stock Dynamics. *Mathematical Methods of Operations Research*, **50**, 271-296. https://doi.org/10.1007/s001860050098

[20] Fleming, W. and Hernández-Hernández, D. (2003) An Optimal Consumption Model with Stochastic Volatility. *Finance and Stochastics*, **7**, 245-262. https://doi.org/10.1007/s007800200083

[21] Fouque, J.P., Papanicolaou, G. and Sircar, K.R. (2000) Derivatives in Financial Markets with Stochastic Volatility. Cambridge University Press, Cambridge.

[22] Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, **81**, 637-654. https://doi.org/10.1086/260062