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Application of G-Brown Motion in the Stock Price

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Abstract

We use the G-geometric Brownian motion and G-quadratic variation process to describe the price change of the asset. We prove that American call options do not pay dividends under G-framework. Finally we can simulate the stock price under the numerical simulation of G-brown motion and G-quadratic variation process.

Keywords

G-Expectation, G-Brown Motion, G-Quadratic Variation

1. Introduction

Asset pricing theory is one of the themes of the financial economy. Lèvy and Paras [1] proposed an uncertain volatility model, but the study could not give a dynamic option price. Peng [2] [3] defines G-expectation and G-Brown motion to provide a solution to this problem. Describing the theoretical basis of the option price. Yang and Zhao [4] simulate the G-normal distribution, and study the numerical simulation of G-Brown motion and the simulation of the second variation of G-Brown motion, then the finite difference method is given to solve the G-heat equation. Xu [5] [6] study the European call option price formula and Girsanov theorem under G-expectation. Wang [7] study the G-Jensen inequality under G-expectation. Wang [8] study the comparison theorem and Asian option pricing under G-expectation. Kang [9] study the Brownian motion martingale representation theorem under G-expectation.

The main purpose of this paper is to introduce the price change of the asset driven by G-geometric Brown. First we give the martingale property of discount value under G-framework. We simulate the stock price $S(t)$. We compare the stock price under normal B_t with the stock price under G- B_t. And we give the stock price under different G- B_t. Then we study the G- $\langle B\rangle_t$ influence on S_t under G-framework.

2. The G-Martingale Property of Discount Value

Definition 1 [2]: B_t is G-brown motion, $0=t_1^N<\cdots<t_N^N=t$ is a division on $[0,t]$, when $\mu(t_j^N)\to 0$, we denote G-quadratic variation process by $\langle B\rangle_t$:

$$\langle B\rangle_t=\sum_{j=0}^{N-1}\left(B_{t_{j+1}^N}-B_{t_j^N}\right)^2=B_t^2-2\int_0^t B_s\mathrm{d}B_s. \tag{1}$$

Definition 2 [2]: A nonlinear expectation E_G is a function $\mathcal{H}\to\mathbb{R}$ satisfying the following properties:

1) Monotonicity: If $X,Y\in\mathcal{H}$ and $X\ge Y$ then $E_G[X]\ge E_G[Y]$;

2) Preserving of constants: $E_G[c]=c$;

3) Sub-additivity $E_G[X]-E_G[Y]\le E_G[X-Y]$, $\forall X,Y\in\mathcal{H}$;

4) Positive homogeneity: $E_G[\lambda X]=\lambda E_G[X]$, $\forall\lambda\ge 0$, $X\in\mathcal{H}$;

5) Constant translatability: $E_G[X+c]=E_G[X]+c$.

Definition 3 [2]: The canonical process B is called a G-Brownian motion under a nonlinear E_G defined on $L_{ip}^0(\mathcal{F})$ if for each $T>0$, $m=1,2,\cdots$ and for each $\phi\in lip(R^m)$, $0\le t_1\le t_2\le\cdots\le t_m\le T$, we have

$$E_G\left[\phi\left(B_{t_1},B_{t_2}-B_{t_1},\cdots,B_{t_M}-B_{t_{m-1}}\right)\right]=\phi_m$$

Lemma 1 [2] [G-Itô formula]: for B_t is G-brownian motion, $\langle B\rangle_t$ is quadratic variation process of G-brownian miton, $\phi(t,x)$ is a function about (t,x), and ϕ_t', ϕ_x', ϕ_{xx}'' are continuous function, we have

$$\phi(t,B_t)-\phi(s,B_s)=\int_s^t\phi_t'(u,B_u)\mathrm{d}u+\int_s^t\phi_x'(u,B_u)\mathrm{d}B_u+\frac{1}{2}\int_s^t\phi_{xx}''(u,B_u)\mathrm{d}\langle B\rangle_u.$$

Lemma 2 [7] [G-Jensen inequality] h is a continuous function defined on R. Then the following two conditions are equivalent:

1) h is a convex function;

2) For $\forall X\in L_G^1(\mathcal{F})$, if $h(X)\in L_G^1(\mathcal{F})$, we have

$$h(E_G[X])\le E_G[h(X)].$$

Lemma 3 [6] [Girsanov under G-framework]: for $H(s,\omega)\in M_G(0,T)$, if existing $\varepsilon_0>0$ and satisfying:

$$E_G\left[\left(\frac{1}{2}+\varepsilon_0\right)\exp\int_0^T H(s,\omega)\mathrm{d}\langle B\rangle_s\right]<\infty,$$

we have

$$\Phi(B_t)=\exp\int_0^t H(s,\omega)\mathrm{d}B_s-\frac{1}{2}\int_0^t H^2(s,\omega)\mathrm{d}\langle B\rangle_s,$$

$\Phi(B_t)$ is a symmetrical martingale under E_G for $\forall t\in[0,T]$, $\Phi(B_t)\in L_G^1(\mathcal{F}_t)$.

In this section, we introduce the American call option, give a G-geometric Brownian motion asset. And we prove that the American call price is the same as the European call price.

Considering a stock whose price process $S(t)$ is given by

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}B_t, \tag{2}$$

where the interest rate r and the volatility σ ($\underline{\sigma}^2 \le \sigma \le \overline{\sigma}^2$) are positive and B_t is a G-brownian motion.

Now we compute (2) through G-Itô formula, in [5] the result is:

$$S(t) = S(0)\mathrm{e}^{\left(rt - \frac{1}{2}\sigma^2 \langle B \rangle_t + \sigma B_t\right)}, \tag{3}$$

where $S(0)$ is the stock value at current moment.

Theorem 1: $h(x) = (x-K)^+$ is a nonnegative and convex function, $K \ge 0$, $h(0) = 0$. Then the discount value $\mathrm{e}^{-rt}h(S(t))$ of American option $h(S(t))$ is a G-submartingale.

Proof: $h(x)$ is a convex, for $0 \le \lambda \le 1$ and $0 \le x_1 \le x_2$, we have

$$h((1-\lambda)x_1 + \lambda x_2) \le (1-\lambda)h(x_1) + \lambda h(x_2). \tag{4}$$

$h(S(t)) = (S(t)-K)^+$. Taking $x_1 = 0$, $x_2 = x$, and using the fact $h(0)=0$, we obtain

$$h(\lambda x) \le \lambda h(x), \text{ for all } x \ge 0, 0 \le \lambda \le 1 \tag{5}$$

for $0 \le u \le t \le T$, we have $0 \le \mathrm{e}^{-r(t-u)} \le 1$, by (5) and G-expectation property

$$E_G\left[\mathrm{e}^{-r(t-u)}h(S(t)) \mid \mathcal{F}(u)\right] \ge E_G\left[h\left(\mathrm{e}^{-r(t-u)}S(t)\right) \mid \mathcal{F}(u)\right]. \tag{6}$$

According to Lemma 2,

$$\begin{aligned} E_G\left[\mathrm{e}^{-r(t-u)}h(S(t)) \mid \mathcal{F}(u)\right] &\ge h\left(E_G\left[\mathrm{e}^{-r(t-u)}S(t) \mid \mathcal{F}(u)\right]\right) \\ &= h\left(\mathrm{e}^{ru}E_G\left[\mathrm{e}^{-rt}S(t) \mid \mathcal{F}(u)\right]\right), \end{aligned} \tag{7}$$

by Lemma 3 we know that $\mathrm{e}^{-rt}S(t)$ is a G-symmetrical martingale, which implies

$$h\left(\mathrm{e}^{ru}E_G\left[\mathrm{e}^{-rt}S(t) \mid \mathcal{F}(u)\right]\right) = h(S(u)). \tag{8}$$

So we conclude that

$$E_G\left[\mathrm{e}^{-r(t-u)}h(S(t)) \mid \mathcal{F}(u)\right] \ge h(S(u)), \tag{9}$$

and

$$E_G\left[\mathrm{e}^{-rt}h(S(t)) \mid \mathcal{F}(u)\right] \ge \mathrm{e}^{-ru}h(S(u)),$$

the $\mathrm{e}^{-rt}h(S(t))$ is a G-submartingale.

The Inequality (9) implies that the European derivative security price always dominates the intrinsic value of American derivative security. This shows that the option to exercise early is worthless, so the American call option agrees with the price of European option under G-framework.

3. Numerical Simulation

We mainly simulate stock price $S(t) = S(0)\mathrm{e}^{\left(rt - \frac{1}{2}\sigma^2 \langle B \rangle_t + \sigma B_t\right)}$ under G- B_t and

G-$\langle B\rangle_t$. The G-B_t and G-$\langle B\rangle_t$ values are simulated in [4]. Yang and Zhao [4] mainly simulate the G-brownian motion by solving a specific HJB equation. Then they give four finite difference methods to solve the HJB equation. Finally they give the numerical algorithms to simulate G-normal distribution, G-brownian motion G-quadratic variation process. The following we give three algorithms.

Algorithm 1 [4] (simulation $\tilde{\mathbb{F}}_X(a)$ and $\rho(a)$):

- For random $a_i \in A^f$, calculating approximation $\tilde{\mathbb{F}}\left(a_i;\underline{\sigma}^2,\bar{\sigma}^2\right)$;
- For $a \in D^f$, calculating the difference $I_h\tilde{\mathbb{F}}\left(a_i;\underline{\sigma}^2,\bar{\sigma}^2\right)$;
- By $I_h\tilde{\mathbb{F}}\left(a_i;\underline{\sigma}^2,\bar{\sigma}^2\right)$ calculating density function $\rho(a)$'s approximation $\tilde{\rho}(a)$.

by the G-heat equation defining the G-normal distribution $\tilde{\mathbb{F}}_X(a)$ and the density function $\rho(a)$. By Algorithm 1 simulating the $\tilde{\mathbb{F}}_X(a)$ and $\rho(a)$, then we apply these in Algorithm 2 and Algorithm 3.

Algorithm 2 [4] (G-brownian motion numerical simulation):

- For random $a_i \in A^f$, using algorithm 1 compute $\tilde{\mathbb{F}}\left(a_i;\underline{\sigma}^2,\bar{\sigma}^2\right)$;
- Produing N random numbers in [0,1] obey uniformly distribution ;
- For $a \in D^f$, calculating $I_h\tilde{\mathbb{F}}(a_i)$;
- By $I_h\tilde{\mathbb{F}}(\bar{a}_k)=\upsilon_k$, solving $\bar{a}_k$, $k=1,\cdots,N$;
- By $\sum_{j=1}^{k}\bar{a}_j$, approaching B_{t_k}, $k=1,\cdots,N$.

We simulate the values of G-brown motion B_t. By simulating the B_t, we use it in Algorithm 3 to get the $\langle B\rangle_t$.

Algorithm 3 [4] (numerical simulation $\langle B\rangle_t$):

- For random $a_i \in A^f$, using algorithm 1 to compute $\tilde{\mathbb{F}}\left(a_i;\underline{\sigma}^2,\bar{\sigma}^2\right)$;
- Generating N random numbers $\{\upsilon_k\}_{k=1,\cdots,N}$ in [0,1] for $a \in D^f$, calculating $I_h\tilde{\mathbb{F}}(a_i)$;
- By $I_h\tilde{\mathbb{F}}(\bar{a}_k)=\upsilon_k$, solving $\bar{a}_k$, $k=1,\cdots,N$;
- By $\sum_{j=1}^{k}\bar{a}_j^2$, approaching $\langle B\rangle_{t_k}$, $k=1,\cdots,N$.

The following we simulate the stock price $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}\sigma^2\langle B\rangle_t+\sigma B_t\right)}$ under the G-B_t and G-$\langle B\rangle_t$ values.

Example 1: we consider stock price $S(t)$ at time t immediately, where interest rate $r=0.2$, the volatility $\sigma=0.3$, $S(0)=100$.

Figure 1 denotes the comparison between $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}\sigma^2\langle B\rangle_t+\sigma B_t\right)}$ under G-framework and $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}\sigma^2 t+\sigma B_t\right)}$ under classical framework. In Figure 1 we can know that the blue line is simulated by $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}\sigma^2 t+\sigma B_t\right)}$, the red line is simulated by $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}\sigma^2\langle B\rangle_t+\sigma B_t\right)}$. Figure 2 simulates the price of $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}\sigma^2\langle B\rangle_t+\sigma B_t\right)}$ based on three different G-B_t in Figure 3. Figure 3 is about G-B_t of simulation. In Figure 3, the three lines are respectively under ($\underline{\sigma}^2=1$, $\bar{\sigma}^2=1$), ($\underline{\sigma}^2=0.8$, $\bar{\sigma}^2=1$), ($\underline{\sigma}^2=0.5$, $\bar{\sigma}^2=1$). Figure 4 is about

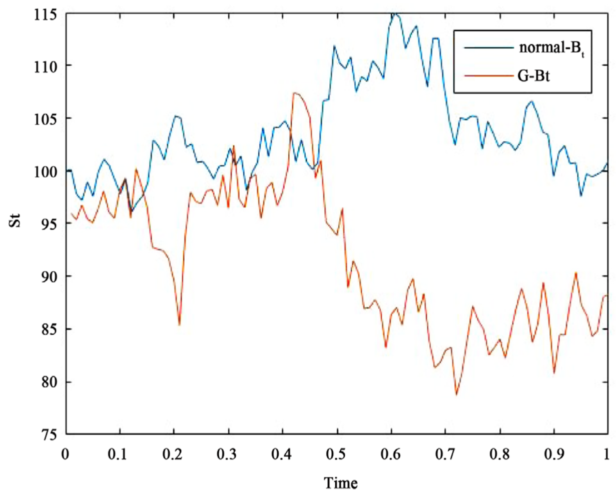

Figure 1. Comparing stock price of simulation between G-expectation framework and classical framework.

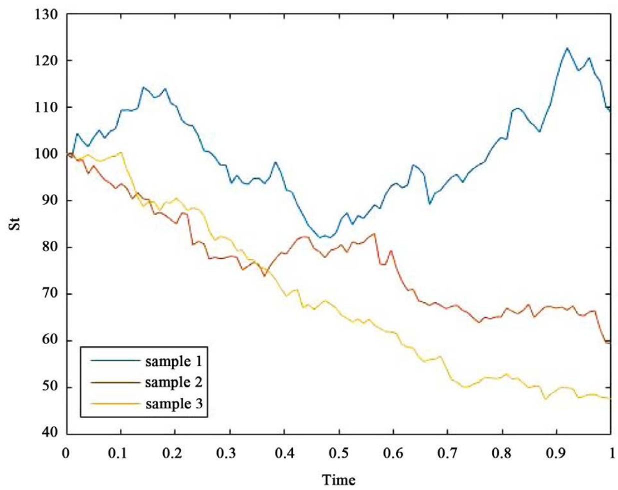

Figure 2. Comparing stock price under different G- B_t .

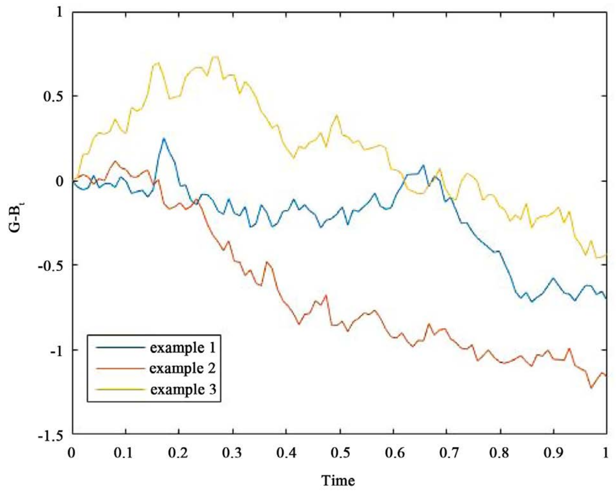

Figure 3. The G- B_t of simulation.

B_t of simulation under classical framework. We can know the G- B_t is different from the B_t according to **Figure 3** and **Figure 4**. And the stock price $S(t)$ is a about G- B_t, G- $\langle B\rangle_t$, t function under G-framework. The stock price $S(t)$ is a function about B_t and t. That is the main reason to cause the difference. We can know that the G- $\langle B\rangle_t$ influence on S_t under G-framework from **Figure 5**. The blue line is function $S(t)=S(0)\mathrm{e}^{(rt+\sigma B_t)}$. The red line is function $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}\sigma^2\langle B\rangle_t+\sigma B_t\right)}$. From **Figure 6**, we can know the G- $\langle B\rangle_t$ of simuation values. According to **Figure 6** when we replace the $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}\sigma^2\langle B\rangle_t+\sigma B_t\right)}$ with the $S(t)=S(0)\mathrm{e}^{\left(rt-\frac{1}{2}t+\sigma B_t\right)}$ under G-framework, it has no impact on stock price fluctuations.

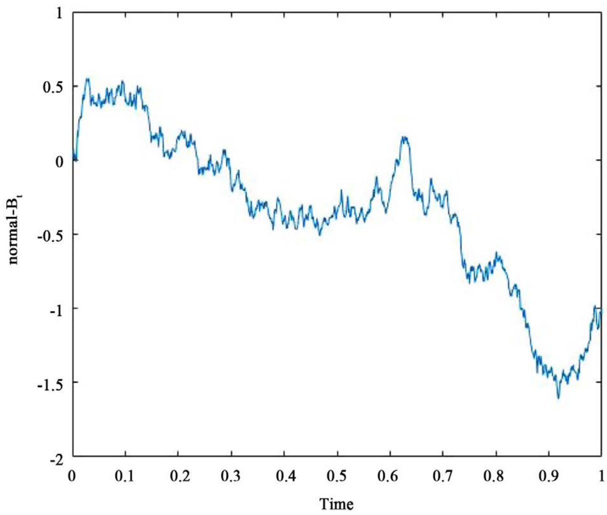

Figure 4. The normal B_t of simulation.

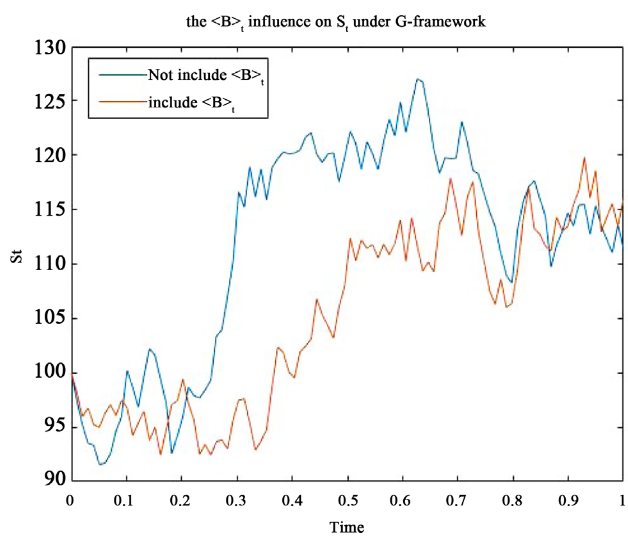

Figure 5. The $\langle B\rangle_t$ influences on $S(t)$.

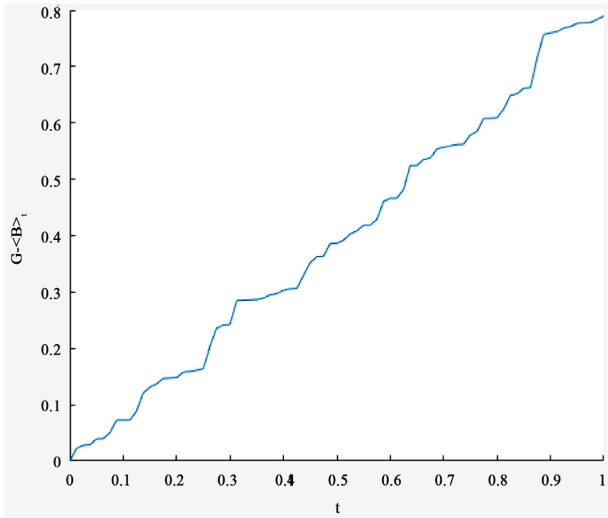

Figure 6. The $\langle B \rangle_t$ of simulation.

4. Conclusion

This article mainly proves that American call options that do not pay dividends under the G-framework are equal to European call options and simulate the G-B_t image. Comparing stock price images under different B_t, G-B_t. There is a restriction on G-B_t. When $\underline{\sigma}^2$ is smaller, the G-B_t of simulation shows a downward fluctuation. We need to find the appropriate range of $\underline{\sigma}^2$ to simulate the stock price.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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