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Testing the Results of Measurements of Neutrino Parameters Using the Dirac CPV Phase Formula

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Abstract

Essentially the main intention of this paper was to test the formula for the Dirac CPV phase and see if it can reflect the results of experimental measurements of neutrino parameters. By knowing the mathematical formula for the Dirac CPV phase, a connection was established with some of the residual symmetry groups, which made it possible to develop a procedure for directly determining the range in which the numerical value for the Dirac CPV phase could be found. In this sense, two different sources of information containing measured data for neutrinos were used for the corresponding calculations, and then a comparative overview of the calculated results was presented. It is particularly emphasized that the formula for the Dirac CPV phase does not depend on the mixing angles that are incorporated into the PMNS matrix, but only on the ratio between the corresponding squares of the neutrino mass difference. All the numerous results obtained from the corresponding calculations for the Dirac CPV phase point to the justified introduction of the theory that is related to three neutrinos, and thus the agreement of our results with the STEREO experiment is justified, so that the hypothesis of the possible existence of a sterile neutrino in nature should be excluded.

Keywords

Ordinary Neutrino, PMNS Matrix, Dirac CPV Phase, Jarlskog Invariant, Residual Discrete Symmetry Group

1. Introduction

We especially emphasize the reasons for the research that we will present in this paper, which refer to the works [1] [2] [3] [4] in which the hypothesis of the existence of the sterile neutrino was challenged. That is why we devoted ourselves to the research of three neutrinos and the results that we obtained at the end of this paper could be considered as a confirmation of agreement with the STEREO experiments [2] that rejected the possibility of the existence of a sterile neutrino in nature.

In previous papers [5] [6], an explicit formula for the Dirac CPV phase was derived. From the form of the formula, it can be seen that it does not depend on the mixing angles embedded in the PMNS mixing matrix, but that it directly depends on the ratio of the corresponding squares of the neutrino masses.

By applying the formula for the Dirac CPV phase with the use of experimental data from measurements of neutrino parameters, the numerical value for the Dirac CPV phase can be calculated, as shown in papers [5] and [6].

In order to create a procedure for determining the range in which the value for $\cos\delta$ could be found, we will achieve a coupling between the formula for the Dirac CPV phase and specially selected forms from the residual discrete symmetry groups.

With this procedure, with the knowledge of the formula for the Dirac CPV phase, the procedure for calculating the range for $\cos\delta$ is significantly simplified compared to the procedure given in the paper [7]. What the application of that procedure looks like is presented in the following sections.

Two sources of information were used [8] [9] and [10] and in the following chapters you can see the results we have reached.

2. Interrelation between the Sum Rule for $\cos\delta$ [7] and the Derived Formula for $\cos\delta$ [6]

All calculations in the following chapters are based on numerical values for $\cos\delta$ using the Dirac CPV phase formula [5] [6] for the $BF, \pm 1\sigma$ ranges and they look like this in normal hierarchy of neutrino masses:

$$\delta_{BF} = 180^{\circ} \times \left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{BF}, \delta_{-1\sigma} = 180^{\circ} \times \left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{-1\sigma}, \delta_{+1\sigma} = 180^{\circ} \times \left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} \quad (1)$$

Ref. [7] shows the connection between the sum rule for $\cos\delta$ and several different groups of residual symmetries that could generate a Dirac CP violation phase.

We especially emphasize the difference between our procedure and that given in Ref. [7] is in Dirac CPV phase display mode, which will be seen in the procedural modes presented in the following sections. In their representation, the Dirac CPV phase can move in a range defined by the relation $|\cos\delta| \le 1$, while we apply a formula to calculate the possible numerical values for the Dirac CPV phase (1) in the basis of $BF, \pm 1\sigma$ by relating them in the same basis with the corresponding mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$. To determine the range in which the value for the Dirac CP violation phase is found, a coupling was established between the formula for the Dirac CPV phase and some selected residual discrete groups of symmetries

Then, in this sense, we selected several residual discrete symmetry groups shown in Ref. [7] and connected the sum rule for $\cos\delta$ and the calculated numerical values for the Dirac CPV phase based on the formula (1).

Data for neutrino parameters from two sources [8] [9] and [10] were used to apply the formula for the Dirac CP violation phase, and the results we arrived at are presented in the following chapters.

3. Reference [8] [9]

In the following **Table 1**, the explicit numerous values that we will use in our procedure are given.

Based on the data from **Table 1**, numerous values are calculated that are shown in the formula (3).

$$\delta_{BF} = \left[180^\circ \times \left(24490/739\right)/360^\circ - 16\right] \times 360^\circ = 205.0879^\circ,$$
$$\cos 205.0879^\circ = -0.9056579;$$
$$\delta_{-1\sigma} = (180^\circ \times \left(24190/719\right)/360^\circ - 16) \times 360^\circ = 295.910^\circ,$$
$$\cos 295.910^\circ = +0.436974;$$

Table 1. Numerous values.

	Range of measured parameters	BF	-1σ	$+1\sigma$
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10^{+0.12}_{-0.11.}$	3.10	2.990	3.220
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.58^{+0.20}_{-0.23}$	5.580	5.350	5.780
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.241^{+0.065}_{-0.065}$	2.2410	2.1760	2.3070
$\sin\theta_{12}$		0.556776	0.5468089	0.567450
$\cos\theta_{12}$		0.830662	0.837257	0.823407
$\sin\theta_{23}$		0.746994	0.7314369	0.760263
$\cos\theta_{23}$		0.664830	0.681909	0.649615
$\sin\theta_{13}$		0.149699	0.1475127	0.151888
$\cos\theta_{13}$		0.988931	0.989060	0.988397
$\frac{\Delta m^2_{21}}{10^{-5}\,\mathrm{eV}^2}$	$7.39^{+0.21}_{-0.20}$	7.z39	7.19	7.60
$\frac{\Delta m^2_{32}}{10^{-3}\,\mathrm{eV}^2}$	$2.449^{+0.032}_{-0.030}$	2.4490	2.4190	2.4810
$\frac{\Delta m^2_{31}}{10^{-3}\,\mathrm{eV}^2}$		2.5229	2.4909	2.5570
δ_{CP} /°	222^{+38}_{-28}	222	194	260

(2)

$$\delta_{+1\sigma} = \left[180^\circ \times (25570/760)/360^\circ - 16\right] \times 360^\circ = 296.052^\circ,$$
$$\cos 296.052^\circ = +0.439196;$$
$$2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} = -0.061650,$$
$$2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} = +0.029746,$$
$$2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} = +0.029897;$$
$$2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} = -0.0610125,$$
$$2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} = +0.029438,$$
$$2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} = +0.029588;$$
$$2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} = -0.063486,$$
$$2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} = +0.030631, \quad (3)$$
$$2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} = +0.030787.$$

4. Sum Rules for $\cos\delta$ of Selected Residual Discrete Symmetry Groups [7]

The following residual symmetry groups shown in Ref. [7] were chosen. And they marked with: C1, C3, C4 and C8.

We will analyze each of them in turn.

4.1. The Case with $U_{12}\left(\theta_{12}^e, \delta_{12}^e\right)$ and $U_{13}\left(\theta_{13}^\nu, \delta_{13}^\nu\right)$ Complex Rotations (Case C1)

In this first example, we will explain the connection between the formula for the Dirac CPV phase and the sum rule for $\cos\delta$ and the procedure for arriving at the regions where it might be found.

First of all, due to the lack of an explicit formula for the Dirac CPV phase in neutrino physics, the determination of the areas where it could be expected to be found was based on varying its absolute value expressed by $|\cos\delta| \le 1$, which can be seen in Ref. [7].

The procedure that we will present for determining the possible areas in which the numerical value of the Dirac CPV phase would be found is based on the formula for the Dirac CPV phase (1) and that in several steps.

First step

The sum rule for $\cos\delta$

For this case of the residual discrete symmetry group we first write the sum rule for $\cos\delta$ as shown in Ref. [7] and it looks like this:

$$\cos\delta = \frac{\left(\sin^2\theta_{23}^0\right) - \left(\cos^2\theta_{12}\sin^2\theta_{23}\right) - \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)}{2\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)} \quad (4)$$

In the expression (4) is an associated relation

$$\left|U_{\tau 2}\right| = \left|\cos\theta_{12}\sin\theta_{23} + \sin\theta_{12}\cos\theta_{23}\sin\theta_{13}e^{i\delta}\right| = \left|\sin\theta_{23}^0\right| \quad (5)$$

which results from the direct parametrization for the PMNS mixing matrix and the detailed procedure for deriving the expression (4) is given in Ref. [7].

And then we find the explicit value for $\left(\sin^2\theta_{23}^0\right)$:

$$\left(\sin^2\theta_{23}^0\right) = 2\cos\delta\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)+\left(\cos^2\theta_{12}\sin^2\theta_{23}\right) +\left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right) \quad (6)$$

Second step

Our main goal is to determine the region where numerical values for the Dirac CPV phase δ could be found.

To achieve this, we will limit our calculations of neutrino parameters to their numerical values in which they appear in experimental measurements in the following three bases: best fit (BF) and $\pm1\sigma$ ranges. Applied to formula (6) it looks like this:

$$\cos\delta = \frac{\left[\left(\sin^2\theta_{23}^0\right)_\delta\right]_{BF} - \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} - \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}}{2\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}} \quad (7)$$

Based on this formula we find:

$$\left[\left(\sin^2\theta_{23}^0\right)_\delta\right]_{BF} = 2\cos\delta\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \quad (8)$$

In the next step, we write the formula for $\cos\delta_{BF}$, but indicate that it is a $\left[\left(\sin^2\theta_{23}^0\right)_{BF}\right]_{BF}$ like this:

$$\cos\delta_{BF} = \frac{\left[\left(\sin^2\theta_{23}^0\right)_{BF}\right]_{BF} - \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} - \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}}{2\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}} \quad (9)$$

Based on this formula we find:

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{BF} = 2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \quad (10)$$

Now we write the sum rule for $\cos\delta_{-1\sigma}$ in the range -1σ and $\left[\left(\sin^2\theta_{23}^0\right)_{-1\sigma}\right]_{BF}$ we write it like this:

$$\cos\delta_{-1\sigma} = \frac{\left[\left(\sin^2\theta_{23}^0\right)_{-1\sigma}\right]_{BF} - \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} - \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}}{2\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}} \quad (11)$$

Based on this formula we find:

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-1\sigma}}\right]_{BF} = 2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \quad (12)$$

Now we write the sum rule for $\cos\delta_{+1\sigma}$ in the range $+1\sigma$ and $\left[\left(\sin^2\theta_{23}^0\right)_{+1\sigma}\right]_{BF}$ we write it like this:

$$\cos\delta_{+1\sigma} = \frac{\left[\left(\sin^2\theta_{23}^0\right)_{+1\sigma}\right]_{BF} - \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} - \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}}{2\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}} \quad (13)$$

Based on this formula we find:

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{BF} = 2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \quad (14)$$

As can be seen in relation (8) there is an unknown quantity $\cos\delta$ to be determined. Using numerical values for data in Table 1 and (3) we calculate numerical values for expressions ((10), (12)-(14)) and they amount to:

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{BF} = 2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \approx 0.3258 \quad (15)$$

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-1\sigma}}\right]_{BF} = 2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \approx 0.41813 \quad (16)$$

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{BF} = 2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \approx 0.41829 \quad (17)$$

Then, from the values calculated in this way, we extract the largest and smallest value from them, and between them we insert the relation (8) and write it like this:

$$\begin{aligned}&\left[\left(\sin^2\theta_{23}^0\right)_{BF}\right]_{BF} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{BF} \le \left[\left(\sin^2\theta_{23}^0\right)_{+1\sigma}\right]_{BF} \to \\ &2\cos\delta_{BF}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \\ &\le 2\cos\delta\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}; \\ &\le 2\cos\delta_{+1\sigma}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}.\end{aligned} \quad (18)$$

In these relations, the common factors cancel each other out so that they are mapped into an extremely simple form:

$$\cos\delta_{BF} \le \cos\delta \le \cos\delta_{+1\sigma} \quad (19)$$

Taking into account the numerical values from T. 2, we write this relationship in the following way, where it is necessary to determine the unknown quantity $\cos\delta$:

$$-0.9056957 \le \cos\delta \le 0.439196 \quad (20)$$

Here we have two inequalities and we will solve them separately.

$$\cos\delta_{BF} \le \cos\delta \to -0.9056579 \le \cos\delta \to \cos\delta + 0.9056579 \ge 0 \to \delta \ge 205.0879^\circ \quad (21)$$

$$\cos\delta \le \cos\delta_{+1\sigma} \to \cos\delta \le 0.439196 \to \cos\delta - 0.439196 \le 0 \to \delta \le 296.052^\circ \quad (22)$$

So, the end result will be:

$$205.0879^\circ \le \delta \le 296.052^\circ \tag{23}$$

These are the theoretical results obtained using the formula for the Dirac CPV phase and when we combine them with the experimental data from T.2 we find the lower and upper limits for δ:

$$\begin{aligned} &205.0879^\circ \le \delta \le 296.052^\circ \to \\ &180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left\{ \left[\left(180^\circ \times \frac{24490}{739} \right) \Big/ 360^\circ - 16 \right] \times 360^\circ \right\}_{th} \\ &\approx 205.00^\circ = \left[\left(222_{-17} \right)^\circ \right]_{\exp} \le \delta \le \left[\left(222^{+38} \right)^\circ \right]_{\exp} = 260^\circ \end{aligned} \tag{24}$$

By further applying the analogous procedure as in the previous example, we perform calculations for two more ranges and show this in the following sections.

Case C1 calculations based on -1σ of the mixing angles

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta} \right]_{-1\sigma} &= 2\cos\delta \times \left(\sin\theta_{13} \sin\theta_{23} \cos\theta_{23} \sin\theta_{12} \cos\theta_{12} \right)_{-1\sigma} \\ &+ \left(\cos^2\theta_{12} \sin^2\theta_{23} \right)_{-1\sigma} + \left(\sin^2\theta_{12} \cos^2\theta_{23} \sin^2\theta_{13} \right)_{-1\sigma} \end{aligned} \tag{25}$$

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{BF}} \right]_{-1\sigma} &= 2\cos\delta_{BF} \times \left(\sin\theta_{13} \sin\theta_{23} \cos\theta_{23} \sin\theta_{12} \cos\theta_{12} \right)_{-1\sigma} \\ &+ \left(\cos^2\theta_{12} \sin^2\theta_{23} \right)_{-1\sigma} + \left(\sin^2\theta_{12} \cos^2\theta_{23} \sin^2\theta_{13} \right)_{-1\sigma} \end{aligned} \tag{26}$$

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{-1\sigma}} \right]_{-1\sigma} &= 2\cos\delta_{-1\sigma} \times \left(\sin\theta_{13} \sin\theta_{23} \cos\theta_{23} \sin\theta_{12} \cos\theta_{12} \right)_{-1\sigma} \\ &+ \left(\cos^2\theta_{12} \sin^2\theta_{23} \right)_{-1\sigma} + \left(\sin^2\theta_{12} \cos^2\theta_{23} \sin^2\theta_{13} \right)_{-1\sigma} \end{aligned} \tag{27}$$

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{+1\sigma}} \right]_{-1\sigma} &= 2\cos\delta_{+1\sigma} \times \left(\sin\theta_{13} \sin\theta_{23} \cos\theta_{23} \sin\theta_{12} \cos\theta_{12} \right)_{-1\sigma} \\ &+ \left(\cos^2\theta_{12} \sin^2\theta_{23} \right)_{-1\sigma} + \left(\sin^2\theta_{12} \cos^2\theta_{23} \sin^2\theta_{13} \right)_{-1\sigma} \end{aligned} \tag{28}$$

From these relations follows the inequality:

$$\left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{BF}} \right]_{-1\sigma} \le \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta} \right]_{-1\sigma} \le \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{+1\sigma}} \right]_{+1\sigma} \tag{29}$$

Because it is:

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{BF}} \right]_{-1\sigma} &= 0.317, \\ \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{-1\sigma}} \right]_{-1\sigma} &= 0.407498, \\ \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{+1\sigma}} \right]_{-1\sigma} &= 0.407647. \end{aligned} \tag{30}$$

And it is mapped to the form:

$$\cos\delta_{BF} \le \cos\delta \le \cos\delta_{+1\sigma}. \tag{31}$$

And from here we find the region of definition for the Dirac CPV phase

$$205.0879^\circ \le \delta \le 296.052^\circ \to$$
$$180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left\{ \left[\left(180^\circ \times \frac{24490}{739} \right) \Big/ 360^\circ - 16 \right] \times 360^\circ \right\}_{th} \tag{32}$$
$$\approx 205.00^\circ = \left[\left(222_{-17} \right)^\circ \right]_{\exp} \le \delta \le \left[\left(222^{+38} \right)^\circ \right]_{\exp} = 260^\circ$$

Case C1 calculations based on $+1\sigma$ of the mixing angles

Applying the complete procedure as in the previous example, which was observed in the BF range, in this case for $+1\sigma$, we obtain the following relations:

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta} \right]_{+1\sigma} &= 2\cos\delta \times \left(\sin\theta_{13} \sin\theta_{23} \cos\theta_{23} \sin\theta_{12} \cos\theta_{12} \right)_{+1\sigma} \\ &+ \left(\cos^2\theta_{12} \sin^2\theta_{23} \right)_{+1\sigma} + \left(\sin^2\theta_{12} \cos^2\theta_{23} \sin^2\theta_{13} \right)_{+1\sigma} \end{aligned} \tag{33}$$

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{BF}} \right]_{+1\sigma} &= 2\cos\delta_{BF} \times \left(\sin\theta_{13} \sin\theta_{23} \cos\theta_{23} \sin\theta_{12} \cos\theta_{12} \right)_{+1\sigma} \\ &+ \left(\cos^2\theta_{12} \sin^2\theta_{23} \right)_{+1\sigma} + \left(\sin^2\theta_{12} \cos^2\theta_{23} \sin^2\theta_{13} \right)_{+1\sigma} \\ &= 0.331532 \end{aligned} \tag{34}$$

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{-1\sigma}} \right]_{+1\sigma} &= 2\cos\delta_{-1\sigma} \times \left(\sin\theta_{13} \sin\theta_{23} \cos\theta_{23} \sin\theta_{12} \cos\theta_{12} \right)_{+1\sigma} \\ &+ \left(\cos^2\theta_{12} \sin^2\theta_{23} \right)_{+1\sigma} + \left(\sin^2\theta_{12} \cos^2\theta_{23} \sin^2\theta_{13} \right)_{+1\sigma} \\ &= 0.425649 \end{aligned} \tag{35}$$

$$\begin{aligned} \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{+1\sigma}} \right]_{+1\sigma} &= 2\cos\delta_{+1\sigma} \times \left(\sin\theta_{13} \sin\theta_{23} \cos\theta_{23} \sin\theta_{12} \cos\theta_{12} \right)_{+1\sigma} \\ &+ \left(\cos^2\theta_{12} \sin^2\theta_{23} \right)_{+1\sigma} + \left(\sin^2\theta_{12} \cos^2\theta_{23} \sin^2\theta_{13} \right)_{+1\sigma} \\ &= 0.425805 \end{aligned} \tag{36}$$

From these relations follows the inequality:

$$\left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{BF}} \right]_{+1\sigma} \le \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta} \right]_{\delta_{+1\sigma}} \le \left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{+1\sigma}} \right]_{+1\sigma} \tag{37}$$

Because it is:

$$\left[\left(\sin^2 \theta_{23}^0 \right)_{\delta_{BF}} \right]_{+1\sigma} = 0.331532, \left[\left(\sin^2 \theta_{23}^0 \right)_{+1\sigma} \right]_{-1\sigma} = 0.425825 \tag{38}$$

and it is mapped to the form:

$$\cos\delta_{BF} \le \cos\delta \le \cos\delta_{+1\sigma}. \tag{39}$$

From these relations we find the region of definition for the Dirac CPV phase

$$205.0879^\circ \le \delta \le 296.052^\circ \to$$
$$180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left\{ \left[\left(180^\circ \times \frac{24490}{739} \right) \Big/ 360^\circ - 16 \right] \times 360^\circ \right\}_{th} \tag{40}$$
$$\approx 205.00^\circ = \left[\left(222_{-17} \right)^\circ \right]_{\exp} \le \delta \le \left[\left(222^{+38} \right)^\circ \right]_{\exp} = 260^\circ$$

4.2. The Case with $U_{12}\left(\theta_{12}^e, \delta_{12}^e\right)$ and $U_{23}\left(\theta_{23}^\nu, \delta_{23}^\nu\right)$ Complex Rotations (Case C3)

Case C3 calculations based on (BF) of the mixing angles

The sum rule for $\cos\delta$

$$\cos\delta=\frac{\left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{BF}-\left(\sin^2\theta_{13}^0\right)+\left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}}{2\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}} \quad (41)$$

In the expression (41) is an associated relation

$$\left|U_{\tau 1}\right|=\left|\sin\theta_{12}\sin\theta_{23}-\cos\theta_{12}\cos\theta_{23}\sin\theta_{13}\mathrm{e}^{i\delta}\right|=\left|\sin\theta_{13}^0\right| \quad (42)$$

which results from the direct parametrization for the PMNS mixing matrix and the detailed procedure for deriving the expression (41) is given in Ref. [7].

Carrying out an analogous procedure as in the first example, we can write the following relations:

$$\begin{aligned}\left[\left(\sin^2\theta_{13}^0\right)_{\delta}\right]_{BF}&=-2\cos\delta\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}\\&\quad+\left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{BF}+\left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}\end{aligned} \quad (43)$$

$$\begin{aligned}\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{BF}&=-2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}\\&\quad+\left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{BF}+\left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}\end{aligned} \quad (44)$$

$$\begin{aligned}\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-1\sigma}}\right]_{BF}&=-2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}\\&\quad+\left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{BF}+\left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}\end{aligned} \quad (45)$$

$$\begin{aligned}\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{BF}&=-2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}\\&\quad+\left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{BF}+\left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{BF}\end{aligned} \quad (46)$$

The inequality follows from these relations:

$$\begin{aligned}&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{BF}\le\left[\left(\sin^2\theta_{13}^0\right)_{\delta}\right]_{BF}\le\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{\delta_{BF}}\\&\to-\cos\delta_{+1\sigma}\le-\cos\delta\le-\cos\delta_{BF}.\\&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{BF}=0.1494,\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-1\sigma}}\right]_{BF}=0.149566,\\&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{BF}=0.242,\end{aligned} \quad (47)$$

From these relations we find the region of definition for the Dirac CPV phase

$$\begin{aligned}&205.0879^\circ\le\delta\le296.052^\circ\to\\&180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=\left\{\left[\left(180^\circ\times\frac{24490}{739}\right)\Big/360^\circ-16\right]\times360^\circ\right\}_{th}\\&\approx205.00^\circ=\left[\left(222_{-17}\right)^\circ\right]_{\exp}\le\delta\le\left[\left(222^{+38}\right)^\circ\right]_{\exp}=260^\circ\end{aligned} \quad (48)$$

Case C3 calculations based on -1σ **of the mixing angles**

We write the corresponding relations in this basis:

$$\begin{aligned}\left[\left(\sin^2\theta_{13}^0\right)_{\delta}\right]_{-1\sigma}&=-2\cos\delta\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma}\\&\quad+\left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{-1\sigma}+\left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma}\end{aligned} \quad (49)$$

$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{-1\sigma} = -2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} + \left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{-1\sigma} + \left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma} \quad (50)$$

$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-1\sigma}}\right]_{-1\sigma} = -2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} + \left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{-1\sigma} + \left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma} \quad (51)$$

$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} = -2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} + \left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{-1\sigma} + \left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma} \quad (52)$$

The inequality follows from these relations:

$$\begin{aligned}
&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} \le \left[\left(\sin^2\theta_{13}^0\right)_{\delta}\right]_{-1\sigma} \le \left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{-1\sigma}, \\
&\to -\cos\delta_{+1\sigma} \le -\cos\delta \le -\cos\delta_{BF} \to \delta_{BF} \le \delta \le \delta_{+1\sigma}; \\
&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} = 0.137470, \left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-1\sigma}}\right]_{-1\sigma} = 0.137619, \\
&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{-1\sigma} = 0.2280.
\end{aligned} \quad (53)$$

From these relations we find the region of definition for the Dirac CPV phase

$$\begin{aligned}
&205.0879^\circ \le \delta \le 296.052^\circ \to \\
&180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left\{\left[\left(180^\circ \times \frac{24490}{739}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \\
&\approx 205.00^\circ = \left[\left(222_{-17}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(222^{+38}\right)^\circ\right]_{\exp} = 260^\circ
\end{aligned} \quad (54)$$

Case C3 calculations based on $+1\sigma$ of the mixing angles

We write the corresponding relations in this basis:

$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta}\right]_{+1\varpi} = -2\cos\delta \times \left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\varpi} + \left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{+1\varpi} + \left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{+1\varpi} \quad (55)$$

$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{+1\varpi} = -2\cos\delta_{BF} \times \left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\varpi} + \left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{+1\varpi} + \left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{+1\varpi} \quad (56)$$

$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-1\sigma}}\right]_{+1\varpi} = -2\cos\delta_{-1\sigma} \times \left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\varpi} + \left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{+1\varpi} + \left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{+1\varpi} \quad (57)$$

$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{+1\varpi} = -2\cos\delta_{+1\sigma} \times \left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\varpi} + \left(\sin^2\theta_{12}\sin^2\theta_{23}\right)_{+1\varpi} + \left(\cos^2\theta_{12}\cos^2\theta_{23}\sin^2\theta_{13}\right)_{+1\varpi} \quad (58)$$

The inequality follows from these relations:

$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{\delta_{+1\sigma}} \le \left[\left(\sin^2\theta_{13}^0\right)_{\delta}\right]_{+1\varpi} \le \left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}$$
$$\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{\delta_{+1\sigma}} = 0.161829, \left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-1\sigma}}\right]_{\delta_{+1\sigma}} = 0.16208,$$
$$\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}} = 0.25620.$$
$$\rightarrow -\cos\delta_{+1\sigma} \le -\cos\delta \le -\cos\delta_{BF} \rightarrow \delta_{BF} \le \delta \le \delta_{+1\sigma}; \tag{59}$$

From these relations we find the region of definition for the Dirac CPV phase

$$205.0879^\circ \le \delta \le 296.052^\circ \rightarrow$$
$$180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left\{\left[\left(180^\circ \times \frac{24490}{739}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \tag{60}$$
$$\approx 205.00^\circ = \left[\left(222_{-17}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(222^{+38}\right)^\circ\right]_{\exp} = 260^\circ$$

4.3. The Case with $U_{13}\left(\theta_{13}^e, \delta_{13}^e\right)$ and $U_{23}\left(\theta_{23}^\nu, \delta_{23}^\nu\right)$ Complex Rotations (Case C4)

Case C4 based on (BF) of the mixing angles

The sum rule for $\cos\delta$

$$\cos\delta = \frac{\left(\sin^2\theta_{12}^0\right) - \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{BF} - \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF}}{2\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}} \tag{61}$$

In the expression (61) is an associated relation

$$\left|U_{\mu 1}\right| = \left|\sin\theta_{12}\cos\theta_{23} + \cos\theta_{12}\sin\theta_{23}\sin\theta_{13}e^{i\delta}\right| = \left|\sin\theta_{12}^0\right| \tag{62}$$

which results from the direct parametrization for the PMNS mixing matrix and the detailed procedure for deriving the expression (61) is given in Ref. [7].

And from here we find:

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta}\right]_{BF} = 2\cos\delta \times \left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \tag{63}$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{BF} = 2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{BF} + \left(\cos^2\theta_{12}\sin s^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \tag{64}$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{-1\sigma}}\right]_{BF} = 2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \tag{65}$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{BF} = 2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{BF} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF} \tag{66}$$

The inequality follows from these relations:

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{BF} \le \left[\left(\sin^2\theta_{12}^0\right)_{\delta}\right]_{BF} \le \left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{\delta_{BF}},$$
$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{BF} = 0.175849, \left[\left(\sin^2\theta_{12}^0\right)_{\delta_{-1\sigma}}\right]_{BF} = 0.175696, \quad (67)$$
$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{\delta_{BF}} = 0.0833;$$
$$\rightarrow \cos\delta_{BF} \le \cos\delta \le \cos\delta_{+1\sigma} \rightarrow \delta_{BF} \le \delta \le \delta_{+1\sigma}.$$

From these relations we find the region of definition for the Dirac CPV phase

$$205.0879^\circ \le \delta \le 296.052^\circ \rightarrow$$
$$180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left\{\left[\left(180^\circ \times \frac{24490}{739}\right)\Big/360^\circ - 16\right] \times 360^\circ\right\}_{th} \quad (68)$$
$$\approx 205.00^\circ = \left[\left(222_{-17}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(222^{+38}\right)^\circ\right]_{\exp} = 260^\circ$$

Case C4 calculations based on -1σ of the mixing angles

We write the corresponding relations in this basis:

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta}\right]_{-1\sigma} = 2\cos\delta\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{-1\sigma} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma} \quad (69)$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{-1\sigma} = 2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{-1\sigma} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma} \quad (70)$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{-1\sigma}}\right]_{-1\sigma} = 2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{-1\sigma} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma} \quad (71)$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{-+\sigma}}\right]_{-1\sigma} = 2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{-1\sigma} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma} \quad (72)$$

The inequality follows from these relations:

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{-1\sigma} \le \left[\left(\sin^2\theta_{12}^0\right)_{\delta}\right]_{-1\sigma} \le \left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{-1\sigma},$$
$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} = 0.176762, \left[\left(\sin^2\theta_{12}^0\right)_{\delta_{-1\sigma}}\right]_{-1\sigma} = 0.176633, \quad (73)$$
$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{-1\sigma} = 0.08618;$$
$$\rightarrow \cos\delta_{BF} \le \cos\delta \le \cos\delta_{+1\sigma} \rightarrow \delta_{BF} \le \delta \le \delta_{+1\sigma}.$$

From these relations we find the region of definition for the Dirac CPV phase

$$205.0879^\circ \le \delta \le 296.052^\circ \rightarrow$$
$$180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left\{\left[\left(180^\circ \times \frac{24490}{739}\right)\Big/360^\circ - 16\right] \times 360^\circ\right\}_{th} \quad (74)$$
$$\approx 205.00^\circ = \left[\left(222_{-17}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(222^{+38}\right)^\circ\right]_{\exp} = 260^\circ$$

Case C4 calculations based on $+1\sigma$ of the mixing angles

We write the corresponding relations in this basis: We write the corresponding relations in this basis:

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta}\right]_{+1\sigma} = 2\cos\delta\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{+1\sigma} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{+1\sigma} \quad (75)$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{+1\sigma} = 2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{+1\sigma} + \left(\cos^2\theta_{12}\sin s^2\theta_{23}\sin^2\theta_{13}\right)_{+1\sigma} \quad (76)$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{-1\sigma}}\right]_{+1\sigma} = 2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{+1\sigma} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{+1\sigma} \quad (77)$$

$$\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma} = 2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} + \left(\sin^2\theta_{12}\cos^2\theta_{23}\right)_{+1\sigma} + \left(\cos^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{+1\sigma} \quad (78)$$

The inequality follows from these relations:

$$\begin{aligned}
&\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{+1\sigma} \le \left[\left(\sin^2\theta_{12}^0\right)_{\delta}\right]_{+1\sigma} \le \left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma}, \\
&\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma} = 0.175711, \left[\left(\sin^2\theta_{12}^0\right)_{\delta_{-1\sigma}}\right]_{+1\sigma} = 0.175355, \\
&\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{+1\sigma} = 0.0814; \\
&\rightarrow \cos\delta_{BF} \le \cos\delta \le \cos\delta_{+1\sigma} \rightarrow \delta_{BF} \le \delta \le \delta_{+1\sigma}.
\end{aligned} \quad (79)$$

From these relations we find the region of definition for the Dirac CPV phase

$$\begin{aligned}
&205.0879^\circ \le \delta \le 296.052^\circ \rightarrow \\
&180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \left\{\left[\left(180^\circ \times \frac{24490}{739}\right)\Big/360^\circ - 16\right]\times 360^\circ\right\}_{th} \\
&\approx 205.00^\circ = \left[\left(222_{-17}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(222^{+38}\right)^\circ\right]_{\exp} = 260^\circ
\end{aligned} \quad (80)$$

4.4. The Case with $U_{13}\left(\theta_{13}^e, \delta_{13}^e\right)$ and $U_{13}\left(\theta_{13}^\nu, \delta_{13}^\nu\right)$ Complex Rotations (Case C8)

Case C8 calculations based on (BF) of the mixing angles

The sum rule for $\cos\delta$

$$\cos\delta = \frac{\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{BF} - \left(\cos^2\theta_{23}^0\right) + \left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF}}{2\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}} \quad (81)$$

In the expression (81) is an associated relation

$$\left|U_{\mu 2}\right| = \left|\cos\theta_{12}\cos\theta_{23} - \sin\theta_{12}\sin\theta_{23}\sin\theta_{13}e^{i\delta}\right| = \left|\cos\theta_{23}^0\right| \quad (82)$$

which results from the direct parametrization for the PMNS mixing matrix and the detailed procedure for deriving the expression (81) is given in Ref. [7].

And from here we find:

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{BF}=1+2\cos\delta\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{BF}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF}\quad(83)$$

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{BF}=1+2\cos\delta_{BF}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{BF}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF}\quad(84)$$

$$\left[\left(\sin^2\theta_{23}^0\right)_{-1\sigma}\right]_{BF}=1+2\cos\delta_{-1\sigma}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{BF}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF}\quad(85)$$

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{BF}=1+2\cos\delta_{+1\sigma}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF}-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{BF}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{BF}\quad(86)$$

The inequality follows from these relations:

$$\begin{aligned}&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{BF}\le\left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{BF}\le\left\{\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right\}_{BF}\\&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{BF}=0.721346,\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-1\sigma}}\right]_{BF}=0.721192,\\&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{BF}=0.628866;\\&\rightarrow\cos\delta_{BF}\le\cos\delta\le\cos\delta_{+1\sigma}\rightarrow\delta_{BF}\le\delta\le\delta_{+1\sigma}.\end{aligned}\quad(87)$$

From these relations we find the region of definition for the Dirac CPV phase

$$\begin{aligned}&205.0879^\circ\le\delta\le296.052^\circ\rightarrow\\&\left(180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{th}=\left\{\left[\left(180^\circ\times\frac{24490}{739}\right)\Big/360^\circ-16\right]\times360^\circ\right\}_{th}\\&\approx205.00^\circ=\left[\left(222_{-17}\right)^\circ\right]_{\exp}\le\delta\le\left[\left(222^{+38}\right)^\circ\right]_{\exp}=260^\circ\end{aligned}\quad(88)$$

Case C8 calculations based on -1σ of the mixing angles

We write the corresponding relations in this basis:

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{-1\sigma}=1+2\cos\delta\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma}-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{-1\sigma}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma}\quad(89)$$

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{-1\sigma}=1+2\cos\delta_{BF}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma}-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{-1\sigma}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma}\quad(90)$$

$$\left[\left(\sin^2\theta_{23}^0\right)_{-1\sigma}\right]_{-1\sigma}=1+2\cos\delta_{-1\sigma}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma}-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{-1\sigma}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma}\quad(91)$$

$$\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma}=1+2\cos\delta_{+1\sigma}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma}-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{-1\sigma}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{-1\sigma}\quad(92)$$

The inequality follows from these relations:

$$\begin{aligned}&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{\delta_{-1\sigma}}\le\left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{-1\sigma}\le\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma},\\&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{\delta_{-1\sigma}}=0.700142,\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-1\sigma}}\right]_{\delta_{-1\sigma}}=0.699993,\\&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{-1\sigma}=0.609543;\\&\to\cos\delta_{BF}\le\cos\delta\le\cos\delta_{+1\sigma}\to\delta_{BF}\le\delta\le\delta_{+1\sigma}.\end{aligned}\tag{93}$$

From these relations we find the region of definition for the Dirac CPV phase

$$\begin{aligned}&205.0879^\circ\le\delta\le296.052^\circ\to\\&180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=\left\{\left[\left(180^\circ\times\frac{24490}{739}\right)\Big/360^\circ-16\right]\times360^\circ\right\}_{th}\\&\approx205.00^\circ=\left[\left(222_{-17}\right)^\circ\right]_{\exp}\le\delta\le\left[\left(222^{+38}\right)^\circ\right]_{\exp}=260^\circ\end{aligned}\tag{94}$$

Case C8 calculations based on $+1\sigma$ of the mixing angles

We write the corresponding relations in this basis: We write the corresponding relations in this basis:

$$\begin{aligned}\left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{+1\sigma}&=1+2\cos\delta\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma}\\&\quad-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{+1\sigma}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{+1\sigma}\end{aligned}\tag{95}$$

$$\begin{aligned}\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{+1\sigma}&=1+2\cos\delta_{BF}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma}\\&\quad-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{+1\sigma}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{+1\sigma}\end{aligned}\tag{96}$$

$$\begin{aligned}\left[\left(\sin^2\theta_{23}^0\right)_{-1\sigma}\right]_{+1\sigma}&=1+2\cos\delta_{-1\sigma}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma}\\&\quad-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{+1\sigma}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{+1\sigma}\end{aligned}\tag{97}$$

$$\begin{aligned}\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma}&=1+2\cos\delta_{+1\sigma}\times\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma}\\&\quad-\left(\cos^2\theta_{12}\cos^2\theta_{23}\right)_{+1\sigma}-\left(\sin^2\theta_{12}\sin^2\theta_{23}\sin^2\theta_{13}\right)_{+1\sigma}\end{aligned}\tag{98}$$

The inequality follows from these relations:

$$\begin{aligned}&\left[\left(\sin^2\theta_{23}^0\right)_{BF}\right]_{+1\sigma}\le\left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{+1\sigma}\le\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma}\\&\left[\left(\sin^2\theta_{23}^0\right)_{+1\sigma}\right]_{+1\sigma}=0.740378,\left[\left(\sin^2\theta_{23}^0\right)_{-1\sigma}\right]_{+1\sigma}=0.740222,\\&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{+1\sigma}=0.646105\\&\to\cos\delta_{BF}\le\cos\delta\le\cos\delta_{+1\sigma}\to\delta_{BF}\le\delta\le\delta_{+1\sigma}.\end{aligned}\tag{99}$$

From these relations we find the region of definition for the Dirac CPV phase

$$\begin{aligned}&205.0879^\circ\le\delta\le296.052^\circ\to\\&180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=\left\{\left[\left(180^\circ\times\frac{24490}{739}\right)\Big/360^\circ-16\right]\times360^\circ\right\}_{th}\\&\approx205.00^\circ=\left[\left(222_{-17}\right)^\circ\right]_{\exp}\le\delta\le\left[\left(222^{+38}\right)^\circ\right]_{\exp}=260^\circ.\end{aligned}\tag{100}$$

5. Interrelation between the Sum Rule for $\cos\delta$ [7] and the Derived Formula for $\cos\delta$ [6]

In this part, we will apply the complete procedure that was shown in the preceding part. The only difference is that here for the corresponding calculations we will use data for neutrino parameters from another source [10].

In the following tables, the explicit numerous values that we will use in our procedure are given.

Ref. [10]

5. Interrelation between the Sum Rule for $\cos\delta$ [7] and the Derived Formula for $\cos\delta$ [6]

In this part, we will apply the complete procedure that was shown in the preceding part. The only difference is that here for the corresponding calculations we will use data for neutrino parameters from another source [10].

In the following **Table 2**, the explicit numerous values that we will use in our procedure are given.

Ref. [10]

Table 2. Numerous values.

	Range of measured parameters	*BF*	-1σ	$+1\sigma$
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.03^{+0.12}_{-0.11}$	3.03	2.920	3.150
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.72^{+0.28}_{-0.20}$	5.720	5.520	6.00
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.203^{+0.056}_{-0.053}$	2.203	2.150	2.261
$\sin\theta_{12}$		0.550454	0.650470	0.561248
$\cos\theta_{12}$		0.834865	0.842427	0.827647
$\sin\theta_{23}$		0.7563068	0.7314369	0.760263
$\cos\theta_{23}$		0.654217	0.669328	0.632455
$\sin\theta_{13}$		0.148425	0.146628	0.150366
$\cos\theta_{13}$		0.9889236	0.989191	0.988630
$\frac{\Delta m^2_{21}}{10^{-5}\,\text{eV}^2}$	$7.41^{+0.21}_{-0.20}$	7.41	7.21	7.62
$\frac{\Delta m^2_{32}}{10^{-3}\,\text{eV}^2}$		2.4369	2.4119	2.4628
$\frac{\Delta m^2_{31}}{10^{-3}\,\text{eV}^2}$	$2.5110^{+0.028}_{-0.027}$	2.5110	2.4840	2.5390
$\delta_{CP}\,/^\circ$	197^{+42}_{-25}	197	172	239

(101)

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Based on the data from Table 2, numerous values are calculated that are shown in the formula (102).

$$\begin{aligned}
&\delta_{BF} = 180^\circ \times (25110/741) = \left(180^\circ \times (25110/741)/360^\circ - 16\right) \times 360^\circ = 339.595^\circ,\\
&\cos 339.595^\circ = 0.937252;\\
&\delta_{-1\sigma} = 180^\circ \times (24119/721) = \left(180^\circ \times (24119/721)/360^\circ - 16\right) \times 360^\circ = 261.386962^\circ,\\
&\cos 261.386962^\circ = -0.149760;\\
&\delta_{+1\sigma} = 180^\circ \times (25390/762) = \left(180^\circ \times (25390/762)/360^\circ - 16\right) \times 360^\circ = 237.637795^\circ,\\
&\cos 237.637795^\circ = -0.535269715;\\
&2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} = +0.063263,\\
&2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} = -0.010108,\\
&2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{BF} = -0.0361299;\\
&2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} = +0.062147,\\
&2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} = -0.009930,\\
&2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{-1\sigma} = -0.035492;\\
&2\cos\delta_{BF}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} = +0.0641415,\\
&2\cos\delta_{-1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} = -0.01024897,\\
&2\cos\delta_{+1\sigma}\left(\sin\theta_{13}\sin\theta_{23}\cos\theta_{23}\sin\theta_{12}\cos\theta_{12}\right)_{+1\sigma} = -0.0366317.
\end{aligned} \tag{102}$$

5.1. Sum Rules for $\cos\delta$ of Selected Residual Discrete Symmetry Groups [7]

5.1.1. Case with $U_{12}\left(\theta_{12}^{e},\delta_{12}^{e}\right)$ and $U_{23}\left(\theta_{23}^{\nu},\delta_{23}^{\nu}\right)$ Complex Rotations (Case C1)

Case C1 calculations based on (BF) of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}
&\left[\left(\sin^2\theta_{23}^{0}\right)_{\delta_{+1\sigma}}\right]_{BF} \le \left[\left(\sin^2\theta_{23}^{0}\right)_{\delta}\right]_{BF} \le \left[\left(\sin^2\theta_{23}^{0}\right)_{\delta_{BF}}\right]_{BF}\\
&\left[\left(\sin^2\theta_{23}^{0}\right)_{\delta_{+1\sigma}}\right]_{BF} = 0.3654, \left[\left(\sin^2\theta_{23}^{0}\right)_{\delta_{-1\sigma}}\right]_{BF} = 0.3914,\\
&\left[\left(\sin^2\theta_{23}^{0}\right)_{\delta_{BF}}\right]_{BF} = 0.4648\\
&\to \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF} \to \delta_{+1\sigma} \le \delta \le \delta_{BF}
\end{aligned} \tag{103}$$

So, the end result will be:

$$237.637^\circ \le \delta \le 339.595^\circ \tag{104}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}
&237.637^\circ \le \delta \le 339.595^\circ \to\\
&\left(180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/360^\circ - 16\right] \times 360^\circ\right\}_{th}\\
&\approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239^\circ
\end{aligned} \tag{105}$$

We do similar calculations in two more bases $\pm 1\sigma$ for the mixing angles and we will show them without detailed calculations as we did in the best fit basis which is shown with (BF).

Case C1 calculations based on -1σ of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}
&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{-1\sigma} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{-1\sigma}, \\
&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} = 0.3581, \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-1\sigma}}\right]_{-1\sigma} = 0.38369, \\
&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{-1\sigma} = 0.4557; \\
&\to \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF} \to \delta_{+1\sigma} \le \delta \le \delta_{BF}.
\end{aligned} \tag{106}$$

So, the end result will be:

$$237.637^\circ \le \delta \le 339.595^\circ \tag{107}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}
&237.637^\circ \le \delta \le 339.595^\circ \to \\
&\left(180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \\
&\approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239^\circ
\end{aligned} \tag{108}$$

Case C1 calculations based on $+1\sigma$ of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}
&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{+1\sigma} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{+1\sigma} \\
&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma} = 0.3772, \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-1\sigma}}\right]_{+1\sigma} = 0.40360, \\
&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{+1\sigma} = 0.477989 \\
&\to \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF} \to \delta_{+1\sigma} \le \delta \le \delta_{BF}
\end{aligned} \tag{109}$$

So, the end result will be:

$$237.637^\circ \le \delta \le 339.595^\circ \tag{110}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}
&237.637^\circ \le \delta \le 339.595^\circ \to \\
&\left(180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \\
&\approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239.0^\circ
\end{aligned} \tag{111}$$

5.1.2. The Case with $U_{12}\left(\theta_{12}^e, \delta_{12}^e\right)$ and $U_{23}\left(\theta_{23}^\nu, \delta_{23}^\nu\right)$ Complex Rotations (Case C3)

Case C3 calculations based on (BF) of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{BF}\le\left[\left(\sin^2\theta_{13}^0\right)_{\delta}\right]_{BF}\le\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{BF}\\&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{BF}=0.116,\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{BF}=0.2160;\\&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-+1\sigma}}\right]_{BF}=0.1899\\&\to-\cos\delta_{BF}\le-\cos\delta\le-\cos\delta_{+1\sigma}.\to\delta_{+1\sigma}\le\delta\le\delta_{BF}.\end{aligned}\tag{112}$$

So, the end result will be:

$$237.637^\circ\le\delta\le339.595^\circ\tag{113}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}&237.637^\circ\le\delta\le339.595^\circ\to\\&\left(180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma}=\left\{\left[\left(180^\circ\times\frac{25390}{762}\right)\Big/360^\circ-16\right]\times360^\circ\right\}_{th}\\&\approx\left(237.637^\circ\right)_{th}\approx\left[\left(197^{+41}\right)^\circ\right]_{\exp}\le\delta\le\left[\left(197^{+42}\right)^\circ\right]_{\exp}=239.0^\circ\end{aligned}\tag{114}$$

Case C3 calculations based on -1σ of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{-1\sigma}\le\left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{-1\sigma}\le\left[\left(\sin^2\theta_{13}^0\right)\delta_{BF}\right]_{-1\sigma}\\&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{-1\sigma}=0.1058,\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma}=0.203495;\\&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-1\sigma}}\right]_{-1\sigma}=0.177933.\\&\to-\cos\delta_{BF}\le-\cos\delta\le-\cos\delta_{+1\sigma}\to\delta_{+1\sigma}\le\delta\le\delta_{BF}.\end{aligned}\tag{115}$$

So, the end result will be:

$$237.637^\circ\le\delta\le339.595^\circ\tag{116}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}&237.637^\circ\le\delta\le339.595^\circ\to\\&\left(180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma}=\left\{\left[\left(180^\circ\times\frac{25390}{762}\right)\Big/360^\circ-16\right]\times360^\circ\right\}_{th}\\&\approx\left(237.637^\circ\right)_{th}\approx\left[\left(197^{+41}\right)^\circ\right]_{\exp}\le\delta\le\left[\left(197^{+42}\right)^\circ\right]_{\exp}=239.0^\circ\end{aligned}\tag{117}$$

Case C3 calculations based on $+1\sigma$ of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{+1\sigma}\le\left[\left(\sin^2\theta_{13}^0\right)_{\delta}\right]_{+1\sigma}\le\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma}\\&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{BF}}\right]_{+1\sigma}=0.13105,\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma}=0.2318,\\&\left[\left(\sin^2\theta_{13}^0\right)_{\delta_{-1\sigma}}\right]_{+1\sigma}=0.2054,\\&\to-\cos\delta_{BF}\le-\cos\delta\le-\cos\delta_{+1\sigma}\to\delta_{+1\sigma}\le\delta\le\delta_{BF}.\end{aligned}\tag{118}$$

So, the end result will be:

$$237.637^{\circ} \le \delta \le 339.595^{\circ} \tag{119}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}
&237.637^{\circ} \le \delta \le 339.595^{\circ} \rightarrow \\
&\left(180^{\circ} \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^{\circ} \times \frac{25390}{762}\right)\Big/360^{\circ} - 16\right] \times 360^{\circ}\right\}_{th} \\
&\approx \left(237.637^{\circ}\right)_{th} \approx \left[\left(197^{+41}\right)^{\circ}\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^{\circ}\right]_{\exp} = 239.0^{\circ}
\end{aligned} \tag{120}$$

5.1.3. The Case with $U_{13}\left(\theta_{13}^{e}, \delta_{13}^{e}\right)$ and $U_{23}\left(\theta_{23}^{\nu}, \delta_{23}^{\nu}\right)$ Complex Rotations (Case C4)

Case C4 calculations based on (BF) of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}
&\left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{BF} \le \left[\left(\sin^2 \theta_{12}^0\right)_{\delta}\right]_{BF} \le \left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{BF}}\right]_{BF}, \\
&\left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{BF} = 0.092339, \left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{-+1\sigma}}\right]_{BF} = 0.128359, \\
&\left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{BF}}\right]_{BF} = 0.201730; \\
&\rightarrow \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF}. \rightarrow \delta_{+1\sigma} \le \delta \le \delta_{BF}.
\end{aligned} \tag{121}$$

So, the end result will be:

$$237.637^{\circ} \le \delta \le 339.595^{\circ} \tag{122}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}
&237.637^{\circ} \le \delta \le 339.595^{\circ} \rightarrow \\
&\left(180^{\circ} \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^{\circ} \times \frac{25390}{762}\right)\Big/360^{\circ} - 16\right] \times 360^{\circ}\right\}_{th} \\
&\approx \left(237.637^{\circ}\right)_{th} \approx \left[\left(197^{+41}\right)^{\circ}\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^{\circ}\right]_{\exp} = 239.0^{\circ}
\end{aligned} \tag{123}$$

Case C4 calculations based on -1σ of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}
&\left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} \le \left[\left(\sin^2 \theta_{12}^0\right)_{\delta}\right]_{-1\sigma} \le \left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{BF}}\right]_{-1\sigma}, \\
&\left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} = 0.103726, \left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{-1\sigma}}\right]_{-1\sigma} = 0.129288, \\
&\left[\left(\sin^2 \theta_{12}^0\right)_{\delta_{BF}}\right]_{-1\sigma} = 0.201365; \\
&\rightarrow \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF} \rightarrow \delta_{+1\sigma} \le \delta \le \delta_{BF}.
\end{aligned} \tag{124}$$

So, the end result will be:

$$237.637^{\circ} \le \delta \le 339.595^{\circ} \tag{125}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}&237.637^\circ \le \delta \le 339.595^\circ \to \\ &\left(180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \\ &\approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239.0^\circ\end{aligned} \tag{126}$$

Case C4 calculations based on $+1\sigma$ of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}&\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma} \le \left[\left(\sin^2\theta_{12}^0\right)_{\delta}\right]_{+1\sigma} \le \left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{+1\sigma} \\ &\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma} = 0.098660, \left[\left(\sin^2\theta_{12}^0\right)_{\delta_{-1\sigma}}\right]_{+1\sigma} = 0.125043, \\ &\left[\left(\sin^2\theta_{12}^0\right)_{\delta_{BF}}\right]_{+1\sigma} = 0.199432; \\ &\to \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF} \to \delta_{+1\sigma} \le \delta \le \delta_{BF}.\end{aligned} \tag{127}$$

So, the end result will be:

$$237.637^\circ \le \delta \le 339.595^\circ \tag{128}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$\begin{aligned}&237.637^\circ \le \delta \le 339.595^\circ \to \\ &\left(180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \\ &\approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239.0^\circ\end{aligned} \tag{129}$$

5.1.4. The Case with $U_{13}\left(\theta_{13}^e, \delta_{13}^e\right)$ and $U_{13}\left(\theta_{13}^\nu, \delta_{13}^\nu\right)$ Complex Rotations (Case C8)

Case C8 calculations based on (BF) of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{aligned}&\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{BF} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{BF} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{BF}, \\ &\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{BF} = 0.661737, \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-1\sigma}}\right]_{BF} = 0.687758, \\ &\left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{BF} = 0.761129; \\ &\to \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF} \to \delta_{+1\sigma} \le \delta \le \delta_{BF}.\end{aligned} \tag{130}$$

So, the end result will be:

$$237.637^\circ \le \delta \le 339.595^\circ \tag{131}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$237.637^\circ \le \delta \le 339.595^\circ \rightarrow \left(180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239.0^\circ \tag{132}$$

Case C8 calculations based on -1σ of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{gathered}\left[\left(\sin^2\theta_{23}^0\right)_{+1\sigma}\right]_{-1\sigma} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta}\right]_{-1\sigma} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{-1\sigma}, \\ \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{-1\sigma} = 0.643859, \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-+1\sigma}}\right]_{-1\sigma} = 0.669421, \\ \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{-1\sigma} = 0.741498; \\ \rightarrow \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF} \rightarrow \delta_{+1\sigma} \le \delta \le \delta_{BF}.\end{gathered} \tag{133}$$

So, the end result will be:

$$237.637^\circ \le \delta \le 339.595^\circ \tag{134}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$237.637^\circ \le \delta \le 339.595^\circ \rightarrow \left(180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239.0^\circ \tag{135}$$

Case C8 calculations based on $+1\sigma$ of the mixing angles

First, we highlight the inequality resulting from the calculation:

$$\begin{gathered}\left[\left(\sin^2\theta_{23}^0\right)_{+1\sigma}\right]_{+1\sigma} \le \left\|\left(\sin^2\theta_{23}^0\right)_{\delta}\right\|_{+1\sigma} \le \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{+1\sigma}, \\ \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{+1\sigma}}\right]_{+1\sigma} = 0.685096, \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{-1\sigma}}\right]_{+1\sigma} = 0.711479, \\ \left[\left(\sin^2\theta_{23}^0\right)_{\delta_{BF}}\right]_{+1\sigma} = 0.785868; \\ \rightarrow \cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF} \rightarrow \delta_{+1\sigma} \le \delta \le \delta_{BF}.\end{gathered} \tag{136}$$

So, the end result will be:

$$237.637^\circ \le \delta \le 339.595^\circ \tag{137}$$

This result combines both theoretical and experimental results, so we can write it like this:

$$237.637^\circ \le \delta \le 339.595^\circ \rightarrow \left(180^\circ \times \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/ 360^\circ - 16\right] \times 360^\circ\right\}_{th} \approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239.0^\circ \tag{138}$$

which matches the result

$$\left(197^{+41}\right)^{\circ} \le \delta \le \left(197^{+42}\right)^{\circ} \tag{139}$$

And at the very end of this paper, if we were to ask whether the formula for Dirac's CPV phase was justifiably applied and withstood the test of its use as a bond with some of the residual discrete symmetry groups, then we could look for the answer in the following computational example.

Calculation task. If the parameters $\left(\Delta m_{21}^2\right)_{+1\sigma} = 0.0000762\ \text{eV}^2$ and $\left(\delta_{CP}\right)_{+1\sigma} = \left[\left(197^{+42}\right)_{+1\sigma}\right]^{\circ} = 239.0^{\circ}$ are measured in experiments, give an answer to the following question: what numerical value should be expected for $\left(\Delta m_{31}^2\right)_{+1\sigma}$ if formula (1) is applied?

Solution: We apply the formula (1) for δ_{CP} in the following form:

$$\left(\delta_{CP}\right)_{+1\sigma} = 180^{\circ} \times \frac{\left(\Delta m_{31}^2\right)_{+1\sigma}}{\left(\Delta m_{21}^2\right)_{+1\sigma}} = 180^{\circ} \times \frac{\left(\Delta m_{31}^2\right)_{+1\sigma}}{0.0000762\ \text{eV}^2}$$

$$= \left(180^{\circ} \times \frac{\left(\Delta m_{31}^2\right)_{+1\sigma}}{0.0000762\ \text{eV}^2} \Big/ 360^{\circ} - 16\right) \times 360^{\circ} = 239^{\circ}$$

where is the unknown quantity $\left(\Delta m_{31}^2\right)_{+1\sigma}$, and we find it from this equation.

$$\left(\Delta m_{31}^2\right)_{+1\sigma} = 2 \times 0.0000762\ \text{eV}^2 \times \left(16 + \frac{239^{\circ}}{360^{\circ}}\right) \approx 0.0025396\ \text{eV}^2.$$

6. Summary

For all performed calculations, we single out the main results that arose from the coupling between the formula for the Dirac CPV phase and the selected residual discrete symmetry groups, and they are as follows:

For experimental data given in Ref. [8] [9]

Common inequality for all applied residual symmetry groups reads:

$$\cos\delta_{BF} \le \cos\delta \le \cos\delta_{+1\sigma}$$

This result combines both theoretical and experimental results for the Dirac CPV phase, so we can write it like this:

$$205.0879^{\circ} \le \delta \le 296.052^{\circ} \to$$

$$180^{\circ} \times \left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{BF} = \left\{\left[\left(180^{\circ} \times \frac{24490}{739}\right) \Big/ 360^{\circ} - 16\right] \times 360^{\circ}\right\}_{th}$$

$$\approx 205.0^{\circ} = \left[\left(222_{-17}\right)^{\circ}\right]_{\exp} \le \delta \le \left[\left(222^{+38}\right)^{\circ}\right]_{\exp} = 260.0^{\circ}.$$

As we have seen, the obtained inequality is based both on the Dirac CPV phases calculated on the basis of formula (1) and on the mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$, and all are shown in (BF) and $\pm 1\sigma$ bases.

However, based on the final result, it is clearly seen that the area of definition for the Dirac CPV phase does not depend on the mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$.

For experimental data given in Ref. [10]

First, we highlight the inequality resulting from the calculation reads:

$$\cos\delta_{+1\sigma} \le \cos\delta \le \cos\delta_{BF}.$$

This result combines both theoretical and experimental results for the Dirac CPV phase, so we can write it like this:

$$237.637^\circ \le \delta \le 339.595^\circ \to$$
$$180^\circ \times \left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)_{+1\sigma} = \left\{\left[\left(180^\circ \times \frac{25390}{762}\right)\Big/360^\circ - 16\right] \times 360^\circ\right\}_{th}$$
$$\approx \left(237.637^\circ\right)_{th} \approx \left[\left(197^{+41}\right)^\circ\right]_{\exp} \le \delta \le \left[\left(197^{+42}\right)^\circ\right]_{\exp} = 239.0^\circ$$

As we have seen, the obtained inequality is based both on the Dirac CPV phases calculated on the basis of formula and on the mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$, and all are shown in (BF) and $\pm 1\sigma$ bases.

However, based on the final result, it is clearly seen that the range of definition for the Dirac CPV phase does not depend on the mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$.

For a more detailed introduction to the theory of symmetry groups, it can be seen in works [11] [12] [13] [14].

7. Conclusions

Essentially the main intention of this paper was to test the formula for the Dirac CPV phase and see if it can reflect the results of experimental measurements of neutrino parameters.

By knowing the mathematical formula for the Dirac CPV phase, a connection was established with some of the residual symmetry groups, which made it possible to develop a procedure for directly determining the range in which the numerical value for the Dirac CPV phase could be found. In this sense, two different sources of information containing measured data for neutrinos were used for the corresponding calculations, and then a comparative overview of the calculated results was presented.

It is particularly emphasized that the formula for the Dirac CPV phase does not depend on the mixing angles that are incorporated into the PMNS matrix, but only on the ratio between the corresponding squares of the neutrino mass difference.

All the numerical results obtained from the corresponding calculations for the Dirac CPV phase point to the justified introduction of the theory that is related to three neutrinos, and thus the agreement of our results with the STEREO experiment is justified, so that the hypothesis of the possible existence of a sterile neutrino in nature should be excluded.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

Examples of applying the formula for the Dirac CP violation phase and the procedure for determining the quadrant of the trigonometric circle in which it is located.

Data source: [8] [9]

In order to avoid confusion regarding the application of the formula for Dirac's CP violation phase, here we provide examples of calculations.

According to the data found in Table 1, we write the following:

$$\tan\left[180^\circ\times\left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right]=\tan\left[180^\circ\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)\right]$$
$$=\frac{\sin\left[180^\circ\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)\right]}{\cos\left[180^\circ\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)\right]}=\frac{-\sin\left[180^\circ\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right]}{-\cos\left[180^\circ\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right]}\to \tag{A1}$$
$$\sin\left[180^\circ\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right]_{BF}=\sin\delta_{BF}=\sin\left[180^\circ\times\frac{24490}{739}\right]=-0.424009,$$
$$\cos\left[180^\circ\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right]_{BF}=\cos\delta_{BF}=\cos\left[180^\circ\times\left(\frac{24490}{739}\right)\right]=-0.9056579.$$

The point of dividing this formula into sine and cosine is to see which quadrant of the trigonometric circle their values belong to. Based on the result (A1), we see that δ_{BF} belongs to the third quadrant.

Another way we can calculate the delta angle is in the following application of the basic formula:

$$\delta_{BF}=180^\circ\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)_{BF}=180^\circ\times\frac{24490}{739}$$
$$=\left[180^\circ\times\frac{24490}{739}\Big/360^\circ-16\right]\times360^\circ=205.0879^\circ \tag{A2}$$

because we see that the angle is in the third quadrant. And it really is

$$\sin 205.0879^\circ=-0.424008,\cos 205.0879^\circ=-0.905658. \tag{A3}$$

$$\delta_{-1\sigma}=\tan^{-1}\left\{\tan\left[180^\circ\times\left(\frac{24909}{719}\right)\right]\right\}$$
$$=\tan^{-1}\left\{\tan\left[180^\circ\times\left(\frac{24190}{719}\right)\right]\right\}=-64.084^\circ\to 295.916^0, \tag{A4}$$
$$\delta_{-1\sigma}=\left[180^\circ\times\frac{24190}{719}\Big/360^\circ-16\right]\times360^\circ\approx 295.910^\circ.$$

We determine the quadrant in which the Dirac CPV phase is located:

$$\tan\left(180^\circ\times\frac{24190}{719}\right)=\frac{\sin\left(180^\circ\times\frac{24190}{719}\right)}{\cos\left(180^\circ\times\frac{24190}{719}\right)}$$

$$\rightarrow \sin\left(180^{\circ}\times\frac{24190}{719}\right)=-0.899473997,\cos\left(180^{\circ}\times\frac{24190}{719}\right)=+0.436974;$$

$$\cos 295.910^{\circ}=\cos\left(180^{\circ}\times\frac{24190}{719}\right)\approx+0.436958, \tag{A5}$$

$$\sin 295.910^{\circ}=\sin\left(180^{\circ}\times\frac{24190}{719}\right)\approx-0.899481.$$

Based on these results, it can be seen that it is in the fourth quadrant.

$$\delta_{+1\sigma}=\tan^{-1}\left\{\tan\left[180^{\circ}\times\left(\frac{24810}{760}\right)\right]\right\}$$

$$=\tan^{-1}\left\{\tan\left[180^{\circ}\times\left(\frac{25570}{760}\right)\right]\right\}=-63.947^{\circ}\rightarrow 296.053^{\circ}, \tag{A6}$$

$$\delta_{+1\sigma}=\left[180^{\circ}\times\frac{25570}{760}\Big/360^{\circ}-16\right]\times360^{\circ}\approx 296.0526^{\circ}.$$

Based on these results, it can be seen that $\delta_{+1\sigma}$ is in the fourth quadrant.

By special calculation of sine and cosine, we see that it is really in the fourth quadrant.

$$\tan\left(180^{\circ}\times\frac{25570}{760}\right)=\frac{\sin\left(180^{\circ}\times\frac{25570}{760}\right)}{\cos\left(180^{\circ}\times\frac{25570}{760}\right)}$$

$$\rightarrow \sin\left(180^{\circ}\times\frac{25570}{760}\right)\approx-0.898390,\cos\left(180^{\circ}\times\frac{25570}{760}\right)\approx+0.439196; \tag{A7}$$

$$\cos 296.0526^{\circ}=\cos\left(180^{\circ}\times\frac{25570}{760}\right)\approx+0.439196,$$

$$\sin 296.0526^{\circ}=\sin\left(180^{\circ}\times\frac{25570}{760}\right)\approx-0.898391.$$

Therefore, these results also confirm that Dirac's CPV phase is located in the fourth quadrant.

Data source: [10]

In order to avoid confusion regarding the application of the formula for Dirac's CP violation phase, here we provide examples of calculations.

According to the data found in Table 2, we write the following:

$$\tan\left[180^{\circ}\times\left(\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right]=\tan\left[180^{\circ}\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+1\right)\right]=\tan\left[\left(180^{\circ}\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+180^{\circ}\right)\right]$$

$$=\frac{\sin\left[\left(180^{\circ}\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+180^{\circ}\right)\right]}{\cos\left[\left(180^{\circ}\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2}+180^{\circ}\right)\right]}=\frac{-\sin\left[180^{\circ}\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right]}{-\cos\left[180^{\circ}\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right]}\rightarrow \tag{A8}$$

$$\sin\left[180^{\circ}\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right]_{-1\sigma}=\sin\left[180^{\circ}\times\frac{24119}{721}\right]=-0.9887223,$$

$$\cos\left[180^{\circ}\times\left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right]_{-1\sigma}=\cos\left[180^{\circ}\times\frac{24119}{721}\right]=-0.149760.$$

The point of dividing this formula into sine and cosine is to see which quadrant of the trigonometric circle their values belong to. Based on the result (A8), we see that $\delta_{-1\sigma}$ belongs to the third quadrant.

Another way we can calculate the delta angle is in the following application of the basic formula:

$$\begin{aligned}\delta_{-1\sigma} &= 180^\circ \times \left(\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)_{-1\sigma} = 180^\circ \times \frac{24119}{721} \\ &= \left[180^\circ \times \frac{24119}{721} \Big/ 360^\circ - 16\right] \times 360^\circ = 261.3869^\circ\end{aligned} \quad \text{(A9)}$$

Because we see that this angle is in the third quadrant. And it really is

$$\sin 261.3869^\circ \approx -0.988722, \cos 261.3869^\circ \approx -0.149760. \quad \text{(A10)}$$

$$\delta_{+1\varpi} = \tan^{-1}\left\{\tan\left[180^\circ \times \left(\frac{25390}{762}\right)\right]\right\} = 57.637795^\circ, \quad \text{(A11)}$$

To see in which quadrant this angle is located, let's divide the tangent angle into sine and cosine in this way:

$$\sin\left[180^\circ \times \left(\frac{25390}{762}\right)\right] = -0.844681, \cos\left[180^\circ \times \left(\frac{25390}{762}\right)\right] = -0.5352697 \quad \text{(A12)}$$

So these results are in the third quadrant.

Then we apply the following formula for calculating the angle:

$$\delta_{+1\sigma} = \left[180^\circ \times \left(\frac{25390}{762}\right) \Big/ 360^\circ - 16\right] \times 360^\circ = 237.6377^\circ \quad \text{(A13)}$$

We calculate the sine and cosine of this angle:

$$\sin 237.6377^\circ = -0.844681, \cos 237.6377^\circ = -0.5352697. \quad \text{(A14)}$$

which matches (A12).

We determine the quadrant in which the Dirac CPV phase δ_{BF} is located:

$$\sin\left(180^\circ \times \frac{25110}{741}\right)_{BF} = -0.348651, \cos\left(180^\circ \times \frac{25110}{741}\right)_{BF} = +0.937252. \quad \text{(A15)}$$

Based on these results, it can be seen that δ_{BF} is in the fourth quadrant. To calculate δ_{BF}, we apply the formula:

$$\delta_{BF} = 180^\circ \times \left(\frac{25110}{741}\right)_{BF} = \left[180^\circ \times \left(\frac{25110}{741}\right) \Big/ 360^\circ - 16\right] \times 360^\circ = 339.595^\circ \quad \text{(A16)}$$

And from here we find:

$$\begin{aligned}\sin\left(180^\circ \times \frac{25110}{741}\right)_{BF} &= \sin\delta_{BF} = \sin 339.595^\circ \approx -0.348651, \\ \cos\left(180^\circ \times \frac{25110}{741}\right)_{BF} &= \cos\delta_{BF} = \cos 339.595^\circ \approx +0.937252.\end{aligned} \quad \text{(A17)}$$

And that belongs to the fourth quadrant.