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An Approach of Price Process, Risk Measures and European Option Pricing Taking into Account the Rating

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Abstract

In this paper, by taking into account the rating in a new concept of economic space, we propose a model of the dynamics of an economic particle, a model of price process, an extension of risk measures, and a new approach of option pricing with associated hedging portfolio.

Keywords

Economic Space, Economic Particle, Rating, Price Process, Risks Measures, Option Pricing

1. Introduction

Risk is inherent to all human activities. In all times, the matter is how to avoid it or how to reduce its impact. In Finance, risk can be classified in two groups: quantifiable risks and unquantifiable risks. Main quantifiable risks are market risk and credit risk. For unquantifiable risks, we can mention, among others, operational risk and legal risk. In this paper, we focus on quantifiable risks. For these risks, many approaches of modeling exist in literature. For these risks, many approaches of modeling exist in literature. Market risks models can be classified in two main groups: dispersion risk measures and capital requirement risk measures. For the first group, main contributors are Markowitz in 1952 [1], Sharpe in 1963 [2], Konno and Yamazaki in 1991 [3] and Hamza and Janssen in 1995 [4]. Capital requirement risk measures have been studied amongst others by Artzner *et al.* in 1999 [5], Föllmer and Shied in 2002 [6], Rockafellar *et al.* in 2002 [7] and Detlefsen and Scandalo in 2006 [8]. In the field of credit risk models, there are two main approaches. Structural models in which the default is an endogenous process linked to the value of firm have been studied amongst others by Merton in 1974 [9], Black and Cox in 1976 [10], Longstaff and Schwartz (1995) [11] and Hsu *et al.* (2004) [12]. Reduced form models where the default can occur at any moment as it is defined as the first jump of some stochastic process have been studied amongst others by Jarrow *et al.* (1992) [13], Jarrow *et al.* (1995) [14] and Duffie *et al.* in 1999 [15]. After the step of modeling, the problem is to control or to mitigate the impact of risk using appropriate financial instruments. The interested reader can consult [16] to have more information on these instruments. In this paper, we focus on market risk mitigation. Let's mention that portfolio selection problem constitutes an important aspect of market risk mitigation and is intensively studied. Recent developments in this field have been made by Bo *et al.* (2015) [17] and Lin *et al.* (2017) [18]. To the best of our knowledge, there is up to now no work on market risk models and option pricing problem which takes explicitly into account the rating of the underlying's issuer. For instance, if the same asset is issued by two economic agents with different ratings, it is not fair to consider that these assets must have the same behavior, because the way each economic agent manages its firm should have significant impact on the evolution of its asset price's process. The rating of an economic agent acts as a measure of its health. It expresses the future capacity to respect its engagements. It was firstly used in 1860 by Henry Varnum Poor in USA to measure the quality of debts issued by railways companies. First agencies of rating appear in 1909 for Moody's and 1910 for Standard Statistics and Poor's. In 1941, Standard Statistic merges with Poor's to form Standard and Poor's. For more information about rating, we refer to the paper of Lawrence [19]. Ratings are available for banks, states, municipalities, corporations, investment funds, pension funds and insurance companies. These ratings act as decision aid for investors who have excess funds needed by these entities. A rating is a string of letters, for example those given by Standard and Poor's have the forms AAA, AA, A and BBB for investment grades; BB, B and CCC which are speculative grades for entities with low capacity of development in the future; CC and C which are ratings for entities near to default and D for entities in default. The question is how to incorporate the rating explicitly in the price's process? This notion is not new in literature. For instance, in the model of credit risk (CreditMetrics) built by Bhatia *et al.* (1997) [20] for the bank J.P Morgan, the authors took into account the rating of the issuer of bonds in the valuation of its forward price. Hackbarth *et al.* (2004) [21] proposed a model of credit risk where the default threshold is rating dependent. However, in these approaches, the matrix of transition giving the probability of moving from one rating to another is calibrated most often for one year. This means that the possibility of instantaneous change of rating is excluded. Another drawback is that they are used only for debts instruments. Shapiro (2015) [22] and Grzegorz (2016) [23] used the notion of risk profile and the quality of credit, but not in the direction of rating approach. Victor Olkhov (2016) [24] [25] proposed a model of asset price's process incorporating effectively the issuer's rating. He supposed that rating methodologies can be extended in order to take values in $\mathbb{R}$. He also supposed that if an economic participant is subject to n risks, its rating should be an n-dimensional vector. He so defined the notion of economic agent and economic space with dimension the number of main risks present in the market. In his paper, he assumed that the evolution of an economic agent or economic particle in economic space is modeled by a Brownian motion. In this paper, we also assume as Victor Olkhov that the rating methodologies can be extended in such a way that they take their value in $\mathbb{R}$. Once this is done, we suppose that

- Each financial variable issued by an economic agent possesses a rating which is the rating of the issuer in the corresponding market. An economic agent is an entity (individual, firm, local governments or state) which has something to exchange in the market. Financial variables are bonds, obligations, goods and services, loans, debts or even work. For example a firm producing two types of goods G_1 and G_2 and issuing an obligation B possesses three rating. For a given good, rating is related on the degree of fulfilment of standard required for the class of product this good belongs to. For debt instruments, rating is related to the degree of creditworthiness of the issuer.
- Rating assumes values in $(0,\infty)\cup\{\infty\}$ with 0 as the best one, acknowledged in the present rating setting as AAA.
- The axis $(0,\infty)$ is oriented in the sense of poorest quality of the rating.
- If an economic agent is absent in a market of a specific financial variable, its rating for this variable is ∞.

Since the number of different financial variables in the market is infinite, the dimension of economic space is thus infinite and for a given economic agent, its coordinates possess just a finite number of rating different from ∞. In the new setting adopted, our first contribution in this paper is the proposition of a new approach of economic particle in economic space combined with a process modeling its dynamics. Our second contribution is to propose a new model of price process taking into account the rating. In our third contribution, we propose new measures of risk which are extensions of capital requirement risk measures, deviation risk measures, quantile-based risk measures and utility-based-risk measures incorporating the rating. The fourth contribution is the derivation of a PDE associated to European call option pricing based on a single asset that incorporates the rating. Using the Feynman-Kac formula, we obtain a closed form solution and we also deduce the hedging portfolio. Our approach of option pricing differs from which proposed by Black and Scholes (1973) in their seminal work [26] by the fact that the payoff is not a function of the underlying's price only, but also of the underlying's rating.

The remainder of the paper is organized as follows. In section 2, we present the notion of economic space and dynamics of economic particle. In Section 3, given an asset, we propose a model of price process that incorporates its rating. In Section 4, by taking into account the notion of rating, we propose extensions of capital requirement risk measures, deviation risk measures, quantile-based risk measures and utility-based-risk measures. In Section 5, we propose a pricing formula of European call option incorporating the rating of the underlying as well as the hedging portfolio associated. In Section 6, we conclude the paper and give perspectives pertaining to this work.

2. On Economic Space and Dynamics of Economic Agent

2.1. On the Economic Space

An economic agent is an entity (individual, firm, local government or state) which has something to exchange in the market (factors of production or goods and services). It is also the basic element of microeconomics studies. Market is understood here as a place (physical or virtual) where trade are made or where there is a confrontation between supply and demand. Up to 2016, the notion of space as defined in Physics was absent in Economy. Victor Olkhov, (2016) [24] after the assumption that rating methodologies can be extended in such a way that rating can assume values in $\mathbb{R}$, defined for the first time an economic space whose dimension is the number of main risks assessed in the market. The author claimed that an economic agent also called economic particle moves in economic space by following a drifted Brownian motion. In this paper, we also assume as Victor Olkhov [24] that the rating methodologies can be extended in such a way that they take their values in $\mathbb{R}$. Furthermore, we claim that each financial variable issued by an economic agent possesses a rating which is the rating of the issuer in the corresponding market. Financial variables are bonds, obligations, goods and services, loans, debts or even work. For example a firm producing two types of goods G_1 and G_2 and issuing an obligation B possesses three rating. For a given good, rating is related on the degree of fulfilment of standard required for the class of product this good belongs to. For debt instruments, rating is related to the degree of creditworthiness of the issuer. For contracts and derivatives, the rating is that of the underlying's issuer in the corresponding underlying's market. Thus the number of ratings (coordinates) for an economic particle depends on the number of financial variables issued. Since the number of different financial variables existing in the market is infinite, the dimension of economic space is infinite. Thus every economic particle moves just in the subspace of the entire economic space. The dimension of this subspace is equal to the number of financial variables issued. We will set the number 0 as the best rating, acknowledged in the present rating setting as AAA. That is in our setting the range of rating is $(0,\infty)\cup\{\infty\}$ and the axis is oriented in the sense of poorest quality of the given financial variable. If it is debt instrument, great ratings mean bad quality of issuer's creditworthiness. But if it is a good or a service, great rating traduce the weakness of the degree of fulfilment to standard required for the class of products to which belongs this good or service. If an economic agent does'nt issue a given financial variable, then its rating for this financial variable is ∞. Otherwise, it is a positive real number.

2.2. The Dynamics of Economic Particle in the Economic Space

In this paper, we assume that economic particles move randomly in the economic space. That is, for a given economic particle, its ratings change randomly.

If an economic particle is present in n ($n \in \mathbb{N}$) different markets, its rating at time t is a vector

$$X(t) = \left(X_t^1, X_t^2, \cdots, X_t^n, \infty, \cdots, \infty\right),$$

where for each i ($i = 1, \cdots, n$),

$$X_t^i = \left(X_t^{i1}, \cdots, X_t^{iq}\right)$$

with $q \in \mathbb{N} \setminus \{0\}$ the number of financial variables issued in a specific market. For example, a given firm can issue more than one type of debt instruments. From now on, we write $X(t) = \left(X_t^1, X_t^2, \cdots, X_t^n\right)$. These ratings are influenced by quantitative and qualitative elements. Quantitative elements are related to financial characteristics of the issuer and qualitative ones are related to the standard in vigor in the market of a specific financial variable. Regulations and standards are dynamic. In the sector of bank industry, there is Basel Committee on Banking Supervision. For certain goods, there are norms (ISO9001 for example). We assume that these regulations and standards can be extended to all financial variables. The set of regulations or standards required at time t for a given financial variable is $R_e(t)$. These regulations and standards are exogenous to economic agents, that is regulators do not have something to exchange in the market and do not intervene directly in market processes. For a given financial variable we suppose that amongst the objectives of its issuer (economic particle), there is the improvement of its rating in the corresponding market.

We make the following assumptions about the variation $\Delta X_t = X_{t+\Delta t} - X_t$ of rating between t and $t + \Delta t$

1) There is a depreciation of the rating due, amongst others, to the evolution of regulations and standards and to the degradation of production's factors. Since regulations and standards are exogenous, this depreciation is measured by the quantity $\alpha(t)\Delta t$, with $\alpha(t)$ a real valued integrable function assuming positive values.

2) The internal effort to ameliorate the rating is proportional to the current one. It is measured by the term $-\lambda(t) X_t \Delta t$, with $\lambda(t)$ a real valued integrable function assuming positive values.

3) The movement of the issuer in economic space is subject to randomness measured by the term $f(t, X_t)(B_{t+\Delta t} - B_t)$ where $(B_t)_{t \geq 0}$ is the standard Brownian motion and f is a function on $\mathbb{R}_+^2$.

Considering the above assumptions, we have

$$\Delta X_t = \left(\alpha(t) - \lambda(t) X_t\right)\Delta t + f(t, X_t)\left(B_{t+\Delta t} - B_t\right). \tag{1}$$

Letting $\Delta t \to 0$, we get

$$\mathrm{d}X_t = \left(\alpha(t) - \lambda(t) X_t\right)\mathrm{d}t + f(t, X_t)\,\mathrm{d}B_t. \tag{2}$$

Remark 2.1. 1) For each financial variable existing in the market, its aggregated value $\bar{F}(t)$ in the entire economy at time t is

$$\bar{F}(t)=\int_0^{\infty}F(t,x)\mathrm{d}x,$$

where $F(t,x)$ is the total value issued by economic particles with rating x at time t.

2) In the case of portfolio P containing n ($n\in\mathbb{N}\setminus\{0\}$) financial variables $A_1,\cdots,A_n$ with issuers' ratings $X_1,\cdots,X_n$ respectively, the rating X_P is the n-dimensional vector $(X_1,\cdots,X_n)$.

3) For financial instruments such as forwards, futures and options, the rating is that of the underlying issuer.

3. A Model of Price Process Taking into Account the Rating of the Issuer

3.1. Motivations

In the pioneer work of Louis Bachelier in 1900 [27], he proposed to model price process of a given asset as a Brownian motion, but his model has a drawback because price can assume negative values. Samuelson [28] proposed a model where the logarithm of price is a Brownian motion and by this way price can only assume positive values, which is more realistic. It has been shown by Mandelbrot in 1966 [29] that the hypothesis of log-normality of price does not fit with experimental data. Other classes of model have been proposed like the Constant Elasticity of Variance Model and models incorporating jumps [30]. Due to the tractability, in the majority of problems in finance, the log-normal approach is used as a first approximation of price process dynamics. The common feature of all these models is that the rating of the asset's issuer is not taking into account. This is not too realistic because for a given financial variable, the specific risk of its issuer is not taken into account in the price's process. This omission can create important damage in the future. An investor who enters in a contract with one of the issuers can suffer a lost due to its rating deterioration. Victor Olkhov [24] proposed a model of asset's price process that incorporates the rating of the issuer. In this paper, we propose a model in which the negative impact of bad (high) rating on the asset price is taken into account.

3.2. The Model

We are given a triplet $\left(\Omega,\mathcal{F}=\bigcup_{t\in[0,T]}\mathcal{F}_t,\mathbb{P}\right)$ modeling the randomness in the market. Ω is the set of all possible states of the market, T is a positive real number, $\mathcal{F}$ is a σ-algebra of subsets of Ω, $(\mathcal{F}_t)_{t\in[0,T]}$ is a filtration and $\mathbb{P}$ is the historical probability. For a given financial variable, with price S_t at time t, Black and Scholes (1973) [26] assumed that S_t follows a log-normal process. If the market is driven by one source of randomness modeled by a standard Brownian motion $(W_t)_{t\in[0,T]}$, we have

$$S_t = S_0 \mathrm{e}^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \tag{3}$$

when there is no dividend payment. Here μ, σ and S_0 are positive real numbers and express respectively the instantaneous rate of return, the volatility and the price at time zero. This model of price doesn't take into account the rating of its issuer. Choy *et al.* in 2006 [31] underline in their work the negative impact of bad rating on the price. We propose to improve the relation (3) in the following way

$$S_t = S_0 \mathrm{e}^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t - kX_t} \tag{4}$$

where k is positive real number and X_t is the rating of the issuer. From relation (2), we have

$$\mathrm{d}X_t = \left(\alpha(t) - \lambda(t) X_t\right)\mathrm{d}t + f(t, X_t)\mathrm{d}B_t \tag{5}$$

and if Brownian motions $(W_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ are independent, relation (4) becomes

$$\begin{aligned} \mathrm{d}S_t = {} & S_t\left(\mu - k\left(\alpha(t) - \lambda(t)X_t\right) + \frac{1}{2}k^2 f^2(t, X_t)\right)\mathrm{d}t \\ & + \sigma S_t \mathrm{d}W_t - kf(t, X_t) S_t \mathrm{d}B_t. \end{aligned} \tag{6}$$

3.3. Graphical Illustration

In **Figure 1**, we illustrate the evolutions of rating and those of the same asset where one is adjusted to rating and the other is not. We can observe that the evolutions can be truly different. At every time *t*, the ratio of these prices is proportional to e^{kX_t}. This emphasizes the fact that neglecting the rating in the process of price can create a disagreement.

We have the following important lemma

Lemma 3.1. *For each finite positive real number T and a fixed* $t < T$, *the rating*

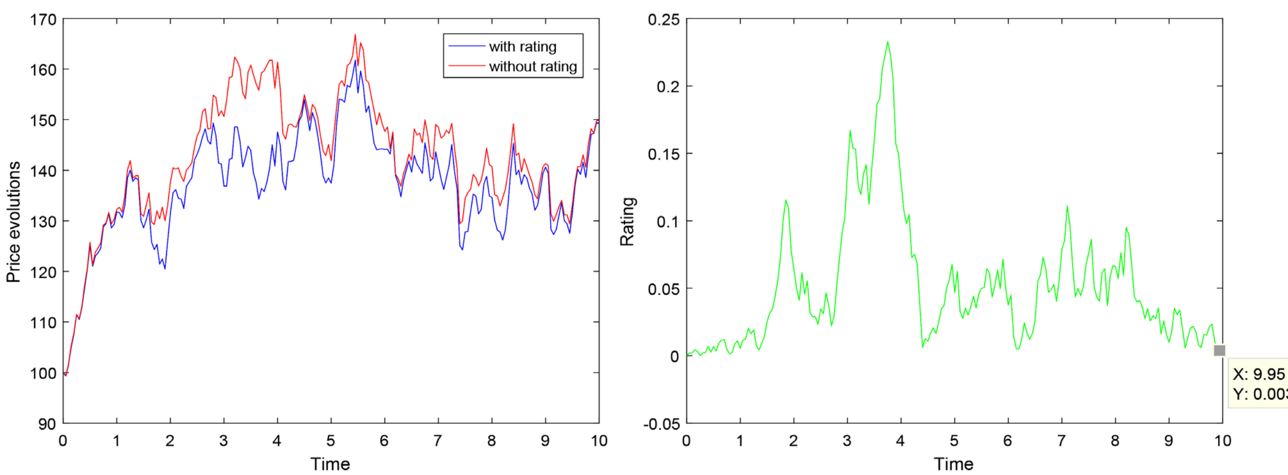

Figure 1. Prices evolution with $f(t, X_t) = \gamma\sqrt{X_t}$ and corresponding rating evolution where for all *t*, $\alpha(t) = 0.07$, $\gamma(t) = 0.2$, $\lambda(t) = 0.08$, $S_0 = 100$, $\sigma = 0.1$, $\mu = 0.05$, $x_0 = 0$ and $k = 0.7$.

X_t *for a given issuer of a financial variable is known. Furthermore, we have the relation* $\lim_{X_t\to\infty} S_t = 0$ almost surely.

To prove Lemma 3.1, we need the following Lemma known as reflection principle of the Brownian motion.

Lemma 3.2. *Let* $(W_t)_{t\in[0,T]}$ *be a standard Brownian motion and w a positive real number. Then*

$$\mathbb{P}\left[\sup_{t\le T} W_t \ge w\right] = \mathbb{P}\left[|W_t| \ge w\right].$$

Proof. Denote by T_w the first time for W_t to reach the level *w*. Then

$$\mathbb{P}\left[\sup_{t\le T} W_t \ge w\right] = \mathbb{P}\left[T_w \le T\right] = \mathbb{P}\left[T_w \le T, W_T < w\right] + \mathbb{P}\left[T_w \le T, W_T > w\right].$$

From Markov and symmetric properties of Brownian motion, we deduce that the events $\{T_w \le T, W_T < w\}$ and $\{T_w \le T, W_T > w\}$ have the same probability. This implies that

$$\mathbb{P}\left[\sup_{t\le T} W_t \ge w\right] = 2\mathbb{P}\left[T_w \le T, W_T > w\right].$$

From the inclusion of events $\{W_T > w\} \subset \{T_w \le T\}$, we deduce that

$$\mathbb{P}\left[\sup_{t\le T} W_t \ge w\right] = 2\mathbb{P}\left[W_T > w\right] = \mathbb{P}\left[|W_t| \ge w\right]. \qquad \square$$

We now give the proof of Lemma 3.1.

Proof. From relation (4) and Lemma 3.1, we have

$\forall t \in [0,T], -5\sqrt{T} \le W_t \le 5\sqrt{T}$ almost surely and we can deduce the relation

$$S_0 \mathrm{e}^{\left(\mu-\frac{1}{2}\sigma^2\right)t-k(X_t-X_0)-5\sqrt{T}} \le S_t \le S_0 \mathrm{e}^{\left(\mu-\frac{1}{2}\sigma^2\right)t-k(X_t-X_0)+5\sqrt{T}} \quad \text{almost surely.} \tag{7}$$

From squeeze Theorem, we obtain the result. □

Remark 3.1. *From Lemma* 3.1 *we deduce that, if the issuer's rating of a financial variable becomes too large or deteriorates considerably, this financial variable is worthless. No economic agent is willing to buy it.*

Up to now in risk measures treatment, to the best of our knowledge, there exists no research work that takes explicitly into account the rating of the issuer of a financial variable. This omission of the specific uncertainty coming from the issuer can entails undervaluation of the actual degree of riskiness associated to a financial instrument and consequently can explain the fact that all risk measures proposed up to now turn out to be insufficient to protect the investors in period of financial crisis against excessive losses. The approach that we propose in the next section, incorporate the rating of the issuer in the valuation of the degree of riskiness. The properties of known risk measures have been preserved in our approach. This can allow to have another approach of portfolio allocation problem.

4. An Extension of Risk Measures

In this section we propose an extension of risk measure in static case. Trading is made at time $t = 0$ and liquidation at time *T*. Firstly, let's define a penalized (or adjusted-to-rating) future wealth or future payoff of a financial instrument. For a given financial variable issued by an economic particle with initial rating X_0 in the corresponding market. We denote by $Y(X)$ its adjusted-to-rating future payoff; where *X* is its random future rating in the corresponding market. We also denote by *Y* the future payoff of the same financial variable if it was issued by an economic particle with rating 0. If S_t is the price at time *t* of this financial variable, from relation (4) we have $S_t = e^{-kX_t} S_t^*$, where S_t^* is the price of the same financial variable if it was issued by an economic particle with rating 0. We have the relation

$$(1+R) = e^{-k(X_T - X_0)}(1+R_0), \tag{8}$$

where R_0 and *R* are respectively the total return of investment on the financial variable issued by economic particle with rating 0 and initial rating X_0. By taking the logarithm in relation (8), we have

$$\ln[(1+R)] = \ln[(1+R_0)] - k(X_T - X_0). \tag{9}$$

The relation (9) agree with works of Ee in 2008 [32], Jorion and Zhang in 2007 [33], Linciano in 2004 [34] and Ditchev and Piotroski in 2001 [35]. These authors emphasize on the negative impact of bad rating on returns. To obtain $Y(X)$, the idea is to penalize the future wealth *Y*. The penalization depends on the sign of $X - X_0$, by subtracting a quantity $\left(\chi_{\{X \le X_0\}}\varrho_1 + \chi_{\{X > X_0\}}\varrho_2\right)p(X - X_0)$ from *Y*, where *p* is an increasing function with $p(0) = 0$, χ is the characteristic function and ϱ_1, ϱ_2 are positive real number such that $\varrho_1 < \varrho_2$ and reflecting the different impact of rating downgrade and rating upgrade as stated by Choy *et al* in 2006 [31].

In this way, the adjusted-to-rating future payoff is less than *Y* if $X > X_0$ and greater than *Y* if $X < X_0$. In the extreme case where *X* is too large, the future payoff can be equal to zero an even less than zero.

Definition 4.1. *The adjusted-to-rating future payoff of a financial variable issued by an economic particle with initial rating* X_0 *in the corresponding market, is given by the relation*

$$Y(X) = Y - \left(\chi_{\{X \le X_0\}}\varrho_1 + \chi_{\{X > X_0\}}\varrho_2\right)p(X - X_0) \tag{10}$$

where p, χ, ϱ_1 and ϱ_2 are defined as above.

Remark 4.1. *The adjusted-to-rating future payoff is equal to Y if* $X = X_0$. *This can be interpreted by the fact that in this case, there is no uncertainty coming from the rating of the issuer. We recover by this way the case where the rating is not taken into account. For the investor, the difference is just on the amount invested at the beginning of trade.*

4.1. Scenario-Based Risk Measures

Before giving an extension of risk measure in scenario-based risk measures, let's recall the definition of coherent risk measure given by Artzner *et al.* [5].

A map $\rho: L^{\infty}(\Omega) \to \mathbb{R}$ is a coherent risk measure if it satisfies the following axioms

- Subadditivity: $\rho(X+Y) \le \rho(X) + \rho(Y), \forall X, Y \in L^{\infty}(\Omega)$.
- Positive homogeneity: if $\lambda \ge 0$, then $\rho(\lambda X) = \lambda \rho(X)$.
- Monotonicity: if $X \le Y$, then $\rho(X) \ge \rho(Y)$.
- Translation invariance: if $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) - m$.

Here $L^{\infty}(\Omega)$ is the set of essentially bounded real value random variables defined on Ω. Given a financial variable issued by an economic particle with initial rating X_0, we defined the associated adjusted-to-rating future wealth $Y(X)$ by the relation (10). In the case of capital requirement risk measures or scenario-based risk measures, a good candidate for $\left(\chi_{\{X \le X_0\}} \varrho_1 + \chi_{\{X > X_0\}} \varrho_2\right) p(X - X_0)$ is a premium to be put aside or invested in a safe way together with the given financial variable in order to rule out possible loss coming from the risk bearing by the rating of the issuer. At the end, if there is an improvement in the rating of the issuer, this amount will be an additional gain, otherwise it will be a buffer reducing potential losses. Given a scenario-based risk measure ρ and a financial variable with adjusted-to-rating future wealth $Y(X)$, we defined the level of riskiness of $Y(X)$ by applying ρ to (10).

Remark 4.2. 1) $\rho(Y(X)) > \rho(Y)$ if almost surely $X > X_0$. This means that bad ratings have positive impact on the level of riskiness.

2) $\rho(Y(X)) < \rho(Y)$ if almost surely $X < X_0$. This means that good ratings have negative impact on the level of riskiness.

3) $\rho(Y(X)) = \rho(Y)$ if almost surely $X = X_0$.

Definition 4.2. *For a risk measure* ρ, *the acceptance set associated is defined by*

$$\mathcal{A} = \left\{ Y \in L^{\infty}(\Omega) \mid \rho(Y) \le 0 \right\}.$$

Remark 4.3. *The acceptability can depend on supervisors like*:

1) The regulator who takes into account the unfavorable states when allowing a risky position which may call for the resources of the government, for example as a guarantor of last resort.

2) The investment manager who gives his portfolio to a trader.

3) The exchange's clearing firm that has to secure transactions between all parties.

Theorem 4.1. *Given a risk measure and its acceptance set* $\mathcal{A}$, *let* $Y(X)$ *be the adjusted-to-rating future wealth associated to a financial variable issued by an economic particle with initial rating* X_0 *and Y the future wealth if it is issued by an economic particle with rating* 0. *Let's suppose that* $Y \in \mathcal{A}$.

1) If $X \le X_0$ almost surely, then $Y(X) \in \mathcal{A}$.

2) If $X > X_0$ almost surely, then $Y(X) \in \mathcal{A}$ if and only if $Y - \varrho_2 p(X - X_0) \in \mathcal{A}$.

Proof. Since risk measure as defined in [5] is static, the premium

$\left(\chi_{\{X\le X_0\}}\varrho_1+\chi_{\{X>X_0\}}\varrho_2\right)p\left(X-X_0\right)$ to be put aside or invested in a safe way together with the given financial variable in order to rule out possible loss coming from the risk bearing by the rating of the issuer, is bounded. In fact this amount cannot exceed the initial investment Y if the financial variable was issued by an economic agent rated 0 (AAA). That is if $Y\in L^\infty(\Omega)$, then $Y(X)\in L^\infty(\Omega)$ too.

1) If $X\le X_0$ almost surely, then from the definitions of the function p and $Y(X)$, we have

$$Y(X)=Y-\varrho_1 p(X-X_0)\ge Y.$$

Using the monotonicity of ρ, we conclude that $\rho(Y(X))\le\rho(Y)\le 0$.

2) If $X>X_0$ almost surely, then from the definitions of p and $Y(X)$, we have

$$Y(X)=Y-\varrho_2 p(X-X_0)$$

and the conclusion follows. □

Theorem 4.2. *The properties of* ρ, *namely subadditivity, positivity, monotonicity, translation invariance* (*and convexity if applicable*) *are preserved.*

Proof. It is obvious. □

Let's now talk about representation theorem. For coherent risk measures, Freddy Delbaen [36] proved that under mild continuity assumptions, they can be represented as worst expected loss with respect to a given set of probability models. In our setting, we have the following representation result. □

Theorem 4.3. *Let* $\rho: L^\infty(\Omega)\to\mathbb{R}$ *be a coherent risk measure. There exists a closed convex set* $\mathcal{P}$ *of P-absolutely continuous probability measures such that*

$$\rho(Y(X))=\sup_{Q\in\mathcal{P}}\mathbb{E}_{\mathbb{Q}}\left[-Y+\left(\chi_{\{X\le X_0\}}\varrho_1+\chi_{\{X>X_0\}}\varrho_2\right)p\left(X-X_0\right)\right],\ \forall Y\in L^\infty(\Omega).$$

E_Q is the expectation under Q. This characterization generalizes the earliest one given by Artzner *et al.* [5] in the case of finite sample spaces.

Proof. Replacing X by $Y(X)$ in [36], Theorem 2, we obtain the expected result.

The law invariance property is also preserved because the motion of an economic particle in economic space follows an Ornstein-Ulhenbeck process with specific characteristics known at every time.

The definition we gave above is for static risk measures. It doesn't take into account additional information that can be granted during the period of trading. Evaluation is done only one time; it's the reason why they are also called single period risk assessment. Cvitanić and Karatzas [37] proposed an approach of multi-period risk assessment that follows the first one proposed by Hakansson [38] as an improvement of Harry Markowitz's model. Other approaches were provided by several authors such as Detlefsen and Scandalo [8], Föllmer and Penner [39] and Bion-Nadal Jocelyne [40]. These authors expressed risk as capital requirement. Detlefsen and Scandalo [8] proposed a suitable way to assess periodically the riskiness of a given financial variable incorporating available additional information. They defined it as conditional risk measure and they provided a robust representation of this class of risk measures under the assumptions of continuity from above. A dynamic risk measure is a family of conditional risk measures. Given a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ modeling the available additional information, let's define the subspace

$$L_{\mathcal{G}}^{\infty} = \left\{ Y \in L^{\infty}(\Omega) \mid Y \text{ is } \mathcal{G}\text{-measurable} \right\}.$$

In their paper, Detlefsen and Scandalo [8] defined a conditional convex risk measure as follows. A function $\rho : L^{\infty}(\Omega) \to L_{\mathcal{G}}^{\infty}$ is a conditional convex risk measure if it satisfies the following relations:

1) Translation invariance. For any $X \in L^{\infty}(\Omega)$ and any Z in $L_{\mathcal{G}}^{\infty}$,

$$\rho(X + Z) = \rho(X) - Z.$$

2) Monotonicity. For any X and Y in $L^{\infty}(\Omega)$:

$$X \le Y \Rightarrow \rho(X) \ge \rho(Y).$$

3) Conditional convexity. For any X and Y in $L^{\infty}(\Omega)$ and Λ in $L_{\mathcal{G}}^{\infty}$ with $0 \le \Lambda \le 1$

$$\rho(\Lambda X + (1-\Lambda)Y) \le \Lambda \rho(X) + (1-\Lambda)\rho(Y).$$

4) $\rho(0) = 0$.

We obtain the conditional measure of risk adjusted-to-rating by applying ρ to $Y(X)$. In our setting, the representation provided by Detlefsen and Scandalo [8] is expressed as follows.

Theorem 4.4. *Let* $\rho : L^{\infty}(\Omega) \to L_{\mathcal{G}}^{\infty}$ *be a conditional convex risk measure which admits a dual representation. When* ρ *is applied to penalized future wealth, it remains representable and*

$$\rho(Y(X)) = \text{ess.sup}_{Q \in \mathcal{P}_{\mathcal{G}}} \left\{ -E_Q \left[Y - \left(\chi_{\{X \le X_0\}} \varrho_1 + \chi_{\{X > X_0\}} \varrho_2 \right) p(X - X_0) \mid \mathcal{G} \right] - \alpha(Q) \right\},$$

where $\mathcal{P}_{\mathcal{G}} = \left\{ Q \mid Q \text{ is absolutly continuous with respect to } P \text{ and } Q = P \text{ in } \mathcal{G} \right\}$, α is a (random) penalty function.

Proof. Replacing X by $Y(X)$ in the proof given by Detlefsen and Scandalo [8], Theorem 1, the expected result is obtained. □

4.2. Deviation Risk Measures

In the category of deviation risk measures we can cite, amongst others, variance and standard deviation of the return introduced by Markowitz [1] and the combination of semi-variance of return introduced by Hamza and Janssen [4]. In the case of deviation risk measure, p can be defined as follows

1) If $X = X_0$ almost surely, $p(X - X_0) = 0$ almost surely. It means that there is no risk of loss coming from the rating of the issuer.

2) If $X \ne X_0$ almost surely, then $\left(\chi_{\{X \le X_0\}} \varrho_1 + \chi_{\{X > X_0\}} \varrho_2 \right) p(X - X_0)$ is the random potential loss or gain on return which can be granted due to the rating's variation of the issuer.

If ρ is a given deviation risk measure, the idea here is the same as before, to apply ρ to adjusted-to-rating future return.

4.3. Quantile-Based Risk Measures

In this class of risk measures, we are concerned with the distribution of future losses. Given a financial variable, issued by an economic particle with initial rating X_0, its adjusted-to-rating future loss $Y(X)$ is given by the relation

$$Y(X) = Y - \left(\chi_{\{X \le X_0\}}\varrho_1 + \chi_{\{X > X_0\}}\varrho_2\right) p(X - X_0).$$

Here $\left(\chi_{\{X \le X_0\}}\varrho_1 + \chi_{\{X > X_0\}}\varrho_2\right) p(X - X_0)$ is the potential loss or gain coming from the change in the rating of the issuer. Quantile-based risk measures include Value at Risk (VaR), expected shortfall (ES), tail conditional expectation (TCE) and conditional value at risk (CVaR). In the existing literature, these risk measures don't take explicitly into account the rating of the issuer. Our idea here is also to apply each of these risk measures to adjusted-to-rating future loss. This allows to incorporate the rating of the issuer in the assessment of the degree of riskiness of the issued financial instrument.

The example of VaR stands as follows. Given a financial instrument issued by an economic agent with initial rating X_0, by applying the VaR on the adjusted-to-rating future loss $Y(X)$, we obtain the relation

$$VaR_\beta\left(Y(X)\right) = VaR_\beta\left(Y - \left(\chi_{\{X \le X_0\}}\varrho_1 + \chi_{\{X > X_0\}}\varrho_2\right) p(X - X_0)\right) = F_{Y(X)}^{-1}(\beta),$$

where $\beta \in (0,1)$ is the degree of confidence and $F_{Y(X)}$ is the cumulative distribution function of $Y(X)$.

4.4. Utility-Based Risk Measures

In this category of risk measures, used mainly in the insurance industry, p has the same interpretation as in scenario-based risk measures. Namely p is a premium to be put aside or invested in a safe way together with the given financial instrument in order to rule out possible loss coming from the adverse change in the rating of the issuer. At the end of period, if there is an improvement in the rating of the issuer, this amount will be an additional gain. Otherwise it will be a buffer reducing potential losses. The idea here is also to apply to adjusted-to-rating future wealth the utility function of the regulator.

For example, given a financial variable issued by an economic agent with initial rating X_0, its adjusted-to-rating future payoff $Y(X)$ is defined by

$$Y(X) = Y - \left(\chi_{\{X \le X_0\}}\varrho_1 + \chi_{\{X > X_0\}}\varrho_2\right) p(X - X_0).$$

If the utility function of regulators is in the form $u(y) = -\mathrm{e}^{-\lambda y}$, where λ is the degree of risk aversion, then the measure of risk obtained is

$$\rho\left(Y(X)\right) = \frac{1}{\lambda}\left(\ln\left(E\left[\mathrm{e}^{\lambda\left(\chi_{\{X \le X_0\}}\varrho_1 + \chi_{\{X > X_0\}}\varrho_2\right)p(X - X_0)}\right]\right) + \ln\left(E\left[\mathrm{e}^{-\lambda Y}\right]\right)\right).$$

5. European Option Pricing in Economic Space

In Section 2, we introduced the notion of economic space and economic particle. In Section 3, we proposed a model of price process taken into account the rating of the underlying issuer. In what follows, we propose a new approach of European option pricing in economic space. An option is a contract in which the writer of the option grants to the buyer of the option the right, but not the obligation, to purchase from or sell to the writer something at a specified price within a specified period of time (or at a specified date). The writer, also referred to as the seller, grants the right to the buyer in exchange for a certain sum of money which is called the option price or option premium. The price at which the asset may be bought or sold is called the exercise price or strike price. The date after which an option is void is called the expiration date. When an option grants the buyer the right to purchase the designed instrument from the seller, it is referred to as a call option, or call. When the option buyer has the right to sell the designed instrument to the writer, the option is called a put option, or put. Buying calls or selling puts allows the investor to gain if the price of the underlying asset rises. Selling calls or buying puts allows the investor to gain if the price of the underlying asset falls. An option is also categorized according to when the option buyer may exercise the option. The option that may be exercised at any time before (including) the expiration date, is referred to as American option whereas the option that may be exercised only at the expiration date is called European option. The formula proposed by Black and Scholes [26] as the price or premium of an European call (or put) didn't take into account the rating of the issuer of the underlying asset. To the traditional characteristics of an option, we add the exercised rating κ that determines the desired quality of the underlying at expiration date. In the case of exercise of the option, the seller must buy or sell to the buyer the underlying financial variable having κ for the specified price K. Hence, for a given financial variable issued by a given economic agent, the option contract should specify the strike, the period or expiration date and the exercise rating. At the maturity, there are four possibilities:

- The quality of the underlying (rating of the issuer) can deteriorate in such a way that even its price is below the strike.
- The quality can be improved, but the underlying's price is less than the strike.
- The quality can be deteriorated but the underlying's price is greater than the strike.
- The quality can be improved and the underlying's price is greater than the strike.

5.1. An Extension of the Black and Scholes' Equation

The Black Scholes and Merton's (BSM) equation is one of the most famous equation in the field of mathematical finance. Its solution which expresses the price of an European option is on the basis of the lightning development of derivatives market and insurance premium. It permitted to develop the first model of credit risk by Merton [9]; namely the first structural model of credit risk. But this work didn't take into account the rating of the issuer of the underlying asset. To derive the Black Scholes and Merton's PDE for an European option in economic space, we make the following assumptions

1) Trading is made on a fixed period of time T.

2) There is no restriction on selling and buying stocks.

3) There is no dividends, frictions and transactions costs.

4) There is no arbitrage opportunity.

5) There exists a risk free savings account (in a bank rated AAA) with constant interest rate r.

6) The option is written on a single financial variable.

Let S_t and X_t be the price and rating of the underlying respectively at time t. In the original Black Scholes model a risk free self financing portfolio is constructed by using the underlying asset and a derivative which is used to hedge the underlying asset. In the present case, due to the existence of additional risk, namely the risk coming from the deterioration of the rating of the underlying's issuer, to hedge this additional risk, we consider another derivative written on the rating of the underlying's issuer. Hence we construct a portfolio containing at any time t one option $V(t, X_t, S_t) \equiv V_t$ on the underlying, $\xi(t)$ units of the financial variable and $\eta(t)$ units of another option $U(t, X_t, S_t) \equiv U_t$ to hedge underlying's issuer rating. The portfolio's value at time t is given by the relation

$$\Pi(t) = V_t + \xi(t) S_t + \eta(t) U_t. \tag{11}$$

Under the self financing assumption, the change in the portfolio is expressed as follows

$$\mathrm{d}\Pi(t) = \mathrm{d}V_t + \xi(t)\mathrm{d}S_t + \eta(t)\mathrm{d}U_t. \tag{12}$$

We assume further that the functions V_t and U_t are $\mathcal{C}^1$ in time and at least $\mathcal{C}^2$ for other variables. If we assume that the variance of the random effect on the variation of underlying's rating is proportional to the current rating, then the function f in relation (5) can be chosen in the form

$$f(t, x) = \gamma(t)\sqrt{x},$$

where $\gamma(t)$ is a real valued integrable function assuming positive values. If the parameters α, λ and γ are positive real numbers, then rating dynamics follows a Cox, Ingersoll and Ross process introduced in 1985 [41] to model the interest rate process. In this case, from Feller [42], an examination of the boundary classification criteria in our setting shows that, for a given economic particle,

1) if $2\alpha < \gamma^2$, then 0 (AAA) is an attracting rating and can be reached if the initial rating is strictly greater than 0,

2) if $2\alpha \geq \gamma^2$, then if the initial rating is strictly positive, then it can no longer reach 0.

For tractability reasons, we suppose in what follows that these parameters are constant.

Under the assumption that $(B_t)_{t \geq [0,T]}$ and $(W_t)_{t \geq [0,T]}$ are independent Brownian motions, applying the Itô's Lemma to the derivatives V_t and U_t and using relations (5) and (6), we obtain the following relations

$$\begin{aligned}\mathrm{d}V_t = {} & \frac{\partial V_t}{\partial t}\mathrm{d}t + \frac{\partial V_t}{\partial S_t}\mathrm{d}S_t + \frac{\partial V_t}{\partial X_t}\mathrm{d}X_t + \frac{1}{2}\sigma_0^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2}\mathrm{d}t \\ & + \frac{1}{2}\gamma^2 X_t \frac{\partial^2 V_t}{\partial X_t^2}\mathrm{d}t - k\gamma^2 S_t X_t \frac{\partial^2 V_t}{\partial S_t \partial X_t}\mathrm{d}t,\end{aligned} \tag{13}$$

$$\begin{aligned}\mathrm{d}U_t = {} & \frac{\partial U_t}{\partial t}\mathrm{d}t + \frac{\partial U_t}{\partial S_t}\mathrm{d}S_t + \frac{\partial U_t}{\partial X_t}\mathrm{d}X_t + \frac{1}{2}\sigma_0^2 S_t^2 \frac{\partial^2 U_t}{\partial S_t^2}\mathrm{d}t \\ & + \frac{1}{2}\gamma^2 X_t \frac{\partial^2 U_t}{\partial X_t^2}\mathrm{d}t - k\gamma^2 S_t X_t \frac{\partial^2 U_t}{\partial S_t \partial X_t}\mathrm{d}t,\end{aligned} \tag{14}$$

where $\sigma_0^2 = \sigma^2 + k^2\gamma^2 X_t$.

Substituting (13) and (14) into (12), the change in the portfolio becomes

$$\begin{aligned}\mathrm{d}\Pi(t) = {} & \left[\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma_0^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} + \frac{1}{2}\gamma^2 X_t \frac{\partial^2 V_t}{\partial X_t^2} - k\gamma^2 S_t X_t \frac{\partial^2 V_t}{\partial S_t \partial X_t}\right]\mathrm{d}t \\ & + \eta(t)\left[\frac{\partial U_t}{\partial t} + \frac{1}{2}\sigma_0^2 S_t^2 \frac{\partial^2 U_t}{\partial S_t^2} + \frac{1}{2}\gamma^2 X_t \frac{\partial^2 U_t}{\partial X_t^2} - k\gamma^2 S_t X_t \frac{\partial^2 U_t}{\partial S_t \partial X_t}\right]\mathrm{d}t \\ & + \left[\frac{\partial V_t}{\partial S_t} + \eta(t)\frac{\partial U_t}{\partial S_t} + \xi(t)\right]\mathrm{d}S_t + \left[\frac{\partial V_t}{\partial X_t} + \eta(t)\frac{\partial U_t}{\partial X_t}\right]\mathrm{d}X_t.\end{aligned} \tag{15}$$

In order for the portfolio to be hedged against movement in financial variable's price and rating of the underlying's issuer, the last two terms of (15) must be zero. This implies that

$$\xi(t) = -\frac{\partial V_t}{\partial S_t} - \eta(t)\frac{\partial U_t}{\partial S_t} \quad \text{and} \quad \eta(t) = -\frac{\dfrac{\partial V_t}{\partial X_t}}{\dfrac{\partial U_t}{\partial X_t}}, \tag{16}$$

where we suppose that $\dfrac{\partial U_t}{\partial X_t} \neq 0$.

The condition that the portfolio is risk free implies that the change in the portfolio is equal to the change in the risk free saving account; that is $\mathrm{d}\Pi(t) = r\Pi(t)\mathrm{d}t$. Equation (12) thus becomes

$$\mathrm{d}\Pi(t) = r\left(V_t + \xi(t)S_t + \eta(t)U_t\right). \tag{17}$$

Now, equating (15) with (17), and using (16), we obtain the following relation

$$\begin{aligned}& \frac{\dfrac{\partial V_t}{\partial t} + \dfrac{1}{2}\sigma_0^2 S_t^2 \dfrac{\partial^2 V_t}{\partial S_t^2} + \dfrac{1}{2}\gamma^2 X_t \dfrac{\partial^2 V_t}{\partial X_t^2} - k\gamma^2 X_t S_t \dfrac{\partial^2 V_t}{\partial S_t \partial X_t} - rV_t + rS_t \dfrac{\partial V_t}{\partial S_t}}{\dfrac{\partial V_t}{\partial X_t}} \\ & = \frac{\dfrac{\partial U_t}{\partial t} + \dfrac{1}{2}\sigma_0^2 S_t^2 \dfrac{\partial^2 U_t}{\partial S_t^2} + \dfrac{1}{2}\gamma^2 X_t \dfrac{\partial^2 U_t}{\partial X_t^2} - k\gamma^2 X_t S_t \dfrac{\partial^2 U_t}{\partial S_t \partial X_t} - rU_t + rS_t \dfrac{\partial U_t}{\partial S_t}}{\dfrac{\partial U_t}{\partial X_t}}.\end{aligned} \tag{18}$$

The left hand side of (18) depends only on the function V_t and the right hand side depends only on the function U_t. Hence, they are equal to a function $h \equiv h(t, X_t, S_t)$. Following Heston [43], we can select the function h having the form

$$h(t, X_t, S_t) = -(\alpha - \lambda X_t) + \delta(t, X_t, S_t), \tag{19}$$

where $\delta(t, S_t, X_t)$ is the price of rating risk. This allows us to obtain the equations to be written as

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_0^2 y^2 \frac{\partial^2 V}{\partial y^2} + \frac{1}{2}\gamma^2 x \frac{\partial^2 V}{\partial x^2} - k\gamma^2 xy \frac{\partial^2 V}{\partial x \partial y} \\ &+ ry\frac{\partial V}{\partial y} - h(t, x, y)\frac{\partial V}{\partial x} - rV = 0\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\frac{\partial U}{\partial t} + \frac{1}{2}\sigma_0^2 y^2 \frac{\partial^2 U}{\partial y^2} + \frac{1}{2}\gamma^2 x \frac{\partial^2 U}{\partial x^2} - k\gamma^2 xy \frac{\partial^2 U}{\partial x \partial y} \\ &+ ry\frac{\partial U}{\partial y} - h(t, x, y)\frac{\partial U}{\partial x} - rU = 0.\end{aligned} \tag{21}$$

In the case of European call option with strike K, maturity T and exercise rating κ, the goal of the buyer is to buy at time T the underlying asset with quality (rating) κ at a price of K. For an European put option with same characteristics, the goal of the buyer is to sell at time T the underlying with quality κ at a price of K.

From relation (4), we have

$$S_T^* = S_T \mathrm{e}^{k(X_T - \kappa)},$$

where S_T^* and S_T are prices at time T of the underlying having rating κ and X_T respectively. Hence, the payoffs of these European options are

$$V(T, \kappa, K) = \max\left(S_T \mathrm{e}^{k(X_T - \kappa)} - K, 0\right)$$

and

$$V(T, \kappa, K) = \max\left(K - S_T \mathrm{e}^{k(X_T - \kappa)}, 0\right)$$

for a call and a put respectively.

The payoff of the function U is the potential prejudice undergone in case of rating deterioration and is given in both case (call and put options) by the relation

$$U(T, \kappa, K) = \max\left(S_T^* - S_T, 0\right) = S_T \max\left(\mathrm{e}^{k(X_T - \kappa)} - 1, 0\right)$$

and it represents the maximum between the difference of prices and 0.

The problems to be solved for the European call option with strike K, maturity T and exercise rating κ are

$$\begin{cases}\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma_0^2 y^2 \dfrac{\partial^2 V}{\partial y^2} + \dfrac{1}{2}\gamma^2 x \dfrac{\partial^2 V}{\partial x^2} - k\gamma^2 xy \dfrac{\partial^2 V}{\partial x \partial y} + ry\dfrac{\partial V}{\partial y} - h(t, x, y)\dfrac{\partial V}{\partial x} - rV = 0, (t, x, y) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}_+^* \\ V(T, x, y) = \max\left(y\mathrm{e}^{k(x - \kappa)} - K, 0\right), (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^*\end{cases} \tag{22}$$

and

$$\begin{cases} \frac{\partial U}{\partial t}+\frac{1}{2}\sigma_0^2 y^2\frac{\partial^2 U}{\partial y^2}+\frac{1}{2}\gamma^2 x\frac{\partial^2 U}{\partial x^2}-k\gamma^2 xy\frac{\partial^2 U}{\partial x\partial y}+ry\frac{\partial U}{\partial y}-h(t,x,y)\frac{\partial U}{\partial x}-rU=0, (t,x,y)\in[0,T)\times\mathbb{R}_+\times\mathbb{R}_+^* \\ U(T,x,y)=y\max\left(e^{k(x-\kappa)}-1,0\right), (x,y)\in\mathbb{R}_+\times\mathbb{R}_+^* \end{cases} \quad (23)$$

For European put option with same characteristics, the problems to be solved are obtained by replacing the payoff in (22) by $V(T,\kappa,K)=\max\left(K-S_T e^{k(X_T-\kappa)},0\right)$ and (23) remains unchanged.

5.2. Closed-Form Solution of European Call Option

In the sequel, we assume that $\delta(t,X_t,S_t)=\delta x$ where δ is a positive real number. Hence, h is defined by

$$h(t,x,y)=-(\alpha-(\lambda+\delta)x).$$

For the problem (22), we will investigate a solution of the form

$$V(t,x,y)=V_1(t,y)+V_2(t,x), \quad (24)$$

where V_1 and V_2 belong to the set $C^{2,1}\left(\mathbb{R}^2\times[0,T]\right)$ of C^2 functions in space variables (x and y) and C^1 functions of time variable t.

The payoff $\max\left(ye^{k(x-\kappa)}-K,0\right)$ can also be written as

$$\max\left(ye^{k(x-\kappa)}-K,0\right)=\frac{ye^{k(x-\kappa)}-K+\left|ye^{k(x-\kappa)}-K\right|}{2}.$$

Using relation (24), problem (22) is transformed into the following problems

1) For all fixed $x\geq 0$

$$\begin{cases} \frac{\partial V_1}{\partial t}+ry\frac{\partial V_1}{\partial y}+\frac{1}{2}\sigma_0^2 y^2\frac{\partial^2 V_1}{\partial y^2}-rV_1=0, (t,y)\in[0,T)\times\mathbb{R}_+^* \\ V_1(T,y)=\frac{\left|ye^{k(x-\kappa)}-K\right|}{2}, y\in\mathbb{R}_+^* \end{cases} \quad (25)$$

2) For all fixed $y>0$

$$\begin{cases} \frac{\partial V_2}{\partial t}+\frac{1}{2}\gamma^2 x\frac{\partial^2 V_2}{\partial x^2}+(\alpha-(\delta+\lambda)x)\frac{\partial V_2}{\partial x}-rV_2=0, (t,x)\in[0,T)\times\mathbb{R}_+ \\ V_2(T,x)=\frac{ye^{k(x-\kappa)}-K}{2}, (x,y)\in\mathbb{R}_+ \end{cases} \quad (26)$$

The following Theorem gives a solution to the problem (22).

Theorem 5.1. *If* $k\gamma^2<2(\delta+\lambda)$, *the function V defined for all* $(t,x,y)\in[0,T)\times\mathbb{R}_+\times\mathbb{R}_+^*$ *by*

$$V(t,x,y)=\left(2N(d_1)-1+e^{-r\tau}\left(\frac{c_T}{c_T-k}\right)^{\frac{2\alpha}{\gamma^2}-1}\exp\left(kx\left(\frac{c_T e^{-(\delta+\lambda)\tau}}{c_T-k}-1\right)\right)\right)\frac{y}{2}e^{k(x-\kappa)} - Ke^{-r\tau}N(d_2) \quad (27)$$

is a solution of the problem (22),

where

$$\tau = T - t,$$

$$d_1 = \frac{\ln\left(\frac{y e^{k(x-\kappa)}}{K}\right) + \left(r + \frac{1}{2}\sigma_0^2\right)\tau}{\sigma_0\sqrt{\tau}},$$

$$d_2 = \frac{\ln\left(\frac{y e^{k(x-\kappa)}}{K}\right) + \left(r - \frac{1}{2}\sigma_0^2\right)\tau}{\sigma_0\sqrt{\tau}},$$

$$c_T = \frac{2(\delta+\lambda)}{\gamma^2\left(1 - e^{-(\delta+\lambda)\tau}\right)} \quad \text{and}$$

N is the cumulative function of the normally distributed random variable.

To give a proof of Theorem 5.1, we need the following result well-known as the Feynman-Kac formula.

Lemma 5.1 [44] *Let* ϕ *and* f *be two real valued continuous functions on* $\mathbb{R}$ *and* $\mathbb{R}\times[0,T]$ *and satisfying the relation*

$$\phi \geq 0, \quad f \geq 0.$$

Let $\mathcal{L}_t = b(z,t)\frac{\partial}{\partial z} + \frac{1}{2}a(z,t)\frac{\partial^2}{\partial z^2}$ be a differential operator on $\mathcal{C}^{2,1}(\mathbb{R}\times[0,T))$. We consider its associated canonical diffusion $\left(\mathcal{C}, \mathcal{M}, (\mathcal{M}_s)_{s\in[t,T]}, (Z_s)_{s\in[t,T]}, \mathbb{P}^{z,t}\right)$ defined by

$$\begin{cases} dZ_s = \rho(s,Z_s)ds + \nu(s,Z_s)dW_s^*, \ s\in[t,T] \\ Z_t = z \end{cases} \tag{28}$$

Let $c(z,t)$ be a continuous function bounded below and w a $\mathcal{C}^{2,1}(\mathbb{R}\times[0,T))$ continuous function on $\mathbb{R}\times[0,T]$ and a solution of the problem

$$\begin{cases} \frac{\partial w}{\partial t} + \mathcal{L}_t w - cw = f, \ (z,t)\in\mathbb{R}\times[0,T) \\ w(z,T) = \phi(z), \ z\in\mathbb{R}. \end{cases} \tag{29}$$

Let us assume further that a and b are Lipschitz continuous on $\mathbb{R}\times[0,T]$ and that q is uniformly elliptic. We also assume that w has a polynomial growth for every t in $[0,T]$. Then we have the representation formula

$$w(z,t) = \mathbb{E}^{z,t}\left[\phi(Z_T)e^{-\int_t^T c(Z_s,s)ds}\right] - \mathbb{E}^{z,t}\left[\int_t^T f(Z_s,s)e^{-\int_t^s c(Z_u,u)du}ds\right].$$

Here $\mathcal{C} = \mathcal{C}([0,T],\mathbb{R})$ is the set of continuous functions on $[0,T]$ with value in $\mathbb{R}$, $\mathcal{M}$ is the set of Borelian subset of $\mathcal{C}([0,T],\mathbb{R})$, $\mathcal{M}_s = \sigma(Z_v, t\leq v\leq s)$ is the σ-algebra generated by Z_v $(t\leq v\leq s)$, $\mathbb{P}^{z,t}$ is the law of the process $(Z_s)_{s\geq t}$ on $\mathcal{C}$ and $\mathbb{E}^{z,t}$ is the expectation with respect to $\mathbb{P}^{z,t}$.

We give now the proof of Theorem 5.1.

Proof. We split problem (22) into problems (25) and (26).

These problems can be written as

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \mathcal{L}_t^1 V_1 - rV_1 = 0, (t, y) \in [0, T) \times \mathbb{R}_+^* \\ V_1(T, y) = \dfrac{\left|y\mathrm{e}^{k(x-\kappa)} - K\right|}{2}, y \in \mathbb{R}_+^* \end{cases} \quad \forall x \geq 0 \tag{30}$$

and

$$\begin{cases} \dfrac{\partial V_2}{\partial t} + \mathcal{L}_t^2 V_2 - rV_2 = 0, (x, t) \in \mathbb{R}_+ \times [0, T) \\ V_2(T, x) = \dfrac{y\mathrm{e}^{k(x-\kappa)} - K}{2}, x \in \mathbb{R}_+ \end{cases} \quad \forall y > 0 \tag{31}$$

where the operators $\mathcal{L}_t^1$ and $\mathcal{L}_t^2$ are defined by

$$\mathcal{L}_t^1 = ry\frac{\partial}{\partial y} + \frac{1}{2}\sigma_0^2 y^2 \frac{\partial^2}{\partial x^2}$$

and

$$\mathcal{L}_t^2 = \frac{1}{2}\gamma^2 x \frac{\partial^2}{\partial x} + \left(\alpha - (\delta + \lambda)x\right)\frac{\partial}{\partial x}.$$

The canonical diffusions associated to $\mathcal{L}_t^1$ and $\mathcal{L}_t^2$ are respectively defined by

$$\begin{cases} \mathrm{d}Y_s = rY_s\mathrm{d}s + \sigma_0 Y_s \mathrm{d}W_s^1, s \geq t \\ Y_t = y \end{cases} \tag{32}$$

and

$$\begin{cases} \mathrm{d}X_s = \left(\alpha - (\delta + \lambda)X_s\right)\mathrm{d}s + \gamma\sqrt{X_s}\mathrm{d}W_s^2, s \geq t \\ X_t = x \end{cases} \tag{33}$$

Here $\left(W_s^i\right)_{s\in[t,T]} (i = 1, 2)$ are two standard Brownian motions.

The process in (32) follows a log-normal distribution with mean r and variance $\sigma_0^2\tau$. From [41], the process in (33) follows a non-central chi-square distribution with $\dfrac{4\alpha}{\gamma^2}$ degrees of freedom and non centrality parameter $\lambda_0 = 2c_T x\mathrm{e}^{-(\delta+\lambda)(T-t)}$ defined by

$$G(z) = \left(\frac{z}{\lambda_0}\right)^{\frac{\alpha}{\gamma^2}-\frac{1}{2}} \mathrm{e}^{-\frac{z+\lambda_0}{2}} \sum_{n=0}^{\infty} \frac{\left(\dfrac{\sqrt{\lambda_0 z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}-1}}{n!\Gamma\left(n + \dfrac{2\alpha}{\gamma^2}\right)},$$

where $\Gamma(u) = \int_0^\infty t^{u-1}\mathrm{e}^{-t}\mathrm{d}t$ and $c_T = \dfrac{2(\delta + \lambda)}{\gamma^2\left(1 - \mathrm{e}^{-(\delta+\lambda)\tau}\right)}$.

Since V_1 and V_2 represent prices, we can reasonably assume that they have polynomial growth. Hence by applying Lemma 5.1 to (30) and (31) respectively, we have

$$V_1(t,y)=\frac{1}{2}y\mathrm{e}^{k(x-\kappa)}\left(2N(d_1)-1\right)-\frac{1}{2}K\mathrm{e}^{-r\tau}\left(2N(d_2)-1\right) \text{ for all } x\geq 0 \tag{34}$$

and

$$V_2(t,x)=\frac{1}{2}\left(y\mathrm{e}^{-r\tau-k\kappa}\int_0^\infty \exp\left(\frac{k}{2c_T}z\right)G(z)\mathrm{d}z-\mathrm{e}^{-r\tau}K\int_0^\infty G(z)\mathrm{d}z\right) \text{ for all } y>0\,. \tag{35}$$

We now have

$$\int_0^\infty \exp\left(\frac{k}{2c_T}z\right)G(z)\mathrm{d}z$$

$$=\int_0^\infty \exp\left(\frac{k}{2c_T}z\right)\left(\frac{z}{\lambda_0}\right)^{\frac{\alpha}{\gamma^2}-\frac{1}{2}}\mathrm{e}^{-\frac{\left(1-\frac{k}{c_T}\right)z+\lambda_0}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_0 z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}-1}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}\right)}\mathrm{d}z.$$

If $1-\frac{k}{c_T}>0$, we can make the change of variable $v=\left(1-\frac{k}{c_T}\right)z$.

But

$$1-\frac{k}{c_T}=1-\frac{k\gamma^2\left(1-\mathrm{e}^{-(\delta+\lambda)\tau}\right)}{2(\delta+\lambda)}.$$

If the condition $k\gamma^2<2(\delta+\lambda)$ is fulfilled, that is $\frac{k\gamma^2}{2(\delta+\lambda)}<1$, then the relation

$$1-\frac{k}{c_T}>0$$

is satisfied and we can make the change of variable $v=\left(1-\frac{k}{c_T}\right)z$ in the first integral of relation (35) and it becomes

$$y\mathrm{e}^{-r\tau}\left(\frac{c_T}{c_T-k}\right)^{\frac{2\alpha}{\gamma^2}-1}\exp\left(k\left(\frac{c_T x\mathrm{e}^{-(\delta+\lambda)\tau}}{c_T-k}\right)-\kappa\right)\int_0^\infty\left(\frac{z}{\lambda_*}\right)^{\frac{\alpha}{\gamma^2}-\frac{1}{2}}\mathrm{e}^{-\frac{z+\lambda_*}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_* z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}-1}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}\right)}\mathrm{d}z$$

where $\lambda_*=\frac{c_T\lambda_0}{c_T-k}$. The proof is completed. □

The following Theorem gives the behavior of V when the rating x tends to ∞.

Theorem 5.2 *Under the hypothesis* $k\gamma^2<2(\delta+\lambda)$, *we have the relation*

$$\lim_{x\to\infty}V(t,x,y)=\frac{y^*}{2} \tag{36}$$

where y^* is the price at time t of the underlying having rating κ.

Proof. Equation (27) can be rewritten as

$$V(t,x,y)=\frac{y^*}{2}\left(2N(d_1)-1+\mathrm{e}^{-r\tau}\left(\frac{c_T}{c_T-k}\right)^{\frac{2\alpha}{\gamma^2}-1}\exp\left(kx\left(\frac{c_T\mathrm{e}^{-(\delta+\lambda)\tau}}{c_T-k}-1\right)\right)\right) \quad (37)$$
$$-K\mathrm{e}^{-r\tau}N(d_2)$$

with $d_1=\frac{\ln\left(\frac{y^*}{K}\right)+\left(r+\frac{1}{2}\sigma_0^2\right)\tau}{\sigma_0\sqrt{\tau}}$ and $d_2=\frac{\ln\left(\frac{y^*}{K}\right)+\left(r-\frac{1}{2}\sigma_0^2\right)\tau}{\sigma_0\sqrt{\tau}}$.

At time *t*, the value y^* is independent of *x*.

Since $\sigma_0^2=\sigma^2+k^2\gamma^2x$, we have $\lim_{x\to\infty}N(d_1)=1$ and $\lim_{x\to\infty}N(d_2)=0$. We also have

$$\frac{c_T\mathrm{e}^{-(\delta+\lambda)\tau}}{c_T-k}-1=\frac{\left(k\gamma^2-2(\delta+\lambda)\right)\left(1-\mathrm{e}^{-(\delta+\lambda)\tau}\right)}{\left(2(\delta+\lambda)-k\gamma^2\right)+k\gamma^2\mathrm{e}^{-(\delta+\lambda)\tau}}<0.$$

This implies that

$$\lim_{x\to\infty}\mathrm{e}^{-r\tau}\left(\frac{c_T}{c_T-k}\right)^{\frac{2\alpha}{\gamma^2}-1}\exp\left(kx\left(\frac{c_T\mathrm{e}^{-(\delta+\lambda)\tau}}{c_T-k}-1\right)\right)=0.$$

□

5.3. Greeks and Hedging

We now talk about the Greeks, which are measures of sensitivity of the option price with respect to its variables. They are another way to measure risk associated to an investment. In fact risk associated to a specific variable is canceled if the partial derivative of the price with respect to this variable vanishes for all *t*.

To obtain Greeks, we need to solve problem (23). We will investigate a solution of the form

$$U(t,x,y)=yW(t,x). \quad (38)$$

Using the relation (38), the problem (23) is transformed as follows

$$\begin{cases} y\left(\frac{\partial W}{\partial t}+\frac{1}{2}\gamma^2x\frac{\partial^2W}{\partial x^2}+\left(\alpha-\left(\delta+\lambda+k\gamma^2\right)x\right)\frac{\partial W}{\partial x}\right)=0,\ (t,x)\in[0,T)\times\mathbb{R}_+ \\ W(T,x)=\max\left(\mathrm{e}^{k(x-\kappa)}-1,0\right),\ x\in\mathbb{R}_+ \end{cases} \quad \forall y>0 \quad (39)$$

We have the following result.

Theorem 5.3 *The function U defined for all* $(t,x,y)\in[0,T)\times\mathbb{R}_+\times\mathbb{R}_+^*$ *by*

$$U(t,x,y)=y\left(\left(\frac{c_T^1}{c_T^1-k}\right)^{\frac{2\alpha}{\gamma^2}-1}\exp\left(k\left(\frac{c_T^1x\mathrm{e}^{-\left(\delta+\lambda+k\gamma^2\right)\tau}}{c_T^1-k}-\kappa\right)\right)\right. \quad (40)$$
$$\left.\times\left(1-G_1\left(2\left(c_T^1-k\right)\kappa\right)\right)-1+G_2\left(2c_T^1\kappa\right)\right)$$

is a solution of the problem (23),
where

$$G_1\left(2\kappa\left(c_T^1-k\right)\right)=\int_0^{2\kappa\left(c_T^1-k\right)}\left(\frac{z}{\lambda_1^*}\right)^{\frac{\alpha}{\gamma^2}-\frac{1}{2}}\mathrm{e}^{-\frac{z+\lambda_1^*}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_1^* z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}-1}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}\right)}\mathrm{d}z,$$

$$G_2\left(2c_T^1\kappa\right)=\int_0^{2c_T^1\kappa}\left(\frac{z}{\lambda_1}\right)^{\frac{\alpha}{\gamma^2}-\frac{1}{2}}\mathrm{e}^{-\frac{z+\lambda_1}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_1 z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}-1}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}\right)}\mathrm{d}z,$$

$$c_T^1=\frac{2\left(\delta+\lambda+k\gamma^2\right)}{\gamma^2\left(1-\mathrm{e}^{-\left(\delta+\lambda+k\gamma^2\right)\tau}\right)},\quad \lambda_1=2c_T^1x\mathrm{e}^{-\left(\delta+\lambda+k\gamma^2\right)(T-t)}\quad\text{and}\quad \lambda_1^*=\frac{\lambda_1 c_T^1}{c_T^1-k}.$$

Proof. The problem (39) can be written as

$$\begin{cases}\dfrac{\partial W}{\partial t}+\mathcal{L}_t^3W=0,\ (t,x)\in[0,T)\times\mathbb{R}_+ & \forall x\geq 0\\ W(T,x)=\max\left(\mathrm{e}^{k(x-\kappa)}-1,0\right),\ x\geq 0 & \end{cases}\tag{41}$$

where the operator $\mathcal{L}_t^3$ is defined by

$$\mathcal{L}_t^3=\frac{1}{2}\gamma^2x\frac{\partial^2}{\partial x}+\left(\alpha-\left(\delta+\lambda+k\gamma^2\right)x\right)\frac{\partial}{\partial x}.$$

The canonical diffusion associated to $\mathcal{L}_t^3$ is defined by

$$\begin{cases}\mathrm{d}X_s=\left(\alpha-\left(\delta+\lambda+k\gamma^2\right)X_s\right)\mathrm{d}s+\gamma\sqrt{X_s}\,\mathrm{d}W_s^3,\ s\geq t,\\ X_t=x.\end{cases}\tag{42}$$

Here $\left(W_s^3\right)_{s\in[t,T]}$ is a standard Brownian motion. From [41], the process in (42) follows a non-central chi-square distribution with $\frac{4\alpha}{\gamma^2}$ degrees of freedom and non centrality parameter $\lambda_1=2c_T^1x\mathrm{e}^{-\left(\delta+\lambda+k\gamma^2\right)(T-t)}$ defined by

$$G(z)=\left(\frac{z}{\lambda_1}\right)^{\frac{\alpha}{\gamma^2}-\frac{1}{2}}\mathrm{e}^{-\frac{z+\lambda_1}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_1 z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}-1}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}\right)}.$$

Since W represents a price, we can reasonably assume that it has polynomial growth. Hence by an appropriate change of variable and application of Lemma 5.1, we have the result. □

Some Greeks are necessary to obtain hedging's portfolio. From relation (16) they are $\frac{\partial V}{\partial y}$, $\frac{\partial U}{\partial y}$, $\frac{\partial V}{\partial x}$ and $\frac{\partial U}{\partial x}$.

These Greeks are given in the following Lemma.

Lemma 5.2. *For an European call option with strike K, maturity T and exercise rating* κ, *principal Greeks are given as follows*

1) $$\frac{\partial V}{\partial y}=\mathrm{e}^{k(x-\kappa)}\left(N(d_1)-\frac{1}{2}+\frac{1}{2}\mathrm{e}^{-r\tau}\left(\frac{c_T}{c_T-k}\right)^{\frac{2\alpha}{\gamma^2}-1}\exp\left(kx\left(\frac{c_T\mathrm{e}^{-(\delta+\lambda)\tau}}{c_T-k}-1\right)\right)\right),$$

2) $$\frac{\partial U}{\partial y}=\left(\frac{c_T 1}{c_T^1-k}\right)^{\frac{2\alpha}{\gamma^2}-1}\exp\left(k\left(\frac{c_T^1 x\mathrm{e}^{-\left(\delta+\lambda+k\gamma^2\right)\tau}}{c_T^1-k}-\kappa\right)\right), \times\left(1-G_1\left(2\left(c_T^1-k\right)\kappa\right)\right)-1+G_2\left(2c_T^1\kappa\right)$$

3) $$\frac{\partial V}{\partial x}=y\left(k\mathrm{e}^{k(x-\kappa)}N(d_1)+\frac{k^2\gamma^2}{2\sigma_0\sqrt{2\pi\tau}}\mathrm{e}^{-\frac{1}{2}d_1^2+k(x-\kappa)}\right)+W_1(t,x,y),$$

4) $$\frac{\partial U}{\partial x}=y\left(\mathrm{e}^{-\left(\delta+\lambda+k\gamma^2\right)\tau}\left(\frac{c_T^1}{c_T^1-k}\right)^{\frac{2\alpha}{\gamma^2}-1}\exp\left(k\left(\frac{c_T^1 x\mathrm{e}^{-\left(\delta+\lambda+k\gamma^2\right)\tau}}{c_T^1-k}-\kappa\right)\right)\right.$$
$$\times\left(k-\frac{c_T^1}{2}+\left(c_T^1-k\right)G^1_{\frac{2\alpha}{\gamma^2}-1}\left(2\kappa\left(c_T^1-k\right)\right)-\frac{1}{2}c_T^1G^1_{\frac{2\alpha}{\gamma^2}}\left(2\kappa\left(c_T^1-k\right)\right)\right),$$
$$\left.+\frac{1}{2}c_T^1\mathrm{e}^{-\left(\delta+\lambda+k\gamma^2\right)\tau}\left(1-2G_{\frac{2\alpha}{\gamma^2}-1}\left(2c_T^1\kappa\right)+G_{\frac{2\alpha}{\gamma^2}}\left(2c_T^1\kappa\right)\right)\right)$$

where

$$W_1(t,x,y)=\frac{1}{2}yk\mathrm{e}^{k(x-\kappa)}\left(-1+\mathrm{e}^{-(r+\delta+\lambda)\tau}\left(\frac{c_T}{c_T-k}\right)^{\frac{2\alpha}{\gamma^2}}\exp\left(kx\left(\frac{c_T x\mathrm{e}^{-(\delta+\lambda)\tau}}{c_T-k}-1\right)\right)\right),$$

$$G^1_{\frac{2\alpha}{\gamma^2}-1}(u)=\int_0^u\left(\frac{z}{\lambda_1^*}\right)^{\frac{\alpha}{\gamma^2}-\frac{1}{2}}\mathrm{e}^{-\frac{z+\lambda_1^*}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_1^* z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}-1}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}\right)}\mathrm{d}z,$$

$$G^1_{\frac{2\alpha}{\gamma^2}}(u)=\int_0^u\left(\frac{z}{\lambda_1^*}\right)^{\frac{\alpha}{\gamma^2}}\mathrm{e}^{-\frac{z+\lambda_1^*}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_1^* z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}+1\right)}\mathrm{d}z,$$

$$G^1_{\frac{2\alpha}{\gamma^2}-1}(u)=\int_0^u\left(\frac{z}{\lambda_1}\right)^{\frac{\alpha}{\gamma^2}-\frac{1}{2}}\mathrm{e}^{-\frac{z+\lambda_1}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_1 z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}-1}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}\right)}\mathrm{d}z,$$

$$G_{\frac{2\alpha}{\gamma^2}}(u)=\int_0^u\left(\frac{z}{\lambda_1}\right)^{\frac{\alpha}{\gamma^2}}\mathrm{e}^{-\frac{z+\lambda_1}{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt{\lambda_1 z}}{2}\right)^{2n+\frac{2\alpha}{\gamma^2}}}{n!\Gamma\left(n+\frac{2\alpha}{\gamma^2}+1\right)}\mathrm{d}z \quad \text{and}$$

$$\lambda_1^*=\frac{c_T^1\lambda_1}{c_T^1-k},$$

Proof. Direct computations give the result. □

Remark 5.1 *Consider an European call option with strike K, maturity T and exercise rating* κ. *At every time* $t<T$, *the hedging portfolio should contain* $\xi(t)$ *quantity of underlying asset and* $\eta(t)$ *quantity of another option on issuer's rating.* $\xi(t)$ *and* $\eta(t)$ *are obtained by using Lemma* 5.2 *and relations* (16).

6. Conclusion

In this paper, we proposed a new notion of economic space similar to notion of space in Physics that by definition has infinite dimension due to the unaccountability of financial variables in the market. The coordinates of an economic particle are represented by its rating on different financial variables issued. The dynamics of an economic particle in economic space has been proposed as well as new price process taking into account the rating of the issuer. We also proposed an extension of known risk measures to adjusted-to-rating future payoff or wealth. This new definition allows to put aside additional capital to rule out the loss of money that can be granted due to the adverse change of the rating of the issuer. This new approach enables reduction of the exposition of the investor towards specific risk coming from the issuer of a financial instrument. We extended the Black Scholes and Merton's equation by deriving new PDE of European option pricing taking into account the underlying's rating and obtained the corresponding hedging portfolio. Our future challenge is to propose another approach of credit risk models and portfolio selection taking explicitly into account the rating.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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