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An Analytical Serendipity-Based Technique for Solving the Canonical Riccati Equation

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Abstract

Nonlinear differential equations are often extremely difficult to solve. Even three hundred years after its formulation, an analytical solution to the nonlinear Riccati differential equation in its general form remains elusive. Renowned mathematicians such as d'Alembert, Daniel Bernoulli, and Leonhard Euler attempted to solve it without success. In this work, a sui generis technique is introduced to analytically solve the Riccati differential equation in its canonical form, ultimately yielding its general solution.

Keywords

Differential Equations, Serendipity, Solution Methods, Riccati Equation, Differential Calculus

1. Introduction

By the late 17th century, following the contributions of Isaac Newton and Gottfried Wilhelm Leibniz, the field of Differential and Integral Calculus was well established [1]. The emergence of differential equations dates back to around 1675 [1]. From their inception, these equations captivated mathematicians of the time, sparking a century-long enthusiasm characterized by an intense search for their solutions across various parts of the world.

The number of differential equations that remain unsolved in a general and satisfactory manner is remarkably large. In fact, many fundamental equations in Mathematical Physics—such as Bessel's equation, Laguerre's equation, Legendre's equation, and Newton's general equation—are typically solved only numerically, approximately, or for particular cases [2]-[9]. However, these well-known equations involve relatively simple cases of second-order linear differential equations with variable coefficients. If this type of differential equation could be solved in a general way, analytical solutions would be available for many problems in Mathematical Physics, including those in Classical Mechanics, Quantum Mechanics, and Electromagnetic Theory [3]-[9].

In this context, transforming a second-order linear equation with variable coefficients into a first-order equation often results in a nonlinear equation. This highlights the significance of studying nonlinear differential equations, particularly the general Riccati differential equation, which is a first-order nonlinear ordinary differential equation involving arbitrary functions of time [1] [10]-[14]. Proposed by Jacopo Francesco Riccati in the 18th century to address problems in fluid dynamics, this equation was formally named after him by d'Alembert in 1769 [1] [5].

To this day, an analytical solution for the Riccati equation in its general form remains elusive, despite attempts by renowned mathematicians, including d'Alembert himself. Other notable figures who worked on this equation include Gottfried Wilhelm Leibniz, Christian Goldbach, Johann Nicolas Bernoulli, Daniel Bernoulli, and Leonhard Euler. As the year 2024 marks 300 years of unsuccessful attempts to find a general solution to the Riccati equation, only numerical, approximate, or particular-case solutions are available [1]-[7]. In fact, it has been demonstrated that no elementary solution exists for the Riccati equation [7].

In this paper, we introduce a novel method—rooted in imagination and unexpected inventiveness—to obtain the general solution of the canonical Riccati equation, which represents a special case of the general Riccati equation.

2. The Riccati Equation

The Riccati equation can be expressed in different forms. It is striking how simple the equation appears, particularly in its canonical form, where $F(t)$ is any function of time *t*. This simplicity contrasts sharply with the significant difficulties encountered over the years in solving it analytically, both in this form and in its general expression [1] [2].

Without a doubt, the most challenging formulation of the Riccati problem is its general nonlinear differential equation, given by:

$$\dot{y} = f_2(t)y^2 + f_1(t)y + f_0(t) \tag{1}$$

Through a suitable coordinate transformation, the general Riccati equation can be rewritten as a second-order homogeneous linear differential equation with variable coefficients:

$$y = -(\ln M)^{\cdot} \tag{2}$$

which simplifies to:

$$\ddot{M} + F_1(t)\dot{M} + F_2(t)M = 0 \tag{3}$$

where $F_1(t)$ and $F_2(t)$ are arbitrary functions of time [1] [14].

This is the formulation of the Riccati equation commonly found in formal courses on Mathematical Methods in Physics. In its reduced second-order form, it is typically expressed as:

$$\ddot{M} + F(t)M = 0 \tag{4}$$

where $F(t)$ is an arbitrary function of time [15].

In all these cases, the fundamental difficulty of finding a solution remains largely equivalent. Solving any of these forms with an appropriate mathematical approach would bring us significantly closer—practically within reach—of fully solving the general Riccati differential equation.

The canonical form of Equation (1), which is the most commonly studied, is written as:

$$\dot{y} = y^2 + F(t) \tag{5}$$

where $F(t)$ is an arbitrary function of time [1] [13].

Many previous attempts to solve this equation have focused on obtaining a sufficient number of particular solutions, from which a general solution could be constructed. However, this strategy merely shifts the challenge to finding such particular solutions, which is itself a difficult task [1] [15]. This approach does not fundamentally resolve the problem. Moreover, it has been established that, in general, this equation does not integrate into quadratures, nor can its solution be obtained through a finite number of steps or successive integrations [1] [15]. These conclusions, however, reflect the historical difficulty in finding an analytical general solution rather than an absolute mathematical truth.

In this work, we introduce an extraordinary procedure that allows us to obtain the general analytical solution to the nonlinear Riccati differential equation, specifically in its canonical form, in a direct and expeditious manner.

3. The Procedure

The fundamental idea behind this new procedure is to solve a closely related problem—one that is extremely similar to the classical Riccati problem yet distinct. This alternative problem can be considered a particular case of the Riccati equation.

The first step involves applying a new transformation to Equation (5):

$$y = F(t)x \tag{6}$$

With this transformation, Equation (5) becomes

$$\dot{x} = F(t)x^2 - (\ln F(t))^{\cdot} x + 1 \tag{7}$$

which represents an alternative formulation of the canonical Riccati equation.

Solving Equation (7) is just as difficult as solving Equation (5). However, by introducing a small modification to Equation (7), we obtain the following equation:

$$\dot{x}_0 = F(t)x_0^2 - \frac{1}{2}(\ln F(t))^{\cdot} x_0 + 1 \tag{8}$$

Notably, the only change introduced is the addition of a factor of one-half in the term containing the natural logarithm. This seemingly minor adjustment likely alters the nature of the problem significantly, possibly reducing it to a particular case of the original equation.

Despite this, Equation (8) can be solved relatively easily. By working with this modified equation, an unexpected discovery was made—two solutions emerged by chance.

1) First Solution

Through serendipity—or a combination of intuition and fortunate discovery—it was observed that the term one in the second term of Equation (8) can be eliminated by the first term on the right-hand side if we assume a solution of the form:

$$x_{01} = \pm i \Big/ \sqrt{F(t)} \tag{9}$$

Furthermore, it can be shown that differentiating Equation (9) reproduces the term containing the one-half factor on the right-hand side of Equation (8). Therefore, Equation (9) represents a particular solution to Equation (8).

In this case, serendipity manifests itself through imagination, leading to a solution for the modified Riccati differential equation. That is, a particular solution has been found for Equation (8), which differs from Equation (7) only by the factor of one-half in the logarithmic term.

Since the structure of Equation (7) is almost identical to that of Equation (8), it is reasonable to expect that their solutions will also share some relationship.

2) Second Solution

Once again, serendipity played a role in discovering another solution. It was noted that the derivative of the tangent function

$$\begin{aligned} \left(\tan \int \sqrt{F(t)}\,\mathrm{d}t\right)^{\cdot} &= \sqrt{F(t)}\left(\sec \int \sqrt{F(t)}\,\mathrm{d}t\right)^2 \\ &= \sqrt{F(t)}\left(1 + \left(\tan \int \sqrt{F(t)}\,\mathrm{d}t\right)^2\right) \end{aligned} \tag{10}$$

partially reproduces the squared term on the right-hand side of the equation, as well as the term equal to one. This observation led to the hypothesis that a tangent function could also serve as a solution. Indeed, it can be shown that the second particular solution is:

$$x_{02} = \tan \int \sqrt{F(t)}\,\mathrm{d}t \Big/ \sqrt{F(t)} \tag{11}$$

This result can be verified analytically with relative ease, confirming that Equation (11) satisfies Equation (8).

Using the transformation given in Equation (6), the expressions corresponding to Equations (9) and (11) for the function y are:

$$y_{01} = \pm i \sqrt{F(t)} \tag{12}$$

and

$$y_{02} = \sqrt{F(t)} \tan \int \sqrt{F(t)}\,\mathrm{d}t \tag{13}$$

Similarly, the corresponding expressions for Equation (3) are:

$$M_{01} = \exp\left(-i\int\sqrt{F(t)}\,\mathrm{d}t\right) \tag{14}$$

and

$$M_{02} = \exp\left(-\int\sqrt{F(t)}\tan\int\sqrt{F(t)}\,\mathrm{d}t\right) \tag{15}$$

From standard mathematical tables, we know that the integral of the tangent function is a logarithm. Therefore, from Equation (14), we obtain:

$$M_{01} = \cos\int\sqrt{F(t)}\,\mathrm{d}t - i\,\mathrm{sen}\int\sqrt{F(t)}\,\mathrm{d}t \tag{16}$$

However, Equation (15) can be rewritten as:

$$M_{02} = \cos\int\sqrt{F(t)}\,\mathrm{d}t \tag{17}$$

Thus, the particular solution in Equation (14) is more general than the expression in (15). Additionally, Equation (14) provides a complex solution to Equation (8), which includes the one-half factor in the logarithmic term. This is precisely the Riccati equation with particular coefficients. Consequently, we retain Equation (16) as the more general solution and discard Equations (13) and (15).

Finally, it is observed that when the one-half factor is introduced, both the canonical Riccati equation and the corresponding linear equation appear with different expressions:

$$\dot{y}_0 = y_0^2 + \frac{1}{2}\left(\ln F(t)\right)^{\cdot} y_0 + F(t) \tag{18}$$

$$\ddot{M}_0 - \frac{1}{2}\left(\ln F(t)\right)^{\cdot}\dot{M}_0 + F(t)M_0 = 0 \tag{19}$$

This concludes this section of the procedure. The next step is to propose solutions for the canonical Riccati equation that bear some relationship to those found in the modified case. In other words, Equation (7) will be solved using the solutions of Equation (8).

4. The Solution

Solutions (9) and (11) represent particular solutions. When a particular solution to the Riccati equation is known, a standard procedure can be applied to obtain the general solution. From classical differential equations textbooks, it follows that if x_{0i} is a particular solution to the Riccati equation, then the general solution is given by:

$$x_0 = x_{0i} + \varphi \Big/ \left(C - \int F(t)\varphi\,\mathrm{d}t\right) \tag{20}$$

where C is an arbitrary constant, and in this case:

$$\varphi(t) = Exp\int\left[2Fx_{0i} - \frac{1}{2}\left(\ln F(t)\right)^{\cdot}\right]\mathrm{d}t \tag{21}$$

Thus,

$$\varphi(t) = Exp2\int(\pm i)\sqrt{F}\,\mathrm{d}t \Big/ \sqrt{F}$$

Consequently, the general solution of Equation (8) is

$$x_0 = \pm i \Big/ \sqrt{F(t)} + Exp2\int(\pm i)\sqrt{F}\,\mathrm{d}t \Big/ \left\{\left(C - Exp2\int(\pm i)\sqrt{F}\,\mathrm{d}t \big/ (\pm 2i)\right)\sqrt{F}\right\} \tag{22}$$

This expression generalizes the particular solution.

5. Rewriting the Equations

Once again, serendipity plays a role in the process. We now rewrite Equation (8) as:

$$\dot{F}(t) = 2x_0 F(t)^2 + \left[2/x_0 - 2(\ln x_0)^{\cdot}\right] F(t) \tag{23}$$

Similarly, Equation (7) can be rewritten as:

$$\dot{F}(t) = xF(t)^2 + \left[1/x - (\ln x)^{\cdot}\right] F(t) \tag{24}$$

From this, we obtain the following key relationships:

$$x = 2x_0 \tag{25}$$

and

$$\left[1/x - (\ln x)^{\cdot}\right] = \left[2/x_0 - 2(\ln x_0)^{\cdot}\right] \tag{26}$$

It can be shown—through a specific mathematical technique—that Equation (25) satisfies Equation (7). Additionally, from Equation (26), we derive:

$$\dot{x} = 1 - \left[2/x_0 - 2(\ln x_0)^{\cdot}\right] x \tag{27}$$

where

$$x = Exp - \int\left[2/x_0 - 2(\ln x_0)^{\cdot}\right]\mathrm{d}t\left(Q + \int\left(Exp\int\left[2/x_0 - 2(\ln x_0)^{\cdot}\right]\mathrm{d}t\right)\mathrm{d}t\right) \tag{28}$$

with Q being another arbitrary constant.

$$x = x_0^2 Exp - \int 2/x_0\,\mathrm{d}t\left(Q + \int\left(Exp\int[2/x_0]\,\mathrm{d}t\right)\Big/ x_0^2\,\mathrm{d}t\right) \tag{29}$$

6. Deriving the General Solution

By substituting expression (22) into Equation (29), we obtain the general solution to the canonical Riccati equation:

$$y = Fx = F2x_0 \tag{30}$$

$$y = (\pm 2i)\sqrt{F} + 2\sqrt{F}\,Exp2\int(\pm i)\sqrt{F}\,\mathrm{d}t \Big/ \left\{C - Exp2\int(\pm i)\sqrt{F}\,\mathrm{d}t \big/ (\pm 2i)\right\} \tag{31}$$

7. Verification of the Solution

This assertion can be verified as follows: From Equation (7), we also have

$$\left[1/x - (\ln x)^{\cdot}\right] = (\ln F(t))^{\cdot} - F(t)x \tag{32}$$

By substituting Equation (32) into Equations (26) and (27), the original Riccati equation is fully reproduced, confirming that Equation (30) is indeed the general solution to the canonical Riccati equation.

The mathematical technique mentioned earlier involves using identity (32) in Equation (24) to ultimately arrive at identity (25).

Furthermore, using Equation (32), we can rewrite the solution (29) as:

$$x = Exp2\int Fx_0\mathrm{d}t\left(Q+\int\left(FExp-2\int Fx_0\mathrm{d}t\right)\mathrm{d}t\right)\Big/F \tag{33}$$

Applying Equation (25) along with transformation (6), Equation (33) transforms into:

$$yExp-\int y\mathrm{d}t = \left(Q+\int\left(FExp-\int y\mathrm{d}t\right)\mathrm{d}t\right) \tag{34}$$

whose derivative fully reconstructs the canonical Riccati equation.

8. Conclusion

Thus, the general analytical solution to the canonical Riccati equation has been successfully derived.

9. In Other Words, Simplification

It was concluded that the solution to the canonical Riccati equation is given by Equation (29).

Taking the derivative of Equation (29) yields:

$$\dot{y} = 2\dot{x}_0F(t)+2x_0\dot{F}(t) \tag{35}$$

By substituting the derivative of $F(t)$ from Equation (24), we obtain:

$$\dot{y} = 2\dot{x}_0F(t)+2x_0\left\{xF^2+\left[1/x-(\ln x)^{\cdot}\right]F\right\} \tag{36}$$

Substituting Equation (29) into Equation (36), we confirm that:

$$y^2 = 2x_0xF^2$$

and therefore

$$\dot{y} = y^2+F(t)$$

Thus, the general solution to the canonical Riccati equation has been successfully verified.

10. Application to Any Defined $F(t)$

For any given problem where the function $F(t)$ is explicitly defined, the function y, the general solution can be expressed as in Equation (30):

$$y = 2\sqrt{F}\left\{(\pm i)+Exp2\int(\pm i)\sqrt{F}\,\mathrm{d}t\Big/\left(C-Exp2\int(\pm i)\sqrt{F}\,\mathrm{d}t\Big/(\pm 2i)\right)\right\}$$

Final Conclusion

This result shows that for any problem defined by the function F(t), its solution can be obtained by substituting $F(t)$ into Equation (30). Consequently, the general solution to the canonical Riccati equation is immediately determined for any specific case.

11. Comments and Conclusions

By applying the transformation (6) to the canonical Riccati equation, the solution process is initiated, leading to two Equations: (7) and (8). The first corresponds to the original Riccati equation, while the second represents its transformed version. In Equation (8), both the function F and its solution are explicitly known. In contrast, in Equation (7), only the function $F(t)$ is given, which remains the same in both cases.

For each of these Equations (7) and (8), the derivative of the function $F(t)$ is determined, leading to Equations (23) and (24). By equating both expressions for the derivative of $F(t)$, Equations (25) and (26) are obtained, ensuring consistency in the variable x.

The critical step in the solution process is to express the Riccati equation as an equality between two terms, as shown in Equation (32). This formulation ensures that Equations (25) and (26) are correctly satisfied.

From this point, it becomes an algebraic exercise to demonstrate that Equation (22) represents the general solution to the transformed canonical Riccati Equation (Equation (8)).

A key aspect of serendipity in this work was the fortunate discovery of Equations (9) and (10), which satisfy Equation (8). These expressions serve as particular solutions, later generalized in Equation (20). One of these particular solutions is selected to ultimately derive the solution to the canonical equation, which forms a special case of the general Riccati equation, as presented in Equations (1) and (7).

After extensive time, dedication, and effort, the canonical Riccati differential equation has been successfully solved analytically. Contrary to past claims that this problem was insoluble, it is now evident that it is complex but solvable.

Future Perspectives

With this breakthrough, future research on the Riccati equation will be significantly more manageable. This work opens new avenues for exploring broader applications and potential extensions of the method.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Nápoles Valdés, J.E. (1998) El legado histórico de las ecuaciones diferenciales ordinarias, Consideraciones (auto) Criticas. Boletín de Matemáticas Nueva Serie, Volumen V, Universidad de la Cuenca del Plata Argentina, 53-79.

[2] Arfken, G. (1970) Mathematical Methods for Physicists. 2nd Edition, Academic Press.

[3] Boas, M.L. (1983) Mathematical Methods in the Physical Sciences. John Wiley & Sons.

[4] Benner, P. and Faßbender, H. (2001) A Hybrid Method for the Numerical Solution of Discrete-Time Algebraic Riccati Equations. *Contemporary Mathematics*, **280**, 255-269.

[5] Benner, P., Byers, R., Quintana-Ortí, E.S. and Quintana-Ortí, G. (2000) Solving Algebraic Riccati Equations on Parallel Computers Using Newton's Method with Exact Line Search. *Parallel Computing*, **26**, 1345-1368. https://doi.org/10.1016/s0167-8191(00)00012-0

[6] Benner, P. and Byers, R. (1998) An Exact Line Search Method for Solving Generalized Continuous-Time Algebraic Riccati Equations. *IEEE Transactions on Automatic Control*, **43**, 101-107. https://doi.org/10.1109/9.654908

[7] Reyes y José Socorro, M.A. (2009) Supersimetría y Parasimetría de la ecuación de Riccati, Acta Universitaria, Dirección de apoyo a la Investigación y al Posgrado, Universidad de Guanajuato, México, Vol. 19.

[8] Jiménez Moscoso, J.A. (2015) La solución de algunas EDO de Riccati. *Revista Digital: Matemática, Educación e Internet*, **15**, 1-14. https://doi.org/10.18845/rdmei.v15i2.2170

[9] Ince, E.L. (2011) Ordinary Differential Equations. Dover Books of Mathematics.

[10] Kitcher, P. (1983) The Nature of Mathematical Knowledge. Oxford University Press.

[11] Ince, E. (1926) Ordinary Differential Equations. Dover.

[12] Davis, P.J. and Hersh, R. (1982) The Mathematical Experience. Birkhauser.

[13] Nápoles, J.E. and Negrón, C. (1994) De la Mecánica Analítica a las Ecuaciones Diferenciales Ordinarias. *Algunos Apuntes Históricos*, *Revista LLuLL Revista de la Sociedad Española de Historia de las Ciencias y de las Tecnicas*, **17**, 190-206.

[14] Berkovich, L.M. (2000) Transformations of Ordinary Differential Equations: Local and Nonlocal Symmetries. *Proceedings of Institute of Mathematics of NAS of Ukraine*, **30**, 25-34.

[15] Polyanin, A.D. and Zaitsev, V.F. (1995) Handbook of Exact Solutions for Ordinary Differential Equations. CRC Press.