

Different Types of Structure Conditions of Semimartingale with Jacod Decomposition

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How to cite this paper: Mwigilwa, W., Aduda, J., Mbele, B.M.L.D. and Kube, A. (2022) Different Types of Structure Conditions of Semimartingale with Jacod Decomposition. *Journal of Mathematical Finance*, 12, 367-381.

<https://doi.org/10.4236/jmf.2022.122021>

Received: February 10, 2022

Accepted: May 23, 2022

Published: May 26, 2022

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Abstract

The objective of this article is to use Jacod decomposition to develop different types of semimartingale structure conditions. We make the following contributions to that end: When a continuous semimartingale meets the structure condition (SC), we prove that there is a minimal martingale density and a predictable variation part. When a special semimartingale meets the minimal structure condition (MSC) and the natural structure condition (NSC), we derive a Radon-Nikodym decomposition and a Natural Kunita-Watanabe decomposition from a given sigma martingale density, which is written under the Jacod decomposition.

Keywords

Structure Condition (SC), Minimal Structure Condition (MSC), Natural Structure Condition (NSC), Jacod Decomposition

1. Introduction

According to [1], the fundamental theorem of asset pricing (FTAP) gives an economic meaning to the no-arbitrage condition: the no-free lunch with vanishing risk (NFLVR). This theorem guarantees that (NFLVR) is necessary and sufficient for the presence of a particular pricing operator, an equivalent σ -martingale measure. The weaker requirements that make up (NFLVR) are the no-arbitrage condition (NA) and the no unbounded profit with bounded risk condition (NUPBR).

Several researchers have proposed reformulations of the (NUPBR) condition that are similar. For example, [2] recently demonstrated the equivalence of (NUPBR)

and the existence of a rigorous sigma-martingale density for one-dimensional but in general semimartingale. The (NUPBR) condition is also equivalent to the existence of a strictly positive σ -martingale density for the underlying semimartingale, as [3] demonstrated. The fundamental benefit of these equivalent reformulations is that they all guarantee the existence of a fair pricing operator to price the terminal wealth of all 1-admissible trading strategies.

However, how can we identify a natural candidate for a strictly positive σ -martingale density for an arbitrary, locally square-integrable semimartingale $S = M + A$, where M denotes the local martingale part and A denotes a predictable variation part? A weak structure condition (SC') that leads to a structure condition (SC) for a continuous semimartingale was proposed by [4] [5]. Structure condition (SC) is a useful tool that leads to the minimal martingale density, which is a natural candidate for a strictly positive σ -martingale density.

Although (SC) and its related structure theorem are stable for a large class of semimartingales, they have some flaws. This is because they're only useful for finding σ -martingale densities for a continuous semimartingale that are strictly positive. Furthermore, under the (proper) measure, the structure condition (SC) is not invariant. To address these flaws, [6] chose to take a fresh look at the special semimartingale and its unique decomposition, resulting in the creation of more than one new type of structure condition: minimal structure condition (MSC) and natural structure condition (NSC).

The goal of this paper is to use Jacod decomposition to establish the concept of several types of semimartingale structure conditions. We'll start by deriving the minimal martingale density from a positive exponential sigma martingale density, which is stated using Jacod decomposition when a continuous semimartingale S meets the structure condition. For the circumstances when P -minimal martingale density and Q -minimal martingale density exist, we will additionally show a predictable variation part of a continuous semimartingale. Finally, when a special semimartingale meets the minimal and natural structure conditions, we will derive Radon-Nikodym decomposition and natural Kunita-Watanabe decomposition from sigma martingale density.

The following is how the rest of the article is structured: Important definitions, theorems, assertions, and Lemmas are discussed in Section 2. In Section 3, we show that a continuous semimartingale has minimal martingale density and drift part. We derive Radon-Nikodym decomposition from a given sigma martingale density and prove a drift variation part of minimal structure condition in Section 4. We derive natural Kunita-Watanabe decomposition (NKWD) and prove the existence of sigma martingale from (NKWD) in Section 5. Finally, conclusions and suggestions are in Section 6.

2. Important Definitions, Notations, Theorems and Propositions

i) According to [7] $L^2(X)$ (resp $L^2_{loc}(X)$) is the set of all d-dimensional

predictable processes B such that the increasing process $\sum_{i,j} (B^i c^{i,j} B^j) \cdot A$ is integrable, where $c = (c^{i,j})_{1 \leq j, j \leq d}$ is predictable taking values in the set of all symmetric nonnegatives $d \times d$ matrices and A is an increasing predictable process and X is a continuous local martingale.

ii) \mathcal{M}_{loc}^2 is the set of all locally square-integrable martingale.

iii) According to ([7], Proposition 1.14) the random measure μ associated with its jumps is defined by $\mu(dt, du) = \sum \mathbf{1}_{\{\Delta S_s \neq 0\}} \delta_{\{s, \Delta S_s\}}(dt, du)$ where δ_a denotes the Dirac measure at point a .

iv) According to ([7], Definition 1.27a) $\mathcal{G}_{loc}(\mu)$ is the set of all $\tilde{\mathcal{P}}$ -measurable real-valued function W on $\tilde{\Omega}$ such that the process

$$\tilde{W}_t(w) = W(w, t, \beta_t(w)) \mathbf{1}_D(w, t) - \hat{W}_t(w) \text{ satisfies } \left[\sum_{s \leq t} (\tilde{W}_s)^2 \right]^{1/2} \in \mathcal{A}_{loc}^+.$$

Therefore the set $\mathcal{G}_{loc}^1(\mu)$ is given by

$$\mathcal{G}_{loc}^1(\mu) = \left\{ W \in \tilde{\mathcal{P}} : \left[\sum_{0 < s \leq t} (W_s(w, s, \beta_s(w)) \mathbf{1}_D(w, s) - \hat{W}_s)^2 \right]^{1/2} \in \mathcal{A}_{loc}^+ \right\}$$

v) According to [8] $\mathcal{H}_{loc}^1(\mu)$ is given by

$$\mathcal{H}_{loc}^1(\mu) := \left\{ V : \Omega \times [0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}, V \in \tilde{\mathcal{O}}, \mathcal{M}_{\mu}^p(V | \mathcal{P}) = 0 \right. \\ \left. \text{and } (V^2 * \mu)^{1/2} \in \mathcal{A}_{loc}^+ \right\}$$

vi) According to ([7], Definition 1.27b) if $W \in \mathcal{G}_{loc}^1(\mu)$ then the stochastic integral $W * (\mu - \nu)$ of W with respect to $(\mu - \nu)$ is the unique element of

$$\mathcal{M}_{loc}^d \text{ where } W = U + \frac{\hat{U}}{1-a} \mathbf{1}_{\{a < 1\}} \text{ is in } \mathcal{G}_{loc}^1(\mu).$$

vii) According to ([9], Lemma 3.72) if $V \in \mathcal{H}_{loc}^1(\mu)$ then the stochastic integral $(V * \mu)$ of V with respect to μ is the unique element of \mathcal{M}_{loc} .

Definition 2.1 (σ -martingale density)

According to [5] a σ -martingale density (or local martingale density) for S is a local P -martingale $Z = (Z_t)_{0 \leq t \leq T}$ with the following properties

- 1) $Z_0 = 1$
- 2) $Z > 0$ up to indistinguishability
- 3) ZS^i is a P - σ -martingale (a local P -martingale) for each $i \in 1, \dots, d$ where S^i is a semimartingale.

Also in addition Z is called in addition strictly positive- σ martingale density if $Z > 0$.

A strictly positive σ -martingale density is in general only a local martingale but not a true P -martingale. But if Z happens to be a true P -martingale (on $[0, T]$ or equivalently if $\mathbb{E}_P[Z_T] = 1$) we can define a probability measure Q equivalent to P via $dQ = Z_T dP$. This Q if it exists is called an equivalent σ -martingale measure (equivalent local martingale measure) for a semimartingale S .

Definition 2.2. (Weak structure condition/Structure condition)

According to [5] if $S = (S_t)_{0 \leq t \leq T}$ is an \mathbb{R}^d valued-continuous semimartingale with canonical decomposition $S = S_0 + M + A$ where the processes $M = (M_t)_{0 \leq t \leq T}$ and $A = (A_t)_{0 \leq t \leq T}$ are both \mathbb{R}^d valued, continuous and null at 0. Then a semimartingale S satisfies the weak structure condition (SC') if A is absolutely continuous with respect to $\langle M \rangle$ in the sense that there exists an \mathbb{R}^d -valued predictable process $\hat{\lambda} = (\hat{\lambda}_t)_{0 \leq t \leq T}$ such that $A = \int d\langle M \rangle \hat{\lambda}$. i.e.

$$A_t^i = \sum_{j=1}^d \int_0^t \hat{\lambda}_u^j d\langle M \rangle_u^{ij} = \sum_{j=1}^d \hat{\lambda}_u^j d\langle M^i, M^j \rangle_u \quad \text{for } i \in 1, \dots, d \text{ and } 0 \leq t \leq T.$$

$\hat{\lambda}$ is called the (instantaneous) market price of risk for S .

Definition 2.3. (Mean-variance tradeoff)

According to [5] if semimartingale S satisfies the weak structure condition (SC'), we define

$$\hat{K}_t = \int_0^t \hat{\lambda}_u^v d\langle M \rangle_u \hat{\lambda}_u = \sum_{i,j=1}^d \int_0^t \hat{\lambda}_u^i \hat{\lambda}_u^j d\langle M^i, M^j \rangle_u$$

$K = (K_t)_{0 \leq t \leq T}$ is called the mean-variance tradeoff process of semimartingale S . Because $\langle M \rangle$ is positive semidefinite, the process \hat{K} is increasing and null at 0, but note that it may take the value ∞ in general.

Therefore we say that a semimartingale S satisfies the structure condition (SC) if S satisfies (SC') and $\hat{K}_T < \infty$ P -a.s.

Proposition 2.4. ([10], Proposition 5a). *If Z satisfies $Z_t = 1 - \int_0^t Z_{u-} \hat{\lambda} d\tilde{M}_u + R_t$ on $[0, T]$ for some $R \in \mathcal{M}_{loc,0}^2(P)$ orthogonal to \tilde{M} . Then Z is a martingale density and locally square-integrable, when $R = 0$. The equation is then reduced to $Z_t = 1 - \int_0^t Z_{u-} \hat{\lambda} d\tilde{M}_u$ whose solution \hat{Z} is given by the stochastic exponential $\mathcal{E}\left(-\int \hat{\lambda} d\tilde{M}\right)$. We call \hat{Z} the minimal martingale density for S . Since it is in a sense the simplest martingale density. Therefore, the perceptive \hat{Z} is minimal because it is obtained from the simplest choice, $R = 0$.*

Theorem 2.5. ([11], Theorem 1). *If S admits a strict martingale density Z^* and that either S is continuous or S is a special semimartingale satisfying $M \in \mathcal{M}_{0,loc}^2(P)$ and $Z^* \in \mathcal{M}_{0,loc}^2(P)$. Then S satisfies the structure condition (SC). Further Z^* can be written as $Z^* = \mathcal{E}\left(-\int \hat{\lambda} dM + L\right)$ where $L \in \mathcal{M}_{0,loc}(P)$ is strongly orthogonal to M . If S is continuous we can simplify $Z^* = \mathcal{E}\left(-\int \hat{\lambda} dM\right)\mathcal{E}(L)$, where $\hat{Z} = \mathcal{E}\left(-\int \hat{\lambda} dM\right)$ where \hat{Z} is minimal martingale density which is obtained for the simplest choice $L = 0$.*

Definition 2.6. (Local martingale)

According to [7] [12], a stochastic process $(X_t)_{t \geq 0}$ is called a local martingale if there exists a localising sequence $(T_n)_{n \in \mathbb{N}}$ such that the stopped process $(X_t^{\tau_n})_{t \geq 0} = (X_{t \wedge \tau_n})_{t \geq 0}$ is martingale for every $n \in \mathbb{N}$.

According to [7] [12] any local martingale M admits a unique decomposition

$$M = M_0 + M^c + M^d$$

where $M_0^c = M_0^d = 0$, M^c is a continuous local martingale and M^d is a purely discontinuous local martingale.

Definition 2.7. (Semimartingale)

According to [7] [12], a cadlag, adapted stochastic process $(X_t)_{t \in [0, T]}$ is called a semimartingale for a given filtration \mathcal{F}_t if it decomposed as $X_t = X_0 + M_t + A_t$ where M_t is a local martingale and A_t is an adapted cadlag process with a finite variation.

A special semimartingale is a semimartingale X_t that admits a decomposition $X_t = X_0 + M_t + A_t$ with a process A_t that is predictable.

The presence of arbitrage is not allowed in modelling asset prices. Therefore it is important that the underlying asset price process be arbitrage-free. The arbitrage possibilities can be eliminated if and only if the underlying asset price process is a semimartingale.

Theorem 2.8. ([12], Lemma 9.6). *Let M be a continuous martingale and H be an optional process. Then there exists a $L \in \mathcal{M}_{loc}$ such that $[L, N] = H \cdot [M, N]$ holds for all $N \in \mathcal{M}_{loc}$ if and only if $H^2 \cdot [M] \in \mathcal{V}^+$. In this case, there exist a predictable process $K \in L_m(M)$ such as that $K \cdot M = L$. We say that H is integrable with respect to M and L is called the stochastic integral of H with respect to M denoted by $H \cdot M$.*

Lemma 2.1. ([6], Key lemma 2.11). *Let S to be an \mathbb{R}^d -valued P -semimartingale and $Q \sim P$. Further, if we let S to be a special Q -semimartingale and denote its canonical decomposition (under Q) by $S = S_0 + M + A$. Moreover, if we let $Z = \mathcal{E}(-N)$ be a strictly positive local Q -martingale. Then Z is a strictly positive Q - σ -martingale (local Q -martingale) density for S if and only if $[N, M^i] - A^i$ is a Q - σ -martingale (local Q -martingale) $\forall i \in 1, \dots, d$.*

Lemma 2.2. *According to [6] if we let $Q \sim P$, $M \in \mathcal{M}_{loc}^2(Q)$ and H be an optional process, such the path-wise Lebesgue-Stieltjes Integral $H \cdot [M]$ exists. Moreover let $H \cdot [M]$ be compensable and denote its compensator by C . There exists $\langle M \rangle^Q$ -a.s unique predictable process $K \in L_m([M^d] - \langle M \rangle^Q)$ under Q such that $C = K \cdot \langle M \rangle^Q$*

Jacod Decomposition

Jacod decomposition was introduced by [8] and has four parameters (B, W, V, N) that belong to the local martingale sets.

Theorem 2.9. ([9], Theorem 3.75).

If we let \mathbb{R}^d -valued semimartingale S to be fixed. Then every (real-valued) local P -martingale M null at 0 can be written as

$$M = W \star (\mu - \nu) + V \star \mu + \bar{N}$$

Then if you apply the Kunita-Watanabe decomposition under P to \bar{N} and the continuous local martingale part of S^c of S to write $\bar{N} = B \cdot S^c + N$, where B is a predictable S^c -integrable process and M a local P -martingale null at 0 and strongly P -orthogonal to S^c . We are going to have the following theorem:

Theorem 2.10. ([13] [14], Theorem 2.4). *If we recall the \mathbb{R}^d -valued semi-*

martingale S is fixed. Then every (real-valued) local P -martingale M null at 0 can be written as

$$M = B \cdot S^c + W \star (\mu - \nu) + V \star \mu + N \tag{1}$$

where

1) $B \in L^2_{loc}(S^c)$ is a predictable S^c -integrable process.

2) A measurable function $W = U + \frac{\hat{U}}{1-a} \mathbf{1}_{\{a < 1\}} \in \mathcal{G}^1_{loc}(\mu)$ in which $U \in \tilde{\mathcal{P}}$

such that the following integral exists $\hat{U} = \int U_t(x) \nu(t \times dx)$.

3) $V \in \mathcal{H}^1_{loc}(\mu)$ is an optional process and.

4) $N \in \mathcal{M}^2_{loc}$ is a local P -martingale null at 0 with $[S, N] = 0$.

The difference between Jacod decomposition (1) and a known local martingale decomposition from [[7], Definition 4.22] is the addition of the term $V \star \mu$.

3. Structure Condition

In order to explain the importance of (SC) as a good tool for finding strictly positive σ -martingale densities in continuous paths cases, we recall its definition. A continuous semimartingale S satisfies the structure condition (SC) if

$$S = M + \int \hat{\lambda} d\langle M^c \rangle$$

where $\hat{\lambda}$ is an \mathbb{R}^d -valued continuous predictable process and M is continuous local martingale. Due to this specific structure of the semimartingale, one can show that all strictly positive σ -martingale densities $\mathcal{E}(-N)$ for S feature a specific Kunita Watanabe decomposition

$$N = \int \hat{\lambda} d\langle M \rangle + L$$

where L is a local martingale orthogonal to M . Therefore since S is continuous it ensures that the minimal martingale density $\mathcal{E}(-\int \hat{\lambda} d\langle M \rangle)$ is a strictly positive σ -martingale density for S .

In this section first, we are going to prove the existence of minimal martingale density from a given sigma martingale density written in Jacod decomposition. Also since the existence of structure condition rely on the existence of the predictable process of quadratic variation of the local martingale part, we are going to prove in the second part the existence of a continuous predictable process

$$A = -\frac{1}{Z_-} \cdot \langle Z, S \rangle = -\frac{1}{Z_-} \cdot \langle Z, M' \rangle \text{ for a continuous semimartingale.}$$

3.1. Minimal Martingale Density

Proposition 3.1. *If we let a sigma martingale density $Z = \mathcal{E}(M)$ where M is a Jacod decomposition then $Z = \hat{Z}^P = \mathcal{E}\left(-\int_0^T B_t d\langle S_t \rangle\right)$ where \hat{Z}^P is a minimal martingale density if and only if a continuous semimartingale S satisfies a structure condition.*

Proof

Given S is a continuous semimartingale and it satisfies the structure condition

(SC). We have $Z = \mathcal{E}(M)$ where $M \in \mathcal{M}_{0,loc}(P)$. We can define $M \in \mathcal{M}_{0,loc}(P)$ by $M = \frac{1}{Z_-} dZ$ which is well defined since Z is strictly positive. In order to get

a required minimal martingale density we are going to multiply by negative both sides of $M = B \cdot S^c + W \star (\mu - \nu) + V \star \mu + N$ then

$$-M = -(B \cdot S^c + W \star (\mu - \nu) + V \star \mu + N).$$

Since any local martingale admits a unique decomposition of two parts: a continuous part and a discontinuous part, we can decompose M as $-M = -(M^1 + M^2)$ with $-M^1 \in \mathcal{M}_{0,loc}^2(P)$ and $-M^2 \in \mathcal{M}_{0,loc}^2(P)$ such that $-M^2$ is strongly orthogonal to each M^i . In this case, we can set $-M^1 = -M^c$ and $-M^2 = -M^d$ as the continuous and purely discontinuous local martingale part of M .

In fact we can choose $-M^2 = 0$ (discontinuous part of local martingale part), if $Z \in \mathcal{M}_{0,loc}^2(P)$ holds and when S is continuous respectively.

By the Galtchouk-Kunita Watanabe decomposition theorem, the continuous part of the local martingale part $-M^1 = -M^c$ can be written as $-M^1 = -M^c = -(B \cdot S^c + N)$ where $B \in L_{loc}^2(S^c)$ and $N \in \mathcal{M}_{0,loc}^2(P)$ null at 0 is strongly orthogonal to S^c , where $\langle S^c, N \rangle = 0$ and $[S, N] = 0$. Since S is continuous, we can simplify

$$Z = \mathcal{E}(-M^c) = \mathcal{E}(-(B \cdot S^c + N)) = \mathcal{E}(-B \cdot S^c) \mathcal{E}(-N) \tag{2}$$

Let $\mathcal{E}(-M^c) = \mathcal{E}(-\int_0^T B_t dS_t^c) = \hat{Z}^{(P)}$ from Equation (2), then we can call $\hat{Z}^{(P)}$ a minimal martingale density obtained for the simplest choice when $-N = 0$ Theorem (2.5).

Let S^c be a semimartingale with representation

$$S^c = S^0 + M_t^{1,c} + A_t^1 \tag{3}$$

where M^c is a continuous local martingale part and A is a finite variation process. Then the stochastic integral:

$$\int_0^T B_t dS_t^c = \int_0^T B_t dM_t^{1,c} + \int_0^T B_t dA_t^1 \tag{4}$$

The representation (3) is not unique therefore the stochastic integral (4) is also not unique. If

$$S^c = S^0 + M_t^{2,c} + A_t^2 \tag{5}$$

is another presentation, then we can have also another stochastic integral

$$\int_0^T B_t dS_t^c = \int_0^T B_t dM_t^{2,c} + \int_0^T B_t dA_t^2 \tag{6}$$

Then:

$$\int_0^T B_t dS_t^c = \int_0^T B_t d(M_t^{1,c} - M_t^{2,c}) = -\int_0^T B_t d(A_t^1 - A_t^2) \tag{7}$$

So that $(M_t^c - M_t^{1,c})$ is a local martingale of finite variation. Since the difference between the two local martingales is also a local martingale then

$$-\int_0^T B_t dS_t^c = -\int_0^T B_t dM_t^c$$

Since S is continuous with integrable quadratic variation then $\langle S \rangle = [S]$

means there is no difference between the sharp and square brackets according to [15]. We are going to have

$$-\int_0^T B_t d\langle S_t \rangle = -\int_0^T B_t d\langle M_t \rangle$$

Therefore, $Z = \hat{Z}^P = \mathcal{E}\left(-\int_0^T B_t d\langle S \rangle\right) = \mathcal{E}\left(-\int_0^T B_t d\langle M_t \rangle\right)$ is a minimal martingale density.

This means our strictly positive σ martingale density is equal to minimal martingale density when a semimartingale S is continuous.

3.2. A Predictable Variation Part When P -Minimal Martingale Density Exists

Structure condition which is defined with continuous semimartingale S of the form $S = S_0 + M' + A$ has two aspects:

i) It requires a predictable variation component A to be absolutely continuous with $\langle M' \rangle$ of martingale component M' with density function Z .

ii) It imposes the square integrability condition on M' and a specific integrability condition on the density function Z .

All two conditions are reinterpretations under square integrability conditions of the equation $A = -\frac{1}{Z_-} \cdot \langle Z, S \rangle = -\frac{1}{Z_-} \cdot \langle Z, M' \rangle$. That is between the a predictable variation part A , the local martingale part M' and a strict σ martingale density Z .

Therefore first we are going to prove the equation

$A = -\frac{1}{Z_-} \cdot \langle Z, S \rangle = -\frac{1}{Z_-} \cdot \langle Z, M' \rangle$ when P -minimal martingale density \hat{Z}^P exists.

Proof

$$d(SZ) = S_- dZ + Z_- dS + d[S, Z] \tag{8}$$

$$S = M' + A \tag{9}$$

$$dS = dM' + dA \tag{10}$$

Substitute Equations (9) and (10) into Equation (8). Then we are going to have the following equation

$$\begin{aligned} d(ZS) &= S_- dZ + Z_- (dM' + dA) + d[Z, M' + A] \\ d(ZS) &= S_- dZ + Z_- dM' + Z_- dA + d[Z, M'] + d[Z, A] \\ d(ZS) &= S_- dZ + Z_- dM' + Z_- dA + d[Z, M'] + d[Z, A] + d\langle Z, S \rangle - d\langle Z, S \rangle \end{aligned} \tag{11}$$

From Proposition (2.4) if Z satisfies $Z_t = Z_0 - \int_0^t Z_{u-} B dS^c + N$ on $[0, T]$ for $N \in \mathcal{M}_{0,loc}^2(P)$ orthogonal to S^c . Then Z is a martingale density and locally square integrable when $N = 0$. The equation then reduced to

$Z_t = Z_0 - \int_0^t Z_{u-} B dS^c$ whose solution \hat{Z}_t^P is given by the stochastic exponential $\mathcal{E}\left(-\int_0^t B dS^c\right)$ where \hat{Z}_t^P is known as minimal density for a semimartingale S . If

we let $Z_0 = 0$ we are going to have $Z_t = -\int_0^t Z_{u-} BdS^c = -Z_t \int_0^t BdS^c = -Z_{t-} \cdot N'$

Therefore from equation (11) since a semimartingale S is decomposed into local martingale part and finite variation part then we are going to have

$$\begin{aligned} d(ZS) &= (S_- (-Z_- dN') + Z_- dM' + [-Z_- dN', M'] + [-Z_- dN', A] \\ &\quad - \langle -Z_- dN', S \rangle) + Z_- dA + d\langle Z, S \rangle \\ d(ZS) &= (S_- (-Z_- dN') + Z_- dM' - Z_- d[N', M'] - Z_- d[N', A] \\ &\quad + Z_- d\langle N', S \rangle) + Z_- dA + d\langle Z, S \rangle \end{aligned} \tag{12}$$

From Equation (12) we have a local martingale part $S_- (-Z_- dN') + Z_- dM' - Z_- d[N', M'] - Z_- d[N', A] + Z_- d\langle N', S \rangle$ and a finite variation part is $Z_- dA + d\langle Z, S \rangle$.

From Equation (12) we are going to have

$$\begin{aligned} d(ZS) &= Z_- (-S_- dN' + dM' - d[N', M'] - d[N', A] + d\langle N', S \rangle) \\ &\quad + Z_- dA + d\langle Z, S \rangle \end{aligned} \tag{13}$$

Due to Yoeurps Lemma [12], the first bracket term on the R.H.S is a P -local martingale, and L.H.S is a (differential of) a P - σ -martingale (a local- P -martingale) if and only if Z is a strictly positive P - σ -martingale (a local P -martingale) density for a semimartingale S .

Therefore by application of the product rule, this shows that ZS has a predictable variation part $Z_- dA + d\langle Z, S \rangle$ which must vanish because ZS is P - σ -martingale.

From this finite variation part

$$\begin{aligned} Z_- dA + d\langle Z, S \rangle &= 0 \\ dA &= -\frac{1}{Z_-} d\langle Z, S \rangle \\ A &= -\frac{1}{Z_-} \cdot \langle Z, S \rangle = -\frac{1}{Z_-} \cdot \langle Z, M' \rangle \end{aligned} \tag{14}$$

3.3. A Predictable Variation Part When Q -Minimal Martingale Density Exists

For the case of Q -minimal martingale density \hat{Z}^Q it depends on the \hat{Z}^P by Bayes Rule.

$$\text{Then } \hat{Z}^Q = \frac{\hat{Z}^P}{\mathcal{E}(M^e)} = \frac{\mathcal{E}(M^p)}{\mathcal{E}(M^e)}.$$

Since we have seen that structure condition is satisfied in \hat{Z}^P it is also be satisfied on \hat{Z}^Q because they depend on each other. According to [6] the predictable quadratic variation of a locally square-integrable martingale is not invariant under equivalent measure changing. Therefore, in this case under \hat{Z}^Q we are going to have $A = -\frac{1}{Z_-} \cdot \langle Z, S^e \rangle = -\frac{1}{Z_-} \cdot \langle Z, M^{e'} \rangle$.

4. Minimal Structure Condition

Although the SC theorem holds for every locally square-integrable semimartin-

gale, it has several limitations in terms of working as an indication for a strictly positive sigma-martingale density. SC lacks the ability to be invariant under equivalent measure changes, and when it comes to arbitrary locally square integrable semimartingales, SC is neither required nor sufficient for the presence of a strictly positive sigma martingale density, as [6] pointed out.

We present a new structure condition in this part that solves the weakness of the structure condition (SC), which is the minimal structure condition (MSC). A MSC is equivalent to the SC for a continuous semimartingale, a property that the SC' does not possess. The MSC definition is derived from the first structure theorem. A theorem that sheds light on the relationship between strictly positive sigma martingale densities, on the one hand, and the structure of locally square integrable semimartingale and potential decomposition of its predictable finite variation part on the other.

In the first part, we are going to prove the existence of Radon-Nikodym decomposition from sigma martingale density Z if we let this density be written under Jacod decomposition with the condition that a semimartingale satisfies minimal structure condition (Proposition 4.3). We are going to prove a predictable variation part of MSC in the second part.

Theorem 4.1 ([6], Theorem 2.24, First Structure theorem). *Let S to be P -semimartingale and $Q \sim P$ be a probability measure such that $S \in \mathcal{S}_{loc}^2(Q)$. And we denote by canonical decomposition of $S = S_0 + M + A$. Further, we denote by M^c and M^d the continuous and purely discontinuous local martingale part of M . Then, if we let $Z = \mathcal{E}(-N)$ to be a positive local Q -martingale, we can denote a Radon-Nikodym decomposition of N with respect to M by equation*

$$N = \lambda \cdot M^c + H \bullet_c M^d + L$$

Definition 4.2. ([6], Definition 2.27). *Let S to be a P -semimartingale and $Q \sim P$ a probability measure. We say that S satisfies the Minimal Structure condition (MSC) under Q if the following properties hold.*

i) S is a locally square-integrable Q -semimartingale with canonical decomposition $S = S_0 + M + A$, where M^c and M^d are continuous and purely discontinuous local martingale parts of M .

ii) There exists a process $\lambda \in L_m(M^c)$ and $\eta \in \left([M^d] - \langle M^d \rangle^Q \right)$ under Q such that $A = \int \lambda d \langle M^c \rangle + \int \eta d \langle M^d \rangle$

Proposition 4.3. *If we let a sigma martingale density $Z = \mathcal{E}(M)$ where M is a Jacod decomposition then there is the existence of Radon-Nikodym decomposition $M = B \cdot S^c + H \bullet_c S^d + N$ from M if a semimartingale S satisfies a minimal structure condition.*

Proof

We are required to prove the existence of Radon-Nikodym decomposition $N = \lambda \cdot M^c + H \bullet_c M^d + L$ from a given Jacod decomposition if a semimartingale S satisfies a minimal structure condition.

If we recall that the \mathbb{R}^d -valued semimartingale S is fixed. Then every (real-valued) local P -martingale M^1 null at 0 can be written as

$$M^1 = W^1 \star (\mu - \nu) + V^1 \star \mu + \bar{N}^1 \tag{15}$$

If we apply the Kunita-Watanabe decomposition under P to \bar{N}^1 and the continuous martingale part S^c of S to write $\bar{N}^1 = B \cdot S^c + N^1$ then every (real-valued) local P -martingale M^1 null at 0 can be written as

$$M^1 = B \cdot S^c + W^1 \star (\mu - \nu) + V^1 \star \mu + N^1 \tag{16}$$

We can modify Equation (15) if we apply the Radon-Nikodym-decomposition under P on \bar{N}^1 and if we let H to be Radon-Nikodym derivative of $d[\bar{N}^1, S^d]$ with respect to $d[S^d]$. We define $N^{11} = \bar{N}^1 - H \bullet_c S^d \in \mathcal{M}_{loc}$. Then $[N^{11}, S^d]$ is a σ -martingale and the decomposition $\bar{N}^1 = H \bullet_c S^d + N^{11}$ is a Radon-Nikodym decomposition of \bar{N}^1 with respect to S^d where \bar{N}^1 is a local martingale and S^d is a discontinuous local martingale part of S . Then a local P -martingale null at 0 can be written as

$$M^2 = H \bullet_c S^d + W^2 \star (\mu - \nu) + V^2 \star \mu + N^{11} \tag{17}$$

now let's add two Equations (16) and (17)

$$M^1 + M^2 = B \cdot S^c + H \bullet_c S^d + W^1 \star (\mu - \nu) + V^1 \star \mu + W^2 \star (\mu - \nu) + V^2 \star \mu + N^1 + N^{11} \tag{18}$$

According to [16] [17] [18] [19] a sum of two local martingales is itself a local martingale. therefore we are going to have the following local martingales from Equation (18)

$$M^1 + M^2 = M \text{ for RHS, and } N^1 + N^{11} = N \text{ for LHS}$$

$$(W^1 \star (\mu - \nu) + W^2 \star (\mu - \nu)) + (V^1 \star \mu + V^2 \star \mu) = W \star (\mu - \nu) + V \star \mu$$

From above we are going to have the following equation

$$M = B \cdot S^c + H \bullet_c S^d + W \star (\mu - \nu) + V \star \mu + N \tag{19}$$

From Equation (19), let multiply both sides by negative sign then we are going to have $-M = -(B \cdot S^c + H \bullet_c S^d + W \star (\mu - \nu) + V \star \mu + N)$.

Our local martingale admits a unique decomposition of two parts: a compensated stochastic integral part and a discontinuous part, we can decompose M as $-M = -(\hat{M}^1 + \hat{M}^2)$ where $-\hat{M}^1 = -(B \cdot S^c + H \bullet_c S^d + N)$ and $-\hat{M}^2 = -(W \star (\mu - \nu) + V \star \mu)$.

In fact, we can choose $-\hat{M}^2 = 0$ if $\sqrt{H^2 \cdot [S]} \in \mathcal{A}_{loc}^+$ then $H \bullet_c \hat{M}^1$ exists and also when S satisfying the Radon-Nikodym decomposition on \hat{M}^1 . Then $-M = -\hat{M}^1$ which implies that $Z = \mathcal{E}(-M) = \mathcal{E}(-\hat{M}^1) = \mathcal{E}\left(-\left(B \cdot S^c + H \bullet_c S^d + N\right)\right)$.

Therefore, there is the existence of Radon-Nikodym decomposition $M = B \cdot S^c + H \bullet_c S^d + N$ from a sigma martingale density.

A Predictable Variation Part of (MSC)

According to the 1st structure theorem and the minimal structure conditions Z is

a strictly positive σ -martingale (local martingale) density for a semimartingale S under Q if there exists a process $\lambda \in L_m(M^c)$ and $\eta \in \left([M^d] - \langle M^d \rangle^Q \right)$ under Q such that $A = \int \lambda d\langle M^c \rangle + \int \eta d\langle M^d \rangle$.

Then from the key Lemma (2.1) Z is a strictly positive σ -martingale (local martingale) density for S if and only if $[\hat{M}^1, S] - A = H \cdot [S] - A$. Then we are going to adapt the proof as from [6]. Due to the Radon-Nikodym decomposition of \hat{M}^1 w.r.t S we are going to have

$$\begin{aligned} [\hat{M}^1, S] - A &= H \cdot [S] - A \\ &= B \cdot [S^c] + H \cdot [S^d] - A \\ &= H \cdot [S^d] + B \cdot \langle S^c \rangle^Q - A \end{aligned} \tag{20}$$

According to Lemma (2.2), we know there exists a $\langle S^d \rangle^Q$ a.s unique predictable process $\eta \in \left([M^d] - \langle M^d \rangle^Q \right)$ such that

$$H \cdot [S^d] - \eta \cdot \langle S^d \rangle^Q = 0 \tag{21}$$

Is a σ -martingale (local martingale).

If you conclude from Equations (20) and (21) then we are going to have

$$A = \int B d\langle S^c \rangle^Q + \eta d\langle S^d \rangle^Q \tag{22}$$

5. Natural Structure Condition

The fact that all strictly positive sigma-martingale semimartingale densities have a natural Kunita-Watanabe decomposition is essentially equal to the NSC. NSC addresses MSC's flaw by demonstrating that if (λ, η) is a version of MSC, we may be certain that $\int \lambda dM^c$ exists. However, we don't know if $\int \eta dM^d$ exists, and it's a local martingale. As a result, NSC concentrates on those parts of MSC.

In this section, when a semimartingale meets the natural structure condition, we will prove the existence of natural Kunita-Watanabe decomposition of sigma martingale density Z stated in terms of Jacod decomposition. (Proposition 5.3). Also, we are going to prove the existence of strictly σ martingale density from (NKWD).

Definition 5.1. ([6], Definition 2.35). *Let a semimartingale S be a locally square-integrable P -semimartingale with canonical decomposition $S = S_0 + M + A$ that satisfy minimal structure condition (MSC). We say S satisfies the natural structure condition (NSC) if there is exist a version (λ, η) of (MSC) such that $\eta \in L_m(M^d)$. A pair (λ, η) satisfying this condition is called a version (NSC).*

Theorem 5.2. ([6], Theorem 2.40, Second structure theorem). *Let a semimartingale S satisfy (NSC) and denote its canonical decomposition by $S = S_0 + M + A$. Moreover, if we let (λ, η) be a version of strictly positive local martingale. Then Z is a strictly positive σ -martingale (local martingale) density for S if and only if N features a natural Kunita-Watanabe decomposition*

$$N = \lambda \cdot M^c + \eta \cdot M^d + L$$

where L is a local martingale and $[L, M]$ is a σ -martingale (local martingale).

Proposition 5.3. *If we let a sigma martingale density $Z = \mathcal{E}(M)$ where M is a Jacod decomposition then there is the existence of a natural Kunita-Watanabe decomposition $M = B \cdot S^c + \eta \cdot S^d + R$ from M if a semimartingale S satisfies a natural structure condition.*

Proof

From Jacod theorem (15) if we apply the natural Kunita-Watanabe decomposition under P to \bar{N} where we have a continuous part S^c of S and the discontinuous part S^d of S then we write $\bar{N} = B \cdot S^c + \eta \cdot S^d + R$ then we are going to have the following decomposition

$$M^3 = B \cdot S^c + \eta \cdot S^d + W^3 \star (\mu - \nu) + V^3 \star \mu + R \tag{23}$$

From Equation (23) let $M^3 = \tilde{M}^1 + \tilde{M}^2$ where $\tilde{M}^1 = B \cdot S^c + \eta \cdot S^d + R$ and $\tilde{M}^2 = W^3 \star (\mu - \nu) + V^3 \star \nu$. In order to satisfy the conditions of second structure theorem and natural structure conditions, a positive σ -martingale density $Z = \mathcal{E}(-M^3)$ can be written in terms of $-M^3 = -\tilde{M}^1 = -(B \cdot S^c + \eta \cdot S^d + R)$ while $-M^3 = -\tilde{M}^2 = -(W^3 \star (\mu - \nu) + V^3 \star \nu = 0)$ if $Z \in \mathcal{M}_{0,loc}(P)$ hold and when S satisfy the natural Kunita-Watanabe decomposition (NKWD) of \tilde{N}^1 with respect to S . Therefore, $Z = \mathcal{E}(-M) = \mathcal{E}(-(B \cdot S^c + \eta \cdot S^d + R))$.

Therefore, there is the existence of Natural Kunita-Watanabe decomposition $M = B \cdot S^c + \eta \cdot S^d + R$ from a given sigma martingale density $Z = \mathcal{E}(M)$.

Existence of Strictly σ Martingale Density from (NKWD)

We are going to prove the existence of strictly σ martingale density if given (NKWD) $M = B \cdot S^c + \eta \cdot S^d + R$.

From the definition of σ -martingale: SZ is a σ -martingale if and only if $(S - [S, M])$ is a σ -martingale.

Therefore

$$\begin{aligned} S - [S, M] &= S - [B \cdot S^c + \eta \cdot S^d + R, S] \\ S - [S, M] &= S - [R, S] - [B \cdot S^c + \eta \cdot S^d, S] \\ S - [S, M] &= S - [R, S] - [B \cdot S^c + \eta \cdot S^d, S^c + S^d] \text{ when } (S = S^c + S^d) \\ S - [S, M] &= S - [R, S] - [B \cdot S^c, S^c] - [B \cdot S^c, S^d] - [\eta \cdot S^d, S^c] - [\eta \cdot S^d, S^d] \end{aligned} \tag{24}$$

From Equation (24) we are going to apply Lemma 2.2 and known properties of quadratic covariation

$$\begin{aligned} S - [S, M] &= S - [R, S] - B \cdot [S^c, S^c] - B \cdot [S^c, S^d] - \eta \cdot [S^d, S^c] - \eta \cdot [S^d, S^d] \\ &= S - [R, S] - B \cdot [S^c] - B \cdot [S^c, S^d] - \eta \cdot [S^d, S^c] - \eta \cdot [S^d] \\ &= S - [R, S] - B \cdot ([S^c] - \langle S^c \rangle^Q) - \eta \cdot ([S^d] - \langle S^d \rangle^Q) \\ &\quad - (B + \eta) \cdot ([S^c, S^d] - \langle S^c, S^d \rangle^Q) \end{aligned} \tag{25}$$

is a σ -martingale (local martingale). Due to natural structure condition (NSC), we can say $B \cdot ([S^c] - \langle S^c \rangle^Q)$, $\eta \cdot ([S^d] - \langle S^d \rangle^Q)$ and

$(B + \eta) \cdot \left([S^c, S^d] - \langle S^c, S^d \rangle^Q \right)$ are local martingales. Therefore $Z = \mathcal{E}(-M)$ is a strictly positive σ -martingale (local martingale) density for S if and only if $S - [R, S]$ is a σ -martingale.

6. Conclusions and Suggestions

There have been no meaningful uses of Jacod decomposition since the introduction of structure condition (SC), which allows for the existence of minimal martingale measure until the introduction of new types of structure conditions (MSC and NSC) of special semimartingale. According to [20] [21] [22], the necessary results can be obtained by writing a sigma martingale density in terms of a Dolean-Dade exponential of Jacod decomposition.

We were able to prove the existence of minimal martingale density when a continuous semimartingale satisfies SC and derive a Radon-Nikodym decomposition and a Natural Kunita-Watanabe decomposition in our case, by writing a sigma martingale density in terms of Dolean-Dade exponential of Jacod decomposition.

We can suggest that, if the sigma martingale density is stated in a different way from the one we studied, the minimal martingale density can be derived. However, this will only be achievable if continuous semimartingale satisfies the structure conditions' features.

Acknowledgements

We acknowledge the African Union through the Pan African University, Institute of Basic Sciences, Technology and Innovation for its consideration and support.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

- [1] Delbaen, F. and Schachermayer, W. (2006) *The Mathematics of Arbitrage*. Springer, Berlin, Heidelberg. <https://doi.org/10.1007/978-3-540-31299-4>
- [2] Kardaras, C. (2012) Market Viability via Absence of Arbitrage of the First Kind. *Finance and Stochastics*, **16**, 651-667. <https://doi.org/10.1007/s00780-012-0172-5>
- [3] Takaoka, K. and Schweizer, M. (2011) A Note on the Condition of no Unbounded Profit with Bounded Risk. *Finance and Stochastics*, **18**, 393-405. <https://doi.org/10.1007/s00780-014-0229-8>
- [4] Choulli, T. and Stricker, C. (1996) Deux applications de la décomposition de galtchouk-kunitawanabe. In: Azéma, J., Yor, M. and Emery, M., Eds., *Séminaire de Probabilités XXX*, Springer, Berlin, Heidelberg, 12-23. <https://doi.org/10.1007/BFb0094638>
- [5] Hulley, H. and Schweizer, M. (2010) M^6 —On Minimal Market Models and Minimal Martingale Measures. In: Chiarella, C. and Novikov, A., Eds., *Contemporary Quan-*

- titative Finance*, Springer, Berlin, Heidelberg, 35-51.
https://doi.org/10.1007/978-3-642-03479-4_3
- [6] Blümmel, T. (2015) Martingale Decomposition Theorems and the Structure of No Arbitrage. Ph.D. Dissertation, Technische Universität Wien, Vienna.
- [7] Jacod, J. and Shiryaev, A. (2013) Limit Theorems for Stochastic Processes. Vol. 288, Springer Science & Business Media, Berlin, Heidelberg.
- [8] Ma, J. (2013) Minimal Hellinger Deflators and Hara Forward Utilities with Applications: Hedging with Variable Horizon. Doctoral Dissertation, University of Alberta, Edmonton.
- [9] Jacod, J. (2006) Calcul stochastique et problemes de martingales. Vol. 714, Springer, Berlin.
- [10] Schweizer, M. (1992) Martingale Densities for General Asset Prices. *Journal of Mathematical Economics*, **21**, 363-378. [https://doi.org/10.1016/0304-4068\(92\)90014-X](https://doi.org/10.1016/0304-4068(92)90014-X)
- [11] Martin, S. (1995) On the Minimal Martingale Measure and the Möllmer-Schweizer Decomposition. *Stochastic Analysis and Applications*, **13**, 573-599. <https://doi.org/10.1080/07362999508809418>
- [12] He, S.-W., Wang, J.-G. and Yan, J.-A. (2019) Semimartingale Theory and Stochastic Calculus. Routledge, New York.
- [13] Choulli, T. and Schweizer, M. (2011) Stability of Sigma-Martingale Densities in Log I under an Equivalent Change of Measure. Research Paper No. 11-67, Swiss Finance Institute, Zürich. <https://doi.org/10.2139/ssrn.1986855>
- [14] Choulli, T. and Martin, S. (2015) Locally Phi-Integrable Sigma-Martingale Densities for General Semimartingales. Research Paper No. 15-15, Swiss Finance Institute, Zürich. <https://doi.org/10.2139/ssrn.2606674>
- [15] Klebaner, F.C. (2012) Introduction to Stochastic Calculus with Applications. World Scientific Publishing Company, Singapore. <https://doi.org/10.1142/p821>
- [16] Protter, P.E. (2005) Stochastic Differential Equations. In: Rozovskii and Yor, M., Eds., *Stochastic Integration and Differential Equations*, Springer, Berlin, Heidelberg, 249-361. https://doi.org/10.1007/978-3-662-10061-5_6
- [17] Revuz, D. and Yor, M. (2013) Continuous Martingales and Brownian Motion. Vol. 293, Springer Science & Business Media, Berlin Heidelberg.
- [18] Cohen, S.N. and Elliott, R.J. (2015) Stochastic Calculus and Applications. Vol. 2, Birkhäuser, New York. <https://doi.org/10.1007/978-1-4939-2867-5>
- [19] Dellacherie, C. and Meyer, P. (1982) Probabilities and Potential B: Theory of Martingales. North-Holland Mathematics Studies, Vol. II, North-Holland, Amsterdam.
- [20] Choulli, T. and Stricker, C. (2005) Minimal Entropy-Hellinger Martingale Measure in Incomplete Markets. *Mathematical Finance*, **15**, 465-490. <https://doi.org/10.1111/j.1467-9965.2005.00229.x>
- [21] Tahir, C. and Stricker, C. (2006) More on Minimal Entropy-Hellinger Martingale Measure. *Mathematical Finance*, **16**, 1-19. <https://doi.org/10.1111/j.1467-9965.2006.00258.x>
- [22] Choulli, T., Stricker, C. and Li, J. (2007) Minimal Hellinger Martingale Measures of Order q . *Finance and Stochastics*, **11**, 399-427. <https://doi.org/10.1007/s00780-007-0039-3>