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Discrete Time Risk Model Financed by Random Premiums

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Abstract

We propose a novel actuarial risk model which, unlike the classical Crámer-Lundberg model, incorporates a stream of random premiums that offset random claims. A key feature of the model is a discrete time accounting of premiums and claims flow, whereby lending itself to random walk type analysis. We derive various estimates of ruin probability thereby providing an effective method of risk assessment over a future time horizon.

Keywords

Risk Process, Kolmogorov Maximal Inequality, Stopped Martingale, Probability of Ruin

1. Introduction

Typical risk considerations in the area of insurance and finance are concerned with the Risk Process

$$U(t) = u + ct - \sum_{i=1}^{N_t} X_i \tag{1.1}$$

where $U(t)$ represents the capital available at time $t > 0$, given the initial capital $U(0) = u \geq 0$, after paying claims X_i which occurred at random times during the interval $(0, t]$ according to a Poisson process N_t. The premium income stream ct is deterministic with premium rate c per unit of time. $U(t)$ is known as the Crámer-Lundberg model and represents the risk reserve of a company at time t. The main objective is to calculate the odds that the company reserve will ever become negative, referred to as the probability of ultimate ruin.

Except a few special cases with closed form solutions, the analysis of this process is usually carried out by numerical inversion of the associated Laplace Transform to solve a renewal equation involving the probability of ruin in infinite time.

Since the joint work of Gerber and Shiu [1] [2] in the late 1990's it has been customary to analyze the process in terms of an expected discounted penalty function.

Various attempts have also been made to add a Lévy Process component to the model in [3] [4] and [5] among many others. Over the years, discrete time versions of the model have been studied, see for example [6] for recent work along these lines. Lately, a stochastic premium income component has been added, see for example [7]. Much of the theory and applications are elucidated in [8] and [9].

We remark, that while it may be reasonable for an insurance company to conveniently collect premiums according to deterministic formula *ct,* given customers contractual obligation to pay premiums to receive coverage for their claims, it certainly is not a reasonable assumption for most models of business income, as the future number of customers and their respective premiums cannot be guaranteed. Furthermore, while it may be true "on average" that an insurance company receives premiums as a continuous stream, it is still possible that the total premiums collected by time t may be substantially smaller than *ct*, at some future times t.

To remedy this drawback, we propose a model in which *ct* is replaced by a stochastic component leading to a shifted discrete time zero-mean random walk representation of the Risk Process that can be analyzed by various tools from probability theory.

The paper is organized as follows. Section 2 introduces our new model. In sections 3 - 5 we derive estimates for the probability of ruin by Kolmogorov's Maximal Inequality, Stopping a Martingale and Large Deviation Principle. Section 6 includes summary conclusions and directions for future research.

2. Derivation of the Model

An extension of the Crámer-Lundberg model to random premiums, by Boikov [7], is as follows

$$U(t) = u + \sum_{i=1}^{N_1(t)} Y_i - \sum_{i=1}^{N_2(t)} X_i,\ Y_i \in [0,\infty),\ X_i \in [0,\infty) \tag{2.1}$$

where $N_1(t), N_2(t)$ are independent Poison processes and $(Y_i),(X_i)$ are independent sequences of i.i.d. representing the premiums and claims respectively.

Our objective is to propose a new model that can be considered a discrete time counterpart to continuous model (2.1) which provides considerable reduction in random complexity through replacing random sums $\sum_{i=1}^{N_1(t)} Y_i, \sum_{i=1}^{N_2(t)} X_i$ by $\sum_{i=1}^{t} Y_i, \sum_{i=1}^{t} X_i$. The significance of such model is that it reflects the actual real-world practice. Namely, "Ruin" is naturally defined as having a negative balance at the end of the day. Likewise, "ruin" has not occurred if the balance at the end of the day is not negative. This is irrespective of whether or not the balance may have been negative at some point before the end of the day.

Dickson and Waters [10] and Dickson [11] studied a discrete model with deterministic premiums

$$U_t = u + t - \sum_{i=1}^{t} X_i,\ \ X_i \in \{0,1,\cdots\},\ t = 1,2,\cdots \tag{2.2}$$

For our model we discretize time in (2.1) whereby generalize (2.2) to random premiums with simultaneous extension of the range of X_i from non-negative integers to non-negative reals as follows

$$U_n = u + \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i = u + \sum_{i=1}^{n} (Y_i - X_i),\ \ Y_i \in [0,\infty),\ X_i \in [0,\infty) \tag{2.3}$$

and can be viewed as a random walk started at initial capital u at time 0.

Recall that the safety loading requires the expected value of the Risk Process gain = $\sum_{i=1}^{n}(Y_i - X_i)$ to be positive, for otherwise the probability of eventual ruin is one. Therefore,

$$E\left[\sum_{i=1}^{n}(Y_i - X_i)\right] = n(EY - EX) = n\theta\mu,\ \text{ where } EY = (1+\theta)EX,\ EX = \mu \tag{2.4}$$

where θ is a safety loading factor.

U_n representation below will play a key role in establishing several estimates for the probability of ruin. Namely, thanks to (2.1) we have

$$U_n = u + \theta\mu n - \sum_{i=1}^{n} Z_i,\ \text{where } Z_i = X_i - Y_i + \theta\mu,\ \text{with } EZ_i = 0,\ n = 1,2,\cdots \tag{2.5}$$

which is a zero-mean random walk $\sum_{i=1}^{n}(-Z_i)$ with linear drift $\theta\mu n$ started at u.

3. Probability of Ruin by Kolmogorov's Maximal Inequality

The results in this section provide an upper estimate on the probability of ultimate ruin in relation to the initial capital. Furthermore, it is shown how to select the initial capital to achieve a low probability of ruin in the finite time interval [0, T].

Theorem 3.1. Let $U_n = u_0 + \theta\mu n - S_n$, where $S_n = Z_1 + Z_2 + \cdots + Z_n$, and Z_i are *i.i.d.* with $EZ_1 = 0$, $EZ_1^2 < \infty$.

Then for every positive integer N there exists an initial capital u_0 such that

$$P(\text{Ultimate Ruin}) = P(\exists\, n > 0 \text{ s.t. } U_n < 0) < \frac{1}{N}$$

Proof. For any subsequence of integers $n_k \nearrow \infty, n_0 = 0$, and an integer l we have

$$\begin{aligned} &P(\text{Ultimate Ruin}) \\ &= P(\text{Ruin occurs in } (0, n_{l-1}]) + \sum_{k=l}^{\infty} P(\text{Ruin occurs in } (n_{k-1}, n_k]) \\ &= P(\exists 1 \le m \le n_{l-1} \text{ s.t. } u_0 + \theta\mu m < S_m) \end{aligned}$$

$$+\sum_{k=l}^{\infty} P\left(\exists\, n_{k-1} < m \le n_k \text{ s.t. } u_0 + \theta\mu m < S_m\right)$$
$$\le P\left(\max_{1\le m\le n_{l-1}} |S_m| > u_0\right) + \sum_{k=l}^{\infty} P\left(\max_{n_{k-1}\le m\le n_k} |S_m| > u_0 + \theta\mu n_k^{2/3}\right)$$
$$\le P\left(\max_{1\le m\le n_{l-1}} |S_m| > u_0\right) + \sum_{k=l}^{\infty} P\left(\max_{1\le m\le n_k} |S_m| > \theta\mu n_k^{2/3}\right)$$

where the last two inequalities follow from the lower bound on $u_0 + \theta\mu m$ depicted in our graph below. (See Figure 1)

Choosing $u_0 = \theta\mu n_{l-1}^{2/3}$ and setting $b_k = P\left(\max_{1\le m\le n_k} |S_m| > \theta\mu n_k^{2/3}\right)$ we obtain

$$P(\text{Ultimate Ruin}) \le \sum_{k=l-1}^{\infty} b_k$$

To complete the proof it suffices to show that for suitably chosen subsequence (n_k)the series $\sum_k b_k$ converges, and consequently there exists an integer $l = l(N)$ such that $\sum_{k=l-1}^{\infty} b_k < \frac{1}{N}$.

To this end, by Kolmogorov's maximal inequality with $n_k = k^6, k = 1, 2, \cdots$

$$P\left(\max_{1\le m\le n_k} |S_m| > \theta\mu n_k^{2/3}\right) \le \frac{Var\left(S_{n_k}\right)}{\left(\theta\mu n_k^{2/3}\right)^2} = \frac{E\left(S_{n_k}^2\right)}{\left(\theta\mu n_k^{2/3}\right)^2} = \frac{n_k EZ_1^2}{\theta^2\mu^2 n_k^{4/3}} = \frac{EZ_1^2}{\theta^2\mu^2 k^2}$$

hence

$$\sum_{k=1}^{\infty} b_k \le \frac{EZ_1^2}{\theta^2\mu^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{EZ_1^2}{\theta^2\mu^2}\frac{\pi^2}{6} < \infty$$

as needed.

Corollary 3.1. For every positive integer N there exist an initial capital u_0 and a finite time T such that

$$P(\text{Ruin by time } T) \le \frac{1}{N}$$

Proof. Choose $u_0 = T\sqrt{EZ_1^2}$ and $T = N$. Then by Kolmogorov's maximal inequality

$$P(\text{Ruin by time } T)$$
$$= P\left(\exists 1 \le m \le T \text{ s.t. } u_0 + \theta\mu m < S_m\right) \le P\left(\exists 1 \le m \le T \text{ s.t. } u_0 < S_m\right)$$
$$\le P\left(\max_{1\le m\le T} |S_m| > u_0\right) \le \frac{Var\left(S_T\right)}{u_0^2} = \frac{TEZ_1^2}{T^2 EZ_1^2} = \frac{1}{T} = \frac{1}{N}$$

Corollary 3.2. Starting with capital $N\sqrt{EZ_1^2}$ the probability of no ruin by time N is at least $1 - \frac{1}{N}$.

4. Probability of Ruin by Stopping a Martingale

We show how martingale method can be applied to calculate the probability of ultimate ruin for our model. Recall that by (2.3) - (2.4) the risk process reads

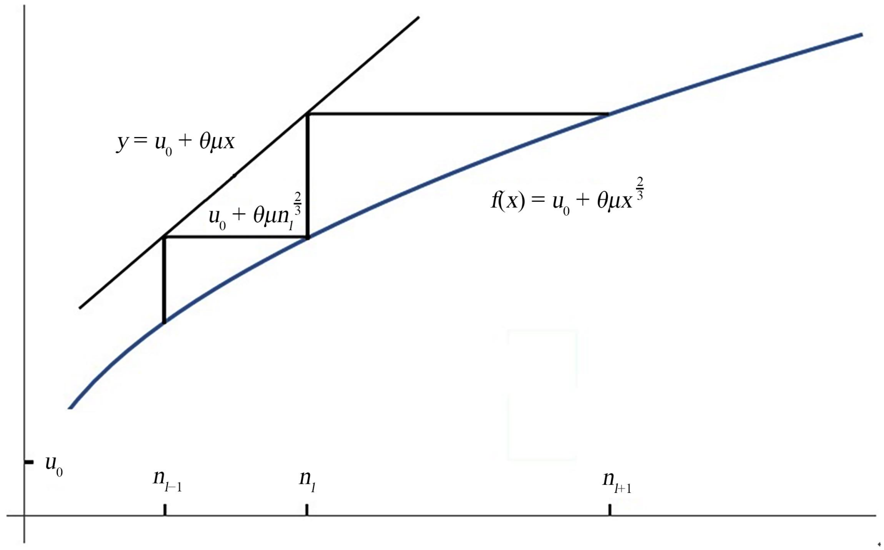

Figure 1. Linear income trend and its lower bound.

$$U_n = u + \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} X_i, \quad EY = (1+\theta)EX, \quad 0 < EX < \infty \tag{4.1}$$

where $X_i \sim X, Y_i \sim Y$ are nonnegative independent random variables.

Theorem 4.1. Suppose $\exists r \neq 0$ such that $Ee^{r(X-Y)} = 1$.

Then $r > 0$ and

$$P(\text{Ultimate Ruin}) \leq e^{-ru}. \tag{4.2}$$

Proof. Let $W_i = X_i - Y_i$ so $W \sim X - Y$ and $EW = -\theta EX < 0$. By Jensen's inequality for $\varphi(x) = e^{rx}$ we have $e^{rEW} \leq Ee^{rW}$ and therefore r must be positive. For any $a, b > 0$ and $S_n = \sum_{n=1}^{n} W_i$ we have

$$P_{a,b} = P\left(\exists n\ S_n \geq a \text{ and } \min_{1\leq i\leq n} S_i > -b\right) = P(S_n \text{ crosses } a \text{ before crossing } -b)$$

and

$$P_{b,a} = P\left(\exists n\ S_n \leq -b \text{ and } \max_{1\leq i\leq n} S_i < a\right) = P(S_n \text{ crosses } -b \text{ before crossing } a)$$

Define a stopping time N by

$$N = \min\{n \mid S_n \geq a \text{ or } S_n \leq -b\} = \text{smallest } n \text{ s.t. } S_n \text{ exits the interval } (a,b)$$

with $N = \infty$ in the case no such n exists.

Then $M_n = e^{rS_n}$ is a martingale as

$$\begin{aligned} E\left[e^{rS_n} \mid W_1, W_2, \cdots, W_{n-1}\right] &= E\left[e^{rS_{n-1} + rW_n} \mid W_1, W_2, \cdots, W_{n-1}\right] \\ &= e^{rS_{n-1}} E\left[e^{rW_n} \mid W_1, W_2, \cdots, W_{n-1}\right] \\ &= M_{n-1} \end{aligned}$$

where the conditional expectation becomes expectation, due to indepedence of (W_i) and equals 1 by assumption. It is standard to check that $P(N < \infty) = 1$ ([12]) whence $EM_N = EM_n = 1$. Now

$$1 = E\left[e^{rS_n} \mid S_N \ge a\right]P_{a,b} + E\left[e^{rS_N} \mid S_N \le -b\right]P_{b,a} \ \ge e^{ra}P_{a,b}$$

giving

$$P_{a,b} \le e^{-ra} \tag{4.3}$$

By taking $b = k, k = 1, 2, \cdots$

$$P_{a,k} = P(S_n \text{ crosses } a \text{ before crossing } -k) = P\left(\bigcup_{n=1}^{\infty}\left\{S_n \ge a \cap \min_{1\le i\le n} S_i > -k\right\}\right)$$

and setting

$$C_{n,k} = \bigcup_{n=1}^{\infty}\left\{S_n \ge a \cap \min_{1\le i\le n} S_i > -k\right\} \equiv \bigcup_{n=1}^{\infty} A_n \cap B_{n,k}, \quad B_{n,k} \subset B_{n,k+1}$$

it follows that $C_{n,k}$ is increasing in *k*. Consequently, by continuity of $P(\cdot)$ for monotone sequences

$$\begin{aligned}\lim_{k\to\infty} P(C_{n,k}) &= P\left(\bigcup_{k=1}^{\infty} C_{n,k}\right) = P\left(\bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} A_n \cap B_{n,k}\right)\\ &= P\left(\bigcup_{n=1}^{\infty} A_n \cap \left(\bigcup_{k=1}^{\infty} B_{n,k}\right)\right) = P\left(\bigcup_{n=1}^{\infty} A_n \cap \left\{\min_{1\le i\le n} S_i > -\infty\right\}\right)\\ &= P\left(\bigcup_{n=1}^{\infty} A_n\right), \quad \text{thanks to } P\left(\min_{1\le i\le n} S_i > -\infty\right) = 1\end{aligned}$$

Consequently by (4.3)

$$\begin{aligned}\lim_{k\to\infty} P_{n,k} &= \lim_{k\to\infty} P(C_{n,k}) = P\left(\bigcup_{n=1}^{\infty} A_n\right) = P(\exists\, n\ S_n \ge a)\\ &= P(S_n \text{ ever crosses } a) \le e^{-ra}\end{aligned}$$

Finally by (4.3) with $a = u$

$$\begin{aligned}P(\text{Ultimate Ruin}) &= P(\exists\, n\ U_n < 0) \le P(\exists\, n\ U_n \le 0) = P(\exists\, n\ u \le S_n)\\ &= P(S_n \text{ ever crosses } u) \le e^{-ru}\end{aligned}$$

Example 4.1 (Exponential case). Let the claim size $X \sim$ exponential with mean μ and premium size $Y \sim$ exponential with mean $\lambda = (1+\theta)\mu$. Then the condition

$$Ee^{r(X-Y)} = 1 \tag{4.4}$$

in terms of the moment generating function is as follows

$$M_{X-Y}(r) = \frac{\frac{1}{\mu}}{\frac{1}{\mu} - r} \cdot \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + r} = 1.$$

Solving for r we get

$$r = \frac{\theta}{(1+\theta)\mu} \pm \frac{\sqrt{(2+\theta)^2 - \left(\frac{4}{\mu}\right)^2}}{(1+\theta)\mu} \tag{4.5}$$

Some comments regarding (4.5) are in order and we collect them in the following.

Remark. For solutions r to be well defined and positive, some conditions must be satisfied as follows.

1) $(2+\theta)^2 - \left(\frac{4}{\mu}\right)^2 \geq 0$,

which is always satisfied for $\mu \geq 1$, whereas $0 < \mu < 1$, $\theta \geq 2\left(\frac{1}{\mu} - 1\right) > 0$.

2) r^+ is always a positive solution, however r^- can also be solution if $0 < r^-$ and $2\left(\frac{1}{\mu} - 1\right) \leq \theta \leq \frac{1}{\mu^2} - 1$ with $0 < \mu < 1$.

Consequently $r = r^+$ if $\theta > \frac{1}{\mu^2} - 1$ and $r = r^-$ if $2\left(\frac{1}{\mu} - 1\right) \leq \theta \leq \frac{1}{\mu^2} - 1$.

Example 4.2 (Binomial case). For claim and premium $X \sim Bin(p_X, k)$, $Y \sim Bin(p_Y, k)$ with $0 < p_X < p_Y$, $\theta = \frac{EY - EX}{EX} = \frac{p_Y}{p_X} - 1$. Then condition (4.4)

$$Ee^{r(X-Y)} = 1 \tag{4.6}$$

in terms of the moment generating function reads

$$r = \ln\left[\frac{(1-p_X)p_Y}{(1-p_Y)p_X}\right]$$

and (4.2) has the form

$$P(\text{Ultimate Ruin}) \leq e^{-ru} = \left[\frac{(1-p_Y)p_X}{(1-p_X)p_Y}\right]^u$$

Notice that the assumption $0 < p_X < p_Y$ gives $\alpha \equiv \frac{(1-p_Y)p_X}{(1-p_X)p_Y} < 1$.

For example, $p_X = 0.5, p_Y = 0.67$ gives $\alpha = 0.492$ whence

$$P(\text{Ultimate Ruin with initial capital } u) \leq \left(\frac{1}{2}\right)^u$$

Namely, every extra dollar of initial capital halves the probability of the *Ultimate Ruin*!

5. Probability of Ruin via Large Deviation Principle

This section is concerned with the derivation of the upper bound for the probability of ruin on the interval $[n, \infty)$, which we will refer to as *Tail Ruin* probability. Our arguments are based on the *rate function*—a key ingredient of the Large Deviation Principle, so for the sake of completeness we recall some relevant facts.

Large deviation results show that probabilities of atypical events A_n, away from typical events, all off to zero at an exponential rate. That is, $P(A_n) \sim \exp(-\alpha n)$ for large n where the constant $\alpha > 0$ is directly computable. One of the first and important rates is concerned with the Law of Large Numbers and states $P\left(\frac{X_1 + \cdots + X_n}{n} \in A\right) \sim e^{-\alpha n}$, whenever $EX_i = \mu \notin A$.

A large deviation result we need is attributed to Crámer and stated below without proof.

Large Deviation (Th. I.4, [13]).

Let (X_i) be *i.i.d.* with the moment generating function

$M_{X_1}(t) = E\left(e^{tX_1}\right) < \infty$, $t \in \mathbb{R}$, and $S_n = \sum_{i=1}^{n} X_i$. Then for any $a > EX_1$

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n \geq an) = I(a)$$

and rate function

$$I(a) = \sup_{t\in\mathbb{R}} \left[at - \log M_{X_1}(t) \right].$$

Remark 5.1. The above result has a straightforward extension $M_X(t) < \infty$ for t from some subset of $\mathbb{R}$. Key properties of the rate function $I(\cdot)$ are as follows: $0 \leq I(\cdot)$, $I(\cdot)$ is convex, $I(EX_i) = I(\mu) = 0$, $I(x)$ may assume $+\infty$, $I(x)\searrow$ for $x < \mu$, $I(x)\nearrow$ for $x > \mu$. $I(x)$ is a convex conjugate or Legendre Transform of the convex function $\ln M_X(t)$.

Lemma 5.1 (upper bound). Assume $M_X(t) = E\left(e^{tX}\right) < \infty$, $t \geq 0$. Then for a $x > \mu = EX$

$$P(S_n \geq nx) \leq e^{-nI(x)} \tag{5.1}$$

Proof. By Markov's inequality

$$P(S_n \geq nx) = P\left(e^{tS_n - txn} \geq 1\right) \leq Ee^{tS_n - txn} = e^{-txn}\left(M_X(t)\right)^n = e^{-n(xt - \log M_X(t))}. \tag{5.2}$$

Since t is arbitrary one can optimize this upper bound by maximizing the function $h(t) = xt - \ln M_X(t)$ over t. We have

$$h'(t) = x - \left.\frac{M'_X(t)}{M_X(t)}\right|_{t=0} = x - \mu > 0,$$

and therefore $h(t) > 0$ in some vicinity of t = 0, because $h(0) = 0$. This in turn, since $h(t)$ is concave down, shows that $h(t)$ has a unique strictly positive maxim um, which can be readily obtained by solving $x - \frac{M'_X(t)}{M_X(t)} = 0$ for some $t = t(x)$, whence $\max_t \left[xt - \ln M_X(t) \right] = I(x)$.

Remark 5.2 We would like to point out and emphasize the often overlooked draw-back of the probability of ultimate ruin, which stems from that fact that it does not provide any information as to when the actual ruin occur during the time interval $[0,\infty)$. For this very reason, our theorem below fills this gap and sheds some light on the time window where the ruin is most likely to occur.

Theorem 5.1 Let $U_n = u + \theta\mu n - S_n$, where $S_n = Z_1 + Z_2 + \cdots + Z_n$, and Z_i are *i.i.d.* with $EZ_1 = 0$. Then we have the following upper bounds for the probability of ruin

$$P\left(\text{Ruin in } [m,n]\right) \leq e^{-mI\left(\theta\mu + \frac{u}{n}\right)} \left(\frac{1 - e^{(n-m+1)I\left(\theta\mu + \frac{u}{n}\right)}}{1 - e^{-I\left(\theta\mu + \frac{u}{n}\right)}} \right) \tag{5.3}$$

Proof. We have

$$P(\text{Ruin in }[m,n])$$
$$\le P\left(\exists\ m\le k\le n \mid u+\theta\mu k-S_k<0\right)=P\left(\cup_{k=m}^{n}\{u+k\theta\mu<S_k\}\right)$$
$$\le\sum_{k=m}^{n}P(u+k\theta\mu<S_k)\le\sum_{k=m}^{n}\mathrm{e}^{-kI\left(\theta\mu+\frac{u}{k}\right)}\le\sum_{k=m}^{n}\mathrm{e}^{-kI\left(\theta\mu+\frac{u}{n}\right)}$$
$$=\mathrm{e}^{-mI\left(\theta\mu+\frac{u}{n}\right)}\left(\frac{1-\mathrm{e}^{(n-m+1)I\left(\theta\mu+\frac{u}{n}\right)}}{1-\mathrm{e}^{-I\left(\theta\mu+\frac{u}{n}\right)}}\right)$$

In the example below we will illustrate how the upper bounds (5.3) can be used to estimate the probability of ruin in a $[0,m-1]$, when an upper bound for the probability of ultimate ruin is available.

Example 5.1 Consider our previous Example 4.1 where the claim size X is exponential with mean μ and the premium size Y is exponential with mean $(1+\theta)\mu$.

Then

$$P(\text{Ultimate Ruin})=P(\text{Ruin in }[0,\infty))\le\mathrm{e}^{-ru},\quad r=\frac{\theta}{(1+\theta)\mu}\pm\frac{\sqrt{(2+\theta)^2-\left(\frac{4}{\mu}\right)^2}}{(1+\theta)\mu}. \tag{5.4}$$

Furthermore,

$$M_Z(t)=M_{X-Y+\theta\mu}(t)=\mathrm{e}^{t\theta\mu}M_{X-Y}(t)=\mathrm{e}^{t\theta\mu}\frac{\frac{1}{\mu}}{\frac{1}{\mu}-t}\cdot\frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+t}$$

and for $x=\theta\mu+\frac{u}{n}$ as in (5.2)

$$\max_t\left[xt-\ln M_Z(t)\right]=\max_t\left[\left(\theta\mu+\frac{u}{n}\right)t-\theta\mu t-\ln M_{X-Y}(t)\right]$$
$$=\max_t\left[\frac{u}{n}t-\ln M_{X-Y}(t)\right]=I\left(\frac{u}{n}\right)$$

Therefore, $P(S_k>u+k\theta\mu)=P\left(S_k>k\left(\theta\mu+\frac{u}{k}\right)\right)\le\mathrm{e}^{-kI\left(\frac{u}{k}\right)}$ and gives

$$P(\text{Ruin in }[m,n])\le\mathrm{e}^{-mI\left(\frac{u}{n}\right)}\left(\frac{1-\mathrm{e}^{(n-m+1)I\left(\frac{u}{n}\right)}}{1-\mathrm{e}^{-I\left(\frac{u}{n}\right)}}\right),\quad P(\text{Ruin in }[m,\infty))\le\frac{\mathrm{e}^{-mI\left(\frac{u}{n}\right)}}{1-\mathrm{e}^{-I\left(\frac{u}{n}\right)}}. \tag{5.5}$$

Let us choose $\mu=25,\theta=0.2,u=75$. Then by (5.5)

$$P(\text{Ultimate Ruin})\le\mathrm{e}^{-ru}=\mathrm{e}^{-0.079805\times75}=0.002515$$

or $\frac{1}{4}$ of 1%.

On the other hand

$$P(\text{Ruin between 5th and 10th year})$$
$$= P(\text{Ruin} \in [1825, 3650]) \le \frac{e^{-mI\left(\frac{u}{n}\right)}}{1-e^{-I\left(\frac{u}{n}\right)}} = \frac{e^{-1825\times 0.0083675}}{1-e^{0.0083675}} = 0.000028$$

which is 100 fold smaller than the probability of *Ultimate Ruin*, thus negligible in comparison!

A word about why $I\left(\frac{u}{n}\right) = I\left(\frac{75}{3650}\right) = 0.0083675$ is in order. Given

$$x = \frac{75}{3650}$$

$$\max_t [h(t)] = \max_t \left[xt - \log\left(\frac{\frac{1}{25}}{\frac{1}{25} - t}\right) - \log\left(\frac{\frac{1}{30}}{\frac{1}{30} + t}\right) \right] = I\left(\frac{75}{3650}\right) = 0.0083675$$

was obtained numerically. We include our graph of concave down *h(t)* below. (See **Figure 2**)

Similarly,

$$P(\text{Ruin between 10th and 20-ties year})$$
$$= P(\text{Ruin} \in [3650, 7300]) \le \frac{e^{-mI\left(\frac{u}{n}\right)}}{1-e^{-I\left(\frac{u}{n}\right)}} = \frac{e^{-3650\times 0.008333}}{1-e^{0.0083333}} = 7.443\times 10^{-12}$$

is negligibly small and can be dropped. By comparing the order of smallness of the respective probabilities we infer that, if the ruin occurs, it will most likely happen within the first five years.

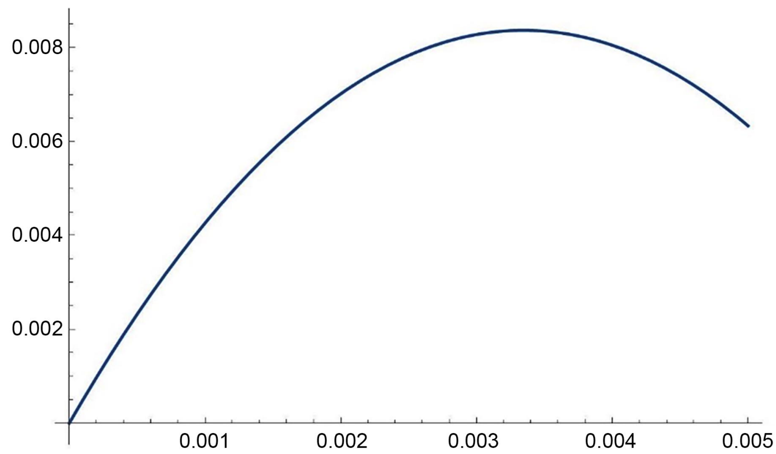

Figure 2. The graph of $h(t) = xt - \log\left(\frac{\frac{1}{25}}{\frac{1}{25} - t}\right) - \log\left(\frac{\frac{1}{30}}{\frac{1}{30} + t}\right)$.

6. Conclusions

We have introduced a discrete time risk model that features a convenient way of maintaining end of the day net balance of company's capital reserve, resulting from the random size premiums income minus the incoming random size claims on the daily basis. Three different methods of estimating the probability of ruin (*i.e.*, negative capital reserve) were presented and illustrated by examples. The key innovation is a reduction of complexity associated with randomness of the Risk Process by modeling random premiums and random claims arriving at discrete deterministic times in our model, as opposed to random claims arriving at random times according to Poisson process in the Crámer-Lundberg model studied in the literature.

Future research will focus on extending the model to allowing investment of the collected premiums into stock market equities.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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