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A Dark Energy Hypothesis VII

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Abstract

Application of the Klein-Gordon equation to a Dark Energy Hypothesis (DEH) leads to the hypothesis that dark matter couples to spacetime structure. The Compton wavelength is the scale at which coupling occurs.

Keywords

Klein-Gordon Equation, Dark Matter, Hyperbolic Spacetime

1. Introduction

By the mathematics of DEH I [1], the uncoupling of conformal time, η, and cosmic latitude, χ, in the early universe produces two products as by a phase change: a variable cosmological parameter, Λ, and a spacetime metric, $\mathrm{d}s^2$.

$$\mathrm{d}s^2 = a^2\left[\mathrm{d}\eta^2 - \mathrm{d}\chi^2 - f(\chi)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{1}$$

The theme of the DEH series of papers is that these represent dark energy and dark matter. In Equation (1), "a" is the scale factor: $a\mathrm{d}\eta = c\mathrm{d}t$. DEH II [2] argues that space is hyperbolic, in which case $f(\chi) = \sinh^2\chi$, and proposes that the dark matter particle is spinless. DEH IV [3] proposes an equation of state for dark matter and DEH VI [4] describes an interaction of dark matter with baryonic matter.

All numerical work in the DEH series of papers supposes that the energy inventory in the current epoch is that dark energy/dark matter/baryonic matter are in the proportions of 70/25/5.

This paper explores a DEH through the Klein-Gordon equation, [5] also known as the relativistic Schrödinger equation for a spinless particle [6].

2. The Klein-Gordon Equation

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \mu^2\psi \tag{2}$$

where $1/\mu = \hbar/mc$ is the Compton wavelength of a particle of mass m, which in this article is that of a dark matter particle.

The sources of this equation are in special relativity and Schrödinger's wave mechanics. Its construction is straightforward. In special relativity, the relationship between the energy, E, of a particle and its momentum, p, is

$$E^2 = p^2c^2 + m^2c^4$$

Combination of this with Schrödinger's prescription for converting E and p into operators leads to the Klein-Gordon equation:

$$E \Rightarrow \mathrm{i}\hbar\frac{\partial}{\partial t} \quad \text{and} \quad p \Rightarrow \frac{\hbar}{\mathrm{i}}\nabla$$

The wave function depends on the three comoving coordinates and the cosmic time: $\psi = \psi(\chi,\theta,\phi,t)$. Given a metric, the construction of the Laplacian is straightforward [7]:

$$\nabla^2 = \frac{\hat{O}(\chi) + \hat{O}(\theta,\phi)}{a^2\sinh^2(\chi)}$$

The Laplacian is for hyperbolic space as suggested by the hyperbolic sine. The symbol $\hat{O}$ is for operators: for the cosmic latitude

$$\hat{O}(\chi) = \frac{\partial}{\partial\chi}\left(\sinh^2\chi\frac{\partial}{\partial\chi}\right)$$

The foci of this paper are the radial (cosmic latitude) and time-dependent wave functions. The operator over angles is the associated Legendre equation and the wave functions are the spherical harmonics, $Y(\theta, \phi)$, but given that angular behavior is not a focus, the angular solutions are acknowledged only in the separation of variables, but not further considered.

3. Separation of Variables

Since the scale factor is a function of time but not of space, multiply through the Klein-Gordon equation by a^2, which sets the stage for separating space and time coordinates, and then space coordinates:

$$\psi(\chi,\theta,\phi,t) = F(\chi,\theta,\phi)G(t) = A(\chi)Y(\theta,\phi)G(t)$$

Two ordinary differential equations result:

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\left(\sinh^2\chi\frac{\mathrm{d}A}{\mathrm{d}\chi}\right) + \left[\beta + \alpha^2\sinh^2\chi\right]A = 0 \tag{3}$$

$$\left(\frac{a}{c}\right)^2\frac{\mathrm{d}^2G}{\mathrm{d}t^2} + \left[(a\mu)^2 + \alpha^2\right]G = 0 \tag{4}$$

where β and α^2 are separation constants.

4. Time-Dependence

In Equation (4), the constant α^2 is negligible compared to the other term in the square brackets and can be dropped. Integration is trivial:

$$G(t) = \exp(-i\omega t)$$

where the angular frequency is

$$\omega = \frac{c}{\lambda_C} = \frac{mc^2}{\hbar} \quad (5)$$

The paper DEH VI explores an example of the interaction of baryonic matter with dark matter. The specific example is the effect of dark matter on the recombination of an electron and proton into a hydrogen atom. The broad conclusion is that WIMPs (Weakly Interacting Massive Particles) do not interact at all, whereas interaction appears over the mass range 5×10^{-38} kg to 1×10^{-36} kg, the lower mass corresponding to about 30 meV, somewhat axion-like. The least massive WIMP has a mass of about 10 GeV, $m = 10^{-26}$ kg, which is about 10 amu, and $\lambda_C = 4 \times 10^{-17}$ m. By contrast for the axion-like particle, $\lambda_C = 7 \times 10^{-6}$ m. From this point of view, the result is not surprising since the Compton wavelength represents the effective size of the particle, which is its interactive range; consequently the axion-like particle has an interactive "radius" 10^{11} times greater than that of the least massive WIMP.

5. The Radial Equation

Harmonic structure of spacetime. The goal here is to argue that spacetime has a harmonic (periodic) structure, and that since hyperbolic functions exhibit no periodicity, the procedure will be to replace them by trigonometric functions. Equation (3) can be rewritten as

$$\sinh^2\chi \frac{\mathrm{d}^2 A}{\mathrm{d}\chi^2} + 2\sinh(\chi)\cosh(\chi)\frac{\mathrm{d}A}{\mathrm{d}\chi} + \left[\beta + \alpha^2 \sinh^2\chi\right]A = 0$$

The parameter β is not just a separation constant but also an eigenvalue of the associated Legendre equation: $\beta = l(l+1)$, where $l = 0,1,2,\cdots$, which is the orbital angular momentum quantum number. (The notation is that of reference [7], Equations (18-12) and (18-25)). Choose $l = 0$, in effect selecting the s-wave from the set of spherical harmonics and set $\alpha^2 = -1$. Then the above equation becomes

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\chi^2} + 2\coth(\chi)\frac{\mathrm{d}A}{\mathrm{d}\chi} - A = 0$$

Trigonometric functions can be generated from hyperbolic by a familiar mapping: make the replacement $\chi \to \mathrm{i}\chi$; then $\sinh(\mathrm{i}\chi) \to \mathrm{i}\sin(\chi)$ and $\cosh(\mathrm{i}\chi) \to \cos(\chi)$. The hyperbolic equation becomes trigonometric:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\chi^2} + 2\cot(\chi)\frac{\mathrm{d}A}{\mathrm{d}\chi} + A = 0$$

To integrate, transform $A(\chi)$ into a wave function $B(\chi)$ that has no first derivative by means of the following algorithm [8]:

$$A(\chi) = B(\chi)\exp\left[-\frac{1}{2}\int \xi \mathrm{d}\chi\right]$$

where ξ is the coefficient of the first derivative. This produces a differential equation of the form

$$\frac{\mathrm{d}^2 B}{\mathrm{d}\chi^2} + f(\chi) B = 0$$

In this problem, the result is simply $f(\chi) = 2$. Hence,

$$B = c_1 \sin\left(\chi\sqrt{2}\right) + c_2 \cos\left(\chi\sqrt{2}\right) \tag{6}$$

$B(\chi)$ is a probability amplitude, which implies that $c_2 = 0$, permitting $B(0) = 0$ as a boundary condition. Let the range of the cosmic latitude be all of the visible universe: $0 \le \chi \le \chi_{\mathrm{PH}}$, where PH means the particle horizon, which in turn is $0 \le \chi \le \eta$, where the upper limit is the conformal time for the epoch in question. This limit is hardly restrictive since it includes everything that can be observed in principle and excludes everything that in principle couldn't be observed anyway. To have a harmonic structure requires both $B(0) = B(\eta) = 0$, so make the replacement $\sqrt{2} \to n\pi/\eta$, where $n = 1, 2, \cdots$

$$B(\chi) = \sqrt{\frac{2}{\eta}} \sin\left(\frac{n\pi\chi}{\eta}\right) \tag{7}$$

This is a formula seen by every undergraduate student of physical chemistry and physics.

<u>Application of harmonicity to dark matter.</u> Operators for position and momentum are

$$\hat{x} = x \quad \text{and} \quad \hat{p} = \frac{\hbar}{\mathrm{i}} \nabla = 3 \frac{\hbar}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x}$$

The factor of three arises from the isotropy of space. The cosmic latitude χ is a dimensionless comoving coordinate. To relate it to position and momentum means multiplying it by an appropriate scale factor. For applications on the cosmological scale that factor would be $a \sim 10^{26}$ m in the current epoch; for applications on the dark matter scale that factor would be the Compton wavelength, λ_C, the latter differing from the former by thirty to forty orders of magnitude. That is the meaning of being coupled to spacetime structure: the microscopic participates in the macroscopic but at a small scale of size.

Hence, for dark matter

$$x = \lambda_c \chi \quad \& \quad \hat{p} = \frac{\hbar}{\mathrm{i}} \nabla = 3 \frac{\hbar}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x} = \frac{3\hbar}{\mathrm{i}\lambda_C} \frac{\mathrm{d}}{\mathrm{d}\chi}$$

The mean momentum vanishes, $\langle p \rangle = 0$, because of the boundary conditions at $\chi = 0$ and $\chi = \eta$. The mean square momentum, however, is an eigenvalue of Equation (7). The expectation value for the energy is

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{1}{2m} \left(\frac{3\pi nmc}{\eta}\right)^2$$

The standard assumption is that dark matter is cold, which permits the use of the equation for non-relativistic momentum. The result is

$$E = \frac{n^2 mc^2}{\eta^2} \frac{9\pi^2}{2} \quad (8)$$

By definition, the dark matter rest mass energy is $E(\mathrm{dm}) = mc^2$, so the particle horizon distance rescales it; $\eta = 5.571$ in the current epoch. The ratio mc^2/η^2 is a ratio of particle to cosmic magnitudes, which illustrates the coupling. The quantum number n defines the particle's momentum state. For $n = 1$, $\langle E \rangle = 1.43E$ (dm) in the current epoch.

Presumably a dark matter particle occupies a specific location (χ, θ, ϕ) on the grid of comoving coordinates, but it must not violate the Uncertainty Principle. A short calculation (details omitted) illustrates this for the $n = 1$ state.

$$\delta\chi^2 = \left[\left\langle \chi^2 \right\rangle - \left\langle \chi \right\rangle^2\right]$$

$$\delta x = \frac{\hbar}{mc}\delta\chi = \frac{\hbar}{mc} \cdot 0.181\eta$$

$$\delta p = \left\langle p^2 \right\rangle^{1/2} = \frac{3\pi mc}{\eta}$$

$$\delta x \delta p = 1.70\hbar > \hbar/2$$

6. Thermodynamics of Dark Matter

Tolman [9] relates thermodynamic properties to those objects subject to the laws of special relativity, and since the Klein-Gordon equation is rooted in special relativity, his analysis is germane here. He notes that pressure is an absolute, meaning that in the context of special relativity its numerical value is independent of the observer's state of inertial motion. That is particularly relevant to a dark energy hypothesis since DEH IV proposes an equation of state for dark matter.

7. Conclusions

The notions of harmonic structure and coupling are hypotheses, but once given hopefully the above analysis from beginning to end is cogent.

Spacetime has a harmonic structure whose size in a given epoch is given by the scale factor, a. Dark matter participates in this structure at a scale of size given by the Compton wavelength, which is called coupling, an interdependence of the large and the small. As a consequence, measurement of the mass of a dark matter particle will be scaled by a spacetime parameter, the dimensionless distance to the particle horizon in the above analysis; in other words, measurement of a microscopic quantity brings with it a cosmological one. The coupling means that dark matter is an inextricable part of spacetime; it does not just happen to be there. In scholastic language, dark matter is present essentially, not accidentally.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Togeas, J. (2024) A Dark Energy Hypothesis I. *Journal of High Energy Physics, Grav-*

itation and Cosmology, **10**, 1138-1141. https://doi.org/10.4236/jhepgc.2024.103068

[2] Togeas, J. (2024) A Dark Energy Hypothesis II. *Journal of High Energy Physics*, *Gravitation and Cosmology*, **10**, 1142-1151. https://doi.org/10.4236/jhepgc.2024.103069

[3] Togeas, J. (2025) A Dark Energy Hypothesis IV. *Journal of High Energy Physics*, *Gravitation and Cosmology*, **11**, 45-55. https://doi.org/10.4236/jhepgc.2025.111006

[4] Togeas, J. (2025) A Dark Energy Hypothesis VI. *Journal of High Energy Physics*, *Gravitation and Cosmology*, **11**, 600-606. https://doi.org/10.4236/jhepgc.2025.112042

[5] Messiah, A. (1966) Quantum Mechanics. Vol. II. John Wiley & Sons, Ch. XX, § 5.

[6] Schrödinger, E. (1956) Expanding Universes. Cambridge University Press, Ch. IV.

[7] Pauling, L. and Wilson, E.B. (1935) Introduction to Quantum Mechanics. McGraw-Hill, Ch. IV, § 16.

[8] Margenau, H. and Murphy, G.M. (1956) The Mathematics of Physics and Chemistry. 2nd Edition, D. Van Nostrand, § 2.10.

[9] Tolman, R.C. (1934) Relativity, Thermodynamics and Cosmology. The Clarendon Press, Ch. V.