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Geometrical Diagnostics for Modified Gravitational Theory with the Different Formalisms

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Abstract

Geometrical diagnostic methods were often applied to distinguish the gravitational models. But it is scarce to investigate the differences between the different formalisms of modified gravitational theories (e.g. the metric formalism and the Palatini formalism). In this paper, we discriminate the gravitational theory with the different formalisms by using the geometrical diagnostic methods. For a considered modified theory of gravity (e.g. the $f(R)$ theory or GBD theory), we can see that the difference between the two formalisms is remarkable according to the diagnostic results. And relative to the ΛCDM model, there are more deviations in metric formalism than those in Palatini formalism, according to the $\{r, s\}$ diagnostic. Given that the GBD (generalized Brans-Dicke theory) is a time-variable Newton gravitational constant (VG) theory, the differences between the VG theory and the constant-G theory are studied. It indicates that the variation of Newton's gravitational constant could induce notable effects on geometrical quantities (e.g. r, s and q) in both metric formalism and Palatini formalism.

Keywords

Time-Variable Gravitational Constant, Metric Formalism, Palatini Formalism, Geometrical Diagnostic

1. Introduction

For exploring properties of gravity or solving questions in the theory of general relativity (GR) [1]-[6], lots of modified theories of GR have been constructed, e.g. the $f(R)$ theory [7] [8], the $f(G)$ theory [9] [10], the Brans-Dicke (BD) theory [11], and so on [12]-[28]. There is a prior assumption that we have to take in studying modified theories of gravity (MG), *i.e.* which one or ones of the dynamical variables should be chosen to describe the gravitational interaction? Correspondingly, the modified gravitational theories could be divided into two classifications: the metric formalism [8] [29] and the Palatini formalism [30] [31]. In the metric formalism, the Levi-Civita connection is related to metric, while in the Palatini formalism, the metric and the connection are regarded as independent dynamical variables. Usually, the different field equations are gained in the metric formalism theories and the Palatini formalism theories [8], respectively.

To find the "final" theory of describing the gravitational interaction, it is significant to explore the differences between the different modified theories (or the different formalisms). Then one can test them according to the observational results. For example, gravitational wave astronomy, which was recently started by the famous LIGO detections [32] [33], could be, in principle, fundamental for testing the effective viability of extended theories of gravity. The key point is that some differences between different gravity theories can be found in linearized gravity by analyzing gravitational wave polarizations via the interferometric response functions [34].

In addition, lots of gravitational models have been differentiated via the geometrical diagnostic methods [35] [36] [37] [38] [39]. But studying the distinctions between the different formalisms is scarce by using the so-called geometrical diagnostic. Obviously, which formalism should be chosen preferentially is an important issue, since it can decrease the uncertainty of the theoretical research of gravity. In this paper, we probe the discrepancy between the different formalisms of modified theory, by selecting the $f(R)$ and the generalized Brans-Dicke (GBD) theories as examples. Also, we try to give an answer, *i.e.* under the observational limits on the Newton gravitational constant G, whether the change of G could lead to the remarkable geometrical effect in the generalized Brans-Dicke theory (a theory with the time-variable Newton gravitational constant). Due to the dynamics of BD field and the coupling between the Brans-Dicke field and the gravitational geometry, we find that the difference of some geometrical quantity (e.g. r, s or q) between the variable-G theory and the constant-G theory could be obvious for both the metric formalism and the Palatini formalism.

The constructions of this paper are as follows. In Section II, we introduce the basic equations for $f(R)$ modified gravitational theory with the different formalisms, and apply the geometrical diagnostics to distinguish the different formalisms. Section III investigates the geometrical diagnostics for the different formalisms of GBD theory. Especially, we study the influences of variable G on the geometrical quantities in this theory. Section IV is the conclusion.

2. The Geometrical Diagnostics for *f*(*R*) Theory with the Different Formalisms

1) Basic equations for $f(R)$ modified theory in both the metric formalism and

the Palatini formalism

In this section, we show the field equations of *f*(*R*) theory in the metric formalism and the Palatini formalism, respectively. *f*(*R*) modified gravitational theory is a simple and popular extension relative to GR. In the metric formalism, the action of *f*(*R*) theory is denoted by

$$S_{met} = S_g\left(g_{\mu\nu}\right) + S_m\left(g_{\mu\nu}, \Psi\right) = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi G} f\left(R\right) + L_M\right], \quad (1)$$

where g denotes the determinant of metric $g_{\mu\nu}$, R is the Ricci scalar, $f(R)$ is an arbitrary function of R, G denotes the Newton gravitational constant, L_m denotes the Lagrangian density of matter, respectively. Using the variation principle, one gets the modified field equation of gravity

$$f_R R_{\mu\nu} - \frac{1}{2} f\left(R\right) g_{\mu\nu} - \left[\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right] f_R = 8\pi G T_{\mu\nu}, \quad (2)$$

Here $f\left(R\right) = \frac{\partial f\left(R\right)}{\partial R}$, and $\Box \equiv \nabla^\mu \nabla_\mu$. $R_{\mu\nu}$ and $T_{\mu\nu}$ denote the Ricci tensor and the energy-momentum tensor of matter, respectively. The trace of Equation (2) is

$$f_R\left(R\right) R - 2 f\left(R\right) + 3\Box f_R = 8\pi G T. \quad (3)$$

In the Palatini formalism, the action of *f*(*R*) theory read as,

$$S_{Pala} = S_g\left(g_{\mu\nu}, \tilde{\Gamma}^\lambda_{\mu\nu}\right) + S_m\left(g_{\mu\nu}, \psi\right) = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi G} f\left(\tilde{R}\right) + L_m\right]. \quad (4)$$

Here the metric $g_{\mu\nu}$ and the connection $\tilde{\Gamma}^\lambda_{\mu\nu}$ are regarded as the independent dynamical variables. $\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}$ and the Ricci tensor $\tilde{R}_{\mu\nu}$ is defined by the independent Palatini connection

$$\tilde{R}_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} \quad (5)$$

Varying the action (4) with respect to $g_{\mu\nu}$, we gain the gravitational field equation in the Palatini formalism

$$F\left(\tilde{R}\right)\tilde{R}_{\mu\nu} - \frac{1}{2} f\left(\tilde{R}\right) g_{\mu\nu} = \kappa T_{\mu\nu}\bar{\nabla}_\lambda, \quad (6)$$

where $F\left(\tilde{R}\right) = \frac{\partial f\left(\tilde{R}\right)}{\partial \tilde{R}}$. The trace of Equation (6) is

$$F\left(\tilde{R}\right)\tilde{R} - 2 f\left(\tilde{R}\right) = 8\pi G T. \quad (7)$$

Varying the action with respect to $\tilde{\Gamma}^\lambda_{\mu\nu}$ gives

$$\tilde{\nabla}_\lambda\left(\sqrt{-g} F\left(\tilde{R}\right) g^{\mu\nu}\right) = 0, \quad (8)$$

where $\tilde{\nabla}$ is the covariant derivative with respect to the Palatini connection. Equation (8) implies that the connection can be represented as the Christoffel symbol associated with the metric $h_{\mu\nu}$ by defining $h_{\mu\nu} = F\left(R\right) g_{\mu\nu}$. Then we arrive at a relation:

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2F}\left[-g_{\mu\nu}\partial^\lambda F + \delta^\lambda_\nu \partial_\mu F + \delta^\lambda_\mu \partial_\nu F\right], \quad (9)$$

where $\Gamma^{\lambda}_{\mu\nu}$ is the Livi-Civita connection associated with the metric $g_{\mu\nu}$. Thus, by using Equation (5) the Ricci tensor and the Ricci scalar in the Palatini formalism are rewritten as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu}(g) + \frac{3}{2F^2}\nabla_\mu F \nabla_\nu F - \frac{1}{F}\nabla_\mu \nabla_\nu F - \frac{1}{2F} g_{\mu\nu}\nabla_\sigma \nabla^\sigma F, \tag{10}$$

where $R_{\mu\nu}$ denotes the Ricci tensor defining in the metric formalism, and all covariant derivatives are taken with respect to the metric $g_{\mu\nu}$. Combining above equations, the modified Einstein equation in the Palatini-formalism $f(\tilde{R})$ theory can be reexpressed as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{\kappa T_{\mu\nu}}{F} + 8\pi T^{eff}_{\mu\nu} \tag{11}$$

with $\kappa = 8\pi G$,

$$8\pi T^{eff}_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\left(\tilde{R} - \frac{f}{F}\right) + \frac{1}{F}\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right)F - \frac{3}{2}\frac{1}{F^2}\left[\nabla_\mu F \nabla_\nu F - \frac{1}{2}g_{\mu\nu}(\nabla F)^2\right].$$

2) The geometrical diagnostics for $f(R)$ theory with the different formalisms

In this part, we utilize the $f(R)$ theory with the different formalisms to cosmology. A flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric has a form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2. \tag{12}$$

Here a is the cosmic scale factor, t is the cosmic time. Inserting the FLRW metric into field equations of $f(R)$ theory with metric formalism and assuming the energy momentum tensor: $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$, we have the cosmological equations [8]

$$H^2 = \frac{\kappa}{3f_R}\left[\rho + \frac{Rf_R - f}{2} - 3H\dot{R}f_{RR}\right], \tag{13}$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f_R}\left[p + \dot{R}^2 f_{RRR} + 2H\dot{R}f_{RR} + \ddot{R}f_{RR} + \frac{1}{2}(f - Rf_R)\right]. \tag{14}$$

Here $H \equiv \frac{\dot{a}}{a}$ is a geometrical quantity, called the Hubble parameter. ρ and p denote the density and pressure of universal matter, respectively. U_μ represents the four-velocity of an observer comoving with the fluid. For solving Equations (13) and (14), we define two dimensionless quantities: $y_H = H^2/m^2 - (1+z)^3$ and $y_R = R/m^2 - 3(1+z)^3$, with the cosmic redshift $z = 1/a - 1$. Then we receive two differential equations:

$$y'_H = -\frac{1}{1+z}\left(\frac{1}{3}y_R - 4y_H\right), \tag{15}$$

$$y'_R = \frac{\left[y_H + (1+z)^3\right]f_R - (1+z)^3 - \frac{f_R}{6}\left[y_R + 3(1+z)^3\right] + \frac{f}{6m^2}}{\left[y_H + (1+z)^3\right](1+z)f_{RR}m^2} - 9(1+z)^2, \tag{16}$$

where $'$ denotes the derivative with respect to z. The initial conditions are provided by: $y_H|_{z=0} = H_0^2/m^2 - 1$ and $y_R|_{z=0} = 6H_0^2(1-q_0)/m^2 - 3$, with

$m^2 = (8315\ \text{Mpc})^{-2}\left(\frac{\Omega_{0m}h^2}{0.13}\right)$. According to the observational results, we consider the current dimensionless energy density of matter $\Omega_{0m} = 0.27$ [40], the current value of deceleration parameter $q_0 = -0.63$ [41], the current value of dimensionless Hubble constant $h = 0.673 \pm 0.010$ [42], with $H_0 = 100\ \text{h·km·s}^{-1}\text{·Mpc}^{-1}$.

In the Palatini-$f(R)$ theory, the cosmological equation could be exhibited as [8]:

$$\left(H + \frac{\dot{f}_R}{2f_R}\right)^2 = \frac{\kappa(\rho + 3p) + f}{6f_R} \tag{17}$$

Combining Equation (17) and the conservation equation, we have

$$\dot{R} = -\frac{3H\left[f_R\tilde{R} - 2f\right]}{f_{RR}\tilde{R} - f_R} \tag{18}$$

Also, Equation (17) can be rewritten as

$$H^2 = \frac{1}{6f_R}\frac{3f - f_R\tilde{R}}{\left[1 - \frac{3f_{RR}\left(f_R\tilde{R} - 2f\right)}{2f_R\left(f_{RR}\tilde{R} - f_R\right)}\right]^2} \tag{19}$$

Furthermore, we gain

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}z} = -\frac{9H_0^2\Omega_{m0}(1+z)^2}{f_{RR}\tilde{R} - f_R} \tag{20}$$

$$\frac{H^2}{H_0^2} = \frac{1}{6f_R}\frac{3\Omega_{m0}(1+z)^3 + f/H_0^2}{\left[1 + \frac{9H_0^2\Omega_{m0}(1+z)^3 f_{RR}}{2f_R\left(f_{RR}R - f_R\right)}\right]^2} \tag{21}$$

Thus, combining with trace equation in Palatini formalism of $f(R)$ theory, we can solve Equation (21).

The geometrical quantities—statefinder parameters $\{r, s\}$ are introduced in Ref. [35], which are defined as follows:

$$r \equiv \frac{\dddot{a}}{aH^3} = \frac{H''(z+1)^2}{H} - \frac{2H'(z+1)}{H} + \frac{H'^2(z+1)^2}{H^2} + 1, \tag{22}$$

$$s \equiv \frac{r-1}{3\left(q - \frac{1}{2}\right)} = \frac{\frac{H''(z+1)^2}{H} - \frac{2H'(z+1)}{H} + \frac{H'^2(z+1)^2}{H^2}}{3\frac{H'(z+1)}{H} - \frac{9}{2}}. \tag{23}$$

Obviously, both r and s are the third-order derivative (the highest) of a with respect to t. Here

$$q \equiv \frac{-\ddot{a}}{aH^2} \tag{24}$$

is another geometrical quantity, called the deceleration parameter, which is the second derivative of a with respect to t.

Lots of models are diagnosed by using the statefinder parameters [36] [37] [38] [39]. But applying the statefinder diagnose to distinguish the modified gravitational theory with the different formalisms are lack. In this paper, we investigate the differences between the Palatini formalism and the metric formalism by using the diagnostic method. From above, we can find that the gravitational field equation is the fourth-order PDE (partial differential equation) in the metric formalism, while the field equation in the Palatini formalism is the second-order PDE which is easier to solve and interpret [43]. Then the different methods are utilized to solve the cosmological equations numerically for the different formalisms. For plotting pictures, we take a viable model $f(R)=R-\beta R_s\left(1-e^{-R/R_s}\right)$, which is proposed and developed by Refs. [44] [45] [46]. Here β and R_s are two constants, and could be related by $\beta R_s \simeq 18H_0^2\Omega_{0m}$. Some viable conditions on this model can be found in Refs. [45] [46]. This model has an important feature that it owns only one more parameter than the ΛCDM model.

The graphs of $\{r, s\}$ geometrical diagnostic are plotted in **Figure 1** with the metric formalism (left) and the Palatini formalism (right), respectively. The values of model parameter β are taken as [0.9, 1.1, 1.3], and marked by β_1, β_2 and β_3, respectively. Considering that for the most popular ΛCDM model, we have $\{r, s\} = \{1, 0\}$. Hence we could find the deviation of $f(R)$ model in both formalisms from the ΛCDM model, which show that the values of $\{r, s\}$ in metric formalism are larger (or more deviation) than those in Palatini formalism according to **Figure 1**. Also, **Figure 1** shows that for the same function of $f(R)$, the difference between the metric formalism and the Palatini formalism are notable according to the $\{r, s\}$ diagnostic. In addition, difference between these two formalisms can be reflected by the values of model parameter β. We can notice that the more larger value of β, the $\{r, s\}$ pictures are more close to ΛCDM in Palatini formalism, while the opposite results are given in metric formalism. The arrow describes the evolution of universe from the early stage to the late stage.

Using the same model-parameter values with those in **Figure 1**, **Figure 2** depicts geometric diagnostic of $\{r, q\}$. We can read that the shapes of $\{r, q\}$ in

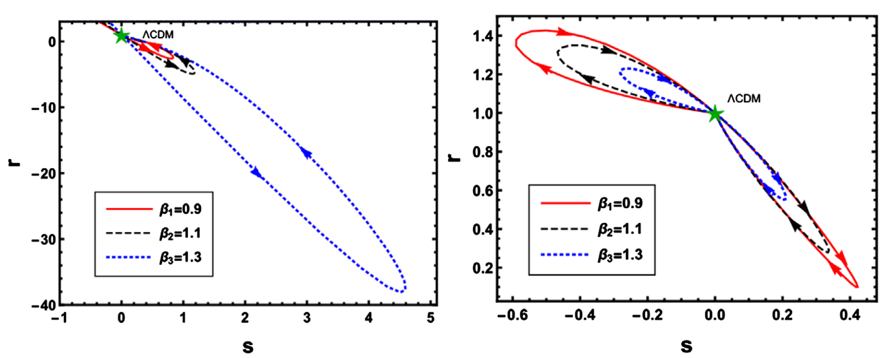

Figure 1. The $\{r, s\}$ geometrical diagnostic for $f(R)$ model with the metric formalism (left) and the Palatini formalism (right), respectively. The selected values of model parameter β are marked on the pictures.

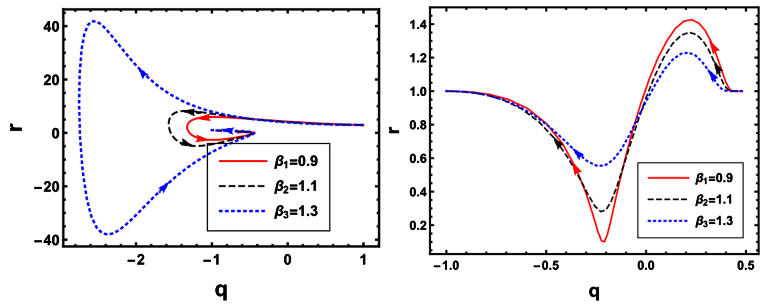

Figure 2. The $\{r, q\}$ geometrical diagnostic for $f(R)$ model with the metric formalism (left) and the Palatini formalism (right).

metric formalism are different from those in Palatini formalism. For seeing the effect of q on the $\{r, q\}$ diagnostic, we illustrate the evolutions of deceleration parameter q for the considered $f(R)$ model. The metric formalism is drawn in Figure 3 (left), while the Palatini formalism is plotted in Figure 3 (right). Given that we have $q(z) = 1/2$ for the matter dominated universe, then there must be $q(z) \le 1/2$ for any cosmological model. For selected model-parameter values in this paper, we can see that the evolutional curves of $q(z)$ in Palatini formalism are consistent with the requirement: $q(z) \le 1/2$.

3. The Geometrical Diagnostics for GBD Theory with the Different Formalisms

1) Basic equations for GBD modified theory in both the metric formalism and the Palatini formalism

Some problems on $f(R)$ theory have been found in some reference [8], such as the inconsistent problem of γ between the theoretical value and the observational value (here γ is the parametrized post-Newtonian parameter). The $f(R)$ theory can be extended by considering some methods, such as the $f(G)$ theory (adding the higher-order terms) [9] [10], the GBD theory [47] [48] [49], etc. In this part, we apply the geometric diagnostic methods to distinguish the different formalisms of GBD modified theory. In addition, given that the time-variable gravitational constant G have been investigated in some theoretical and observational issues [50]-[57], we explore the effects of time-variable G in the modified theory with the different formalisms.

The action of system in the metric-formalism GBD theory is written as

$$\begin{aligned} S &= S_g\left(g_{\mu\nu},\phi\right)+S_\phi\left(g_{\mu\nu},\phi\right)+S_m\left(g_{\mu\nu},\psi_m\right) \\ &= \frac{1}{2}\int L_T \mathrm{d}^4x = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[\phi f\left(R\right)-\frac{\omega}{2\phi}\partial_\mu\phi\partial^\mu\phi+\frac{16\pi}{c^4}L_m\right]. \end{aligned} \tag{25}$$

Using the variational principle, in the metric-formalism GBD theory we obtain the gravitational field equation and the BD scalar field equation as follows

$$\begin{aligned} &\phi\left[f_R R_{\mu\nu}-\frac{1}{2}f\left(R\right)g_{\mu\nu}\right]-\left(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box\right)\left(\phi f_R\right) \\ &+\frac{1}{2}\frac{\omega}{\phi}g_{\mu\nu}\partial_\sigma\phi\partial^\sigma\phi-\frac{\omega}{\phi}\partial_\mu\phi\partial_\nu\phi = 8\pi T_{\mu\nu}. \end{aligned} \tag{26}$$

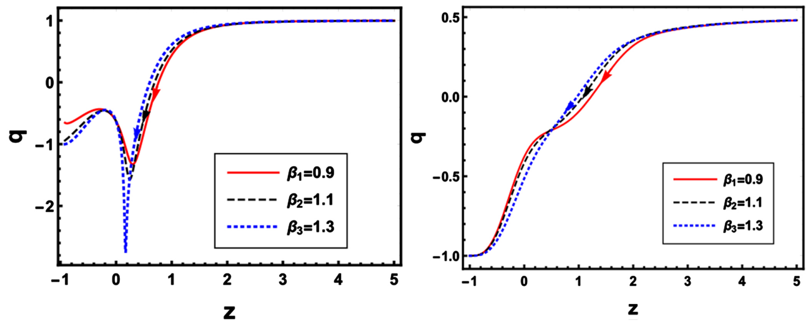

Figure 3. The evolutions of deceleration parameter q for $f(R)$ model with the metric formalism (left) and the Palatini formalism (right).

$$f(R)+2\omega\frac{\Box\phi}{\phi}-\frac{\omega}{\phi^2}\partial_\mu\phi\partial^\mu\phi=0. \tag{27}$$

where ϕ is the BD scalar field, ω is the coupling constant, ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric. The trace of Equation (26) is

$$f_R R-2f(R)+\frac{3\Box(\phi f_R)}{\phi}+\frac{\omega}{\phi^2}\partial_\mu\phi\partial^\mu\phi=\frac{8\pi T}{\phi}. \tag{28}$$

The action of GBD theory in the Palatini formalism read as,

$$S_p=S_g\left(g_{\mu\nu},\tilde{\Gamma}^\lambda_{\mu\nu},\phi\right)+S_\phi\left(g_{\mu\nu},\phi\right)+S_m\left(g_{\mu\nu},\psi\right)=\frac{1}{2}\int d^4xL_T, \tag{29}$$

with the total Lagrange quantity $L_T=\sqrt{-g}\left[\phi f\left(\tilde{R}\right)-\frac{\omega}{\phi}\partial_\mu\phi\partial^\mu\phi+\frac{16\pi}{c^4}L_M\right]$. In the Palatini formalism, varying the action (29) with respect to $g_{\mu\nu}$ and ϕ, we gain two field equations as follows

$$\phi F\left(\tilde{R}\right)\tilde{R}_{\mu\nu}-\frac{1}{2}\phi f\left(\tilde{R}\right)g_{\mu\nu}-\frac{\omega}{\phi}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}g_{\mu\nu}\frac{\omega}{\phi}\partial_\sigma\phi\partial^\sigma\phi=8\pi T_{\mu\nu} \tag{30}$$

$$\frac{2\omega}{\phi}\Box\phi-\frac{\omega}{\phi^2}\partial_\mu\phi\partial_\nu\phi+f\left(\tilde{R}\right)=0 \tag{31}$$

The trace of Equation (30) is

$$F\left(\tilde{R}\right)\tilde{R}-2f\left(\tilde{R}\right)+\frac{\omega}{\phi^2}\partial_\mu\phi\partial^\mu\phi=\frac{8\pi T}{\phi}. \tag{32}$$

Varying the action with respect to $\tilde{\Gamma}^\lambda_{\mu\nu}$ gives

$$\tilde{\nabla}_\lambda\left(\sqrt{-g}\phi F\left(\tilde{R}\right)g^{\mu\nu}\right)=0, \tag{33}$$

where $\tilde{\nabla}$ is the covariant derivative with respect to the Palatini connection.

2) The geometrical diagnostics for GBD theory with the different formalisms

For a flat FLRW universe, using Equations (26) and (27) we can derive the evolutional equations of background universe in the metric-GBD theory as,

$$3f_RH^2=\frac{8\pi\rho_m}{\phi}+\frac{f_RR-f(R)}{2}-3H\dot{f}_R+\frac{1}{2}\omega\left(\frac{\dot{\phi}}{\phi}\right)^2-3Hf_R\frac{\dot{\phi}}{\phi}, \tag{34}$$

$$-2f_R\dot{H}=\frac{8\pi}{\phi}\left(\rho_m+p_m\right)+\ddot{f}_R-H\dot{f}_R+\omega\left(\frac{\dot{\phi}}{\phi}\right)^2-Hf_R\frac{\dot{\phi}}{\phi}+f_R\frac{\ddot{\phi}}{\phi}+2\frac{\dot{\phi}}{\phi}\dot{f}_R, \quad (35)$$

$$f\left(R\right)-\omega\left(\frac{\dot{\phi}}{\phi}\right)^2+2\omega\frac{\ddot{\phi}}{\phi}+6\omega H\frac{\dot{\phi}}{\phi}=0. \quad (36)$$

Here $R=6\left(2H^2+\dot{H}\right)$, and "dot" denotes the derivative with respect to cosmic time *t*. For ϕ = constant, Equations (34) - (36) are reduced to the cases of *f*(*R*) theory.

Using Equations (30)-(33), we can get the evolutional equations of background universe in the Palatini-GBD theory,

$$3FH^2=\frac{8\pi\left(\rho+3p\right)}{2\phi}+\frac{1}{2}f+\omega\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{3}{4}\left(\frac{\dot{\phi}}{\phi}\right)^2F-\frac{3}{2}\frac{\dot{\phi}}{\phi}\dot{F}$$
$$-\frac{3}{4}\frac{\dot{F}^2}{F}-3HF\frac{\dot{\phi}}{\phi}-3H\dot{F}, \quad (37)$$

$$-2F\dot{H}=\frac{8\pi\left(\rho+p\right)}{\phi}+\omega\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{3}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2F-3\frac{\dot{\phi}}{\phi}\dot{F}-\frac{3}{2}\frac{\dot{F}^2}{F}$$
$$+\frac{\ddot{\phi}}{\phi}F+2\dot{F}\frac{\dot{\phi}}{\phi}+\ddot{F}-HF\frac{\dot{\phi}}{\phi}-H\dot{F} \quad (38)$$

$$-f\left(\tilde{R}\right)-\frac{\omega}{\phi^2}\dot{\phi}^2+2\omega\frac{\ddot{\phi}}{\phi}+6\omega H\frac{\dot{\phi}}{\phi}=0 \quad (39)$$

In the following, we solve the cosmological equations of GBD theory with the different formalisms. For solving Equations (34) - (36) in the framework of metric formalism, we define the dimensionless variables: $y_H=H^2/m^2-\left(1+z\right)^3$, $y_R=R/m^2-3\left(1+z\right)^3$, $y_\phi=\phi/\phi_0$, $y'_\phi=\phi'/\phi_0$. Then Equations (34) - (36) provide the differential equations for $\left\{y_H,y_R,y_\phi,y'_\phi\right\}$ as follows

$$y'_H=-\frac{1}{1+z}\left(\frac{1}{3}y_R-4y_H\right), \quad (40)$$

$$y'_R=\frac{1}{\left[y_H+\left(z+1\right)^3\right]\left(z+1\right)m^2f_{RR}}\left\{\left[y_H+\left(z+1\right)^3\right]f_R\right.$$
$$-\left[y_R+3\left(z+1\right)^3\right]\frac{f_R}{6}+\frac{f}{6m^2}-\frac{\omega}{6}\left(z+1\right)^2\left[y_H+\left(z+1\right)^3\right]\left(\frac{y'_\phi}{y_\phi}\right)^2 \quad (41)$$
$$\left.-\left[y_H+\left(z+1\right)^3\right]\left(z+1\right)f_R\frac{y'_\phi}{y_\phi}-\frac{\left(z+1\right)^3}{\phi}\right\}-9\left(z+1\right)^2$$

$$y'_\phi=\phi'/\phi_0, \quad (42)$$

$$y''_\phi=\frac{y_\phi}{2\omega\left[y_H+\left(z+1\right)^3\right]\left(z+1\right)^2}\left\{\frac{f}{m^2}+\omega\left[y_H+\left(z+1\right)^3\right]\left(z+1\right)^2\left(\frac{y'_\phi}{y_\phi}\right)^2\right.$$
$$\left.+\omega\left(z+1\right)\frac{y'_\phi}{y_\phi}\left[\frac{1}{3}y_R-4y_H-3\left(z+1\right)^3\right]+4\omega\left(z+1\right)\frac{y'_\phi}{y_\phi}\left[y_H+\left(z+1\right)^3\right]\right\} \quad (43)$$

To solve above differential equations, the initial conditions are selected as respectively: $y_\phi\big|_{z=0}=1$, $y'_\phi\big|_{z=0}=0.01$. The initial conditions of y_H and y_R are taken the same values with those in *f*(*R*) theory. The value of initial condition $y'_\phi\big|_{z=0}$ can be indicated by the following observations. For example, the limits on the variation of *G* can be exhibited by: $\left|\frac{\dot G}{G}\right| = \left|\frac{\dot\phi}{\phi}\right| \le 4.1\times 10\ \text{y}^{-1}$ from Pulsating white dwarf G117-B15A [51], $-4 \times 10\ \text{y}^{-1} \le \frac{\dot\phi}{\phi} \le 2.5 \times 10^{-10}\ \text{y}^{-1}$ from Nonradial pulsations of white dwarfs [52], $\left|\frac{\dot\phi}{\phi}\right| \le 2.3 \times 10^{-11}\ \text{y}^{-1}$ from Millisecond pulsar PSR J0437-4715 [53], $\left|\frac{\dot\phi}{\phi}\right| \le 10^{-11}\ \text{y}^{-1}$ from Type-Ia supernovae [54], $\frac{\dot\phi}{\phi} = (0.6 \pm 0.42) \times 10^{-12}\ \text{y}^{-1}$ from Neutron star masses [55], $\left|\frac{\dot\phi}{\phi}\right| \le 1.6 \times 10^{-12}\ \text{y}^{-1}$ from Helioseismology [56], and $\frac{\dot\phi}{\phi} = (4 \pm 9) \times 10^{-13}\ \text{y}^{-1}$ from Lunar laser ranging experiment [57], etc. Taking a stringent bound $\left|\frac{\dot\phi}{\phi}\right| \le 10^{-12}\ \text{y}^{-1}$, we can calculate to limit $\left|y'_\phi(z=0)\right| \le 0.015$ by using the center value $H_0 = 67.3\ \text{kms}^{-1}\cdot\text{Mpc}^{-1} = 6.87 \times 10^{-11}\ \text{y}^{-1}$. Here we take $y'_\phi(z=0)=0.01$ as an initial condition in Equation (42). Then the pictures of geometrical diagnostic are illustrated in **Figure 4** and **Figure 5** for the metric formalism of GBD theory.

For the Palatini formalism of GBD theory, using the trace equation in GBD theory we have

$$\dot{\tilde R} = \frac{-8\pi\left(\dot\rho_m\phi - \dot\phi\rho_m\right)\phi + \omega\left(2\ddot\phi\dot\phi\phi - 2\dot\phi^3\right)}{\phi^3\left(f_{\tilde R\tilde R}R - f_{\tilde R}\right)}. \tag{44}$$

Changing the variable from *t* to *z*, with $\frac{\text{d}}{\text{d}t} = -H(1+z)\frac{\text{d}}{\text{d}z}$, we receive $\dot\phi = \phi'\left(-(1+z)H\right)$, and $\ddot\phi = \phi'' H^2(z+1)^2 + \phi' H^2(z+1) + \phi' H' H (z+1)^2$. Using the definition of geometrical quantities, we can plot the diagnostic pictures of {*r*, *s*} and {*r*, *q*} in the Palatini formalism of GBD theory.

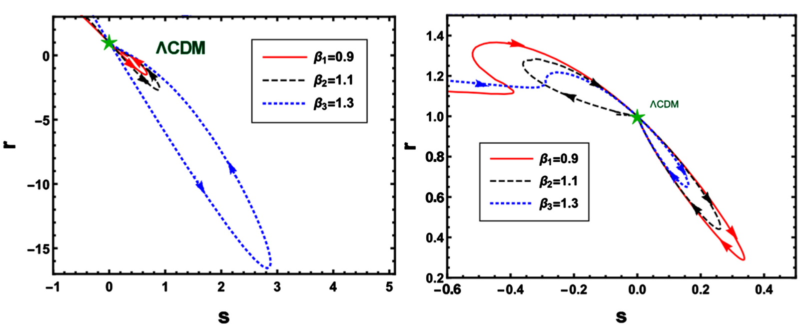

Figure 4. The {*r*, *s*} geometrical diagnostic for GBD theory with the metric formalism (left) and the Palatini formalism (right).

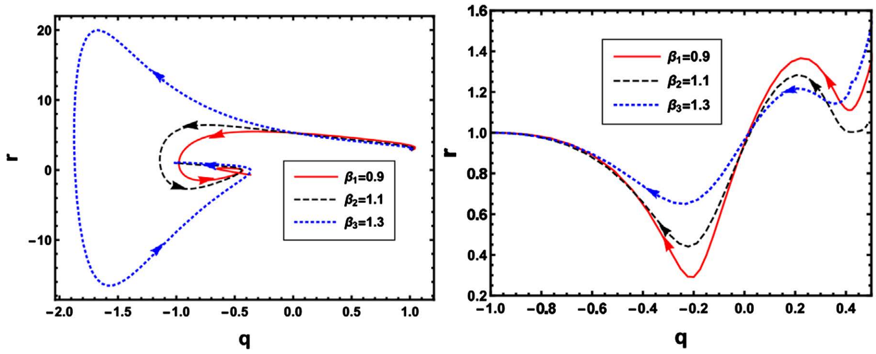

Figure 5. The {r, q} geometrical diagnostic for GBD theory with the metric formalism (left) and the Palatini formalism (right).

In this part, we distinguish the different formalisms of GBD modified gravitational theory. For comparison, we take the same $f(R)$ function as above. From Figure 4 and Figure 5, we can see that the difference between the metric formalism and the Palatini formalism are still conspicuous according to the {r, s} and {r, q} diagnostics, respectively. In the GBD theory, the evolutions of deceleration parameter q are plotted in Figure 6, which exhibit the different evolutions between two formalisms. According to Figure 6, for case of β = 1.1 in Palatini formalism, it is satisfied with the requirement of $q(z) \le 1/2$. And for this case we have $z_T = 0.11$ (at where $q = 0$) for metric formalism, and $z_T = 1.09$ for Palatini formalism. is called the transition redshift, which describes the universal expansion from deceleration toacceleration. Obviously, the significantly different values of z_T are indicated in these two formalisms.

In order to explore the effects of time-variable G, we also compare the $f(R)$ theory (G is a constant) with the GBD theory (G is a variable quantity) according to the results of geometrical diagnostics. Some results could be exhibited as follows: 1) According to Figure 1 (left) and Figure 4 (left), in the metric formalism the variation of G affects obviously on the values of r (and s). For example, for case of β = 1.3, the value of r vary from 0 to −38.55 for $f(R)$ theory, and from 0 to −16.54 for GBD theory, respectively; the largest value of s vary from 2.88 (corresponding to GBD) to 4.61 (corresponding to $f(R)$ theory). Obviously, for the influence of variable G, the values of r (or s) in GBD are smaller than those in $f(R)$ theory, which indicates that the variation of Newton gravitational constant can induce remarkable effects on geometrical quantities for the existence of BD scalar field (including the terms of its dynamics and the coupling between the BD field and the $f(R)$ in the action). 2) In the Palatini formalism (see Figure 1 (right) and Figure 4 (right)), for β = 1.1 case the difference between the variable-G theory (GBD) and the constant-G theory ($f(R)$) are not obvious, while for other cases the shapes of r-s curves are different. 3) According to Figure 3 and Figure 6, in the metric formalism we find that the values of q decay more early (about $z \sim 2$) in $f(R)$ theory than that (about $z \sim 0.8$) in GBD theory; while in the Palatini formalism, the difference between the GBD theory and the $f(R)$ theory are more reflected at the earlier stage of universe (higher redshift).

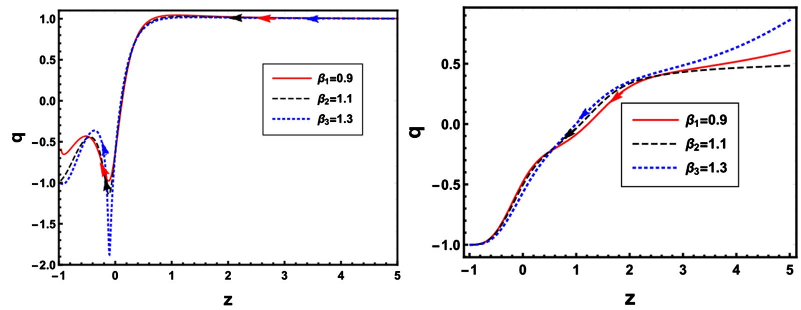

Figure 6. The evolutions of a geometrical quantity-deceleration parameter q for GBD theory with the metric formalism (left) and the Palatini formalism (right).

4. Conclusions

Several observational and theoretical motivations require us to investigate the modified theories of GR. In modified gravitational theory, there exist the metric formalism and the Palatini formalism. In these two formalisms, the dynamical variables are considered to be different. Geometrical-diagnostic methods were often applied to distinguish the gravitational models. But it is scarce to investigate the differences between the different formalisms (e.g. the metric formalism and the Palatini formalism) in the modified theories of gravity. In this paper, we discriminate the different formalisms of the modified gravity by using the diagnostic methods. For considered modified theories of gravity (including the $f(R)$ theory and the GBD theory), we can see that the difference between the two formalisms is notable according to the geometrical diagnostics. And relative to the ΛCDM model, there are more deviations in metric formalism than those in Palatini formalism, according to the $\{r, s\}$ diagnostic.

Given that the GBD is a time-variable Newton gravitational constant theory, the differences between the variable-G theory and the constant-G theory are studied by using the diagnostic methods. According to the observational limits on G, we plot some pictures on the geometrical quantities. For the influence of variable G, the values of r (or s) in metric-formalism GBD are smaller than those in $f(R)$ theory, which indicates that the variation of Newton's gravitational constant can induce remarkable effects on geometrical quantities for the existence of BD scalar field. In the Palatini formalism, the shapes of r-s curves between the GBD theory and the $f(R)$ theory could be obviously different, depending on the values of model parameter β. In addition, in the metric formalism we can find that the values of q decay more early (about $z \sim 2$) in $f(R)$ theory than that (about $z \sim 0.8$) in GBD theory; while in the Palatini formalism, the difference between the $f(R)$ theory and the GBD theory are more reflected at the higher-redshift universe. In summary, according to our study, the effects of variable G could be found in both the metric formalism and the Palatini formalism, by testing the geometrical quantities (e.g. r, s and q).

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Zwicky, F. (1933) Die Rotverschiebung von extragalaktischen Nebeln. *Helvetica Physica Acta*, **6**, 110-127.

[2] Smith, S. (1936) The Mass of the Virgo Cluster. *The Astrophysical Journal*, **83**, 23-30. https://doi.org/10.1086/143697

[3] Guth, A.H. (1981) Inflationary Universe: A Possible Solution to the Horizon and Flatness Problem. *Physical Review D*, **23**, 347-356. https://doi.org/10.1103/PhysRevD.23.347

[4] La, D. and Steinhardt, P.J. (1989) Extended Inflationary Cosmology. *Physical Review Letters*, **62**, 376-378. https://doi.org/10.1103/PhysRevLett.62.376

[5] Perlmutter, S., *et al.* (1999) Measurements of Ω and Λ from 42 High-Redshift Supernovae. *The Astrophysical Journal*, **517**, 565-586. https://doi.org/10.1086/307221

[6] Riess, A.G., *et al.* (1998) Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. *The Astronomical Journal*, **116**, 1009-1038. https://doi.org/10.1086/300499

[7] De Felice, A. and Tsujikawa, S. (2010) f(R) Theories. *Living Reviews in Relativity*, **13**, Article No. 3. https://doi.org/10.12942/lrr-2010-3

[8] Sotiriou, T.P. and Faraoni, V. (2010) f(R) Theories of Gravity. *Reviews of Modern Physics*, **82**, 451-497. https://doi.org/10.1103/RevModPhys.82.451

[9] Nojiri, S., Odintsov, S.D. and Sami, M. (2006) Dark Energy Cosmology from Higher-Order, String-Inspired Gravity, and Its Reconstruction. *Physical Review D*, **74**, Article ID: 046004. https://doi.org/10.1103/PhysRevD.74.046004

[10] Cartier, C., Copeland, E.J. and Madden, R. (2000) The Graceful Exit in String Cosmology. *JHEP*, **1**, 35. https://doi.org/10.1088/1126-6708/2000/01/035

[11] Brans, C. and Dicke, R.H. (1961) Mach's Principle and a Relativistic Theory of Gravitation. *Physical Review*, **124**, 925-935. https://doi.org/10.1103/PhysRev.124.925

[12] Maartens, R. and Koyama, K. (2010) Brane-World Gravity. *Living Reviews in Relativity*, **13**, Article No. 5. https://doi.org/10.12942/lrr-2010-5

[13] Petrov, A.Yu. (2004) Introduction to Modified Gravity.

[14] Clifton, T., Ferreira, P.G., Padilla, A. and Skordis, C. (2012) Modified Gravity and Cosmology. *Physics Reports*, **513**, 1-189. https://doi.org/10.1016/j.physrep.2012.01.001

[15] Ishak, M. (2019) Testing General Relativity in Cosmology. *Living Reviews in Relativity*, **22**, 1-204. https://doi.org/10.1007/s41114-018-0017-4

[16] Capozziello, S. and De Laurentis, M. (2011) Extended Theories of Gravity. *Physics Reports*, **509**, 167-321. https://doi.org/10.1016/j.physrep.2011.09.003

[17] De Felice, A. and Tsujikawa, S. (2010) f(R) Theories. *Living Reviews in Relativity*,

13, Article No. 3. https://doi.org/10.12942/lrr-2010-3

[18] Lu, J. and Chee, G. (2016) Cosmology in Poincare Gauge Gravity with a Pseudoscalar Torsion. *JHEP*, **5**, 24. https://doi.org/10.1007/JHEP05(2016)024

[19] Lu, J., Xu, L., Tan, H. and Gao, S. (2014) Extended Chaplygin Gas as a Unified Fluid of Dark Components in Varying Gravitational Constant Theory. *Physical Review D*, **89**, Article ID: 063526. https://doi.org/10.1103/PhysRevD.89.063526

[20] Lu, J., Xu, Y. and Wu, Y. (2015) Cosmic Constraint on the Unified Model of Dark Sectors with or without a Cosmic String Fluid in the Varying Gravitational Constant Theory. *The European Physical Journal C*, **75**, Article No. 473. https://doi.org/10.1140/epjc/s10052-015-3691-3

[21] Huang, Q.G. (2014) An Analytic Calculation of the Growth Index for f(R) Dark Energy Model. *The European Physical Journal C*, **74**, Article No. 2964. https://doi.org/10.1140/epjc/s10052-014-2964-6

[22] Hohmann, M., Jarv, L., Kuusk, P., Randla, E. and Vilson, O. (2016) Post-Newtonian Parameter γ for Multiscalar-Tensor Gravity with a General Potential. *Physical Review D*, **94**, Article ID: 124015. https://doi.org/10.1103/PhysRevD.94.124015

[23] Nojiri, S., Odintsov, S.D. and Oikonomou, V.K. (2017) Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-Time Evolution. *Physics Reports*, **692**, 1-104. https://doi.org/10.1016/j.physrep.2017.06.001

[24] de la Cruz-Dombriz, A., Elizalde, E., Odintsov, S.D. and Saez-Gomez, D. (2016) Spotting Deviations from R^2 Inflation. *JCAP*, **5**, 60. https://doi.org/10.1088/1475-7516/2016/05/060

[25] Sotiriou, T.P. (2006) f(R) Gravity and Scalar-Tensor Theory. *Classical and Quantum Gravity*, **23**, Article No. 5117. https://doi.org/10.1088/0264-9381/23/17/003

[26] Giacchini, B.L. and Shapiro, I.L. (2018) Light Bending in f(R) Extended Gravity Theories. *Physics Letters B*, **780**, 54-60. https://doi.org/10.1016/j.physletb.2018.02.055

[27] De Laurentis, M., De Martino, I. and Lazkoz, R. (2018) Modified Gravity Revealed Along Geodesic Tracks. *The European Physical Journal C*, **78**, Article No. 916. https://doi.org/10.1140/epjc/s10052-018-6401-0

[28] De Martino, I., Lazkoz, R. and De Laurentis, M. (2018) Analysis of the Yukawa Gravitational Potential in f(R) Gravity. I. Semiclassical Periastron Advance. *Physical Review D*, **97**, Article ID: 104067. https://doi.org/10.1103/PhysRevD.97.104067

[29] Geng, C.Q., Lee, C.C. and Shen, J.L. (2015) Matter Power Spectra in Viable f(R) Gravity Models with Massive Neutrinos. *Physics Letters B*, **740**, 285-290. https://doi.org/10.1016/j.physletb.2014.11.061

[30] Sefiedgar, A.S., Atazadeh, K. and Sepangi, H.R. (2009) Generalized Virial Theorem in Palatini f(R) Gravity. *Physical Review D*, **80**, Article ID: 064010. https://doi.org/10.1103/PhysRevD.80.064010

[31] Faraoni, V. (2008) Palatini f(R) Gravity as a Fixed Point. *Physics Letters B*, **665**, 135-138. https://doi.org/10.1016/j.physletb.2008.06.002

[32] Abbott, B.P., *et al.* (2016) Directly Comparing GW150914 with Numerical Solutions of Einstein's Equations for Binary Black Hole Coalescence. *Physical Review D*, **94**, Article ID: 064035.

[33] Abbott, B.P., *et al.* (2017) GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral. *Physical Review Letters*, **119**, Article ID: 161101.

[34] Corda, C. (2009) Interferometric Detection of Gravitational Waves: The Definitive

Test for General Relativity. *International Journal of Modern Physics D*, **18**, 2275-2282. https://doi.org/10.1142/S0218271809015904

[35] Sahni, V., Saini, T.D., Starobinsky, A.A. and Alam, U. (2003) Statefinder—A New Geometrical Diagnostic of Dark Energy. *Soviet Journal of Experimental and Theoretical Physics Letters*, **77**, 201-206. https://doi.org/10.1134/1.1574831

[36] Alam, U., Sahni, V., Deep Saini, T. and Starobinsky, A.A. (2003) Exploring the Expanding Universe and Dark Energy Using the Statefinder Diagnostic. *Monthly Notices of the Royal Astronomical Society*, **344**, 1057-1074. https://doi.org/10.1046/j.1365-8711.2003.06871.x

[37] Zhang, X. (2005) Statefinder Diagnostic for Holographic Dark Energy Model. *International Journal of Modern Physics D*, **14**, 1597-1606. https://doi.org/10.1142/S0218271805007243

[38] Setare, M.R., Zhang, J. and Zhang, X. (2007) Statefinder Diagnosis in a Non-Flat Universe and the Holographic Model of Dark Energy. *JCAP*, **3**, 7. https://doi.org/10.1088/1475-7516/2007/03/007

[39] Yi, Z.L. and Zhang, T.J. (2007) Statefinder Diagnostic for the Modified Polytropic Cardassian Universe. *Physical Review D*, **75**, Article ID: 083515. https://doi.org/10.1103/PhysRevD.75.083515

[40] Komatsu, E., *et al.* (2009) Five-Year Wilkinson Microwave Anisotropy Probe Observations: Cosmological Interpretation. *The Astrophysical Journal Supplement Series*, **180**, 330-376. https://doi.org/10.1088/0067-0049/180/2/330

[41] Ade, P.A.R., Aghanim, N., Arnaud, M., *et al.* (2016) Planck 2015 Results XIII. Cosmological Parameters. *Astronomy & Astrophysics*, **594**, A13.

[42] Riess, A.G., *et al.* (2009) A Redetermination of the Hubble Constant with the Hubble Space Telescope from a Differential Distance Ladder. *The Astrophysical Journal*, **699**, 539-563. https://doi.org/10.1088/0004-637X/699/1/539

[43] Vollick, D.N. (2003) 1/R Curvature Corrections as the Source of the Cosmological Acceleration. *Physical Review D*, **68**, Article ID: 063510. https://doi.org/10.1103/PhysRevD.68.063510

[44] Zhang, P. (2006) Testing Gravity against the Early Time Integrated Sachs-Wolfe Effect. *Physical Review D*, **73**, Article ID: 123504. https://doi.org/10.1103/PhysRevD.73.123504

[45] Linder, E.V. (2009) Exponential Gravity. *Physical Review D*, **80**, Article ID: 123528. https://doi.org/10.1103/PhysRevD.80.123528

[46] Bamba, K., Geng, C.G. and Lee, C.C. (2010) Cosmological Evolution in Exponential Gravity. *JCAP*, **8**, 21. https://doi.org/10.1088/1475-7516/2010/08/021

[47] Lu, J., Wang, Y. and Zhao, X. (2019) Linearized Modified Gravity Theories and Gravitational Waves Physics in the GBD Theory. *Physics Letters B*, **795**, 129-134. https://doi.org/10.1016/j.physletb.2019.05.051

[48] Lu, J., Wu, Y., Yang, W., Liu, M. and Zhao, X. (2019) The Generalized Brans-Dicke Theory and Its Cosmology. *The European Physical Journal Plus*, **134**, Article No. 318. https://doi.org/10.1140/epjp/i2019-12684-0

[49] Lu, J., Li, J., Guo, H., Zhuang, Z. and Zhao, X. (2020) Linearized Physics and Gravitational-Waves Polarizations in the Palatini Formalism of GBD Theory. *Physics Letters B*, **811**, Article ID: 135985. https://doi.org/10.1016/j.physletb.2020.135985

[50] Wei, H., Qi, H.-Y. and Ma, X.-P. (2012) Constraining f(T) Theories with the Varying Gravitational Constant. *The European Physical Journal C*, **72**, Article No. 2117.

https://doi.org/10.1140/epjc/s10052-012-2117-8

[51] Biesiada, M. and Malec, B. (2004) A New White Dwarf Constraint on the Rate of Change of the Gravitational Constant. *Monthly Notices of the Royal Astronomical Society*, **350**, 644-648. https://doi.org/10.1111/j.1365-2966.2004.07677.x

[52] Benvenuto, O.G., *et al.* (2004) Asteroseismological Bound on $\dot{G}/G$ from Pulsating White Dwarfs. *Physical Review D*, **69**, Article ID: 082002. https://doi.org/10.1103/PhysRevD.69.082002

[53] Verbiest, J.P.W., *et al.* (2008) Precision Timing of PSR J0437-4715: An Accurate Pulsar Distance, a High Pulsar Mass, and a Limit on the Variation of Newton's Gravitational Constant. *The Astrophysical Journal*, **679**, 675-680. https://doi.org/10.1086/529576

[54] Gaztanaga, E., *et al.* (2002) Bounds on the Possible Evolution of the Gravitational Constant from Cosmological Type-Ia Supernovae. *Physical Review D*, **65**, Article ID: 023506. https://doi.org/10.1103/PhysRevD.65.023506

[55] Thorsett, S.E. (1996) The Gravitational Constant, the Chandrasekhar Limit, and Neutron Star Masses. *Physical Review Letters*, **77**, 1432-1435. https://doi.org/10.1103/PhysRevLett.77.1432

[56] Guenther, D.B., Krauss, L.M. and Demarque, P. (1998) Testing the Constancy of the Gravitational Constant Using Helioseismology. *The Astrophysical Journal*, **498**, 871-876. https://doi.org/10.1086/305567

[57] Williams, J.G., Turyshev, S.G. and Boggs, D.H. (2004) Progress in Lunar Laser Ranging Tests of Relativistic Gravity. *Physical Review Letters*, **93**, Article ID: 261101. https://doi.org/10.1103/PhysRevLett.93.261101