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A Nontrivial Math Error in "Sophisticated Monetary Policies"

Rongyu Wang[1,2]

[1]School of Economics, The University of Edinburgh, Edinburgh, United Kingdom
[2]Institute of Information Research, Qilu University of Technology (Shandong Academy of Sciences), Jinan, China
Email: v1rwang7@ed.ac.uk

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Abstract

[1] is an important literature in the field of the fiscal theory of price level. However, I prove that the equations of (39) in [1] are wrong and provide the correct results. I also find that the mistake of equations of (39) affects the rest of [1], which is based on the establishment of equations of (39). I list all the results in [1] that cannot be established due to the mistake of equations of (39). Considering the importance of [1] in the field of fiscal theory of price level, all knowledge developed by far based on the results that cannot be established in [1] should be re-examined.

Keywords

Staggered-Pricing Model, New Keynesian Model, Fiscal Theory of Price Level

1. Introduction

In the *Quarterly Journal of Economics* article "Sophisticated Monetary Policies", [1] use two models, a sticky-price model with one-period price setting and a sticky-price model with staggered-price setting, the latter of which is also called the New Keynesian model. I have discovered that there is an error in their math, an error that might be damaging to some of the main ideas of their article.

According to ([2], p. 29), in the field of fiscal theory of price level, there have been two prominent papers by far that study the off-equilibrium behaviour with game theoretic foundations and [1] is one of the two. As of 4 February 2024, [1] had 108 Google Scholar citations and 40 Web of Science citations.

The complete abstract to [1] 2010 reads as follows:

> In standard monetary policy approaches, interest-rate rules often produce indeterminacy. A sophisticated policy approach does not. Sophisticated policies depend on the history of private actions, government policies, and

exogenous events and can differ on and off the equilibrium path. They can uniquely implement any desired competitive equilibrium. When interest rates are used along the equilibrium path, implementation requires regime-switching. These results are robust to imperfect information. Our results imply that the Taylor principle is neither necessary nor sufficient for unique implementation. They also provide a direction for empirical work on monetary policy rules and determinacy. ([1] 2010, abstract)

In the article [1] use the expression "pure interest-rate rule" to mean [the King rule, which is a rule using interest rates for all histories (p. 51 in [1])]. For the staggered-pricing model with pure interest-rate rules, the economy has a continuum of competitive equilibria, which is obtained by solving the equation group composed by [1]'s Equations (37) and (38). [1]'s equations of (39) are the solutions of the continuum of solutions. However, I find that the solutions expressed by equations of (39) are not correct. In the next section, I exhibit the correct solutions of the equation group composed by Equations (37) and (38). Also, I show that the incorrect solutions of (39) in [1] affect the establishment of all following results that are based on the staggered-pricing model with pure interest-rate rules in the rest of the paper. That means, due to the mistake of equations of (39) in [1], all results based on the staggered-pricing model with pure interest rules in [1] are not assured. In the final section I briefly treat the issue of how damaging the error is, confessing that I am not sure myself, because I have not fully worked through the implications of the error. Even though the math error in [1] is non-trivial and has comprehensive and fundamental impacts in particular in the field of game theory approach to fiscal theory of price level, it does not deny the entire contribution of the three authors, Andrew Atkeson, Varadarajan V. Chari and Patrick J. Kehoe, in economics and my respect to them.

2. The Correction of Equations of (39)

Equations (37) and (38) in [1] are as follows:

$$\begin{cases} \tilde{y}_{t+1} + \psi\tilde{\pi}_{t+1} = \tilde{y}_t + \psi\phi\tilde{\pi}_t \\ \tilde{\pi}_t = k\tilde{y}_t + \beta\tilde{\pi}_{t+1} \end{cases}$$

Equations (37) and (38) can be recaset in the following form:

$$\begin{pmatrix} 1 & \psi \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \tilde{y}_{t+1} \\ \tilde{\pi}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & \psi\phi \\ -k & 1 \end{pmatrix}\begin{pmatrix} \tilde{y}_t \\ \tilde{\pi}_t \end{pmatrix}$$

And the above matrix form of Equations (37) and (38) results in the following discrete dynamic system:

$$\begin{pmatrix} \tilde{y}_{t+1} \\ \tilde{\pi}_{t+1} \end{pmatrix} = A\begin{pmatrix} \tilde{y}_t \\ \tilde{\pi}_t \end{pmatrix} \tag{1}$$

where $A = \begin{pmatrix} a & b \\ -\frac{k}{\beta} & \frac{1}{\beta} \end{pmatrix}$ and $a = 1 + \frac{k\psi}{\beta}$ and $b = \psi\left(\phi - \frac{1}{\beta}\right)$.

Denote the determinant of matrix A by Δ. In the following, I solve the dynamic system (1) to obtain the continuum of equilibria. First, I obtain the eigenvalues of matrix A. Denote λ as the notation for eigenvalues. By solving the following quadratic equation, I obtain the eigenvalues λ_1 and λ_2:

$$|A-\lambda I| = \lambda^2 - \left(1+\frac{1+\psi k}{\beta}\right)\lambda + \frac{1}{\beta}\left(1+\frac{\psi k}{\beta}\right) + \frac{k}{\beta}\psi\left(\phi-\frac{1}{\beta}\right) = 0$$

Solving the above equation, I obtain that

$$\lambda_1 = \frac{\frac{1+k\psi}{\beta}+1-\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2-4(\phi-1)\frac{k\psi}{\beta}}}{2}$$

and

$$\lambda_2 = \frac{\frac{1+k\psi}{\beta}+1+\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2-4(\phi-1)\frac{k\psi}{\beta}}}{2}$$

A large enough ϕ can obviously make λ_1 and λ_2 imaginary numbers. However, in [1], they only discuss the case where λ_1 and λ_2 are real numbers. [1] never mention that there should be a restriction on the value of ϕ to ensure λ_1 and λ_2 real. Apparently, they have implicitly assumed that the value of ϕ will not make λ_1 and λ_2 imaginary. In this paper, I also implicitly assume that the value of ϕ ensures λ_1 and λ_2 real and hence focus on the case where λ_1 and λ_2 are real numbers.

Next, I will solve the equation group $(A-\lambda I)x=0$ to obtain the eigenvectors associated with each eigenvalue:

$$(A-\lambda I)x = \begin{pmatrix} 1+\frac{k\psi}{\beta}-\lambda & \psi\left(\phi-\frac{1}{\beta}\right) \\ -\frac{k}{\beta} & \frac{1}{\beta}-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

where $\lambda \in \{\lambda_1, \lambda_2\}$. Putting λ_1 into $(A-\lambda I)x=0$ and solving the corresponding $(A-\lambda I)x=0$, I obtain the eigenvector associated with λ_1, which is

$$v_1 = k_1\begin{pmatrix} 1 \\ -\frac{1+\frac{\psi k}{\beta}-\lambda_1}{\psi\left(\phi-\frac{1}{\beta}\right)} \end{pmatrix} = k_1\begin{pmatrix} 1 \\ -\frac{a-\lambda_1}{b} \end{pmatrix}$$

where k_1 is an arbitrary constant associated with the eigenvector.

Likewise, putting λ_2 into $(A-\lambda I)x=0$ and solving the corresponding $(A-\lambda I)x=0$, I obtain the eigenvector associated with λ_2, which is

$$v_2 = k_2\begin{pmatrix} 1 \\ -\frac{1+\frac{\psi k}{\beta}-\lambda_2}{\psi\left(\phi-\frac{1}{\beta}\right)} \end{pmatrix} = k_2\begin{pmatrix} 1 \\ -\frac{a-\lambda_2}{b} \end{pmatrix}$$

where k_2 is an arbitrary constant associated with the eigenvector.

Denote the following matrices:

$$D=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

and

$$C=(v_1,v_2)=\begin{pmatrix}k_1 & k_2\\ -\frac{a-\lambda_1}{b}k_1 & -\frac{a-\lambda_2}{b}k_2\end{pmatrix}$$

The initial values of the dynamic system are denoted by $\tilde{y}_0$ and $\tilde{\pi}_0$. Solving the dynamic system of (1), I obtain that

$$\begin{pmatrix}\tilde{y}_t\\ \tilde{\pi}_t\end{pmatrix}=CD^tC^{-1}\begin{pmatrix}\tilde{y}_0\\ \tilde{\pi}_0\end{pmatrix} \tag{2}$$

Equation (2) is a standard result in linear algebra and it can be found in many textbooks such as ([3], pp. 333 to 344). It can be obtained that

$$D^t=\begin{pmatrix}\lambda_1^t & 0\\ 0 & \lambda_2^t\end{pmatrix}$$

$$C^{-1}=\frac{1}{k_1k_2\frac{\lambda_2-\lambda_1}{b}}\begin{pmatrix}-\frac{a-\lambda_2}{b}k_2 & -k_2\\ \frac{a-\lambda_1}{b}k_1 & k_1\end{pmatrix}$$

Therefore, according to (2), I obtain the solution of the dynamic system (1) in the following steps:

$$\begin{aligned}
\begin{pmatrix}\tilde{y}_t\\ \tilde{\pi}_t\end{pmatrix}&=\frac{1}{k_1k_2\frac{\lambda_2-\lambda_1}{b}}\begin{pmatrix}k_1 & k_2\\ -\frac{a-\lambda_1}{b}k_1 & -\frac{a-\lambda_2}{b}k_2\end{pmatrix}\begin{pmatrix}\lambda_1^t & 0\\ 0 & \lambda_2^t\end{pmatrix}\begin{pmatrix}-\frac{a-\lambda_2}{b}k_2 & -k_2\\ \frac{a-\lambda_1}{b}k_1 & k_1\end{pmatrix}\begin{pmatrix}\tilde{y}_0\\ \tilde{\pi}_0\end{pmatrix}\\
&=\frac{1}{k_1k_2\frac{\lambda_2-\lambda_1}{b}}\begin{pmatrix}k_1\lambda_1^t & k_2\lambda_2^t\\ -\frac{a-\lambda_1}{b}k_1\lambda_1^t & -\frac{a-\lambda_2}{b}k_2\lambda_2^t\end{pmatrix}\begin{pmatrix}-\frac{a-\lambda_2}{b}k_2\tilde{y}_0-k_2\tilde{\pi}_0\\ \frac{a-\lambda_1}{b}k_1\tilde{y}_0+k_1\tilde{\pi}_0\end{pmatrix}\\
&=\frac{1}{k_1k_2\frac{\lambda_2-\lambda_1}{b}}\begin{pmatrix}\left(-\frac{a-\lambda_2}{b}k_2\tilde{y}_0-k_2\tilde{\pi}_0\right)k_1\lambda_1^t+\left(\frac{a-\lambda_1}{b}k_1\tilde{y}_0+k_1\tilde{\pi}_0\right)k_2\lambda_2^t\\ -\frac{a-\lambda_1}{b}\left(-\frac{a-\lambda_2}{b}k_2\tilde{y}_0-k_2\tilde{\pi}_0\right)k_1\lambda_1^t-\frac{a-\lambda_2}{b}\left(\frac{a-\lambda_1}{b}k_1\tilde{y}_0+k_1\tilde{\pi}_0\right)k_2\lambda_2^t\end{pmatrix}\\
&=\frac{1}{\frac{\lambda_2-\lambda_1}{b}}\begin{pmatrix}\left(\frac{\lambda_2-a}{b}\tilde{y}_0-\tilde{\pi}_0\right)\lambda_1^t+\left(\frac{a-\lambda_1}{b}\tilde{y}_0+\tilde{\pi}_0\right)\lambda_2^t\\ \frac{\lambda_1-a}{b}\left(\frac{\lambda_2-a}{b}\tilde{y}_0-\tilde{\pi}_0\right)\lambda_1^t+\frac{\lambda_2-a}{b}\left(\frac{a-\lambda_1}{b}\tilde{y}_0+\tilde{\pi}_0\right)\lambda_2^t\end{pmatrix}\\
&=\frac{1}{\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2-4(\phi-1)\frac{k\psi}{\beta}}}{\psi\left(\phi-\frac{1}{\beta}\right)}}\begin{pmatrix}\left(\frac{\lambda_2-a}{b}\tilde{y}_0-\tilde{\pi}_0\right)\lambda_1^t+\left(\frac{a-\lambda_1}{b}\tilde{y}_0+\tilde{\pi}_0\right)\lambda_2^t\\ \frac{\lambda_1-a}{b}\left(\frac{\lambda_2-a}{b}\tilde{y}_0-\tilde{\pi}_0\right)\lambda_1^t+\frac{\lambda_2-a}{b}\left(\frac{a-\lambda_1}{b}\tilde{y}_0+\tilde{\pi}_0\right)\lambda_2^t\end{pmatrix} \quad (3)\\
&=\begin{pmatrix}\omega_1\lambda_1^t+\omega_2\lambda_2^t\\ \omega_1\left(\frac{\lambda_1-a}{b}\right)\lambda_1^t+\omega_2\left(\frac{\lambda_2-a}{b}\right)\lambda_2^t\end{pmatrix}
\end{aligned}$$

where

$$\omega_1 = \frac{\frac{\lambda_2 - a}{b}\tilde{y}_0 - \tilde{\pi}_0}{\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2 - 4(\phi-1)\frac{k\psi}{\beta}}}{\psi\left(\phi - \frac{1}{\beta}\right)}}$$

and

$$\omega_2 = \frac{\frac{a - \lambda_1}{b}\tilde{y}_0 + \tilde{\pi}_0}{\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2 - 4(\phi-1)\frac{k\psi}{\beta}}}{\psi\left(\phi - \frac{1}{\beta}\right)}}$$

The vector (3) is the solution of the equation group composed by Equations (37) and (38) and hence the continuum of equilibria.

However, in [1], in their equations of (39), $\omega_1 = \frac{\frac{\lambda_2 - a}{b}\tilde{y}_0 - \tilde{\pi}_0}{\Delta}$ and $\omega_2 = \frac{\frac{a - \lambda_1}{b}\tilde{y}_0 + \tilde{\pi}_0}{\Delta}$. It can be shown that

$$\Delta \neq \frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2 - 4(\phi-1)\frac{k\psi}{\beta}}}{\psi\left(\phi - \frac{1}{\beta}\right)}$$

unless

$$\begin{cases} \Delta = \dfrac{-2 - \sqrt{4 + b^2\left(a + \frac{1}{\beta}\right)^2}}{b^2} \\ \phi < \dfrac{1}{\beta} \end{cases} \tag{4}$$

or

$$\begin{cases} \Delta = \dfrac{-2 + \sqrt{4 + b^2\left(a + \frac{1}{\beta}\right)^2}}{b^2} \\ \phi > \dfrac{1}{\beta} \end{cases} \tag{5}$$

The procedures to derive the above results are presented in Appendix. However, it can be shown that neither Equation (4) nor Equation (5) can be satisfied in [1]'s context.

First, let us examine Equation (4). Because $\phi < \frac{1}{\beta}$, therefore $b < 0$. The determinant of A $\Delta = \frac{a+kb}{\beta}$. The equation $\Delta = \frac{-2-\sqrt{4+b^2\left(a+\frac{1}{\beta}\right)^2}}{b^2}$ can be equivalently reformulated to

$$\frac{a+kb}{\beta} = \frac{-2-\sqrt{4+b^2\left(a+\frac{1}{\beta}\right)^2}}{b}$$

Because $a+kb = 1+k\psi\phi > 1$, therefore the LHS of above equation is positive and the RHS of the above equation is negative. Therefore, the Equation (4) cannot be established in [1]'s context.

Second, let us examine Equation (5). In this case, because $\phi > \frac{1}{\beta}$, therefore $b > 0$. The equation $\Delta = \frac{-2+\sqrt{4+b^2\left(a+\frac{1}{\beta}\right)^2}}{b^2}$ can be equivalently reformulated to

$$\beta = (a+kb)\frac{\sqrt{4+b^2\left(a+\frac{1}{\beta}\right)^2}+2}{a+\frac{1}{\beta}}$$

It is known that discount factor $\beta \in (0,1)$ and $a+kb > 1$. Therefore, to ensure Equation (5) can be held, it requires that $\frac{\sqrt{4+b^2\left(a+\frac{1}{\beta}\right)^2}+2}{a+\frac{1}{\beta}} < 1$, which can be equivalently reformulated to

$$\sqrt{4+b^2\left(a+\frac{1}{\beta}\right)^2} < a+\frac{1}{\beta}-2 \tag{6}$$

1) If $a+\frac{1}{\beta}-2 < 0$, then inequality (6) cannot be held;

2) If $a+\frac{1}{\beta}-2 > 0$, then inequality (6) can be equivalently reformulated to

$$\frac{4}{a+\frac{1}{\beta}} < 1-b^2 \tag{7}$$

2-1) If $1 < b^2$, because $\frac{4}{a+\frac{1}{\beta}} > 0$, then inequality (7) cannot be held;

2-2) Suppose $1 > b^2$. The prerequisite to ensure inequality (7) held is

$\frac{4}{a+\frac{1}{\beta}}<1$, which can be equivalently reformulated to

$$\gamma\psi > (3\beta - 1)\frac{1-\alpha}{\alpha} \tag{8}$$

In the context of [1], ψ is the intertemporal elasticity in the Euler Equation (Equation (1) of [1]) and γ is the elasticity of equilibrium real wage with respect to output, *i.e.* Taylor's γ. However, by far, there is no empirical evidence to support inequality (8) can be established. Therefore, inequality (6) and hence Equation (5) cannot be established in [1]'s context.

Therefore, the correct result to express the continuum of competitive equilibria under the staggered price-setting of the model with pure interest-rate rules of [1] is my vector (3).

The mistake of equations of (39) has comprehensive implications to the rest of [1]. First, the equations of (42) in Proposition 5 cannot be obtained. Conditions (41) are obtained according to (42) and therefore conditions (41) cannot be established as well. Second, the core result of Proposition 6 is that the staggered-pricing model with King-money hybrid rules implements a unique equilibrium, but the proof of the result relies on equations of (39) and (40). Therefore, Proposition 6 cannot be established. Third, the establishment of Proposition 7 is partly based on Proposition 6. Because Proposition 6 cannot be held, therefore Proposition 7 cannot apply to the staggered-pricing model with King-money hybrid rules. Fourth, the explanation that adherence to Taylor principle is neither necessary nor sufficient to implement a unique equilibrium in Section V.B. depends on the establishment of conditions (41) and Proposition 6. Therefore, due to the failure of the establishment of conditions (41) and Proposition 6, the explanation and hence its related result cannot be established. Fifth, discussing section V.C., implications for estimation, depends on the establishment of Proposition 6. Due to the failure of the establishment of Proposition 6, the results presented in this section cannot be established as well.

In the following, I list the results in [1] that cannot be established due to the mistake of equations of (39). Specifically, [1] cannot support the following results under staggered-price setting. These results are said by [1] but they cannot be established due to the wrong equations of (39) in [1]:

1) Proposition 5: the pure interest-rate rules cannot uniquely implement bounded competitive equilibrium (Section III.B.);

2) Proposition 6: a King-money hybrid rule can uniquely implement any bounded competitive equilibrium (Section III.B.);

3) Proposition 7 does not apply, *i.e.* under trembles and imperfect information, the King-money hybrid rule implements a unique equilibrium and the equilibrium converges to the desired outcome as the variance of measurement errors (*i.e.* trembles and imperfect information) goes to zero (Section IV);

4) Adherence to Taylor principle is neither necessary nor sufficient to implement a unique equilibrium (Section V.B.);

5) Estimating economies with perfect information cannot identify which regions (determinate or indeterminate regions) monetary policies incentivize the economy going to (Section V.C.);

6) With imperfect information, there exist some estimation procedures that can uncover some critical parameters for determinacy, even if researchers are willing to accept some strong assumptions to do the estimation (Section V.C.).

3. The Consequence Due to the Mistake of Equations (39) in [1]

As I have presented, all results related to the staggered-pricing model or the New Keynesian model with pure interest-rate rules in [1] cannot be maintained. Therefore, all knowledge developed based on the results related to the New Keynesian model with pure interest-rate rules in [1] needs to be re-examined.

The paper [1] is very influential. As presented, by far it has 108 citations. However, some papers and book citing [1] have much more citations than 108, where readers can see how influential [1] is. For example, [4] has 545 citations. Professor John Cochrane's famous book [2] has 213 citations. [5] has 371 citations. In the Handbook of Monetary Economics, [6] has 152 citations. Apparently, the reason why those papers cite [1] is that the paper is thought to have a significant influence in this field. However, the prerequisite to underpin a paper's influence is its validity. Therefore, the impact of [1] is very extensive. All following researches built on [1] are expected to be shaken due to the math error in their paper.

Based on my knowledge, there has been no empirical literature on game theory approach to fiscal theory of price level. Therefore, the existing literature in this field focuses on theoretical discussion. Considering that only two prominent papers so far, *i.e.* [1] and [7], characterizing the off-equilibrium behaviour with game theoretic foundations, which emphasizes the importance of [1] in the development of the field of fiscal theory of price level, especially ([2], p. 521) stating that `This approach (game theoretic approach to fiscal theory of price level) is surely right in a deep sense', therefore the re-examination of all results related to the New Keynesian model with pure interest rules in [1] are expected to have a fundamental and comprehensive impact on the existing knowledge of the field.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Atkeson, A., Chari, V.V. and Kehoe, P.J. (2010) Sophisticated Monetary Policies.

Quarterly Journal of Economics, **125**, 47-89.
https://doi.org/10.1162/qjec.2010.125.1.47

[2] Cochrane, J. (2023) The Fiscal Theory of Price Level. Princeton University Press, Princeton. https://doi.org/10.1353/book.109854

[3] Margalit, D., Rabinoff, J. and Williams, B. (2024) Interactive Linear Algebra.
https://personal.math.ubc.ca/~tbjw/ila/ila.pdf

[4] Cochrane, J. (2011) Determinacy and Identification with Taylor Rules. *Journal of Political Economy*, **119**, 565-615. https://doi.org/10.1086/660817

[5] Mertens, K. and Raven, M. (2014) Fiscal Theory in an Expectations-Driven Liquidity Trap. *Review of Economic Studies*, **81**, 1637-1667.
https://doi.org/10.1093/restud/rdu016

[6] Canzoneri, M., Cumby, R., and Diba, B. (2010) The Interaction Between Monetary and Fiscal Policy. *Handbook of Monetary Economics*, **3**, 935-999.
https://doi.org/10.1016/B978-0-444-53454-5.00005-0

[7] Bassetto, M. (2002) A Game-Theoretic View of the Fiscal Theory of the Price Level. *Econometrica*, **70**, 2167-2195. https://doi.org/10.1111/1468-0262.00373

Appendix: Derivation of Equation (4) and Equation (5)

The fraction $\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2-4(\phi-1)\frac{k\psi}{\beta}}}{\psi\left(\phi-\frac{1}{\beta}\right)}$ can be equivalently expressed by

$$\frac{\sqrt{\left(a+\frac{1}{\beta}\right)^2-4\Delta}}{b}.$$

The proof is as following:

$$\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2-4(\phi-1)\frac{k\psi}{\beta}}}{\psi\left(\phi-\frac{1}{\beta}\right)}$$

$$=\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2+4\left(1-\phi-\frac{1}{\beta}+\frac{1}{\beta}\right)\frac{k\psi}{\beta}}}{\psi\left(\phi-\frac{1}{\beta}\right)}$$

$$=\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2+4\left(1-\phi-\frac{1}{\beta}+\frac{1}{\beta}\right)\frac{k\psi}{\beta}-4\left(1-\frac{1}{\beta}\right)^2+4\left(1-\frac{1}{\beta}\right)^2}}{\psi\left(\phi-\frac{1}{\beta}\right)}$$

$$=\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2+4\left(1-\frac{1}{\beta}\right)\frac{k\psi}{\beta}-4\left(1-\frac{1}{\beta}\right)^2+4\left(1-\frac{1}{\beta}\right)^2-4\frac{k\psi}{\beta}\left(\phi-\frac{1}{\beta}\right)}}{\psi\left(\phi-\frac{1}{\beta}\right)}$$

$$=\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2+4\left(1-\frac{1}{\beta}\right)\left(\frac{k\psi}{\beta}+\frac{1}{\beta}-1\right)+4\left(1-\frac{1}{\beta}\right)^2-4\frac{k\psi}{\beta}\left(\phi-\frac{1}{\beta}\right)}}{\psi\left(\phi-\frac{1}{\beta}\right)}$$

$$=\frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2+4\left(1-\frac{1}{\beta}\right)\left(\frac{1+k\psi}{\beta}-1\right)+4\left(1-\frac{1}{\beta}\right)^2-4\frac{k\psi}{\beta}\left(\phi-\frac{1}{\beta}\right)}}{\psi\left(\phi-\frac{1}{\beta}\right)}$$

$$=\frac{\sqrt{\left[2\left(1-\frac{1}{\beta}\right)+\frac{k\psi+1}{\beta}-1\right]^2-4\frac{k\psi}{\beta}\left(\phi-\frac{1}{\beta}\right)}}{\psi\left(\phi-\frac{1}{\beta}\right)}$$

$$=\frac{\sqrt{\left(a-\frac{1}{\beta}\right)^2-4\frac{kb}{\beta}}}{b}=\frac{\sqrt{a^2+2\frac{a}{\beta}+\frac{1}{\beta^2}-4\frac{a}{\beta}-4\frac{kb}{\beta}}}{b}$$

$$= \frac{\sqrt{\left(a+\frac{1}{\beta}\right)^2 - 4\frac{a+kb}{\beta}}}{b} = \frac{\sqrt{\left(a+\frac{1}{\beta}\right)^2 - 4\Delta}}{b}$$

The equation $\Delta = \frac{\sqrt{\left(\frac{1+k\psi}{\beta}-1\right)^2 - 4(\phi-1)\frac{k\psi}{\beta}}}{\psi\left(\phi-\frac{1}{\beta}\right)}$ therefore can be equivalently expressed by

$$\Delta = \frac{\sqrt{\left(a+\frac{1}{\beta}\right)^2 - 4\Delta}}{b}.$$

Suppose the signs of both sides of $\Delta = \frac{\sqrt{\left(a+\frac{1}{\beta}\right)^2 - 4\Delta}}{b}$ are same. Under this prerequisite, the equation $\Delta = \frac{\sqrt{\left(a+\frac{1}{\beta}\right)^2 - 4\Delta}}{b}$ can be equivalently reformulated to a quadratic function

$$b^2\Delta^2 + 4\Delta - \left(a+\frac{1}{\beta}\right)^2 = 0$$

which results the following solutions:

$$\Delta = \frac{-2 \pm \sqrt{4 + b^2\left(a+\frac{1}{\beta}\right)^2}}{b^2}$$

It can be observed that one solution of Δ is negative and one solution of Δ is positive.

Therefore, only when $\phi < \frac{1}{\beta}$, the negative solution of Δ that supports the equation $\Delta = \frac{\sqrt{\left(a+\frac{1}{\beta}\right)^2 - 4\Delta}}{b}$ can exist; only when $\phi > \frac{1}{\beta}$, the positive solution of Δ that supports the equation $\Delta = \frac{\sqrt{\left(a+\frac{1}{\beta}\right)^2 - 4\Delta}}{b}$ can exist, which indicates that

$$\begin{cases} \Delta = \dfrac{-2 - \sqrt{4 + b^2\left(a+\frac{1}{\beta}\right)^2}}{b^2} \\ \phi < \dfrac{1}{\beta} \end{cases}$$

or

$$\begin{cases} \Delta = \dfrac{-2 + \sqrt{4 + b^2 \left(a + \dfrac{1}{\beta}\right)^2}}{b^2} \\ \phi > \dfrac{1}{\beta} \end{cases}$$

The above equations are Equation (4) and Equation (5) respectively.