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Three Neutrinos and the Formula for the Dirac CP Violation Phase

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Abstract

Based on the derived equations of three neutrinos, especially for motion through a physical vacuum and for space with a constant density of matter, the same formula for Dirac's CP-violating phase was obtained. The main property of this formula is that it does not depend on mixing angles $\theta_{12},\theta_{13},\theta_{23}$ and remains unchanged for the spaces through which the neutrino beam moves. Using that formula, the final form for the Jarlskog invariant formula was formed. Knowing the Dirac CPV phase would have the following consequences: 1) By obtaining an explicit mathematical formula for the Dirac CPV phase, it would no longer be necessary to perform computer simulations to draw areas where it could be found. 2) At the same time, the Dirac CPV phase does not depend on the mixing angles $\theta_{12},\theta_{13},\theta_{23}$ that make up the elements of the PMNS matrix, but depends only on the ratio of the corresponding differences of the squares of the neutrino masses.

Keywords

Ordinary Neutrino, PMNS Matrix, Dirac CPV Phase, Jarlskog Invariant

1. Introduction

In this paper, we devoted ourselves to researching the physical characteristics of three neutrinos, especially since it was recently published in several papers [1] [2] [3] [4] that there is experimental evidence that the possibility of the existence of sterile neutrinos has been disputed.

In essence the results of our research we presented only for the existence of three neutrons should be understood that they agree to the STEREO experiment [2], and that the hypothesis of sterile neutrinos should be abandoned.

In this regard, we focused our research on the fact that there could be only three neutrinos in nature, such as: the electron neutrino, the muon neutrino and the tau neutrino, as we have already shown in previous works [5] [6].

And in this paper, we will show the procedure for arriving at the formula for the Dirac CPV phase and in those cases when approximation methods are applied for determining the probability of neutrino oscillation, as stated in Refs. [7] [8]. In this sense, the cases when the neutrino beam propagates through a vacuum as well as through a medium with a constant density of matter are analyzed in particular.

In order to understand the essence of the mathematical process and its use, we made a comparison between the oscillation relations of three neutrinos in two cases: The first case is related to mathematical relations without approximations as shown in Refs. [5] [6] and the second case is related to mathematical relations with approximations as shown in Refs. [7] [8].

In Refs. [5] [6] analyzed the case when a neutrino beam propagates through a physical vacuum. In Refs. [7] [8] two cases are presented: propagation of a neutrino beam through a physical vacuum and propagation of a neutrino beam through space with a constant density of matter.

In all the conducted analyses, the identical mathematical formula for the Dirac CPV phase was arrived at. And what should be highlighted is the result for the formula for the Dirac CPV phase, in which two facts can be seen:

1) The mathematical formula for the Dirac CPV phase does not depend on the mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$.

2) The form of the mathematical formula for the Dirac CPV phase remains unchanged regardless of the characteristics of the medium through which the neutrino beam propagates.

Then, for the sake of comparison, the derivation of the Dirac CPV phase found in Ref. [6] is given again based on the equations for the probability of oscillation of three neutrinos in which no approximations are represented.

That is why all our research is directed towards finding a procedure with which we could give an answer to some still open questions such as:

1) Explicit mathematical formula for the Dirac CPV phase, δ_{CP}.

2) The final form of the formula for the Jarlskog invariant.

All the listed items still represent open questions that seek answers in Neutrino Physics.

In the paper [6] we presented the procedure by which we arrived at the explicit formula for the Dirac CP-violation phase, which we used to find its explicit numerical value.

Even after several decades of research in theoretical and experimental physics, especially related to Dirac's CP-violation phase, which is still shown through computer simulations in which you can see which areas it could belong to. This way of presenting Dirac's CP-violation phase tells us that it still remains an unsolved question.

First of all, because the unitary U_{PMNS} matrix does not change its unitary property for any arbitrarily taken value for it from the set $[0, 2\pi)$.

For this reason, the conclusion follows that in such a way of observing the unitary property of the U_{PMNS} matrix, it would not be possible to reach a solution that would make physical sense.

So, based on the unitary property of the U_{PMNS} matrix, we showed that it was necessary to take another step in order to arrive at the possibility of determining the formula for the Dirac CP-violation phase, and it is devoted to this in the following chapters.

2. Calculation of the Dirac CP-Violation Phase

Here, at the very beginning, we state the formulas for the Dirac CPV phase for both neutrino mass hierarchies that we derived in Ref. [6] considering the case without approximations and that for a neutrino beam that spreads through physical vacuums. In the following sections it will be seen that all analyzes and for cases where mathematical approximations are performed in which the oscillation probabilities for three neutrinos are derived, whether the neutrino beam moves through a physical vacuum or through a medium with a constant density of matter, the same formula for the Dirac CPV phase is reached.

In Ref. [7] the following formulas for the Dirac CP-violation phase were obtained:

Normal neutrino mass hierarchy

$$
\begin{aligned}
\delta_{CP(NO)} &= \tan^{-1}\left(\frac{2W_{(NO)}}{V_{(NO)}}\right) = \tan^{-1}\left[2\frac{\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)}{\sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)}\right] = \tan^{-1}\left[2\frac{\sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)}{\sin\left(2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)}\right] \\
&= \tan^{-1}\left[\tan\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right] = \tan^{-1}\left[\tan\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right] \\
&= \tan^{-1}\left[\tan\left(180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right] = \tan^{-1}\left[\tan\left(180^\circ\times\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right] \qquad (1)\\
&= \left\langle\delta_{CP(NO)}\right\rangle + \left|\Delta\left(\delta_{CP(NIO)}\right)\right| = 180^\circ\times\frac{\left\langle\Delta m_{31}^2\right\rangle}{\left\langle\Delta m_{21}^2\right\rangle} + \left|\Delta\left(180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right|,
\end{aligned}
$$

$$\Delta m_{31}^2 = \left\langle\Delta m_{31}^2\right\rangle \pm \Delta\left(\Delta m_{31}^2\right), \Delta m_{21}^2 = \left\langle\Delta m_{21}^2\right\rangle \pm \Delta\left(\Delta m_{21}^2\right)$$

which represents the solution of the equations:

$$
\begin{aligned}
&\left(2W_{(NO)}\cos\delta_{CP(NO)} - V_{(NO)}\sin\delta_{CP(NO)}\right)*0 \\
&= \left(2\sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\cos\delta_{CP(NO)} - \sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\sin\delta_{CP(NO)}\right)*0 \\
&= 2\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\left[\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\cos\delta_{CP(NO)} - \cos\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\sin\delta_{CP(NO)}\right]*0 \qquad (2)\\
&= 2\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\left[\sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} - \delta_{CP(NO)}\right)\right]*0 = 0 \\
&\to \sin\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} - \delta_{CP(NO)}\right) = 0 \to \pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} = \delta_{CP(NO)}.
\end{aligned}
$$

The first point that can be stated is that this equation is always satisfied for any value of $\delta_{CP(NO)} \in [0, 2\pi)$, so such solutions make no physical sense. It is apparent that among those solutions in the range $\delta_{CP(NO)} \in [0, 2\pi)$ there is the right unique solution for the value $\delta_{CP(NO)}$. From such set of countless values, the real and unique value for $\delta_{CP(NIO)}$ is drawn from the set $[0, 2\pi)$ by solving the particular equation

$$2W_{(NO)} \cos\delta_{CP(NO)} - V_{(NO)} \sin\delta_{CP(NO)} = 0 \tag{3}$$

Wherein

$$W_{(NO)} = \sin^2\left(\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right) = \sin^2\left(\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right), V_{(NO)} = \sin\left(2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right) = \sin\left(2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right) \tag{4}$$

We especially point out that the results (1) and (3) obtained in Ref [7] by analyzing the case where no approximations were introduced.

To avoid ambiguities regarding the appearance of these parameters, see relations (44)-(61).

Inverted neutrino mass hierarchy

$$\begin{aligned}
\delta_{CP(IO)} &= \tan^{-1}\left(\frac{2W_{(IO)}}{V_{(IO)}}\right) = \tan^{-1}\left[2\frac{\sin^2\left(\pi\frac{\Delta m_{23}^2}{\Delta m_{21}^2}\right)}{\sin\left(2\pi\frac{\Delta m_{23}^2}{\Delta m_{21}^2}\right)}\right] = \tan^{-1}\left[2\frac{\sin^2\left(\pi\frac{\Delta m_{13}^2}{\Delta m_{21}^2}\right)}{\sin\left(2\pi\frac{\Delta m_{13}^2}{\Delta m_{21}^2}\right)}\right] \\
&= \tan^{-1}\left[\tan\left(\pi\frac{\Delta m_{23}^2}{\Delta m_{21}^2}\right)\right] = \tan^{-1}\left[\tan\left(\pi\frac{\Delta m_{13}^2}{\Delta m_{21}^2}\right)\right] \\
&= \tan^{-1}\left[\tan\left(180^\circ \times \frac{\Delta m_{23}^2}{\Delta m_{21}^2}\right)\right] = \tan^{-1}\left[\tan\left(180^\circ \times \frac{\Delta m_{13}^2}{\Delta m_{21}^2}\right)\right] \\
&= \left\langle \delta_{CP(IO)} \right\rangle + \left|\Delta\left(\delta_{CP(IO)}\right)\right| = 180^\circ \times \frac{\left\langle \Delta m_{23}^2 \right\rangle}{\left\langle \Delta m_{21}^2 \right\rangle} + \left|\Delta\left(180^\circ \times \frac{\Delta m_{23}^2}{\Delta m_{21}^2}\right)\right|, \\
\Delta m_{23}^2 &= \left\langle \Delta m_{23}^2 \right\rangle \pm \Delta\left(\Delta m_{23}^2\right), \Delta m_{21}^2 = \left\langle \Delta m_{21}^2 \right\rangle \pm \Delta\left(\Delta m_{21}^2\right).
\end{aligned} \tag{5}$$

which represents the solution of the equations:

$$\left(2W_{(IO)} \cos\delta_{CP(IO)} - V_{(IO)} \sin\delta_{CP(IO)}\right) * 0 = 0 \tag{6}$$

The first point that can be stated is that this equation is always satisfied for any value of $\delta_{CP(IO)} \in [0, 2\pi)$, so such solutions make no physical sense. It is apparent that among those solutions in the range $\delta_{CP(IO)} \in [0, 2\pi)$ there is the right unique solution for the value $\delta_{CP(IO)}$. From such set of countless values, the real and unique value for $\delta_{CP(IO)}$ is drawn from the set $[0, 2\pi)$ by solving the particular equation

$$2W_{(IO)} \cos\delta_{CP(IO)} - V_{(IO)} \sin\delta_{CP(IO)} = 0 \tag{7}$$

Wherein

$$W_{(IO)} = \sin^2\left(\pi\frac{\Delta m_{23}^2}{\Delta m_{21}^2}\right) = \sin^2\left(\pi\frac{\Delta m_{13}^2}{\Delta m_{21}^2}\right), V_{(IO)} = \sin\left(2\pi\frac{\Delta m_{23}^2}{\Delta m_{21}^2}\right) = \sin\left(2\pi\frac{\Delta m_{13}^2}{\Delta m_{21}^2}\right) \tag{8}$$

We especially point out that the results (5) and (7) obtained in Ref. [6] by analyzing the case where no approximations were introduced.

In this paper, we extend that procedure but to the case when certain approximations are introduced in the theoretical consideration for three-flavor neutrino oscillations as shown in Ref. [5].

So our goal is to derive an equation in a similar way as it was done in Ref. [6] in which the Dirac CP-violation phase will appear as an unknown quantity.

2.1. Application of the Approximate Procedure Based on Series Expansion up to Second Order in $\alpha_{(NO)} = \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ and $s_{13(NO)}$ on Neutrino Oscillation Probabilities [7]

In the following chapters, we will show how the equations of motion for three neutrinos are derived. We will see that they completely coincide with the equation of type (2). Namely, we will use the mathematical expressions for the oscillation probabilities for three neutrinos that are presented in the paper [7] and show how the equation for the Dirac CPV phase is arrived

In order to achieve that goal, it is necessary to use the following general relations: $P_{e\mu} + P_{e\tau} + P_{ee} = 1$ or $P_{\mu e} + P_{\mu\tau} + P_{\mu\mu} = 1$ which we will illustrate in the following examples.

In the following sections, it is shown how the results for the probability of neutrino oscillation are used for the purposes of deriving the Dirac CPV phase. Here you can see how to arrive at the equation in which the Dirac CPV phase is an unknown quantity, and considering the specificity of the form of that equation, the procedure for arriving at it is given in a solution that makes physical sense.

The case when neutrinos move through a physical vacuum

The research that follows is based on the following assumption: It should be borne in mind that the length of the neutrino oscillation $L = L_{12}$ which is given by the expressions (C6) and (C17) was taken as a common parameter.

In the work [5], the following parameters were adopted:

$$\Delta = \frac{\Delta m_{31}^2 L c^3}{4E\hbar}, A = \frac{2EV}{\Delta m_{31}^2} = \frac{VL}{2\Delta} \tag{9}$$

The potential $V(x)$ is given by

$$V(x) = 7.56 \times 10^{-14} \left(\frac{\rho(x)}{\mathrm{g/cm^3}} \right) \mathrm{Y}_e(x)\,\mathrm{eV} \tag{10}$$

where $\rho(x)$ is the matter density along the neutrino path, and $\mathrm{Y}_e(x)$ is the number of electrons per nucleon. For the matter of the Earth one has, to a very accuracy $\mathrm{Y}_e(x) \approx 0.5$. And L represents the length of the traveled path of the neutrino beam, measured from the neutrino source. If the neutrinos travel through a vacuum, the length of the neutrino journey is calculated from the neutrino source. And this in the case when the source emits electron neutrinos.

Also, in the case with a constant density of matter, the same neutrino path length is taken into account. And it is equal to the electron neutrino oscillation length in both cases.

As will be seen later in the text, we denote that length by $L = L_{12}$.

In this case too, we will investigate the cases for the normal hierarchy and the inverse hierarchy of neutrino masses. We will perform the calculations for the distance traveled by the neutrino beam from the emission source to the distance $L = L_{12}$. Let's first write the parameters that we will adopt for the purposes of our analysis, especially for each hierarchy of masses.

For both mass hierarchies, we adopt the following notations:

Normal ordering

$$\Delta_{(NO)} = \frac{\Delta m_{31}^2 L c^3}{4E\hbar} = \Delta_{(NO)}(L), A_{(NO)} = \frac{2EV}{\Delta m_{31}^2} = \frac{VL}{2\Delta_{(NO)}}, \alpha_{(NO)} = \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \tag{11}$$

Inverted ordering

$$\Delta_{(IO)} = \frac{\Delta m_{23}^2 L c^3}{4E\hbar} = \Delta_{(IO)}(L), A_{(IO)} = \frac{2EV}{\Delta m_{23}^2} = \frac{VL}{2\Delta_{(IO)}}, \alpha_{(IO)} = \frac{\Delta m_{21}^2}{\Delta m_{23}^2} \tag{12}$$

In further considerations it will be seen that the introduced parameters $\Delta_{(NO)}, \Delta_{(IO)}$ are of crucial importance and that practically they determine the value for the Dirac CPV phase.

Normal ordering

We will derive the equation of motion for three neutrinos, and then we will analyze the meaning of the adopted parameters $\Delta_{(NO)} = \left(\Delta m_{31}^2 L c^3\right)/4E\hbar$ and $\Delta_{(IO)} = \left(\Delta m_{23}^2 L c^3\right)/4E\hbar$ in those equations. And especially for each neutrino mass hierarchy.

In the paper [7], the following formulas for three-flavor neutrino oscillation probabilities were derived, for the physical vacuum:

$$\begin{aligned}
P_{ee(NO)}^{vac} &= 1 - \alpha_{(NO)}^2 \sin^2 2\theta_{12(NO)} \Delta_{(NO)}^2 - 4 s_{13(NO)}^2 \sin^2 \Delta_{(NO)}, \\
P_{e\mu(NO)}^{vac} &= \alpha_{(NO)}^2 \sin^2 2\theta_{12(NO)} c_{23(NO)}^2 \Delta_{(NO)}^2 + 4 s_{13(NO)}^2 s_{23(NO)}^2 \sin^2 \Delta_{(NO)} \\
&\quad + 2\alpha_{(NO)} s_{13(NO)} \sin 2\theta_{12(NO)} \sin 2\theta_{23(NO)} \cos\left(\Delta_{(NO)} - \delta_{CP(NO)}\right) \Delta_{(NO)} \sin \Delta_{(NO)}, \\
P_{e\tau(NO)}^{vac} &= \alpha_{(NO)}^2 \sin^2 2\theta_{12(NO)} s_{23(NO)}^2 \Delta_{(NO)}^2 + 4 s_{13(NO)}^2 c_{23(NO)}^2 \sin^2 \Delta_{(NO)} \\
&\quad - 2\alpha_{(NO)} s_{13(NO)} \sin 2\theta_{12(NO)} \sin 2\theta_{23(NO)} \cos\left(\Delta_{(NO)} - \delta_{CP(NO)}\right) \Delta_{(NO)} \sin \Delta_{(NO)}
\end{aligned} \tag{13}$$

Arriving at the equation of motion of three neutrinos is achieved in two steps. In the first step, we separately add the left and right sides of the Equation (13) and get the equation

$$P_{ee(NO)}^{vac} + P_{e\mu(NO)}^{vac} + P_{e\tau(NO)}^{vac} = 1 \tag{14}$$

By writing the Equation (14) in an explicit form, we arrive at the equation of motion of three neutrinos and it reads:

$$\begin{aligned}&-\alpha_{(NO)}^{2}\sin^{2}2\theta_{12(NO)}\Delta_{(NO)}^{2}-4s_{13(NO)}^{2}\sin^{2}\Delta_{(NO)}\\&+\alpha_{(NO)}^{2}\sin^{2}2\theta_{12(NO)}c_{23(NO)}^{2}\Delta_{(NO)}^{2}+4s_{13(NO)}^{2}s_{23(NO)}^{2}\sin^{2}\Delta_{(NO)}\\&+2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\cos\left(\Delta_{(NO)}-\delta_{CP(NO)}\right)\Delta_{(NO)}\sin\Delta_{(NO)}\\&+\alpha_{(NO)}^{2}\sin^{2}2\theta_{12(NO)}s_{23(NO)}^{2}\Delta_{(NO)}^{2}+4s_{13(NO)}^{2}c_{23(NO)}^{2}\sin^{2}\Delta_{(NO)}\\&-2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\cos\left(\Delta_{(NO)}-\delta_{CP(NO)}\right)\Delta_{(NO)}\sin\Delta_{(NO)}=0\end{aligned}\tag{15}$$

In this equation, all parameters are known quantities except for one, which represents the Dirac CP-violation phase, $\delta_{(CP)(NO)}$, and it represents an unknown quantity in this equation. What we can first notice is that the Equation (15) is extremely simple in its structure:

It is obvious that this equation represents the identity in which the measured parameters are found except for $\delta_{(CP)(NO)}$ which represents the unknown quantity. And at first one could think that the unknown quantity in this equation could have all possible values from the set $[0,2\pi)$ which does not have any physical sense. And of course, the unitarity of the PMNS matrix would be satisfied in any case, but on the other hand, it was considered that in such an equation there is no unique solution for $\delta_{(CP)(NO)}$.

And that would be fine if the coefficients along $\cos\left(\Delta_{(NO)}-\delta_{(CP)(NO)}\right)$ were different.

However, the coefficients with $\cos\left(\Delta_{(NO)}-\delta_{(CP)(NO)}\right)$ represent algebraic expressions that are identical to each other, and we can separate them as a common factor, while all the other members of this identity cancel each other out.

Therefore, we can write the following equation on the basis of what has been said:

$$\begin{aligned}&\cos\left(\Delta_{(NO)}-\delta_{CP(NO)}\right)*\left(+2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\Delta_{(NO)}\sin\Delta_{(NO)}\right.\\&\left.-2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\Delta_{(NO)}\sin\Delta_{(NO)}\right)=0\end{aligned}\tag{16}$$

And the structure of this equation is reduced to the simplest possible form:

$$\cos\left[\Delta_{(NO)}(L)-\delta_{CP(NO)}\right]*0=0\tag{17}$$

In this equation we have two physical quantities, one of which is a variable $\Delta_{(NO)}(L)$ and it depends on the distance *L*, and the other which represents an unknown quantity $\delta_{(CP)(NO)}$ of this equation.

As the general equation has unlimited solutions for the unknown quantity $\delta_{(CP)(NO)}$, we can say the same for the solutions for the particular equation.

The Equation (17) defined in this way does not make physical sense, so as in the previous cases, we define the parameter $\Delta_{(NO)}(L)=\Delta_{(NO)}(L_{12})$ in terms of the oscillation wavelength equal to $L=L_{12}$. In that case, we write the Equation (17) in the form:

$$\cos\left[\Delta_{(NO)}(L_{12})-\delta_{CP(NO)}\right]*0=0\tag{18}$$

This equation has a general and a particular solution. The general solution contains all possible values from the set $\delta_{CP}\in[0,2\pi)$, and there are countless of them. So such a general solution has no physical meaning.

However, based on the general Equation (18), it is possible to form two particular equations whose solutions would satisfy the general equation. And each of those particular equations reads:

$$\cos\left(\Delta_{(NO)}\left(L_{12}\right)-\delta_{CP(NO)}\right)=0 \tag{19}$$

$$\cos\left(\Delta_{(NO)}\left(L_{12}\right)-\delta_{CP(NO)}\right)=1 \tag{20}$$

The solution of Equation (19) is:

$$\left(\Delta_{(NO)}\left(L_{12}\right)-\delta_{CP(NO)}\right)=\pm\frac{\pi}{2} \tag{21}$$

If we were to adopt this solution we would have:

$$0*0=0 \tag{22}$$

And that, at least apparently, from the point of view of mathematics, would be fine.

However, if we recall that the Equation (18) was created based on the application of the approximation method, then we would have to ignore this solution. That is why it (22) cannot be equated with the solution that was created without using the approximation method (2, 6).

But if we include another possible particular Equation (20), then its solution would look like this:

$$\delta_{CP(NO)}=\Delta_{(NO)}\left(L_{12}\right)=\frac{\Delta m_{31}^2 L_{12} c^3}{4E\hbar}=\frac{\Delta m_{31}^2 c^3}{4E\hbar}\frac{4\pi E\hbar}{c^3\Delta m_{21}^2}=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2} \tag{23}$$

which coincides with the solution obtained based on the procedure without approximations (1) and this result makes physical sense.

If we consider Equation (2) and its solution that is $\sin(\delta-\Delta)=0$ then if we ask how much it is $\cos(\delta-\Delta)$ we can get to the answer if we use the formula $\cos(\delta-\Delta)=1$.

This expression satisfies the approximation only if the solution in this development is equal to $(\delta-\Delta)=0$ which coincides with the solution of the initial Equation (20).

If we take into account that Equation (20) was obtained by the development process which is approximate in nature, then we could also use the formula for the development of the function into a series.

So, if we look at the other mathematical side, which refers to the development of a function into a series, we will have:

$$\begin{aligned}\exp\left[i(\delta-\Delta)\right]&=\sum_{n=0}^{\infty}\frac{\left[i(\delta-\Delta)\right]^n}{n!}=\cos(\delta-\Delta)+i\sin(\delta-\Delta)\\&=1+i\frac{(\delta-\Delta)}{1!}-\frac{(\delta-\Delta)^2}{2!}-\cdots\approx 1\\&\to\cos(\delta-\Delta)=1\to(\delta-\Delta)=0\to\delta=\Delta=\pi\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2}.\end{aligned}$$

The case when neutrinos move through a matter of constant density

In the paper [7], the following formulas for three-flavor neutrino oscillation probabilities were derived, for an environment with a constant density of matter:

$$P^{mat}_{ee(NO)} = 1-\alpha^2_{(NO)}\sin^2 2\theta_{12(NO)}\frac{\sin^2 A_{(NO)}\Delta_{(NO)}}{A^2_{(NO)}} - 4s^2_{13(NO)}\frac{\sin^2\left(A_{(NO)}-1\right)\Delta_{(NO)}}{\left(A_{(NO)}-1\right)^2},$$

$$P^{mat}_{e\mu(NO)} = \alpha^2_{(NO)}\sin^2 2\theta_{12(NO)}c^2_{23(NO)}\frac{\sin^2 A_{(NO)}\Delta_{(NO)}}{A^2_{(NO)}} + 4s^2_{13(NO)}s^2_{23(NO)}\frac{\sin^2\left(A_{(NO)}-1\right)\Delta_{(NO)}}{\left(A_{(NO)}-1\right)^2}$$

$$+2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\cos\left(\Delta_{(NO)}-\delta_{CP(NO)}\right)\frac{\sin A_{(NO)}\Delta_{(NO)}}{A_{(NO)}}\frac{\sin\left(A_{(NO)}-1\right)\Delta_{(NO)}}{\left(A_{(NO)}-1\right)}, \quad (24)$$

$$P^{mat}_{e\tau(NO)} = \alpha^2_{(NO)}\sin^2 2\theta_{12(NO)}s^2_{23(NO)}\frac{\sin^2 A_{(NO)}\Delta_{(NO)}}{A^2_{(NO)}} + 4s^2_{13(NO)}c^2_{23(NO)}\frac{\sin^2\left(A_{(NO)}-1\right)\Delta_{(NO)}}{\left(A_{(NO)}-1\right)^2}$$

$$-2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\cos\left(\Delta_{(NO)}-\delta_{CP(NO)}\right)\frac{\sin A_{(NO)}\Delta_{(NO)}}{A_{(NO)}}\frac{\sin\left(A_{(NO)}-1\right)\Delta_{(NO)}}{\left(A_{(NO)}-1\right)}.$$

where

$$\Delta_{(NO)} = \Delta_{(NO)}(L) = \frac{\Delta m^2_{31}Lc^3}{4E\hbar}, A_{(NO)} = \frac{2EV}{\Delta m^2_{31}} = \frac{VL}{2\Delta_{(NO)}}, \alpha_{(NO)} = \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \quad (25)$$

We will also use Equation (24) to derive the equation of motion of three neutrinos in the case when a beam of neutrinos passes through a medium with a constant density of matter.

By adding the left and right sides of the system of Equation (24), we get:

$$P^{mat}_{ee(NO)} + P^{mat}_{e\mu(NO)} + P^{mat}_{e\tau(NO)} = 1 \quad (26)$$

Also, we see that these equations contain the Dirac CP-violating phase $\delta_{(CP)(NO)} \in (0, 2\pi)$, which is an unknown quantity. We derive the equation of motion of three neutrinos in a neutrino beam from relation (26) and it reads:

$$-\alpha^2_{(NO)}\sin^2 2\theta_{12(NO)}\frac{\sin^2 A_{(NO)}\Delta_{(NO)}(L)}{A^2_{(NO)}} - 4s^2_{13(NO)}\frac{\sin^2\left(A_{(NO)}-1\right)\Delta_{(NO)}(L)}{\left(A_{(NO)}-1\right)^2}$$

$$+\alpha^2_{(NO)}\sin^2 2\theta_{12(NO)}c^2_{23(NO)}\frac{\sin^2 A_{(NO)}\Delta_{(NO)}(L)}{A^2_{(NO)}} + 4s^2_{13(NO)}s^2_{23(NO)}\frac{\sin^2\left(A_{(NO)}-1\right)\Delta_{(NO)}(L)}{\left(A_{(NO)}-1\right)^2}$$

$$+2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\cos\left[\Delta_{(NO)}(L)-\delta_{CP(NO)}\right]\frac{\sin A_{(NO)}\Delta_{(NO)}(L)}{A_{(NO)}}\frac{\sin\left(A_{(NO)}-1\right)\Delta_{(NO)}(L)}{\left(A_{(NO)}-1\right)}$$

$$+\alpha^2_{(NO)}\sin^2 2\theta_{12(NO)}s^2_{23(NO)}\frac{\sin^2 A_{(NO)}\Delta_{(NO)}(L)}{A^2_{(NO)}} + 4s^2_{13(NO)}c^2_{23(NO)}\frac{\sin^2\left(A_{(NO)}-1\right)\Delta_{(NO)}(L)}{\left(A_{(NO)}-1\right)^2}$$

$$-2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\cos\left[\Delta_{(NO)}(L)-\delta_{CP(NO)}\right]\frac{\sin A_{(NO)}\Delta_{(NO)}(L)}{A_{(NO)}}\frac{\sin\left(A_{(NO)}-1\right)\Delta_{(NO)}(L)}{\left(A_{(NO)}-1\right)} = 0. \quad (27)$$

Also, the structure of this equation is reduced to the simplest possible form:

$$\cos\left[\Delta_{(NO)}(L)-\delta_{CP(NO)}\right]$$
$$\times\left[2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\frac{\sin A_{(NO)}\Delta_{(NO)}(L)}{A_{(NO)}}\frac{\sin\left(A_{(NO)}-1\right)\Delta_{(NO)}(L)}{\left(A_{(NO)}-1\right)}\right.$$
$$\left.-2\alpha_{(NO)}s_{13(NO)}\sin 2\theta_{12(NO)}\sin 2\theta_{23(NO)}\frac{\sin A_{(NO)}\Delta_{(NO)}(L)}{A_{(NO)}}\frac{\sin\left(A_{(NO)}-1\right)\Delta_{(NO)}(L)}{\left(A_{(NO)}-1\right)}\right]=0,$$
$$\rightarrow\cos\left[\Delta_{(NO)}(L)-\delta_{CP(NO)}\right]\times 0=0.$$

From here we extract the final form of the equation:

$$\cos\left[\Delta_{(NO)}(L)-\delta_{CP(NO)}\right]*0=0 \tag{28}$$

In this equation we have two physical quantities, one of which is a variable $\Delta_{(NO)}(L)$ and depends on the distance L and the other which represents an unknown quantity $\delta_{(CP)(NO)}$ of this equation.

As the general equation has unlimited solutions for the unknown quantity $\delta_{(CP)(NO)}$, we can say the same for the solutions for the particular equation.

The Equation (28) defined in this way does not make physical sense, so as in the previous cases, we define the parameter $\Delta_{(NO)}(L)=\Delta_{(NO)}(L_{12})$ in terms of the oscillation wavelength, equal to $L=L_{12}$. In that case, we write the Equation (28) in the form:

$$\cos\left[\Delta_{(NO)}(L_{12})-\delta_{CP(NO)}\right]*0=0 \tag{29}$$

This equation has a general and a particular solution. The general solution contains all possible values from the set $\delta_{CP(NO)}\in(0,2\pi)$, and there are countless of them. So such a general solution has no physical meaning.

However, based on the general Equation (29), it is possible to form two particular equations whose solutions would satisfy the general equation. And each of those particular equations reads:

$$\cos\left[\Delta_{(NO)}(L_{12})-\delta_{CP(NO)}\right]=0 \tag{30}$$

$$\cos\left[\Delta_{(NO)}(L_{12})-\delta_{CP(NO)}\right]=1 \tag{31}$$

The solution of the Equation (30) is:

$$\left(\Delta_{(NO)}(L_{12})-\delta_{CP(NO)}\right)=\pm\frac{\pi}{2} \tag{32}$$

If we were to adopt this solution we would have:

$$0*0=0 \tag{33}$$

And that, at least apparently, from the point of view of mathematics, would be fine.

However, if we recall that the Equation (29) was created based on the application of the approximation method, then we would have to ignore this solution.

That is why it (33) cannot be equated with the solution that was created without using the approximation method (2, 6).

But if we include another possible particular Equation (31), then its solution would look like this:

$$\delta_{CP(NO)} = \Delta_{(NO)}\left(L_{12}\right) = \frac{\Delta m_{31}^2 L_{12} c^3}{4E\hbar} = \frac{\Delta m_{31}^2 c^3}{4E\hbar}\left(\frac{4\pi E\hbar}{c^3 \Delta m_{21}^2}\right) = \pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} = 180\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2} \quad (34)$$

which coincides with the solution (1).

Comment. To the analysis we performed, we add the formulas shown in Ref. [8] for the probabilities of neutrino oscillations when the neutrino beam passes through space with a constant density of matter:

$$\begin{aligned}
P_{ee} &= 1-\frac{8}{9}\alpha^2\frac{\sin^2 A\Delta}{A^2}-2r^2\frac{\sin^2\left(A-1\right)\Delta}{\left(A-1\right)^2},\\
P_{e\mu} &= \frac{4}{9}\alpha^2\frac{\sin^2 A\Delta}{A^2}+r^2\frac{\sin^2\left(A-1\right)\Delta}{\left(A-1\right)^2}+\frac{4}{3}r\alpha\cos\left(\Delta-\delta\right)\frac{\sin A\Delta}{A}\frac{\sin\left(A-1\right)\Delta}{\left(A-1\right)},\\
P_{e\tau} &= \frac{4}{9}\alpha^2\frac{\sin^2 A\Delta}{A^2}+r^2\frac{\sin^2\left(A-1\right)\Delta}{\left(A-1\right)^2}-\frac{4}{3}r\alpha\cos\left(\Delta-\delta\right)\frac{\sin A\Delta}{A}\frac{\sin\left(A-1\right)\Delta}{\left(A-1\right)}
\end{aligned} \quad (35)$$

Applying the rule that the sum of these neutrino oscillation probabilities is equal to unity, we will have:

$$P_{ee}+P_{e\mu}+P_{e\tau}=1 \quad (36)$$

We got the equation:

$$\begin{aligned}
&-\frac{8}{9}\alpha^2\frac{\sin^2 A\Delta}{A^2}-2r^2\frac{\sin^2\left(A-1\right)\Delta}{\left(A-1\right)^2}\\
&+\frac{4}{9}\alpha^2\frac{\sin^2 A\Delta}{A^2}+r^2\frac{\sin^2\left(A-1\right)\Delta}{\left(A-1\right)^2}+\frac{4}{3}r\alpha\cos\left(\Delta-\delta\right)\frac{\sin A\Delta}{A}\frac{\sin\left(A-1\right)\Delta}{\left(A-1\right)}\\
&+\frac{4}{9}\alpha^2\frac{\sin^2 A\Delta}{A^2}+r^2\frac{\sin^2\left(A-1\right)\Delta}{\left(A-1\right)^2}-\frac{4}{3}r\alpha\cos\left(\Delta-\delta\right)\frac{\sin A\Delta}{A}\frac{\sin\left(A-1\right)\Delta}{\left(A-1\right)}=0
\end{aligned} \quad (37)$$

So it is obvious that this equation reduces to the form:

$$\begin{aligned}
&\cos\left(\Delta-\delta\right)\times\left(\frac{4}{3}r\alpha\frac{\sin A\Delta}{A}\frac{\sin\left(A-1\right)\Delta}{\left(A-1\right)}-\frac{4}{3}r\alpha\frac{\sin A\Delta}{A}\frac{\sin\left(A-1\right)\Delta}{\left(A-1\right)}\right)\\
&=\cos\left(\Delta-\delta\right)\times 0=0
\end{aligned} \quad (38)$$

Carrying out the analysis as in the previous cases, we arrive at a particular solution that makes physical sense

$$\delta=\Delta=\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2}=180^\circ\times\frac{\Delta m_{31}^2}{\Delta m_{21}^2} \quad (39)$$

which coincides with the solutions already obtained.

In today's neutrino physics, an explicit formula for the Dirac CPV phase has not yet been derived because it is considered that the unitarity of the PMNS matrix is satisfied for any arbitrarily taken value for $\delta_{CP(NO)}$ from the set $\left(0,2\pi\right)$, of which there are practically infinitely many.

Well, in this connection, equations can be written as we have shown them in the forms ((15), (27), (39)). And really, those equations are essentially identities that are satisfied for any arbitrary value for $\delta_{CP(NO)}$ taken from the set $(0, 2\pi)$, and of which there can be infinitely many. Well in this regard, it is considered superfluous to think at all about a possible explicit value for $\delta_{CP(NO)}$ in the form of some mathematical formula.

However, when you look at all those equations that obviously represent identities, the fact is overlooked that along with the factors that contain an unknown quantity in the form of Dirac's CPV phase, there are completely identical mathematical algebraic expressions that can be extracted as a common factor.

And that's what we did, as shown by Equations ((1), (2), (18), (27), (37)), and noted that this equation can have two solutions: general and particular.

The general solution offers all possible values for the Dirac CPV phase from the set $(0, 2\pi)$, of which there are infinitely many. We also agree with the position of today's neutrino physics, and of course such solutions do not make physical sense.

However, we find a particular solution by solving particular Equations ((2), (20), (31)). Those particular solutions are mutually identical and they are extracted from the set $(0, 2\pi)$ through the explicit formula ((1), (23), (34), (38)).

And such a unique solution gives only one unique value for the Dirac CPV phase. And we think that value for the Dirac CPV phase makes physical sense, and we will show it in the next chapter when we connect that formula for the Dirac CPV phase with the sum rule for $\cos\delta$ of residual discrete symmetry groups.

Derivation of the equation without approximations for three neutrinos during the motion of the neutrino beam through the vacuum

In order to eliminate any ambiguities and for the sake of comparison between the case without approximations and the example with approximations, we will show again the way in which the final Equations ((1), (2)) was derived, which is shown in Ref. [6].

Based on the derived final formulas for the Dirac CPV phase, it can be seen that they all coincide with each other, which shows that the Dirac CPV phase does not depend on the medium through which the neutrino beam propagates.

And what is very important to point out is that the Dirac CPV phase depends exclusively on the ratio between the differences of the squares of the neutrino masses and does not depend on the mixing angles.

To explain why we write the final equation in the form ((1), (18), (27), (37)) we need to repeat the complete procedure given in Ref. [5] and this is given in the following text.

In the processes known as neutrino flavor oscillations, the Dirac CP violation phase δ_{CP} is singled out as the cause of those oscillations in the propagation of the neutrino beam through the physical vacuum. For that reason, there arises the question of writing the equation in which δ_{CP} would appear as an unknown quantity. On the basis of that equation, it would be possible to determine that unknown quantity. So far, there appears to be only one way to derive that equations for a neutrino beam, and it is related to the use of the equations of the neutrino oscillations probabilities. The procedure for deriving those equations is given here.

$$\begin{aligned} U_{PMNS}^{PDG} &= \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{i\delta_{CPe}} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{i\delta_{CPe}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{i\delta_{CPe}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{i\delta_{CP}} & c_{13}c_{23} \end{pmatrix} \\ &= \begin{pmatrix} U_{e1} & U_{e2} & J\mathrm{e}^{-i\delta_{CPe}} \\ -A - B\mathrm{e}^{i\delta_{CPe}} & C - D\mathrm{e}^{i\delta_{CPe}} & U_{\mu 3} \\ E - F\mathrm{e}^{i\delta_{CPe}} & -G - H\mathrm{e}^{i\delta_{CPe}} & U_{\tau 3} \end{pmatrix} \end{aligned} \tag{40}$$

where the mixing angles from the (44) are taken into consideration
$c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}; i, j = 1,2,3.$

In our considerations, we will use the general formula for neutrino oscillations given in [7] [8] [9] [10] [11]

$$\begin{aligned} P\left(\nu_\alpha \to \nu_\beta\right) &= \delta_{\alpha\beta} - 4\sum_{i<j} R_e\left(U_{\alpha i}U_{\beta i}^{*}U_{\alpha j}^{*}U_{\beta j}\right)\sin^2\left(\frac{\Delta m_{ji}^2 Lc^3}{4E\hbar}\right) \\ &\quad + 2\sum_{i<j}\mathrm{Im}\left(U_{\alpha i}U_{\beta i}^{*}U_{\alpha j}^{*}U_{\beta j}\right)\sin\left(\frac{\Delta m_{ji}^2 Lc^3}{2E\hbar}\right); i, j = 1,2,3. \end{aligned} \tag{41}$$

We will derive the equations of motion of the three neutrinos in such a way that we will use the property of the mixing matrix which is expressed by the following relation:

$$P\left(\nu_e \to \nu_\mu\right) + P\left(\nu_e \to \nu_\tau\right) + P\left(\nu_e \to \nu_e\right) = 1 \tag{42}$$

The main goal of this work is to derive the equations of three neutrinos and then to determine their root from those equations, which represents the solution for Dirac's CPV phase. In this sense, we will analyze the normal and inverted hierarchy of neutrinos in particular. We have chosen two ways to calculate the probability oscillations of three neutrinos. The first method refers to the calculation without any approximations, for a neutrino beam moving in a physical vacuum. For the second method, we used the formulas derived in the paper [5], which were obtained on the basis of the series expansion formulas for three-flavor neutrino oscillation probabilities in both mass ordering, especially for vacuum and constant matter density.

In all cases we will use transition channels $\nu_e \to \nu_\mu, \nu_e \to \nu_\tau, \nu_e \to \nu_e, L = L_{12}$

$$U_{PMN(NO)}^{PDG} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-i\delta_{CP(NO)}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{i\delta_{CP(NO)}} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{i\delta_{CP(NO)}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{i\delta_{CP(NO)}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{i\delta_{CP(NO)}} & c_{13}c_{23} \end{pmatrix} \quad (43)$$

$$= \begin{pmatrix} U_{e1} & U_{e2} & J\mathrm{e}^{-i\delta_{CP(NO)}} \\ -A - B\mathrm{e}^{i\delta_{CP(NO)}} & C - D\mathrm{e}^{i\delta_{CP(NO)}} & U_{\mu 3} \\ E - F\mathrm{e}^{i\delta_{CP(NO)}} & -G - H\mathrm{e}^{i\delta_{CP(NO)}} & U_{\tau 3} \end{pmatrix}$$

In order to obtain an explicit numerical value of δ_{CP}, the following unconditional rule will be applied: The sum of the probabilities of three neutrino oscillations during the transition $\nu_e \to \nu_\mu, \nu_e \to \nu_\tau, \nu_e \to \nu_e$, at a distance from the source equal to the entire wavelength of oscillations in the value of $L = L_{12}$, during the process of the disappearance in transition $\nu_e \to \nu_\mu \to \nu_e$, in the propagation of the neutrino beam through vacuum (as it can be seen, the matter effect is excluded in these considerations), is equal to one. In this case, the parameters of the matrix are taken for the normal neutrino mass hierarchy.

On the basis of formula (41), the total probability of neutrino oscillations is shown through the equation [6]

$$\begin{aligned}
&P(\nu_e \to \nu_\mu) + P(\nu_e \to \nu_\tau) + P(\nu_e \to \nu_e) \\
&= 1 - 4R\left\{U_{e1}U_{\mu 1}^* U_{e2}^* U_{\mu 2} \sin^2\left(\pi \frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{e1}U_{\mu 1}^* U_{e2}^* U_{\mu 2} \sin\left(2\pi \frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{e1}U_{\mu 1}^* U_{e3}^* U_{\mu 3} \sin^2\left(\pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{e1}U_{\mu 1}^* U_{e3}^* U_{\mu 3} \sin\left(2\pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{e2}U_{\mu 2}^* U_{e3}^* U_{\mu 3} \sin^2\left(\pi \frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{e2}U_{\mu 2}^* U_{e3}^* U_{\mu 3} \sin\left(2\pi \frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{e1}U_{\tau 1}^* U_{e2}^* U_{\tau 2} \sin^2\left(\pi \frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{e1}U_{\tau 1}^* U_{e2}^* U_{\tau 2} \sin\left(2\pi \frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{e1}U_{\tau 1}^* U_{e3}^* U_{\tau 3} \sin^2\left(\pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{e1}U_{\tau 1}^* U_{e3}^* U_{\tau 3} \sin\left(2\pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4R\left\{U_{e2}U_{\tau 2}^* U_{e3}^* U_{\tau 3} \sin^2\left(\pi \frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} + 2\,\mathrm{Im}\left\{U_{e2}U_{\tau 2}^* U_{e3}^* U_{\tau 3} \sin\left(2\pi \frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right)\right\} \\
&\quad - 4|U_{e1}|^2 |U_{e2}|^2 \sin^2\left(\pi \frac{\Delta m_{21}^2}{\Delta m_{21}^2}\right) - 4|U_{e1}|^2 |U_{e3}|^2 \sin^2\left(\pi \frac{\Delta m_{31}^2}{\Delta m_{21}^2}\right) \qquad (44) \\
&\quad - 4|U_{e2}|^2 |U_{e3}|^2 \sin^2\left(\pi \frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right) \\
&= 1
\end{aligned}$$

And, from the Equation (44), the equation of neutrino motion is formed with a condition that the travelled distance of the neutrino beam, moving through a vacuum from the source, equals the neutrino wavelength $L = L_{12}$. So, it can be written as

$$\begin{aligned}
&-4R\left\{U_{e1}U_{\mu 1}^{*}U_{e3}^{*}U_{\mu 3}W_{(NO)}\right\}+2\operatorname{Im}\left\{U_{e1}U_{\mu 1}^{*}U_{e3}^{*}U_{\mu 3}V_{(NO)}\right\}\\
&-4R\left\{U_{e2}U_{\mu 2}^{*}U_{e3}^{*}U_{\mu 3}W_{(NO)}\right\}+2\operatorname{Im}\left\{U_{e2}U_{\mu 2}^{*}U_{e3}^{*}U_{\mu 3}V_{(NO)}\right\}\\
&-4R\left\{U_{e1}U_{\tau 1}^{*}U_{e3}^{*}U_{\tau 3}W_{(NO)}\right\}+2\operatorname{Im}\left\{U_{e1}U_{\tau 1}^{*}U_{e3}^{*}U_{\tau 3}V_{(NO)}\right\}\\
&-4R\left\{U_{e2}U_{\tau 2}^{*}U_{e3}^{*}U_{\tau 3}W_{(NO)}\right\}+2\operatorname{Im}\left\{U_{e2}U_{\tau 2}^{*}U_{e3}^{*}U_{\tau 3}V_{(NO)}\right\}\\
&-4\left|U_{e1}\right|^{2}\left|U_{e3}\right|^{2}W_{(NO)}-4\left|U_{e2}\right|^{2}\left|U_{e3}\right|^{2}W_{(NO)}=0
\end{aligned}\tag{45}$$

where the following notations shown in the equation of motion of three neutrinos (45) are given by the following expressions:

$$\begin{aligned}
V_{(NO)}&=\sin\left(2\pi\frac{\Delta m_{31}^{2}}{\Delta m_{21}^{2}}\right)=\sin\left(2\pi\frac{\Delta m_{32}^{2}}{\Delta m_{21}^{2}}\right),\\
W_{(NO)}&=\sin^{2}\left(\pi\frac{\Delta m_{31}^{2}}{\Delta m_{21}^{2}}\right)=\sin^{2}\left(\pi\frac{\Delta m_{32}^{2}}{\Delta m_{21}^{2}}\right).
\end{aligned}\tag{46}$$

Algebraic rearrangement of the Equation (45) gives it the following complex form:

$$\begin{aligned}
&4W_{(NO)}J\left[U_{e1}\left(AU_{\mu 3}-EU_{\tau 3}\right)-U_{e2}\left(CU_{\mu 3}-GU_{\tau 3}\right)\right]\cos\delta_{CP(NO)}\\
&-2V_{(NO)}J\left[U_{e1}\left(AU_{\mu 3}-EU_{\tau 3}\right)-U_{e2}\left(CU_{\mu 3}-GU_{\tau 3}\right)\right]\sin\delta_{CP(NO)}\\
&-4W_{(NO)}U_{e1}^{2}J^{2}-4W_{(NO)}U_{e2}^{2}J^{2}+4W_{(NO)}JU_{e1}\left(BU_{\mu 3}+FU_{\tau 3}\right)\\
&+4W_{(NO)}JU_{e2}\left(DU_{\mu 3}+HU_{\tau 3}\right)\\
&=\left(4W_{(NO)}J\cos\delta_{CP(NO)}-2V_{(NO)}J\sin\delta_{CP(NO)}\right)\\
&\quad\times\left(U_{e1}AU_{\mu 3}-U_{e1}EU_{\tau 3}-U_{e2}CU_{\mu 3}+U_{e2}GU_{\tau 3}\right)\\
&\quad-4W_{(NO)}U_{e1}^{2}J^{2}-4W_{(NO)}U_{e2}^{2}J^{2}+4W_{(NO)}JU_{e1}\left(BU_{\mu 3}+FU_{\tau 3}\right)\\
&\quad+4W_{(NO)}JU_{e2}\left(DU_{\mu 3}+HU_{\tau 3}\right)\\
&=0
\end{aligned}\tag{47}$$

And this structure is reduced to an extremely simple form:

$$\left(4W_{(NO)}J\cos\delta_{CP(NO)}-2V_{(NO)}J\sin\delta_{CP(NO)}\right)\varsigma-\xi=0\tag{48}$$

In this equation, the following expressions equal zero:

$$\varsigma=\left(U_{e1}AU_{\mu 3}-U_{e1}EU_{\tau 3}-U_{e2}CU_{\mu 3}+U_{e2}GU_{\tau 3}\right)=0\tag{49}$$

Because

$$U_{\mu 3}A-U_{\tau 3}E=S_{23}C_{13}\times S_{12}C_{23}-C_{23}C_{13}\times S_{12}S_{23}=0\tag{50}$$

$$U_{\tau 3}G-U_{\mu 3}C=C_{23}C_{13}\times C_{12}S_{23}-S_{23}C_{13}\times C_{12}S_{23}=0\tag{51}$$

and

$$\begin{aligned}\xi &= -4W_{(NO)}U_{e1}^2J^2 - 4W_{(NO)}U_{e2}^2J^2 + 4W_{(NO)}U_{e1}J\left(BU_{\mu 3} + FU_{\tau 3}\right) \\ &\quad + 4W_{(NO)}U_{e2}J\left(DU_{\mu 3} + HU_{\tau 3}\right) \\ &= 0\end{aligned} \tag{52}$$

Because

$$U_{\mu 3}B + U_{\tau 3}F - U_{e1}J = S_{23}C_{13} \times C_{12}S_{23}S_{13} + C_{23}C_{13} \times C_{12}C_{23}S_{13} - C_{12}C_{13}S_{13} = 0 \tag{53}$$

$$U_{\mu 3}D + U_{\tau 3}H - U_{e2}J = S_{23}C_{13} \times S_{12}S_{23}S_{13} + C_{23}C_{13} \times S_{12}C_{23}S_{13} - S_{12}C_{13}S_{13} = 0 \tag{54}$$

And this equation reduces to an extremely simple form:

$$\left(2W_{(NO)}\cos\delta_{CP(NO)} - V_{(NO)}\sin\delta_{CP(NO)}\right) * 0 = 0 \tag{55}$$

We wrote this equation in that form because the coefficients next to $\cos\delta_{CP(NO)}$ and $\sin\delta_{CP(NO)}$ are completely identical.

However, that equation can also be written in this form:

$$2W_{(NO)}\cos\delta_{CP(NO)} \times 0 - V_{(NO)}\sin\delta_{CP(NO)} \times 0 = 0 \tag{56}$$

For the equation written this way, we would say that it is satisfied for every solution for the Dirac CP-violation phase from the set $[0, 2\pi)$, which we explained that it does not make physical sense.

Obviously, the equation written in the form (56) has lost its complete mathematical meaning, because the particular equation and its solution have been lost.

However, it is crucial in this way of writing the equation, in what relation the coefficients with $\cos\delta_{CP(NO)}$ and $\sin\delta_{CP(NO)}$ are related to each other.

If the coefficients with $\cos\delta_{CP(NO)}$ and $\sin\delta_{CP(NO)}$ were mutually different in algebraic form and each equal to zero, then the equation written like this (56) is the only possible mathematical form. And the solutions of such an equation have no physical meaning.

But as you can see, the coefficients with $\cos\delta_{CP(NO)}$ and $\sin\delta_{CP(NO)}$ are in the algebraic sense completely identical as expressions, and what is important is that each one is equal to zero, so there is a complete justification for writing the equation in the form (55).

And as we have seen, in such an equation a particular solution appears that makes physical sense.

So, based on the unitarity property of the U_{PMNS} matrix, we showed that it was necessary to take another step in order to arrive at the possibility of determining the formula for the Dirac CP-violation phase.

The final result of the analysis gave a very simple and unusual equation consisting of a general equation and a particular one. The general equation is in agreement with the position in Neutrino Physics that every solution from the set $\delta_{CP} \in [0, 2\pi)$ satisfies the unitarity property of the U_{PMNS} matrix, so such a solution no physical sense.

However, thanks to the particular equation, we are now able, by solving it, to arrive at an explicit formula for the Dirac CP-violation phase:

$$\left(2W_{(NO)}\cos\delta_{CP(NO)} - V_{(NO)}\sin\delta_{CP(NO)}\right) = 0 \tag{57}$$

The solution of this equation for the normal mass hierarchy is given by formula (1), while for the inverted mass hierarchy it is given by formula (5).

So, with that formula and using experimental data, we now extract only one value from that set of countless values $(0, 2\pi)$ and it is the only one that makes physical sense.

Note. The entire procedure that was done for the normal hierarchy of neutrino masses is also applied for the inverted hierarchy of neutrino masses, as long as the corresponding labeling should be adjusted accordingly.

Thus, we will obtain the Dirac CP-phase of the neutrino for the case of the inverted hierarchy as shown with formula (6).

3. The Final Form of the Formula for the Jarlskog Invariant

The Jarlskog invariant has been published in papers [9] in general form by formula $J_{CP}^{Sym} = s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin\delta_{CP} = J_{CP}^{\max} \sin\delta_{CP}$. Due to the ignorance of the explicit numerical value for the Dirac CP-violating phase in neutrino physics, it is represented by computer simulations with graph as shown in **Figure 1**.

The influence of the medium with constant mass density through which the neutrino beam spreads was analyzed in Ref. [12] and they derived the following mathematical relationship:

$$\sin 2\theta_{23}^{m} \sin\delta^{m} = \sin 2\theta_{23} \sin\delta \tag{58}$$

This equality means that the product $\sin 2\theta_{23} \sin\delta$ does not depend on the matter potential, *i.e.*, it is the same for neutrino oscillations occurring in a vacuum as well as in a medium with a constant matter density.

This statement is also valid for neutrino and antineutrino oscillations in a medium with matter, regardless of the nature of the neutrino mass hierarchy.

The following conclusions can also be drawn from this:

$$\theta_{23}^{m} = \theta_{23}, \delta^{m} = \delta \tag{59}$$

And as for the sign for Jarlskog invariant remains unchanged:

$$\operatorname{sgn}\left(J_{CP}^{m}\right) = \operatorname{sgn}\left(J_{CP}\right) \tag{60}$$

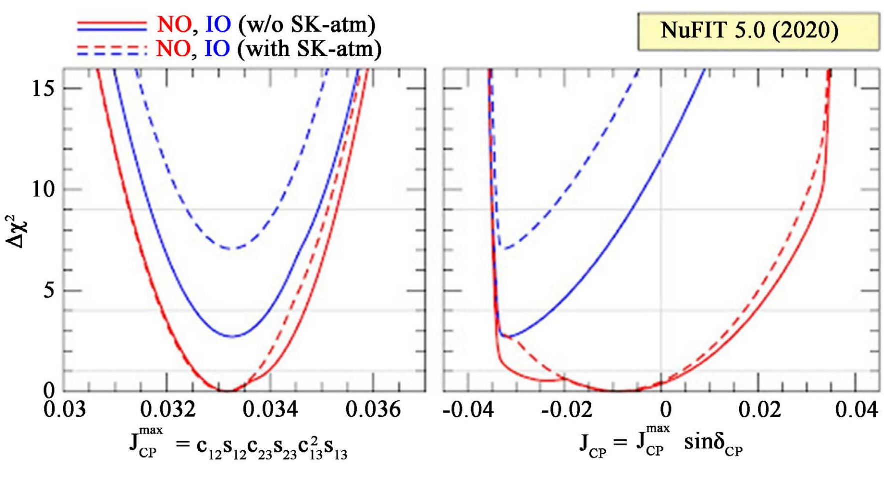

Figure 1. CP violation: Jarlskog invariant [10].

So with such statements we can write the following relations for the Jarlskog invariant:

$$\begin{aligned}J_{CP}^{m} &= \operatorname{Im}\left[\left(U_{\mu_2}^{*}\right)^{m}\left(U_{e2}\right)^{m}\left(U_{\mu3}\right)^{m}\left(U_{e_3}^{*}\right)^{m}\right] = \operatorname{Im}\left[\left(U_{\mu3}^{*}\right)^{m}\left(U_{e3}\right)^{m}\left(U_{\mu1}\right)^{m}\left(U_{e1}^{*}\right)^{m}\right]\\ &= \operatorname{Im}\left[\left(U_{\mu2}^{*}\right)^{m}\left(U_{e2}\right)^{m}\left(U_{\mu1}\right)^{m}\left(U_{e1}^{*}\right)^{m}\right] = \left(J_{CP}^{\max}\right)^{m}\sin\delta^{m}\\ &= J_{CP} = \operatorname{Im}\left[U_{\mu_2}^{*}U_{e2}U_{\mu3}U_{e_3}^{*}\right] = \operatorname{Im}\left[U_{\mu3}^{*}U_{e3}U_{\mu1}U_{e1}^{*}\right] = \operatorname{Im}\left[U_{\mu2}^{*}U_{e2}U_{\mu1}U_{e1}^{*}\right]\\ &= J_{CP}^{\max}\sin\delta = J_{CP}^{\max}\sin\left\{\tan^{-1}\left[\tan\left(\pi\times\frac{\Delta m_{31}^{2}}{\Delta m_{21}^{2}}\right)\right]\right\} = J_{CP}^{\max}\sin\left(\pi\times\frac{\Delta m_{31}^{2}}{\Delta m_{21}^{2}}\right).\end{aligned} \tag{61}$$

Using the formulas ((1), (4), (5), (8)) we can write the final forms for the formula for the Jarlskog invariant for both neutrino mass hierarchies.

In the paper [11], the following data from experimental measurements are given:

$$\begin{aligned}&\frac{\Delta m_{21}^{2}}{10^{-5}\ \text{eV}^{2}} = 7.41_{-0.20}^{+0.21}, \left(\frac{\Delta m_{21}^{2}}{10^{-5}\ \text{eV}^{2}}\right)_{BF} = 7.41, \left(\frac{\Delta m_{21}^{2}}{10^{-5}\ \text{eV}^{2}}\right)_{-1\sigma} = 7.21,\\ &\left(\frac{\Delta m_{21}^{2}}{10^{-5}\ \text{eV}^{2}}\right)_{+1\sigma} = 7.62; \left(\frac{\Delta m_{32}^{2}}{10^{-3}\ \text{eV}^{2}}\right)_{BF} = 2.4369, \left(\frac{\Delta m_{32}^{2}}{10^{-3}\ \text{eV}^{2}}\right)_{-1\sigma} = 2.4119,\\ &\left(\frac{\Delta m_{32}^{2}}{10^{-3}\ \text{eV}^{2}}\right)_{+1\sigma} = 2.4628; \frac{\Delta m_{31}^{2}}{10^{-3}\ \text{eV}^{2}} = 2.5110_{-0.027}^{+0.028}, \left(\frac{\Delta m_{31}^{2}}{10^{-3}\ \text{eV}^{2}}\right)_{BF} = 2.5110,\\ &\left(\frac{\Delta m_{31}^{2}}{10^{-3}\ \text{eV}^{2}}\right)_{-1\sigma} = 2.4840, \left(\frac{\Delta m_{31}^{2}}{10^{-3}\ \text{eV}^{2}}\right)_{+1\sigma} = 2.5390; \delta_{CP}/^{\circ} = 197_{-25}^{+42}.\end{aligned} \tag{62}$$

and we will use this data to apply formula (1), which reads:

$$\delta_{CP} = 180^{\circ}\times\frac{\Delta m_{31}^{2}}{\Delta m_{21}^{2}} \tag{63}$$

One example for a calculation problem to apply the Dirac CPV phase formula (63)

If the parameters $\left(\Delta m_{21}^{2}\right)_{+1\sigma} = 0.0000762\ \text{eV}^{2}$ and $\left(\delta_{CP}\right)_{+1\sigma} = \left[\left(197^{+42}\right)_{+1\sigma}\right]^{\circ} = 239.0^{\circ}$ are measured in experiments (62), give an answer to the following question: what numerical value should be expected for $\left(\Delta m_{31}^{2}\right)_{+1\sigma}$ if formula (1) is applied?

Solution: We apply the formula (1) for δ_{CP} in the following form:

$$\begin{aligned}\left(\delta_{CP}\right)_{+1\sigma} &= 180^{\circ}\times\frac{\left(\Delta m_{31}^{2}\right)_{+1\sigma}}{\left(\Delta m_{21}^{2}\right)_{+1\sigma}} = 180^{\circ}\times\frac{\left(\Delta m_{31}^{2}\right)_{+1\sigma}}{0.0000762\ \text{eV}^{2}}\\ &= \left(180^{\circ}\times\frac{\left(\Delta m_{31}^{2}\right)_{+1\sigma}}{0.0000762\ \text{eV}^{2}}/360^{\circ} - 16\right)\times 360^{\circ} = 239^{\circ}\end{aligned}$$

where is the unknown quantity $\left(\Delta m_{31}^{2}\right)_{+1\sigma}$ and we find it from this equation.

$$\left(\Delta m_{31}^{2}\right)_{+1\sigma} = 2\times 0.0000762\ \text{eV}^{2}\times\left(16+\frac{239^{\circ}}{360^{\circ}}\right)\approx 0.0025396\ \text{eV}^{2}.$$

4. Conclusions

The content of the paper consists of two main parts. The first part is dedicated to the procedure of deriving the formula for the Dirac CPV phase, considering the case when the neutrino beam moves through a vacuum and the case when it spreads through a medium filled with a constant density of matter.

For the purposes of calculating the Dirac CP-violating phase, we derived the equations of motion of three neutrinos for the following cases:

1) When expressions for the oscillation probabilities of three neutrinos are obtained without any approximations [5] [6].

2) When expressions for the oscillation probabilities of three neutrinos are obtained with some approximations [7] [8].

The case under item 1 refers to the motion of a neutrino beam through a physical vacuum.

In the case under item 2 two possibilities were analyzed:

a) The motion of a neutrino beam through a physical vacuum.

b) The motion of a neutrino beam through space with a constant density of matter.

Based on the oscillation probabilities for three neutrinos, we derived the formula for the Dirac CPV phase, and its calculation value is obtained by using the parameters obtained from experimental measurements.

Based on the obtained formula for the Dirac CP-violation phase, the following can be concluded:

1) It is especially emphasized that the derived formula for the value of the Dirac CPV phase does not depend on the mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$ that make up the elements of the PMNS matrix.

2) Regardless of whether the neutrino beam propagates through a vacuum or through a medium with a constant density of matter, the Dirac phase remains unchanged.

On the other hand, in conclusion, we could highlight the following:

- By obtaining an explicit mathematical formula for the Dirac CPV phase, it would no longer be necessary to perform computer simulations to draw areas where it could be found.

- At the same time, the Dirac CPV phase does not depend on the mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$ that make up the elements of the PMNS matrix, but depends only on the ratio of the corresponding differences of the squares of the neutrino masses.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

From a physical point of view, microscopic and macroscopic quantities participate in the process of neutrino oscillation.

The microscopic sizes of neutrinos as quantum-mechanical particles are characterized by their momentums while their macroscopic sizes represent their oscillation wavelengths.

Those two physical properties of neutrinos can be united and represented by an extremely simple relation $\left(p_i - p_j\right) L_{ij} = \Delta p_{ij} L_{ij} = h, i \neq j = 1,2,3$, which can be described in words as follows: The difference between the momentum of two neutrinos, which participate in the oscillation process, multiplied by their oscillation wavelength is equal to Planck's constant *h*.

We will use this relation in the following texts to analyze the cases for different hierarchies of neutrino masses.

C. Defining basic relation in Neutrino Physics

Let the wavelengths of oscillations be denoted by $L_{ij}\left(i \neq j = 1,2,3\right)$, linking them to the differences of the appropriate phases $\phi_{ij}\left(i \neq j = 1,2,3\right)$, and then relations for the processes of disappearances can be written as follows:

Normal mass ordering: $m_1 < m_2 < m_3$

$$\begin{aligned} &\left(\nu_e \to \nu_\mu \to \nu_e\right) \to \phi_1 - \phi_2 = \phi_{12} = \frac{L_{12}}{\hbar}\left(p_1 - p_2\right) = 2\pi \\ &\left(\nu_e \to \nu_\tau \to \nu_e\right) \to \phi_1 - \phi_3 = \phi_{13} = \frac{L_{13}}{\hbar}\left(p_1 - p_3\right) = 2\pi, \\ &\left(\nu_\mu \to \nu_\tau \to \nu_\mu\right) \to \phi_2 - \phi_3 = \phi_{23} = \frac{L_{23}}{\hbar}\left(p_2 - p_3\right) = 2\pi \end{aligned} \tag{C1}$$

The first relation presents the process of oscillation of the electron neutrino through muon neutrino when one full oscillation is performed L_{12}.

The second relation presents the process of oscillation of the electron neutrino through tau neutrino when one full oscillation is performed L_{13}.

The third relation presents the process of oscillation of the muon neutrino through tau neutrino when one full oscillation is performed L_{23}.

The momentum p_1 is linked to mass eigenstate m_1, the momentum p_2 is linked to mass eigenstate m_2, the momentum p_3 is linked to mass eigenstate m_3. From these equations (C1), the link between the wavelengths of oscillations is obtained and the corresponding difference of the momentums with the Planck constant:

$$L_{12}\left(p_1 - p_2\right) = h \tag{C2}$$

$$L_{13}\left(p_1 - p_3\right) = h \tag{C3}$$

$$L_{23}\left(p_2 - p_3\right) = h \tag{C4}$$

where it can be seen that the product of wavelengths of neutrino oscillations and corresponding differences of the momentums equals the Planck constant.

From these equations, a link between wavelengths of oscillations for normal mass ordering (NO) is obtained:

$$\frac{1}{L_{13}} = \frac{1}{L_{12}} + \frac{1}{L_{23}}; L_{12} > L_{23} > L_{13} \tag{C5}$$

In further research, we form the differences of phases of mass eigenstates on the distance L from the source of the neutrino beam, moving through a physical vacuum, and they can be described by following equation:

$$\begin{aligned}
\phi_{12}(L) &= \frac{L}{\hbar}(p_1 - p_2) = \frac{L}{\hbar}\left[E/c(1-\delta_1) - E/c(1-\delta_2)\right] = \frac{L}{\hbar}\left[E/c(\delta_2 - \delta_1)\right] \\
&= \frac{L}{\hbar}\frac{E}{c}\left(\frac{m_2^2c^4}{2E^2} - \frac{m_1^2c^4}{2E^2}\right) = \frac{Lc^3}{2\hbar E}\Delta m_{21}^2, \\
\phi_{12}(L_{12}) &= \frac{L_{12}c^3}{2\hbar E}\Delta m_{21}^2 = 2\pi, L_{12} = \frac{4\pi\hbar E}{\Delta m_{21}^2 c^3}, \\
m_3 &> m_2 > m_1; \delta_1 = \frac{m_1^2c^4}{2E^2} \ll 1, \delta_2 = \frac{m_2^2c^4}{2E^2} \ll 1, \delta_3 = \frac{m_3^2c^4}{2E^2} \ll 1.
\end{aligned} \tag{C6}$$

$$\phi_{23}(L_{23}) = \frac{L_{23}}{\hbar}(p_2 - p_3) = \frac{L_{23}c^3}{2\hbar E}\Delta m_{32}^2 = 2\pi \tag{C7}$$

$$\phi_{13}(L_{13}) = \frac{L_{13}}{\hbar}(p_1 - p_3) = \frac{L_{13}c^3}{2\hbar E}\Delta m_{31}^2 = 2\pi \tag{C8}$$

where c is the speed of light, and $\hbar = h/2\pi$, and one more equation can be written:

$$\Delta m_{21}^2 + \Delta m_{32}^2 = \Delta m_{31}^2 \tag{C9}$$

If we calculate the phases for distances that do not match their oscillation wavelengths, for example $(L_{23}, L_{13}) \to L_{12}$ then we could write the following expressions:

$$\phi_{23}(L_{12}) = \frac{(L_{12})c^3}{2\hbar E}\Delta m_{32}^2 = 2\pi\frac{\Delta m_{32}^2}{\Delta m_{21}^2} \tag{C10}$$

$$\phi_{13}(L_{12}) = \frac{(L_{12})c^3}{2\hbar E}\Delta m_{31}^2 = 2\pi\frac{\Delta m_{31}^2}{\Delta m_{21}^2} \tag{C11}$$

Inverted mass ordering: $m_3 < m_1 < m_2$

$$\begin{aligned}
(\nu_e \to \nu_\mu \to \nu_e) &\to \phi_1 - \phi_2 = \phi_{12} = \frac{L_{12}}{\hbar}(p_1 - p_2) = 2\pi \\
(\nu_e \to \nu_\tau \to \nu_e) &\to \phi_3 - \phi_1 = \phi_{31} = \frac{L_{31}}{\hbar}(p_3 - p_1) = 2\pi, \\
(\nu_\mu \to \nu_\tau \to \nu_\mu) &\to \phi_3 - \phi_2 = \phi_{32} = \frac{L_{32}}{\hbar}(p_3 - p_2) = 2\pi.
\end{aligned} \tag{C12}$$

The first relation describes the process of oscillations of the electron neutrino through muon neutrino when one full oscillation is performed $L = L_{12}$.

The second relation presents the process of oscillations of the electron neutrino through tau neutrino when one full oscillation is performed $L = L_{31}$.

The third relation presents the process of oscillations of the muon neutrino through tau neutrino when one full oscillation is performed $L = L_{32}$.

The momentum p_1 is linked to mass eigenstate m_1, the momentum p_2 is linked to mass eigenstate m_2, the momentum p_3 is linked to mass eigenstate m_3. The equations signify that the product of wavelengths of neutrino oscillations and corresponding differences of the momentums equals the Planck constant.

From the relations (C12), the following equations directly follow:

$$L_{12}\left(p_1 - p_2\right) = h \tag{C13}$$

$$L_{31}\left(p_3 - p_1\right) = h \tag{C14}$$

$$L_{32}\left(p_3 - p_2\right) = h \tag{C15}$$

From which it can be seen that the product of wavelengths of neutrino oscillations and corresponding differences of the momentums equals the Planck constant h.

From these equations, the link between wavelengths of oscillations for inverted mass ordering (IMO) is obtained:

$$\frac{1}{L_{32}} = \frac{1}{L_{12}} + \frac{1}{L_{31}}; L_{32} < L_{31} < L_{12} \tag{C16}$$

Since wavelengths of oscillations are directly proportional to the neutrino energy, these relations apply to any neutrino energy, and they change in proportion to the energy, which should be taken into account when this relation is applied.

Phase differences of mass eigenstates on the distance L from the source of the neutrino beam, moving through a vacuum, can be described by following equations:

$$\begin{aligned}
&\phi_{12}\left(L\right) = \frac{L}{\hbar}\left(p_1 - p_2\right) = \frac{L}{\hbar}\left[E/c\left(1-\delta_1\right) - E/c\left(1-\delta_2\right)\right] = \frac{L}{\hbar}\left[E/c\left(\delta_2 - \delta_1\right)\right] \\
&\qquad = \frac{L}{\hbar}\frac{E}{c}\left(\frac{m_2^2c^4}{2E^2} - \frac{m_1^2c^4}{2E^2}\right) = \frac{Lc^3}{2\hbar E}\Delta m_{21}^2, \\
&\phi_{12}\left(L_{12}\right) = \frac{L_{12}c^3}{2\hbar E}\Delta m_{21}^2 = 2\pi, L_{12} = \frac{4\pi\hbar E}{\Delta m_{21}^2 c^3}, \\
&m_3 < m_1 < m_2; \delta_1 = \frac{m_1^2c^4}{2E^2} \ll 1, \delta_2 = \frac{m_2^2c^4}{2E^2} \ll 1, \delta_3 = \frac{m_3^2c^4}{2E^2} \ll 1.
\end{aligned} \tag{C17}$$

$$\phi_{32}\left(L_{32}\right) = \frac{L_{32}}{\hbar}\left(p_3 - p_2\right) = \frac{L_{32}}{\hbar}\left[E/c\left(\delta_3 - \delta_2\right)\right] = \frac{L_{32}c^3}{2\hbar E}\Delta m_{23}^2 = 2\pi \tag{C18}$$

$$\phi_{31}\left(L_{31}\right) = \frac{L_{31}}{\hbar}\left(p_3 - p_1\right) = \frac{L_{31}}{\hbar}\left[E/c\left(\delta_3 - \delta_1\right)\right] = \frac{L_{31}c^3}{2\hbar E}\Delta m_{13}^2 = 2\pi \tag{C19}$$

where $\boldsymbol{c}$ is the speed of light, and $\hbar = h/2\pi$, and

$$\Delta m_{23}^2 = \Delta m_{21}^2 + \Delta m_{13}^2 \tag{C20}$$

If we calculate the phases for distances that do not match their oscillation wavelengths, for example $\left(L_{32}, L_{13}\right) \to L_{12}$ then we could write the following expressions:

$$\phi_{32}\left(L_{12}\right)=\frac{\left(L_{12}\right)c^{3}}{2\hbar E}\Delta m_{23}^{2}=2\pi\frac{\Delta m_{23}^{2}}{\Delta m_{21}^{2}} \tag{C21}$$

$$\phi_{31}\left(L_{12}\right)=\frac{\left(L_{12}\right)c^{3}}{2\hbar E}\Delta m_{13}^{2}=2\pi\frac{\Delta m_{13}^{2}}{\Delta m_{21}^{2}} \tag{C22}$$

Comment. It should be borne in mind that the length of the neutrino oscillation $L = L_{12}$ which is given by the expression (C6) and (C17) was taken as a common parameter.

For example: The value for Δm_{21}^2 is the same for both the normal and the inverted neutrino mass hierarchy. Therefore, the introduction of the parameter $L = L_{12}$ as common in the theoretical consideration is justified.

We also suggest paying attention to the relations that connect the neutrino oscillation wavelengths for both mass hierarchies (C6) and (C17) in cases where the neutrino energies are mutually equal.

Perhaps, in experimental measurements, those relations would give an answer about the nature of neutrino mass hierarchies.