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Noncommutative Geometry and MOND

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Abstract

Modified Newtonian dynamics (MOND) is a hypothesized modification of Newton's law of universal gravitation to account for the flat rotation curves in the outer regions of galaxies, thereby eliminating the need for dark matter. Although a highly successful model, it is not a self-contained physical theory since it is based entirely on observations. It is proposed in this paper that noncommutative geometry, an offshoot of string theory, can account for the flat rotation curves and thereby provide an explanation for MOND. This paper extends an earlier heuristic argument by the author.

Keywords

Noncommutative Geometry, MOND, Flat Galactic Rotation Curves, Dark Matter

1. Introduction

It is generally assumed that dark matter is needed to account for the flat galactic rotation curves in the outer regions of galactic halos. A well-known alternative is a hypothesis called modified Newtonian dynamics (MOND) due to M. Milgrom [1]. Although highly successful, it is not a self-contained physical theory, but rather a purely ad hoc variant based entirely on observations.

The purpose of this paper is to show that noncommutative geometry, an offshoot of string theory, can provide an explanation for the theory. This possibility had already been suggested in [2] based on a heuristic argument. The dark-matter problem can also be addressed by means of $f(R)$ and other modified gravitational theories. See, for example, [3] and references therein.

2. Noncommutative Geometry and the Dark-Matter Hypothesis

In this section we take a brief look at noncommutative geometry, as discussed in

[4] [5]. One outcome of string theory is that coordinates may become noncommuting operators on a *D*-brane [6] [7]; the commutator is $\left[\mathbf{x}^{\mu},\mathbf{x}^{\nu}\right]=i\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix. The main idea, discussed in [4] [5], is that noncommutativity replaces point-like structures by smeared objects, thereby eliminating the divergences that normally occur in general relativity. The smearing effect can be accomplished in a natural way by means of a Gaussian distribution of minimal length $\sqrt{\beta}$ instead of the Dirac delta function [8] [9] [10]. A simpler, but equivalent, way is proposed in [11] [12]: we assume that in the neighborhood of the origin, the energy density of the static and spherically symmetric and particle-like gravitational source has the form

$$\rho_{\beta}(r)=\frac{m\sqrt{\beta}}{\pi^{2}\left(r^{2}+\beta\right)^{2}}. \tag{1}$$

The point is that the mass m of the particle is diffused throughout the region of linear dimension $\sqrt{\beta}$ due to the uncertainty. According to [8], we can keep the standard form of the Einstein field equations in the sense that the Einstein tensor retains its original form, but the stress-energy tensor is modified. It follows that the length scales can be macroscopic despite the small value of β.

As already noted, the gravitational source in Equation (1) results in a smeared mass. According to [4] [5], the Schwarzschild solution of the field equations involving a smeared source leads to the line element

$$\mathrm{d}s^{2}=-\left(1-\frac{2M_{\beta}(r)}{r}\right)\mathrm{d}t^{2}+\frac{\mathrm{d}r^{2}}{1-\frac{2M_{\beta}(r)}{r}}+r^{2}\left(\mathrm{d}\theta^{2}+\sin^{2}\theta\mathrm{d}\phi^{2}\right). \tag{2}$$

The smeared mass is given by

$$M_{\beta}(r)=\int_{0}^{r}4\pi(r')^{2}\rho(r')\mathrm{d}r'=\frac{2M}{\pi}\left(\tan^{-1}\frac{r}{\sqrt{\beta}}-\frac{r\sqrt{\beta}}{r^{2}+\beta}\right), \tag{3}$$

where M is now the total mass of the source. Observe that the mass of the particle is zero at the center and rapidly rises to M. As a result, from a distance, the smearing is no longer observed and we get an ordinary particle: $\lim_{\beta\to 0}M_{\beta}(r)=M$.

Returning now to the dark-matter hypothesis, despite its origin in the 1930's, the implications of this hypothesis were not fully understood until the 1970's with the discovery of flat galactic rotation curves, *i.e.*, constant velocities sufficiently far from the galactic center [13]. This led to the conclusion that matter in a galaxy increases linearly in the outward radial direction. To recall the reason for this seemingly strange behavior, suppose m_1 is the mass of a star, v its constant velocity, and m_2 the mass of everything else in the outer region, *i.e.*, the region characterized by flat rotation curves. Multiplying m_1 by the centripetal acceleration, we get

$$m_{1}\frac{v^{2}}{r}=m_{1}m_{2}\frac{G}{r^{2}}, \tag{4}$$

where G is Newton's gravitational constant. Using geometrized units, $c=G=1$, we obtain the linear form

$$m_2 = rv^2. \tag{5}$$

So v^2 is independent of *r*, the distance from the center of the galaxy. The purpose of MOND is to account for this outcome without hypothesizing dark matter.

3. Explaining MOND

Consider next a particle located on the spherical surface $r = r_0$. The density of the smeared particle now becomes

$$\rho_s(r) = \frac{M\sqrt{\beta}}{\pi^2\left[(r-r_0)^2+\beta\right]^2}, \tag{6}$$

valid for any surface $r = r_0$. [If $r_0 = 0$, we return to Equation (1)] Equation (6) can also be interpreted as the density of the spherical surface, yielding the smeared mass of the shell in the outward radial direction, the analogue of the smeared mass at the origin. Integration then yields

$$m_0(r) = \frac{2M}{\pi}\left[\tan^{-1}\frac{r-r_0}{\sqrt{\beta}} - \frac{(r-r_0)\sqrt{\beta}}{(r-r_0)^2+\beta}\right]. \tag{7}$$

In particular, $\lim_{\beta\to 0} m_0(r) = M$. So while $m_0(r)$ is zero at $r = r_0$, it rapidly rises to *M* in the outward radial direction. We can readily show that this mass is completely independent of r_0 by rewriting Equation (7) as

$$m_0(r) = \frac{2M}{\pi}\left[\tan^{-1}\frac{1-\frac{r_0}{r}}{\frac{\sqrt{\beta}}{r}} - \frac{\left(1-\frac{r_0}{r}\right)\frac{\sqrt{\beta}}{r}}{\left(1-\frac{r_0}{r}\right)^2+\left(\frac{\sqrt{\beta}}{r}\right)^2}\right]. \tag{8}$$

Indeed, as $\frac{\sqrt{\beta}}{r} \to 0$, $m_0(r) \to M$ for every r_0. To retain the smearing, we would normally require that $\beta > 0$. So for $m_0(r)$ to get close to *M*, $r = r_0$ would have to be sufficiently large. In other words, we would have to be in the outer region of the galaxy, *i.e.*, the region characterized by flat rotation curves.

Now consider the finite sequence $\{r_i\}$ of such radii. Then the smeared mass of every spherical shell becomes

$$m_i(r) = \frac{2M}{\pi}\left[\tan^{-1}\frac{r-r_i}{\sqrt{\beta}} - \frac{(r-r_i)\sqrt{\beta}}{(r-r_i)^2+\beta}\right] \tag{9}$$

with $\lim_{\beta\to 0} m_i(r) = M$ for every *i*. To obtain the total mass M_T of the outer region, we can think of $m_i(r)$ as the increase in M_T per unit length in the outward radial direction, making *M* a dimensionless constant. If we denote the thickness of each smeared spherical shell by Δr, then $m_i(r)\Delta r$ becomes the mass of the shell. However, we cannot simply integrate $m_i(r)$ over the entire region, since each shell has a different r_i. Instead, we proceed as follows:

$$\Delta M_T = \int_{r_i}^{r_i+\Delta r} m_i(r)\,\mathrm{d}r = \int_{r_i}^{r_i+\Delta r} \frac{2M}{\pi}\left[\tan^{-1}\frac{r-r_i}{\sqrt{\beta}} - \frac{(r-r_i)\sqrt{\beta}}{(r-r_i)^2+\beta}\right]\mathrm{d}r$$
$$= \frac{2M}{\pi}\left[(r-r_i)\tan^{-1}\frac{r-r_i}{\sqrt{\beta}} - \sqrt{\beta}\ln\left[(r-r_i)^2+\beta\right]\right]\Bigg|_{r_i}^{r_i+\Delta r} \tag{10}$$
$$= \frac{2M}{\pi}\Delta r\left[\tan^{-1}\frac{\Delta r}{\sqrt{\beta}} - \sqrt{\beta}\,\frac{\ln\left[(\Delta r)^2+\beta\right]}{\Delta r} + \frac{\sqrt{\beta}\ln\beta}{\Delta r}\right].$$

Given that Δr is large compared to β and that $\lim_{\beta\to 0}\sqrt{\beta}\ln\beta = 0$, it follows that

$$\Delta M_T \approx M\,\Delta r \tag{11}$$

for all r in the outer region. So in this region, all shells of thickness Δr have approximately the same mass ΔM_T due to the noncommutative-geometry background. Since $\beta > 0$ is fixed, we can now safely let $\Delta r \to 0$. This leads to our main conclusion: from $\lim_{\Delta r\to 0}\Delta M_T/\Delta r = M$, we obtain the linear form

$$M_T = Mr. \tag{12}$$

By Equation (5), $M = v^2$, showing that v^2 is indeed independent of *r*. So the noncommutative-geometry background provides an explanation for MOND in the outer region of the galaxy. This region is characterized by extremely low accelerations, also referred to as the deep-MOND regime.

4. Conclusions

The existence of flat rotation curves in the outer regions of galaxies can be accounted for by the presence of dark matter or by the use of modified gravitational theories. An example of the latter is M. Milgrom's modified Newtonian dynamics or MOND. Although a highly successful model, MOND is a purely ad hoc theory based entirely on observations. So it cannot be called a self-contained theory.

It is proposed in this paper that noncommutative geometry, an offshoot of string theory, can account for the flat rotation curves, thereby providing an explanation for MOND.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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