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Limits on Application of the Formula for Potential Energy of a Hydrogen Atom and a Previously Unknown Formula

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Abstract

The author has previously shown that ultra-low energy levels exist among the energy levels of a hydrogen atom, in addition to the energy levels predicted by quantum mechanics. The author has also pointed out that the reduction in rest mass energy of the electron $m_e c^2$ corresponds to potential energy of an electron inside a hydrogen atom. Based on this idea, when an electron that has been taken into the region of a hydrogen atom comes to the point of the classic electron radius r_e, its potential energy becomes $-m_e c^2$, and the electron's rest mass energy is depleted. Therefore, $r_e \le r < \infty$ is the region where the well-known formula for potential energy of an electron is applicable. Rest mass energy is not sufficient for an electron to approach closer than that to the atomic nucleus. Under these conditions, an electron cannot attain ultra-low energy levels. Thus, taking a hint from renormalization theory, the author has predicted that the energy of a stationary electron is actually not $m_e c^2$, and instead that the electron has a photon energy of $2m_e c^2$ and a negative energy specific to the electron of $-m_e c^2$. When this model is used, it becomes possible for the electron to approach the point $r = r_e/4$ regarded as the radius of the proton. This paper derives a new formula for potential energy of an electron in regions outside the scope of application of the existing formula for potential energy.

Keywords

Potential Energy, Einstein's Energy-Momentum Relationship, Energy-Momentum Relationship in a Hydrogen Atom, Ultra-Low Energy Levels in a Hydrogen Atom, Negative Energy Specific to the Electron, $n = 0$ Energy Level

1. Introduction

This paper discusses the limits on application of the formula for potential energy of an electron in a hydrogen atom. Prior to that, let us review the energy levels of a hydrogen atom at the level of classical quantum theory.

Earlier, Bohr derived the following formula for energy levels by assuming the quantum condition [1].

$$E_{\mathrm{BO},n} = -\frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{m_{\mathrm{e}}e^4}{\hbar^2}\cdot\frac{1}{n^2}, \qquad n = 1, 2, \cdots. \tag{1}$$

here, $E_{\mathrm{BO},n}$ are the energy levels of a hydrogen atom derived by Bohr. Also, n is the principal quantum number.

Formula (1) can be written as follows.

$$E_{\mathrm{BO},n} = -\frac{m_{\mathrm{e}}c^2}{2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2\cdot\frac{1}{n^2} = -\frac{\alpha^2 m_{\mathrm{e}}c^2}{2n^2}. \tag{2}$$

here, α is the following fine-structure constant.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = 7.2973525643\times10^{-3}. \tag{3}$$

Bohr also derived the following radius for the electron's circular orbit.

$$r_{\mathrm{BO},n} = 4\pi\varepsilon_0\frac{\hbar^2}{m_{\mathrm{e}}e^2}\cdot n^2. \tag{4}$$

Formula (4) can be written as follows.

$$r_{\mathrm{BO},n} = \frac{e^2}{4\pi\varepsilon_0 m_{\mathrm{e}}c^2}\left(\frac{4\pi\varepsilon_0\hbar c}{e^2}\right)^2\cdot n^2 = \frac{r_{\mathrm{e}}}{\alpha^2}\cdot n^2. \tag{5}$$

r_{e} is the classical electron radius of the electron, given by the following formula.

$$r_{\mathrm{e}} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_{\mathrm{e}}c^2}. \tag{6}$$

Since $\alpha^{-2}\approx 18779$, r_n is far larger than r_{e}.

When the product of $E_{\mathrm{BO},n}$ and $r_{\mathrm{BO},n}$ is found here, the result is the following constant value.

$$E_{\mathrm{BO},n}r_{\mathrm{BO},n} = -m_{\mathrm{e}}c^2\cdot\frac{r_{\mathrm{e}}}{2}. \tag{7}$$

In classical quantum theory, the total mechanical energy of a hydrogen atom is defined as the sum of the potential energy and kinetic energy of the electron. That is,

$$E_n = V(r_n) + K_n,\ \ E_n < 0. \tag{8}$$

Also, the potential energy of an electron is given by the following formula.

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{9}$$

According to the Virial theorem, $2K = -V(r)$ in the case of a circular orbit, and thus the energy can be written as follows.

$$E_n = V(r_n) + K_n = \frac{V(r_n)}{2} = -K_n = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n}. \tag{10}$$

Now, if $E_{\mathrm{ph},n}$ is used to represent the photon energy emitted when an electron placed an infinite distance away from the atomic nucleus (proton) of the hydrogen atom is taken into the hydrogen atom, then the following law of energy conservation holds for the electron.

$$V(r_n) + K_n + E_{\mathrm{ph},n} = 0. \tag{11}$$

here, the "ph" subscript of $E_{\mathrm{ph},n}$ stands for "photon".

Normally, the energy of a photon is written as $h\nu$, but when multiple photons are emitted, $E_{\mathrm{ph},n}$ indicates the total energy of those photons.

Formula (11) shows that the energy source for the kinetic energy acquired by an electron and the photon energy emitted by the electron is the potential energy of the electron.

The relationship between the rest mass energy of the electron $m_{\mathrm{e}}c^2$ and the relativistic energy of the electron $m_n c^2$ is as follows.

$$m_n c^2 = m_{\mathrm{e}}c^2 + E_{\mathrm{re},n} = \left(m_{\mathrm{e}}c^2 + V(r_n)\right) + K_{\mathrm{re},n}. \tag{12}$$

here, $m_n c^2$ is the sum of the residual part of the rest mass energy of the electron $\left(m_{\mathrm{e}}c^2 + V(r_n)\right)$ and the relativistic kinetic energy $K_{\mathrm{re},n}$. $E_{\mathrm{re},n}$ are the relativistic energy levels of a hydrogen atom [2].

The relationship between $E_{\mathrm{re},n}$ and other energy is as follows.

$$E_{\mathrm{re},n} = -E_{\mathrm{ph},n} = -K_{\mathrm{re},n}. \tag{13}$$

Incidentally, the author has previously pointed out that the reduction in rest mass energy of an electron corresponds to the potential energy of the electron.

Here, if the reduction in rest mass energy of the electron is represented as $-\Delta m_{\mathrm{e}}c^2$, then the potential energy of the electron can be defined as follows [2] [3].

$$V(r) = -\Delta m_{\mathrm{e}}c^2. \tag{14}$$

In classical quantum theory, it was promised that the potential energy of an electron placed at the position $r = \infty$ would be zero. It was thought that the energy of an electron in this state would also be zero.

However, the view of the author is that the potential energy of an electron placed at the position $r = \infty$ will actually be zero. Also, this electron has a rest mass energy of $m_{\mathrm{e}}c^2$.

However, the existence of the rest mass energy of an electron has not been considered in the Bohr model of the hydrogen atom. This is a shortcoming of the Bohr model.

The r where potential energy of an electron becomes $-m_{\mathrm{e}}c^2$ can be derived from the following formula. That is,

$$-m_e c^2 = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{15}$$

Hence,

$$r = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_e c^2} = r_e. \tag{16}$$

If we think about this simply, without knowing Bohr's quantum condition, the electron can approach the atomic nucleus up to the point r_e where the rest mass energy is depleted.

Incidentally, the potential energy of an electron is given by the following formula in classical quantum theory.

$$V(r_n) = 2E_{BO,n} = -\frac{\alpha^2 m_e c^2}{n^2}. \tag{17}$$

Since $\alpha^2 \approx 1/18779$, the rest mass energy of the electron is hardly consumed, even if the electron drops to the ground state.

For the potential energy of an electron in a hydrogen atom in classical quantum theory, it is sufficient to consider the region $r_1 < r < \infty$. Since $r_e \ll r_1$, there is no need to discuss the limits on application of Formula (9) in classical quantum theory.

2. Ultra-Low Energy Levels of a Hydrogen Atom

The following is the most famous formula discovered by Einstein [4].

$$E = mc^2. \tag{18}$$

A body with mass m has an energy of mc^2.

According to the special theory of relativity, the following relationship holds between the energy and momentum of a body moving in free space [5].

$$\left(mc^2\right)^2 = \left(m_0 c^2\right)^2 + c^2 p^2. \tag{19}$$

here, $m_0 c^2$ is the rest mass energy of the body. And mc^2 is the relativistic energy.

Formula (19), which is called Einstein's energy-momentum relationship, holds when the energy absorbed by a body is all converted to kinetic energy of that body.

Also, Einstein and Sommerfeld defined the relativistic kinetic energy as follows [6].

$$K_{re} = mc^2 - m_0 c^2. \tag{20}$$

The "re" subscript of K_{re} stands for "relativistic".

However, an electron in an atom acquires kinetic energy through emission of energy. Therefore, Einstein's relationship (19) cannot be applied to an electron in an atom.

Thus, the author derived the following relationship applicable to an electron in a hydrogen atom.

$$\left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{21}$$

$p_{\mathrm{re},n}$ is the momentum of an electron whose principal quantum number is in the state n.

The author has previously derived Formula (21) using five methods [6]-[12].

Here, the relativistic kinetic energy of an electron inside a hydrogen atom is defined as follows by referring to Formula (20) [8].

$$K_{\mathrm{re},n} = m_{\mathrm{e}} c^2 - m_n c^2. \tag{22}$$

When Formula (21) is solved, it is evident that ultra-low energy levels $E_{\mathrm{ab},n}^{-}$ exist in a hydrogen atom in addition to the known energy levels $E_{\mathrm{ab},n}^{+}$. If the energy of an electron when it is placed at a position infinitely far from the atomic nucleus is taken to be $m_{\mathrm{e}} c^2$, then $E_{\mathrm{ab},n}^{+}$ and $E_{\mathrm{ab},n}^{-}$ can be described as follows [13] [14].

$$\begin{aligned} E_{\mathrm{ab},n}^{+} &= m_n c^2 = m_{\mathrm{e}} c^2 + V\left(r_n\right) + K_n = m_{\mathrm{e}} c^2 + E_{\mathrm{re},n} \\ &= m_{\mathrm{e}} c^2 \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}, \quad n = 0,1,2,\cdots. \end{aligned} \tag{23}$$

$$E_{\mathrm{ab},n}^{-} = -m_n c^2 = -m_{\mathrm{e}} c^2 \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}, \quad n = 0,1,2,\cdots. \tag{24}$$

The "ab" subscript of $E_{\mathrm{ab},n}$ stands for "absolute".

It has already been pointed out that a state with $n = 0$ exists in the energy levels of a hydrogen atom [15] [16].

The energy levels of a hydrogen atom $E_{\mathrm{re},n}$ are given by the following formula.

$$E_{\mathrm{re},n} = m_n c^2 - m_{\mathrm{e}} c^2 = m_{\mathrm{e}} c^2 \left[\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2} - 1\right], \quad n = 0,1,2,\cdots. \tag{25}$$

In addition, Butto, N. has also discussed electron spin when discussing momentum of the electron. However, electron spin is not incorporated into the formula derived in this paper.

Therefore, it may not be the final formula [17].

Next, when the part of Formula (25) in parentheses is expressed as a Taylor expansion,

$$E_{\mathrm{re},n} \approx m_{\mathrm{e}} c^2 \left[\left(1 - \frac{\alpha^2}{2n^2} + \frac{3\alpha^4}{8n^4} - \frac{5\alpha^6}{16n^6}\right) - 1\right] \approx -\frac{\alpha^2 m_{\mathrm{e}} c^2}{2n^2}. \tag{26}$$

From this, it is evident that Formula (1) is an approximation of Formula (25).

Incidentally, it was once pointed out by Dirac that Equation (19) has a negative solution [18]. In the same way, the author has pointed out that Equation (21) has a negative solution [13]. The mass of an electron at negative energy levels becomes negative.

Next, the following table summarizes the energies of a hydrogen atom obtained from Formulas (1) and (25) (**Table 1**).

Table 1. Comparison of the energies of a hydrogen atom predicted by Bohr's classical quantum theory and this paper.

n	Bohr's Energy Levels, $E_{\mathrm{BO},n}$	This Paper, $E_{\mathrm{re},n}$
0	-	**−511 KeV**
1	−13.6057 eV	−13.6052 eV
2	−3.4014 eV	−3.4014 eV
3	−1.51174 eV	−1.51174 eV

Now, Formula (23) absolutely and relativistically describes the photon energy of an electron constituting a hydrogen atom. In contrast, Formula (24) indicates previously unknown energy levels.

Next, if the electron orbital radii corresponding to the energy levels in Formulas (23) and (24) are taken to be, respectively, r_n^+ and r_n^- [19].

$$r_n^+ = \frac{r_\mathrm{e}}{2}\left[1+\frac{n}{\left(n^2+\alpha^2\right)^{1/2}-n}\right]. \tag{27}$$

$$r_n^- = \frac{r_\mathrm{e}}{2}\left[1-\frac{n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right]. \tag{28}$$

In Formula (28), the electron approaches toward $r_\mathrm{e}/4$ as n increases.

The domain of the ordinary hydrogen atom that we all know starts from $r = r_\mathrm{e}/2\ \left(E_\mathrm{ab}=0\right)$.

The author has previously called matter formed from a proton and an electron with negative mass at the energy levels $E_{\mathrm{ab},n}^-$ a "dark hydrogen atom", and presented that as a candidate for dark matter, the strange matter whose true nature is currently unknown [20] [21].

3. Limits on Application of the Formula for Potential Energy of an Electron in a Hydrogen Atom

This section explains why it is necessary to discuss limits on application of the formula for potential energy of an electron in a hydrogen atom.

The first reason is because it was shown by the author that the minimum energy level of a hydrogen atom is not the ground state (n = 1) predicted by quantum mechanics. Since ultra-low energy levels exist in a hydrogen atom, an explanation must be considered which enables the electron to approach the point $r = r_\mathrm{e}/4$ thought to be the proton radius.

The second reason is because the author pointed out that the reduction of the rest mass energy of the electron corresponds to the potential energy of the electron in a hydrogen atom.

Formula (9), the existing formula for potential energy, can be written as follows.

$$V\left(r_n\right) = -\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\frac{m_e c^2}{r_n} = -m_e c^2 \cdot \frac{r_e}{r_n}. \tag{29}$$

Therefore,

$$V\left(r_n\right) r_n = -m_e c^2 \cdot r_e. \tag{30}$$

When the electron approaches the points $r_e, r_e/2, r_e/4$ from the center of an atomic nucleus, the potential energy becomes as follows.

$$V\left(r_e\right) = -m_e c^2,\ V\left(r_e/2\right) = -2m_e c^2,\ V\left(r_e/4\right) = -4m_e c^2. \tag{31}$$

The following well-known curve of Coulomb potential (**Figure 1**) illustrates the situation.

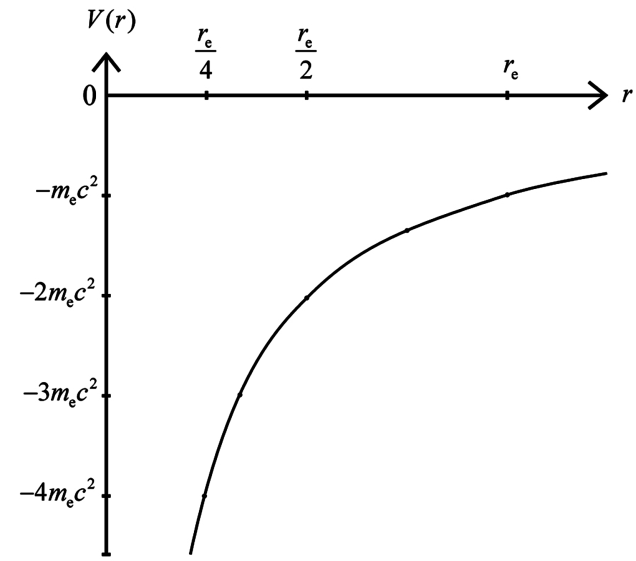

Figure 1. Potential energy curve of an electron in a hydrogen atom based on classical theory.

According to Einstein's special theory of relativity, the rest mass energy of the electron is $m_e c^2$. Inside a hydrogen atom, the rest mass energy of the electron is depleted when the electron approaches the atomic nucleus up to the point $r = r_e$. However, the electron acquires a kinetic energy of $m_e c^2/2$ at this time.

In the finished form of quantum mechanics, there is no discussion of the type of energy possessed by the electron. However, in classical quantum theory, the energy was discriminated.

Thus, this paper too discriminates the energy of the electron into the residual part of the rest mass energy and kinetic energy. In this case, $E_{\mathrm{ab},n}^{+}$ in Formula (25) can be written as follows.

$$E_{\mathrm{ab},n}^{+} = m_n c^2 = m_e c^2 + V\left(r_n\right) + K_n = m_e c^2 + 2E_{\mathrm{re},n} - E_{\mathrm{re},n}. \tag{32}$$

The sum of the first and second terms on the right side of Formula (32) are the residual part of the rest mass energy of the electron. The third term is the kinetic energy of the electron.

When the types of energy of the electron are taken into account, an electron which has approached the atomic nucleus to the point $r = r_e$ next approaches the point $r = r_e/2$ by reducing the acquired kinetic energy $m_e c^2/2$.

Therefore, according to this paper, the energy of an electron which has approached the atomic nucleus to the point $r = r_e/2$ is as follows.

$$V\left(r_e/2\right) = -m_e c^2,\ K = 0,\ E_{ab} = 0. \tag{33}$$

However, under these conditions, the electron cannot approach closer than this to the atomic nucleus. We must consider how the electron can reach ultra-low energy levels.

Thus, taking a hint from the idea of renormalization theory, the author has previously assumed that the energy of an electron placed at the point $r = r_e/2$ is not actually zero, and that this electron additionally has a photon energy $m_e c^2$ and a negative energy specific to the electron of $-m_e c^2$ [10] [11] (**Figure 2**) (**Appendix**).

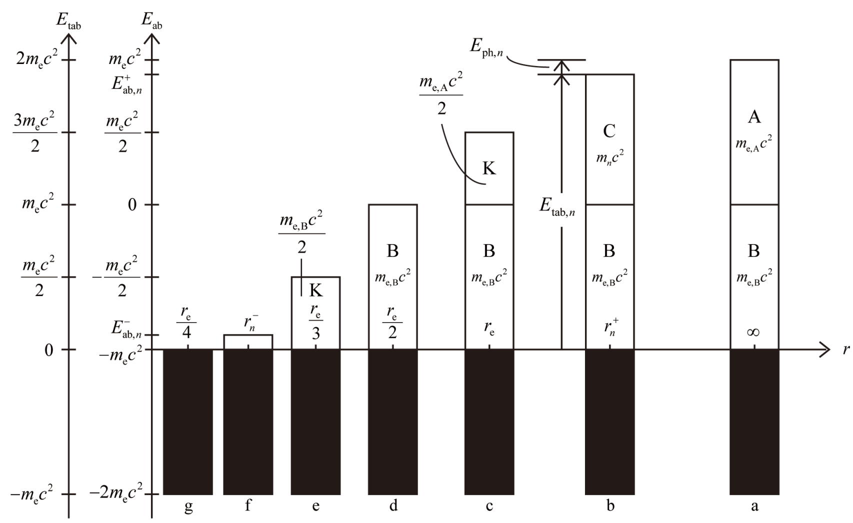

Figure 2. Photon energies of electrons in different states, and negative energy. Energy A is an energy we understand well. This paper asserts the existence of the B part. Also, the negative energy specific to the electron $-m_e c^2$ corresponds to the black rectangle. This figure shows that the original photon energy of an electron with rest mass energy $m_e c^2$ is $2m_e c^2$. (However, this figure is just a conceptual illustration. The r coordinate on the x-axis is not accurate). Also, the energy K of state c and e is kinetic energy of the electron. The electron in state e is in strange state where it has negative mass but positive kinetic energy.

Incidentally, Daviau, C. has already discussed the cloud of photons of an electron. For details, please see that paper [22].

In the state $E_{ab} = 0$, the photon energy $m_{e,B} c^2$ and negative energy $-m_e c^2$ cancel each other out, resulting in a state where energy is zero. An electron in the state where $E_{ab} = 0$ still has photon energy, so it can emit another photon and drop to a negative energy level.

The author has previously defined the residual energy $E_{\text{tab},n}$ of an electron that has emitted the photon energy $E_{\text{ph},n}$ as follows [12].

$$E_{\text{tab},n} = \left(m_{\text{e,A}} + m_{\text{e,B}}\right)c^2 - E_{\text{ph},n} = m_{\text{e,B}}c^2 + E_{\text{ab},n} = \left(m_{\text{e,B}} + m_n\right)c^2. \tag{34}$$

The "tab" subscript of this energy indicates the true, absolute photon energy. The descriptor "tab" is applied because absolute energy $E_{\text{ab},n}$ has already been defined.

In a previous paper, existence of the energy $m_{\text{e,B}}c^2$ was predicted.

When this model is used, the lower limit on potential energy of the electron becomes $-2m_{\text{e}}c^2$, not the $-4m_{\text{e}}c^2$ predicted by classical theory.

When the electron moves from the point $r = r_{\text{e}}/2$ and reaches the point $r = r_{\text{e}}/3$, all of the rest mass energy $m_{\text{e,B}}c^2$ is consumed, and the electron acquires kinetic energy of $m_{\text{e,B}}c^2/2$ (state e).

This electron nears the point $r = r_{\text{e}}/4$ while reducing its kinetic energy. The energy when the electron has reached the point $r = r_{\text{e}}/4$ is as follows.

$$V\left(r_{\text{e}}/4\right) = -2m_{\text{e}}c^2,\ K = 0,\ E_{\text{ab}} = -m_{\text{e}}c^2. \tag{35}$$

The states of this electron can be summarized as in the following table (**Table 2**).

Table 2. The electron reduces its energy in the process of approaching the atomic nucleus (proton), but the reduced energy differs depending on the position where the electron exists.

State	r	$V(r)$	K_{re}	E_{re}	E_{ph}	E_{ab}
a	∞	0	0	0	0	$m_{\text{e,A}}c^2$
a′	$\infty \to r_n^+$	Decrease	Increase	Decrease	Increase	Decrease
b	r_n^+	$2E_{\text{re},n}^+$	$-E_{\text{re},n}^+$	$E_{\text{re},n}^+$	$-E_{\text{re},n}^+$	$m_{\text{e,A}}c^2 + E_{\text{re},n}^+$
b′	$r_n^+ \to r_{\text{e}}$	Decrease	Increase	Decrease	Increase	Decrease
c	r_{e}	$-m_{\text{e,A}}c^2$	$\frac{m_{\text{e,A}}c^2}{2}$	$-\frac{m_{\text{e,A}}c^2}{2}$	$\frac{m_{\text{e,A}}c^2}{2}$	$\frac{m_{\text{e,A}}c^2}{2}$
c′	$r_{\text{e}} \to \frac{r_{\text{e}}}{2}$	Constant	Decrease	Decrease	Increase	Decrease
d	$\frac{r_{\text{e}}}{2}$	$-m_{\text{e,A}}c^2$	0	$-m_{\text{e,A}}c^2$	$m_{\text{e,A}}c^2$	0
d′	$\frac{r_{\text{e}}}{2} \to \frac{r_{\text{e}}}{3}$	Decrease	Increase	Decrease	Increase	Decrease
e	$\frac{r_{\text{e}}}{3}$	$-2m_{\text{e}}c^2$	$\frac{m_{\text{e,B}}c^2}{2}$	$-\frac{3m_{\text{e}}c^2}{2}$	$\frac{3m_{\text{e}}c^2}{2}$	$-\frac{m_{\text{e,B}}c^2}{2}$

Continued

e′	$\frac{r_\mathrm{e}}{3} \to r_n^-$	Constant	Decrease	Decrease	Increase	Decrease
f	r_n^-	$-2m_\mathrm{e}c^2$	$-E_{\mathrm{re},n}^+$	$-2m_\mathrm{e}c^2 - E_{\mathrm{re},n}^+$	$2m_\mathrm{e}c^2 + E_{\mathrm{re},n}^+$	$-m_{\mathrm{e,B}}c^2 - E_{\mathrm{re},n}^+$
f′	$r_n^- \to \frac{r_\mathrm{e}}{4}$	Constant	Decrease	Decrease	Increase	Decrease
g	$\frac{r_\mathrm{e}}{4}$	$-2m_\mathrm{e}c^2$	0	$-2m_\mathrm{e}c^2$	$2m_\mathrm{e}c^2$	$-m_{\mathrm{e,B}}c^2$

That is,

Electron movement: $\infty \to r_\mathrm{e}$, $\frac{r_\mathrm{e}}{2} \to \frac{r_\mathrm{e}}{3}$. Potential energy decreases.

Electron movement: $r_\mathrm{e} \to \frac{r_\mathrm{e}}{2}$, $\frac{r_\mathrm{e}}{3} \to \frac{r_\mathrm{e}}{4}$. Kinetic energy decreases.

In the state d, which serves as the starting point for the two types of energy levels, positive and negative, $r = r_\mathrm{e}/2$, $E = -m_\mathrm{e}c^2$. The product of the two is $Er = -m_\mathrm{e}c^2 \cdot r_\mathrm{e}/2$, and this matches with Formula (7).

Illustrating this situation, the result is as follows (**Figure 3**).

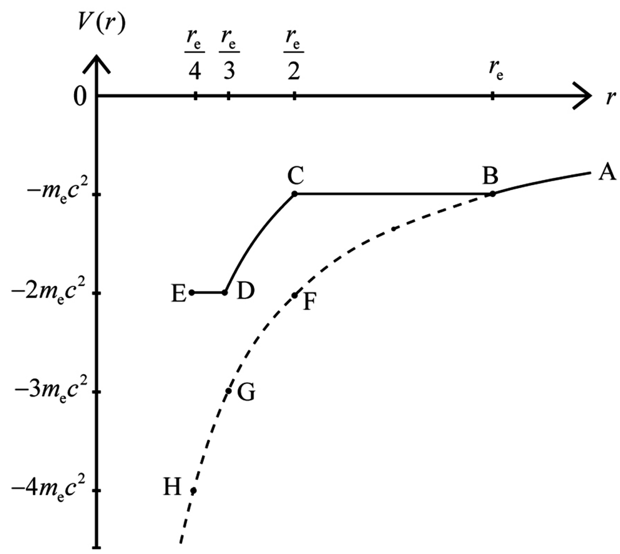

Figure 3. The potential energy of an electron predicted by this paper becomes the solid line connecting the points A, B, C, D, and E. The dotted line from point B to H is the curve of the Coulomb potential in **Figure 1**. The states of the electron at points B, C, D, and E are as follows: B: $(r_\mathrm{e}, -m_\mathrm{e}c^2)$, C: $(r_\mathrm{e}/2, -m_\mathrm{e}c^2)$, D: $(r_\mathrm{e}/3, -2m_\mathrm{e}c^2)$, E: $(r_\mathrm{e}/4, -2m_\mathrm{e}c^2)$.

The straight line BC and the straight line DE are parallel to the r-axis. It is also possible to overlay curve CD and curve FG.

4. Discussion

In the previous section it was shown that there are limits on the application of

Formula (9), and the region where this formula is applicable is $r_e \le r < \infty$. So what form does the formula for potential energy take when $r < r_e$?

In the interval BC and the interval DE in **Figure 3**, potential energy of the electron is constant. Therefore, the scope of application of the unknown formula to be derived is $r_e/3 \le r \le r_e/2$.

The difference in energy between points C and F, and between points D and G, is $m_e c^2$. Therefore, this paper proposes the following as the formula for the curve CD.

$$V(r) = m_e c^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{36}$$

Referring here to Formula (29), Formula (36) becomes as follows.

$$V(r) = m_e c^2 - m_e c^2 \frac{r_e}{r} = m_e c^2\left(1 - \frac{r_e}{r}\right), \quad \frac{r_e}{3} < r < \frac{r_e}{2}. \tag{37}$$

In this paper, Formula (37) was derived by assuming that the potential energy curves CD and FG overlap.

5. Conclusions

A. The following summarizes the results of the discussion in the previous sections.

$$V(r) = -m_e c^2 \frac{r_e}{r}, \quad r_e \le r < \infty. \tag{38}$$

$$V(r) = -m_e c^2, \quad \frac{r_e}{2} \le r \le r_e. \tag{39}$$

$$V(r) = m_e c^2\left(1 - \frac{r_e}{r}\right), \quad \frac{r_e}{3} \le r \le \frac{r_e}{2}. \tag{40}$$

$$V(r) = -2m_e c^2, \quad \frac{r_e}{4} \le r \le \frac{r_e}{3}. \tag{41}$$

Previously, issues such as the limits on application of the existing Formula (9) have not been discussed. However, in this paper, Formula (9) becomes Formula (38). Also, Formula (40) is a formula newly derived in this paper.

B. Electrons at the two types of energy levels $E^{+}_{\mathrm{ab},n}$ and $E^{-}_{\mathrm{ab},n}$ of the hydrogen atom differ in how they transition. When the energy levels transition within $E^{+}_{\mathrm{ab},n}$, the electron's potential energy and kinetic energy vary together. However, when the energy levels transition within $E^{-}_{\mathrm{ab},n}$, potential energy is constant, and only the kinetic energy of the electron varies. An electron at an $E^{-}_{\mathrm{ab},n}$ level has negative mass but positive kinetic energy.

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Author Contributions

The author confirms sole responsibility.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

The idea that the electron originally has a specific negative mass is not something proposed independently by the author. Hints for the idea in this paper were obtained from renormalization theory. In renormalization theory, the electron has a specific negative mass that it possesses from the outset, in addition to mass originating from the energy of the electromagnetic field. That is,

(Observed electron mass) = (Electromagnetic field energy) + (Negative mass specific to the electron)

If the electron is assumed to be a point particle with no size, the problem arises that the electron's mass becomes infinite. To avoid this problem, renormalization theory assumes that the electron's specific mass is negative infinity, and adjustment is done so the observed mass of the electron is infinite. However, this paper asserts that the magnitude of the electron's negative mass is not negative infinity, and instead is $-m_{\mathrm{e}}$.

The electron originally has a specific negative mass of $-m_{\mathrm{e}}$. In this paper, the rest mass of the electron m_{e} is given as the sum of the following two types of masses.

$$m_{\mathrm{e}} = m_{\mathrm{e,A}} + m_{\mathrm{e,B}} + \left(-m_{\mathrm{e}}\right) = 2m_{\mathrm{e}} + \left(-m_{\mathrm{e}}\right).$$