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Conformally Compactified Minkowski Spacetime and Planck Constant

Miguel Socolovsky

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Cd. Universitaria, Ciudad de Mexico, Mexico
Email: socolovs@nucleares.unam.mx

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Abstract

If the geometrical system of units $c = G = 1$ and the Planck length as a natural length scale are used in the construction of the Penrose space (diagram) corresponding to Minkowski spacetime, the presence of the Planck constant $\hbar$ in the Penrose dimensionless time (τ) and radial (ρ) coordinates is unavoidable. This fact suggests that there could be a deep and still unknown relation between the spacetime of special relativity and quantum mechanics.

Keywords

Minkowski Spacetime, Quantum Mechanics

1. Introduction

Minkowski spacetime, the spacetime of special relativity (S.R.), is the simplest solution of the vacuum Einstein equations for general relativity (G.R.). As such, it is subject to the mathematical process of conformal compactification [1] [2], leading to a new—and therefore distinct—spacetime known as the *Penrose diagram* of the original space. This process is applied to many other solutions of the Einstein equations, in particular to black hole solutions, in the vacuum case or with the presence of matter [3]. In all cases, the obtained spaces are distinct from the starting ones and could, in principle, be considered as non-physical. However they globally exhibit with total clarity the light cones or causal structure of the original spaces. We shall restrict our discussion to the 4-dimensional Minkowski spacetime, here denoted by *M*.

The conformal compactification process necessarily requires the introduction, at an intermediate step, of a length scale *L* which can not be arbitrary, since unavoidably it appears in the definition of the dimensionless coordinates of the final Penrose space (diagram) *P*. Though not unique [4], the mostly accepted as natural unit of length is the *Planck lenght* given by

$$L_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \cong 1.6\times 10^{-35}\ \text{m}. \quad (1)$$

(G is Newton constant, $\hbar$ is the reduced Planck constant, and c the velocity of light in vacuum, respectively associated with gravitation, quantum mechanics, and S.R.). In G.R., which is a classical theory—and therefore also in S.R.—the most commonly used system of units is the *geometrical system of units* (g.s.u.) defined by

$$c = G = 1 \quad (2)$$

which leads to

$$L_{Pl} = \sqrt{\hbar}. \quad (3)$$

Notice that the g.s.u. is not the *natural system of units* (n.s.u.) in which $c = G = \hbar = k_B = 1$ (k_B is the Boltzmann constant associated with thermodynamics and statistical mechanics).

The systematic and well known step by step derivation of the Penrose space P corresponding to M (Section 2) exhibits, beyond the causal structure of M, a hidden and/or a novel fact: the unavoidable presence of $\hbar$, the essence of quantum physics, at least if the g.s.u. is used and L_{Pl} is considered the fundamental length scale.

Conclusions (remarks and questions) are presented in Section 3.

2. Penrose Space (P) of Minkowski Space (M)

Let M_{cc}, M_{sc}, and M_{lc} be M in cartesian, spherical, and lightcone coordinates; the corresponding coordinate transformations are:

$$(t,x,y,z) \xrightarrow{\text{(i)}} (t,r,\theta,\varphi) \xrightarrow{\text{(ii)}} (v,u,\theta,\varphi) \quad (4)$$

with (except for the usual coordinate singularities):

$$\text{(i)}: r = \sqrt{x^2+y^2+z^2} \ge 0,\ \theta = \operatorname{arctg}\left(\sqrt{x^2+y^2}/2\right),\ \varphi = \operatorname{arctg}(y/x), \quad (5)$$

$$\text{(ii)}: v = t+r,\ u = t-r,\ \Leftrightarrow t = (v+u)/2,\ r = (v-u)/2, \quad (6)$$

$[t]=[r]=[v]=[u]=[length]^1$, $[\theta]=[\varphi]=[length]^0$, $t\in(-\infty,+\infty)$, $r\in[0,+\infty)$, $u,v\in(-\infty,+\infty)$, $v\ge u$, with metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right) = \mathrm{d}t^2 - \left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2\right) = \mathrm{d}v\mathrm{d}u - \frac{(v-u)^2}{4}\mathrm{d}\Omega_2^2, \quad (7)$$

where $\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$. Since the metric coefficients of $\mathrm{d}u^2$ and $\mathrm{d}v^2$ are null *i.e.* $g_{uu} = g_{vv} = 0$, u and v are null coordinates: constant u (v) lines represent outgoing (ingoing) light rays.

Rescaling the metric with $L = L_{Pl} = \sqrt{\hbar}$ we have the constant conformal transformation

$$\text{(ii)}': (\tilde{v},\tilde{u}) = \left(v/\sqrt{\hbar}, u/\sqrt{\hbar}\right),\ \mathrm{d}\tilde{s} = \mathrm{d}s/\sqrt{\hbar},\ [\tilde{v}] = [\tilde{u}] = [length]^0, \quad (8)$$

leading to $\tilde{M} \neq M_{lc}$ with metric

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}s^2/\hbar = \mathrm{d}\tilde{v}\mathrm{d}\tilde{u} - \frac{(\tilde{v}-\tilde{u})^2}{4}\mathrm{d}\Omega_2^2. \tag{9}$$

The coordinate transformation:

$$\text{(iii)}: v' = 2\operatorname{arctg}(\tilde{v}),\ u' = 2\operatorname{arctg}(\tilde{u}),\ v',u' \in (-\pi,+\pi) \tag{10}$$

leads to $M' = \tilde{M}$ with metric

$$\mathrm{d}\tilde{s}^2 = \frac{1}{4\cos^2(u'/2)\cos^2(v'/2)}\left(\mathrm{d}v'\mathrm{d}u' - \sin^2\left(\frac{v'-u'}{2}\right)\mathrm{d}\Omega_2^2\right). \tag{11}$$

We pass from two null coordinates v',u' and two spacelike coordinates θ,φ to one timelike coordinate τ and three spacelike coordinates ρ,θ,φ through

$$\text{(iv)}: \tau = \frac{v'+u'}{2},\ \rho = \frac{v'-u'}{2} \Leftrightarrow v' = \tau+\rho,\ u' = \tau-\rho,\ -\pi < \tau < +\pi,\ 0 \le \rho < +\pi \tag{12}$$

leading to $M'' = M'$ with metric

$$\mathrm{d}\tilde{s}^2 = \frac{1}{(\cos\tau+\cos\rho)^2}\left(\mathrm{d}\tau^2 - \mathrm{d}\rho^2 - \sin^2\rho\,\mathrm{d}\Omega_2^2\right). \tag{13}$$

Through the conformal transformation

$$\text{(v)}: (\cos\tau+\cos\rho)^2\,\mathrm{d}\tilde{s}^2 \tag{14}$$

We obtain $M_{conf} \neq M'$ with metric

$$\mathrm{d}s_{conf}^2 = \mathrm{d}\tau^2 - \mathrm{d}\rho^2 - \sin^2\rho\,\mathrm{d}\Omega_2^2, \tag{15}$$

Which is represented in **Figure 1**.

$\times = (\rho,\tau)$ in **Figure 1** is a 2-sphere with radius $\sin\rho : S^2(\sin\rho)$.

(vi) The final step is to add the boundary of M_{conf},

$$Bry = \iota^+ \cup \iota^- \cup \iota^0 \cup J^+ \cup J^- \tag{16}$$

with $\iota^\pm = (0,\pm\pi)$: future (past) timelike infinity, $\iota^0 = (\pi,0)$: spatial infinity, and $J^+ : \tau = -\rho+\pi$, $J^- : \tau = \rho-\pi$, $0 < \rho < \pi$, respectively the future (past) null infinity, leading to the *conformally compactified Minkowski spacetime*

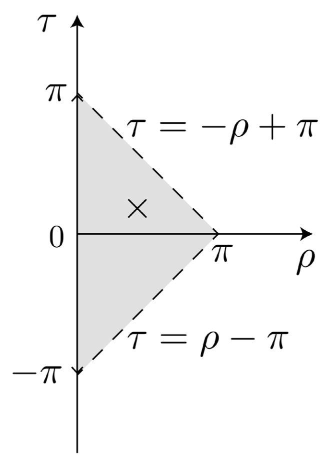

Figure 1. Conformal Minkowski space.

$P = M_{conf.cpt} \neq M_{conf}$, with

$$ds_P^2 = ds_{conf}^2 \tag{17}$$

including the boundary, and topology

$$P \cong I \times S^3 \tag{18}$$

where $I = [-\pi, +\pi]$ and S^3 the 3-sphere with unit radius ($S^3(1)$). P is represented in **Figure 2**.

Remarks: 1) In the whole procedure we passed successively through four distinct spacetimes: M_{cc}, $\tilde{M}$, M_{conf}, and P. 2) P has the dimensionless 4-volume $V_4 = 2\pi \times 2\pi^2 = 4\pi^3$.

3. Discussion and Conclusion

Using the coordinate transformations (ii), (iii) and (iv), the constant conformal transformation (ii)′, and the trigonometric identities
$\text{arctg}(z_1) + \text{arctg}(z_2) = \text{arctg}((z_1 + z_2)/(1 - z_1 z_2))$ and
$\text{arctg}(z_1) - \text{arctg}(z_2) = \text{arctg}((z_1 - z_2)/(1 + z_1 z_2))$ [5], we can express τ and ρ in terms of t, r and $\hbar$:

$$(\tau(t,r;\hbar), \rho(t,r;\hbar)) = \left(\text{arctg}\left(\frac{2\sqrt{\hbar}t}{\hbar - (t^2 - r^2)}\right), \text{arctg}\left(\frac{2\sqrt{\hbar}r}{\hbar + (t^2 - r^2)}\right)\right). \tag{19}$$

Any choice of length scale L distinct from L_{Pl} would leave unchanged the global picture of P. However, the requirement of a physical origin for L (and L_{PL} is the most natural and universal) and the use of the g.s.u. in special and general relativity, force the appearance of $\hbar$ in the time (τ) and radial (ρ) coordinates of the Penrose diagram.

If one insists with a dimensionful metric for P, one should multiply both sides of Equation (15) by L^2 obtaining

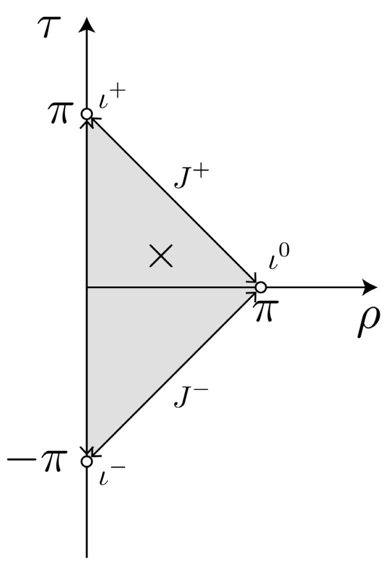

Figure 2. Penrose space of Minkowski space.

$$\mathrm{d}s_P'^2 = \mathrm{d}T^2 - \mathrm{d}R^2 - L^2 \sin^2\left(\frac{R}{L}\right)\mathrm{d}\Omega_2^2 \tag{20}$$

with $T = L\tau$, $R = L\rho$, and $[T] = [R] = [lenght]$. The presence of L is unavoidable which, unless one chooses a universal scale, leaves behind a degree of arbitrariness in the metric for *P*, contrary with the statement in Ref. [6] that the scale L is irrelevant. With $L = L_{Pl}$ in the g.s.u., (20) becomes

$$\mathrm{d}s_P'^2 = \mathrm{d}T^2 - \mathrm{d}R^2 - \hbar \sin^2\left(\frac{R}{\sqrt{\hbar}}\right)\mathrm{d}\Omega_2^2. \tag{21}$$

We conclude that a question remains open: Does the above discussed facts indicate the existence of some hidden and/or unknown relation between Minkowski spacetime and quantum mechanics?

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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