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Light-Cone Coordinate System in General Relativity

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Abstract

If there exists a null gradient field in 3 + 1 dimensional space-time, we can set up a kind of light-cone coordinate system in the space-time. In such coordinate system, the metric takes a simple form, which is helpful for simplifying and solving the Einstein's field equation. This light-cone coordinate system has wonderful properties and has been used widely in astrophysics to calculate parameters. We discuss the structure of space-time with light-cone coordinate system in detail. We show how to construct the light-cone coordinate system and obtain the conditions of its existence, and then explain their geometrical and physical meanings.

Keywords

Light-Cone, Null Gradient Field, Coordinate Transformation, Geodesic, Einstein's Field Equation

1. Introduction

A good choice of coordinate system for the space-time is important to discuss the property of space-time and to solve the Einstein's equation. The usual choices are the Gaussian normal coordinates and the harmonic coordinates [1]. These coordinate systems bring about some convenience for theoretical analysis. However, such coordinate systems are no help to solve the Einstein's field equation. The conventional method to get the exact solution of Einstein's equation is based on the symmetry of the space-time. Many well-known solutions such as the Friedmann-Lemaitre-Robertson-Walker metric, Bianchi universe, Lemaitre-de Sitter universe, Schwarzschild metric and Kerr metric, Taub-NUT solution [2] [3] [4] [5] [6], are all related with some special symmetry of the space-time.

In this paper, we study the structure of light-cone coordinate system (LCS). In such LCS, some partial differentials in Einstein's tensor $G_{\mu\nu}$ can be converted into ordinary derivatives. This property is very helpful to solve the exact vacuum solutions of Einstein's field equation [7] and to simplify the dynamics of an evolving star [8]. This coordinate system can be constructed from a set of null geodesics. There were some coordinate systems related to light-cone used in the previous study. In Minkowski space-time we have "light-cone coordinate". In Schwarzschild space-time, we have "Eddington-Finklestein coordinates". The Newman-Penrose formalism is also based on null tetrad [9]. However it is a little different from LCS and the practical calculation in this formalism is not easy.

In recent years, the geodesic light-cone coordinates (GLC) is introduced to derive explicit expressions for averaging the redshift to luminosity-distance relation in a generic inhomogeneous universe [10]. It is also related to the light-cone coordinates and shares many common properties of LCS. The advantages and wonderful properties of GLC were recognized by many researchers. Some pedagogical introduction to GLC and brief review on its applications are provided in [11] [12] [13]. The GLC is exploited to perform light cone averages in a perturbed Friedmann-Lemaitre-Robertson-Walker space-time, in order to determine the effect of inhomogeneities on the distance-redshift relation [13]-[19], and therefore on the interpretation of the Hubble diagram [18] [20] [21]. GLC is also applied to gravitational lensing in general [22] [23], to galaxy number counts [24], and to the propagation of ultra-relativistic particles [25]. The presence of additional degrees of freedom in the GLC was considered later [26] [27]. In [28], the correct prediction of GLC approach in the conformal Newtonian gauge is compared with other approaches. After the correction suggested in [26], the GLC approach has been successfully used to calculate the expressions of the light-cone observables up to second order in perturbation theory in the Poisson gauge [19] [23] [24] [29]. The consistency of GLC approach with the previous results in an inhomogeneous universe is considered [30].

In this paper, we discuss the structure of space-time with light-cone coordinate system in detail. We establish the relationship between LCS with ordinary coordinate system. This paper is a modification of the early version arXiv:0708.2962v1. The conditions for an LCS are derived, and the differential equations to construct an LCS from usual coordinate system are obtained. Some typical examples to set up an LCS are given.

2. Construction of Light-Cone Coordinate System

Under some conditions, the metric in an LCS has the following simple form

$$g_{\mu\nu} = \begin{pmatrix} u & v & p & q \\ v & 0 & 0 & 0 \\ p & 0 & -a & 0 \\ q & 0 & 0 & -b \end{pmatrix}. \tag{2.1}$$

At first, we give some general analysis for the coordinate transformation to get this canonical metric (2.1). The line element of space-time is generally given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}\xi^\mu \mathrm{d}\xi^\nu = \mathrm{d}Z^+ G\mathrm{d}Z, \quad \mathrm{d}Z = \left(\mathrm{d}\xi^0, \mathrm{d}\xi^1, \mathrm{d}\xi^2, \mathrm{d}\xi^3\right)^+, \tag{2.2}$$

where $G = \left(g_{\mu\nu}\right)$ is matrix form of metric, and index "+" represents transpose. In this paper, we use Greek characters such as $\mu, \nu \in \{0,1,2,3\}$ to denote 4-dimensional indices, and Latin characters $k, l \in \{1,2,3\}$ for spatial indices. Making transformation $\xi^\mu = \xi^\mu\left(x^\nu\right)$ and denoting

$$Y_\mu = \left(\frac{\partial \xi^0}{\partial x^\mu}, \frac{\partial \xi^1}{\partial x^\mu}, \frac{\partial \xi^2}{\partial x^\mu}, \frac{\partial \xi^3}{\partial x^\mu}\right)^+, \quad J = \left(Y_0, Y_1, Y_2, Y_3\right), \tag{2.3}$$

where J is the Jacobian matrix of transformation, we get

$$\mathrm{d}s^2 = \mathrm{d}X^+ J^+ GJ\mathrm{d}X, \quad \mathrm{d}X = \left(\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\right)^+. \tag{2.4}$$

If x^μ forms light-cone coordinate system, by (2.1) and (2.4) we have

$$\left(Y_1, Y_2, Y_3\right)^+ GY_1 = 0, \tag{2.5}$$

$$Y_2^+ GY_3 = 0. \tag{2.6}$$

Let $Y_0^+ GY_1 = K \neq 0$, then by (2.5) we get

$$J^+ GY_1 = K\left(1,0,0,0\right)^+, \quad Y_1 = KG^{-1}J^*\left(1,0,0,0\right)^+, \tag{2.7}$$

where $J^* = \left(J^{-1}\right)^+$.

Noticing the unidirectionality of time, we assume

$$x^0 = T\left(\xi^\mu\right), \quad \partial_0 T > 0. \tag{2.8}$$

In component form, (2.7) becomes

$$\frac{\partial \xi^\mu}{\partial x^1} = Kg^{\mu\nu}\frac{\partial x^0}{\partial \xi^\nu} = Kg^{\mu\nu}\partial_\nu T. \tag{2.9}$$

Since x^0 and x^1 are two independent variables in new coordinate system x^μ, we have

$$0 = \frac{\partial x^0}{\partial x^1} = \partial_\mu T\frac{\partial \xi^\mu}{\partial x^1} = Kg^{\mu\nu}\partial_\mu T\partial_\nu T. \tag{2.10}$$

This means the time coordinate transformation $T\left(\xi^\mu\right)$ is a null gradient field. (2.10) is a necessary condition for LCS.

Let $V_\mu = \partial_\mu T$, if $V^0 \neq 0$, by (2.9) as $\mu = 0$, and then using (2.10) we have

$$K = \frac{1}{V^0}\frac{\partial \xi^0}{\partial x^1} = -\frac{V_k}{V_0 V^0}\frac{\partial \xi^k}{\partial x^1}. \tag{2.11}$$

Substituting (2.11) into (2.9) we get a homogeneous linear equation for $\partial_1 \xi^k$

$$\left(\delta_n^k V^0 V_0 + V^k V_n\right)\frac{\partial \xi^n}{\partial x^1} = 0. \tag{2.12}$$

The determinant of the coefficient matrix is given by

$$\det\left(\delta_n^k V^0 V_0 + V^k V_n\right) = \left(V_0 V^0\right)^2 V_\mu V^\mu = 0. \tag{2.13}$$

The solution to (2.12) reads

$$\frac{\partial}{\partial x^1}\xi^k = f\left(\xi^\mu\right)V^k. \tag{2.14}$$

(2.14) is also in the form of (2.9), but in (2.14) $f \neq 0$ is an arbitrary function which can be selected according to requirement. Solving ξ^0 from (2.8), we have $\xi^0 = t\left(x^0, \xi^k\right)$. Substituting it into (2.14), we get an ordinary differential equation system of $\xi^k\left(x^1\right)$ for any given *f*. We have a unique solution for initial problem

$$\xi^k = F^k\left(x^0, x^1, X^l\right), \tag{2.15}$$

where X^k is the initial values of ξ^k. Making any differentiable and invertible transformation

$$X^k = X^k\left(x^0, x^2, x^3\right), \tag{2.16}$$

substituting it into (2.15), and then substituting the results into $\xi^0 = t\left(x^0, \xi^k\right)$, we get a transformation

$$\xi^\mu = \xi^\mu\left(x^0, x^1, x^2, x^3\right). \tag{2.17}$$

In new coordinate system $\left(x^0, x^1, x^2, x^3\right)$, Equation (2.5) holds. However, we still have two problems. First, does the null gradient field $g^{\mu\nu}\partial_\mu T \partial_\nu T = 0$ has nontrivial solution $\partial_\mu T \not\equiv 0$, and what condition is satisfied to have nontrivial solution? The second is under what conditions (2.6) holds. In what follows, by means of light cone we discuss the problem in detail. The analysis shows, the existence of nontrivial solution of $T\left(\xi^\mu\right)$ is equivalent to the existence of a series of global null geodesics $\xi^\mu\left(\tau\right)$ in the space-time, and we have

$$\partial^\mu T \propto \frac{\mathrm{d}\xi^\mu}{\mathrm{d}\tau}. \tag{2.18}$$

It is difficult to solve the null gradient field $T\left(\xi^\mu\right)$ from (2.10) directly. However, $T\left(\xi^\mu\right)$ can be equivalently derived from null geodesics and the LCS can be constructed as follows.

Theorem 1. *There is an LCS in a space-time, i.e., the metric can be transformed into the following form*

$$g_{\mu\nu} = \begin{pmatrix} u & v & p & q \\ v & 0 & 0 & 0 \\ p & 0 & -a & s \\ q & 0 & s & -b \end{pmatrix}, \tag{2.19}$$

if and only if there exists a null vector field V^μ *in the space-time satisfying* $V_\mu V^\mu = 0$, *and the 1-form*

$$\omega = g_{\mu\nu}V^\mu \mathrm{d}\xi^\nu \tag{2.20}$$

is integrable.

Proof. For necessary part, since $\left(t, z, x, y\right)$ is the light-cone coordinate system, solving the null geodesics along the *z* axis in the space-time with metric

(2.19), we get

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} = -\frac{\partial_z v}{v}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2, \quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}y}{\mathrm{d}\tau} = 0. \tag{2.21}$$

The solution of the null vector field is given by

$$V^\mu \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}x^\mu = \left(0, \frac{\kappa}{v}, 0, 0\right), \tag{2.22}$$

where κ is a constant. The 1-form (2.20) becomes

$$\omega = g_{tz}V^z \mathrm{d}t = \kappa \mathrm{d}t, \tag{2.23}$$

which is an exact differential form.

For the sufficient part, assume $\mathrm{d}\xi^0$ to be time-like. Define

$$\mathrm{d}t = \tilde{K} g_{\mu\nu} V^\mu \mathrm{d}\xi^\nu, \tag{2.24}$$

where $\tilde{K}$ is a factor to make the 1-form (2.24) become an exact differential form, it satisfies

$$\frac{\partial t}{\partial \xi^0} = \tilde{K} g_{\mu 0} V^\mu > 0, \quad \left(\forall \xi^0\right). \tag{2.25}$$

Then we have a regular coordinate transformation for t

$$t = T\left(\xi^\mu\right), \quad V_\mu = \tilde{K}^{-1}\partial_\mu T. \tag{2.26}$$

Along any null geodesic with tangent vector $V^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\xi^\mu(\tau)$, where τ is the parameter of the geodesic, we have

$$\mathrm{d}t(\tau) = \partial_\mu T \frac{\mathrm{d}\xi^\mu(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau = \tilde{K} V_\mu V^\mu \mathrm{d}\tau = 0. \tag{2.27}$$

So for any given constant t_0, the hypersurface $T\left(\xi^\mu\right) = t_0$ is a propagating light wave front orthogonal to V^μ. That is to say, the geometrical meaning of hypersurface $T\left(\xi^\mu\right) = t_0$ is a light wave front scanning the space.

Now we construct the coordinate $z = z\left(\xi^\mu\right)$, which describes the distance of the light wave front $T\left(\xi^\mu\right) = t_0$ moving through

$$\mathrm{d}z = \frac{\partial z}{\partial \xi^\mu}\mathrm{d}\xi^\mu. \tag{2.28}$$

Taking the trajectories of the null geodesic, namely the light rays, as the z axes, then along these z axes we have $V^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\xi^\mu(\tau)$, Substituting it into (2.28) we get

$$V^\mu \partial_\mu z = \frac{\mathrm{d}z}{\mathrm{d}\tau} \equiv K, \tag{2.29}$$

where $K \neq 0$ is a smooth function to be determined, which acts as the scale of z axis. If $K < 0$, make an inversion transformation $\tilde{z} = -z$, we get $V^\mu \partial_\mu \tilde{z} = |K|$. So not lose generality, we always assume $K > 0$.

If we choose K, such that the 2-dimensional surface $t\left(\xi^\mu\right)\big|_{z=const.} = t_0$ is always a fixed light wave front. We denote it by $S(t_0, z)$. The initial surface is $S_0 = S(t_0, z_0)$, where (t_0, z_0) are given constants. By the definition, S is orthogonal to null vector V^μ, that is, S is always orthogonal to the light rays---z axes. Solving (2.29) with boundary condition $z|_{S_0} = z_0$ on surface S_0, we obtain the coordinate transformation $z = z(\xi^\mu)$. The moving distance of the propagating light wave front $S(t_0, z_0) \to S(t_0, z)$ defines the new coordinate *z*.

For the 2-dimensional surface $S(t_0, z_0)$, not loss generality, we can assume the parameter coordinates (x, y) are orthogonal grid. Otherwise, we can take the 2 principal curves of the surface as coordinate grid of (x, y) to get orthogonal coordinates. If we set each null geodesic with unique parameter coordinate (x, y), then the coordinates (x, y) become global coordinates. The metric in new coordinate system (t, z, x, y) takes the following form

$$g_{\mu\nu} = \begin{pmatrix} u & v & p & q \\ v & -w & 0 & 0 \\ p & 0 & -a & s \\ q & 0 & s & -b \end{pmatrix}. \tag{2.30}$$

For light travels along the z lines, we have $\mathrm{d}x = \mathrm{d}y = 0$, and the line element becomes

$$0 = \mathrm{d}s^2 = u\mathrm{d}t^2 - w\mathrm{d}z^2 + 2v\mathrm{d}t\mathrm{d}z. \tag{2.31}$$

By the definition of t in (2.27), we have $\mathrm{d}t = 0$ for the same propagating light wave front $S(t_0, z_0) \to S(t_0, z)$. In this case $\mathrm{d}z \neq 0$, so we get $w = 0$ from (2.31). Considering the arbitrary of (t, z, x, y), we have $w \equiv 0$, and then we get the metric (2.19). The proof is finished.

The selection of K in (2.29) is quite arbitrary. For convenience of solving (2.29), we can usually take $K = 1$ or $K = z$ or some factors of vector V^μ to make the equation simpler.

Theorem 2. *Assuming in LCS* $(t, z, \tilde{x}, \tilde{y})$ *the metric takes the form* (2.19), *then we have*

1° *Let* $A = sa^{-1}, B = sb^{-1}$, *If* $\partial_z A = 0$ *or* $\partial_z B = 0$, *there exists a regular coordinate transformation, such that* (2.19) *can be converted into canonical form* (1).

2° *In the general case with* $\partial_z a \neq 0$, $\partial_z b \neq 0$ *and* $\partial_z s \neq 0$, *metric* (2.19) *can be converted into canonical form* (2.1) *if and only if there exist* (A, B) *independent of z, such that s satisfies*

$$s = \frac{Aa + Bb}{1 + AB}, \quad \partial_z A = \partial_z B = 0, \tag{2.32}$$

and the following partial differential equation system for transformation $\tilde{x}(t, x, y)$, $\tilde{y}(t, x, y)$ *has regular solution,*

$$\partial_y \tilde{x} = A\partial_y \tilde{y}, \quad \partial_x \tilde{y} = B\partial_x \tilde{x}. \tag{2.33}$$

Proof. In the case $\partial_z B = 0$, substituting transformation $\tilde{x} = x$, $\tilde{y} = \tilde{y}(t, x, y)$ into the line element $\mathrm{d}s^2$, we get metric (2.1) by

$$\frac{\partial \tilde{y}}{\partial x} = B(t, x, \tilde{y}). \tag{2.34}$$

Taking t as an independent parameter, (2.34) becomes an ordinary differential equation for $\tilde{y}(x)$. Solving it we get a unique solution for initial value problem $\tilde{y} = f(t, x, y_0)$. Making any regular transformation $y_0 \leftrightarrow y$, or concretely $y_0 = f(t, y)$, we get the total transformation $\tilde{y} = \tilde{y}(t, x, y)$. Similarly we can check the case $\partial_z A = 0$ by transformation $\tilde{x} = \tilde{x}(t, x, y)$, $\tilde{y} = y$.

For the case in 2˚, since z axes are the light rays which have been selected, the coordinates transformation $(\tilde{x}, \tilde{y}) \leftrightarrow (x, y)$ must be independent of z. Under some transformation $\tilde{x} = \tilde{x}(t, x, y)$, $\tilde{y} = \tilde{y}(t, x, y)$, the metric should be converted into (2.1). By straightforward calculation, we find s should take the form of (2.32), and the solution of (2.33) gives the transformation to convert metric (2.19) into (2.1). The proof is finished.

The condition (2.32) is similar to a conformal condition for the 2-dimensional surface $S(t, z)$ for different z. Since $S(t_0, z_0) \to S(t_0, z)$ is an equidistant translation of grid (x, y) along geodesics, (2.32) is a natural requirement for the space-time with LCS. Whether (2.32) can be proved by geometry or derived from vacuum Einstein's equation $G_{\mu\nu} = 0$ is still a problem. If the metric satisfies the conditions in Theorem 2, all space-like coordinates can be orthogonalized, and then the spatial coordinates (z, x, y) form a global orthogonal coordinate grid. The new metric (2.19) becomes the canonical form (2.1).

Theorem 3. *Assuming the coordinate system* (t, z, x, y) *is LCS and the metric takes the canonical form* (2.1), *under the following coordinate transformation*,

$$t = f_0(t'),\ z = f_1(t', z'),\ x = f_2(t', x'),\ y = f_3(t', y'), \tag{2.35}$$

where f_μ *are any given smooth functions*, *the metric also takes the canonical form* (2.1) *in new coordinate system* (t', z', x', y').

Theorem 3 can be directly checked.

From the above proof, we find that the new coordinate system (t, z, x, y) is induced from a global null geodesic series, so it is worthy of the name "light-cone coordinate system". In such LCS, the structure of the space-time becomes simpler, and the exact solutions to the Einstein's field equation can be more easily obtained [7] [8]. For an evolving star with spherical symmetry, in LCS the Einstein's field equation can be reduced to some ordinary differential equations [8].

The GLC introduced in [10] [11] [12] [13] has a little difference from (2.19). In GLC the signature of metric is chosen as $(-, +, +, +)$ and the line element is given by

$$\mathrm{d}s^2_{\mathrm{GLC}} = -2\Upsilon \mathrm{d}\tau \mathrm{d}w + \Upsilon^2 \mathrm{d}w^2 + \gamma_{kl}\left(\mathrm{d}\theta^k - U^k \mathrm{d}w\right)\left(\mathrm{d}\theta^l - U^l \mathrm{d}w\right), (k, l) \in \{1, 2\}. \tag{2.36}$$

The LCS version is inclined to theoretical discussion but GLC is inclined to applications in astrophysics. The basic properties of LCS and GLC are quite similar and can refer to each other. However, one constraint of coordinate condition in GLC is given by $g_{ww} = \Upsilon^2 + \gamma_{kl} U^k U^l$. The specific relations between two kinds light-cone coordinate system need to be clarified in details.

Theorem 4. *If there is a null gradient field* $V_\mu V^\mu = 0$ *in space-time* ξ^μ, (t,z,x,y) *is an LCS. Then the coordinate transformation functions* $\xi^\mu \leftrightarrow (t,z,x,y)$ *satisfy the following linear partial differential equations*

$$V^\mu \partial_\mu (t,z,x,y) = (0, f, 0, 0), \tag{2.37}$$

in which $f(t,z) > 0$ *is any given function with suitable smoothness.*

Proof. For the function $t(\xi^\mu)$, by $\partial_\mu t \propto V_\mu$ we have

$$V^\mu \partial_\mu t(\xi^\nu) \propto V^\mu V_\mu = 0. \tag{2.38}$$

So $t = t(\xi^\mu)$ satisfies (2.37).

The coordinate $z(\xi^\mu)$ is defined by (2.29), so it also satisfies (2.37).

For the coordinate function *x*, along *z* axes we have

$$0 = \mathrm{d}x = \partial_\mu x \mathrm{d}\xi^\mu = \tilde{K}^{-1} V^\mu \partial_\mu x \mathrm{d}\tau, \tag{2.39}$$

that is $x = x(\xi^\mu)$ satisfies (2.37). Similarly, we can check $y = y(\xi^\mu)$ also satisfies (2.37). The proof is finished.

(2.37) forms the basic differential equation system to determine the light-cone coordinate system (t,z,x,y). The above derivation shows the physical and geometrical meanings of LCS and the corresponding metric (2.19). It also provides a method to solve null gradient field equation $\partial_\mu T \partial^\mu T = 0$.

3. Examples and Applications

At first, we take some simple cases in Minkowski space-time as examples to show concepts of LCS. We have line element

$$\mathrm{d}s^2 = \mathrm{d}\xi_\mu \mathrm{d}\xi^\mu = \left(\mathrm{d}\xi^0\right)^2 - \left(\mathrm{d}\xi^1\right)^2 - \left(\mathrm{d}\xi^2\right)^2 - \left(\mathrm{d}\xi^3\right)^2. \tag{3.1}$$

The simplest case corresponds to the plane wave moving along $z = \xi^1$, we have

$$(t,z,x,y) = \left(\xi^0 - \xi^1, \xi^1, \xi^2, \xi^3\right), \tag{3.2}$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 + 2\mathrm{d}t\mathrm{d}z - \mathrm{d}x^2 - \mathrm{d}y^2. \tag{3.3}$$

$\mathrm{d}t = 0$ means $\xi^0 = \xi^1 + t_0$, which stands for a propagating wave front $S(t_0, z)$. $\mathrm{d}t = \mathrm{d}z = 0$ corresponds to a fixed wave front $S(t_0, z_0) = \left\{\xi^\mu \mid \xi^0 = t_0 + z_0, \xi^1 = z_0\right\}$.

$$V_\mu = K\partial_\mu t = K(1,-1,0,0), \quad V^\mu = K(1,1,0,0), \quad V_\mu V^\mu = 0. \tag{3.4}$$

Let $V_0 V^0 = 1$ we get $K = 1$.

The second case corresponds to cylindrical wave moving along ρ, we have

$$(t,\rho,\phi,z) = \left(\xi^0 - \rho, \sqrt{\left(\xi^1\right)^2 + \left(\xi^2\right)^2}, \arctan\left(\xi^1/\xi^2\right), \xi^3\right), \tag{3.5}$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 + 2\mathrm{d}t\mathrm{d}\rho - \rho^2 \mathrm{d}\phi^2 - \mathrm{d}z^2. \tag{3.6}$$

The third case corresponds to the spherical wave moving along *r*, we have

$$t = \xi^0 - r, \quad \mathrm{d}s^2 = \mathrm{d}t^2 + 2\mathrm{d}t\mathrm{d}r - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right). \tag{3.7}$$

In what follows, we take Schwarzschild space-time and Kerr-like one as examples to explain the geometrical meaning of LCS and show how to construct the LCS.

For the Schwarzschild metric

$$g_{\mu\nu}=\mathrm{diag}\left[1-\frac{2m}{r},-\left(1-\frac{2m}{r}\right)^{-1},-r^2,-r^2\sin^2\theta\right],\quad (r>2m) \tag{3.8}$$

with the coordinate system (t,r,θ,φ), the radial null geodesic satisfies

$$g_{00}\dot{t}^2-g_{11}\dot{r}^2=\left(1-\frac{2m}{r}\right)\dot{t}^2-\left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2=0, \tag{3.9}$$

where $\dot{t}=\frac{\mathrm{d}t}{\mathrm{d}\tau}$ and $\dot{r}=\frac{\mathrm{d}r}{\mathrm{d}\tau}$. Taking the null vector orthogonal to the 2-dimensional surface (θ,ϕ) as follows

$$V^\mu=\left(\left(1-\frac{2m}{r}\right)^{-1},\pm1,0,0\right), \tag{3.10}$$

it is easy to check that the corresponding 1-form (2.20) is an exact differential form. The initial light wave front $S(t_0,r_0)$ is simply a sphere in the domain $r>2m$. $V^r=1$ corresponds to the outward light rays and $V^r=-1$ corresponds to the inward light rays. In what follows we only calculate the case $V^r=1$.

By (2.24), we get the coordinate function $\tilde{t}$

$$\tilde{t}=\int g_{\mu\nu}V^\mu\mathrm{d}\xi^\nu=t-r-2m\ln(r-2m)+t_0. \tag{3.11}$$

By (2.37), for x,y we have

$$V^\mu\partial_\mu F=\left(1-\frac{2m}{r}\right)^{-1}\partial_t F+\partial_r F=0. \tag{3.12}$$

The general solution is given by $F=H(\tilde{t},\theta,\phi)$, where $H(\tilde{t},\theta,\phi)$ is arbitrary smooth function. By boundary condition, we get

$$x=\theta,\quad y=\phi. \tag{3.13}$$

By (2.29), we have

$$V^\mu\partial_\mu z=\left(1-\frac{2m}{r}\right)^{-1}\partial_t z+\partial_r z=\left(1-\frac{2m}{r}\right)^{-1}f. \tag{3.14}$$

For (3.14), we get typical solutions independent of (x,y)

$$z=\begin{cases} r+2m\ln(r-2m)+Z(\tilde{t}), & \text{if } f=1,\\ Z(\tilde{t})(r-2m)^{2m}\mathrm{e}^r, & \text{if } f=z,\\ Z(\tilde{t})+r, & \text{if } f=\left(1-\frac{2m}{r}\right), \end{cases} \tag{3.15}$$

where $Z(\tilde{t})$ is an arbitrary function of $\tilde{t}$, we can set $Z=z_0$ according to Theorem 3. In fact, we can choose any given monotone increasing function $z(r)$ in this case,

$$f=\left(1-\frac{2m}{r}\right)z'(r). \tag{3.16}$$

So the option of $f > 0$ is quite arbitrary. The Eddington-Finklestein coordinates are similar to these coordinate system.

In the case of the metric generated by rotating source similar to the Kerr ones [5], we cannot generally construct a null vector field V^μ satisfying the integrable 1-form (2.20), so the corresponding metric cannot be generally converted into the canonical form (2.1). Now we examine the following metric in the coordinate system (t, r, θ, ϕ),

$$g_{\mu\nu} = \begin{pmatrix} u^2 & 0 & 0 & uw \\ 0 & -a & 0 & 0 \\ 0 & 0 & -b & 0 \\ uw & 0 & 0 & w^2 - v \end{pmatrix}, \tag{3.17}$$

where u, v, w, a, b are smooth functions of (r, θ), but independent of (t, ϕ). For speed

$$V^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}), \tag{3.18}$$

after some arrangement, the geodesic equation $\dot{V}^\alpha = -\Gamma^\alpha_{\mu\nu} V^\mu V^\nu$ becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\dot{t} = &-\frac{1}{uv}\left(\left(2v - w^2\right)\frac{\mathrm{d}u}{\mathrm{d}\tau} + uw\frac{\mathrm{d}w}{\mathrm{d}\tau}\right)\dot{t} \\ &-\frac{1}{u^2 v}\left(w\left(v - w^2\right)\frac{\mathrm{d}u}{\mathrm{d}\tau} + u\left(v + w^2\right)\frac{\mathrm{d}w}{\mathrm{d}\tau} - uw\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)\dot{\phi}, \end{aligned} \tag{3.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\dot{\phi} = -\frac{1}{v}\left(w\frac{\mathrm{d}u}{\mathrm{d}\tau} - u\frac{\mathrm{d}w}{\mathrm{d}\tau}\right)\dot{t} - \frac{1}{uv}\left(w^2\frac{\mathrm{d}u}{\mathrm{d}\tau} - uw\frac{\mathrm{d}w}{\mathrm{d}\tau} + u\frac{\mathrm{d}v}{\mathrm{d}\tau}\right)\dot{\phi}, \tag{3.20}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(a\dot{r}^2\right) = &-\left(2u\partial_r u\dot{t}^2 + \left(2w\partial_r u + 2u\partial_r w\right)\dot{t}\dot{\phi} + \left(2w\partial_r w - \partial_r v\right)\dot{\phi}^2\right)\dot{r} \\ &+ \partial_r b\dot{r}\dot{\theta}^2 - \partial_\theta a\dot{r}^2\dot{\theta}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(b\dot{\theta}^2\right) = &-\left(2u\partial_\theta u\dot{t}^2 + \left(2w\partial_\theta u + 2u\partial_\theta w\right)\dot{t}\dot{\phi} + \left(2w\partial_\theta w - \partial_\theta v\right)\dot{\phi}^2\right)\dot{\theta} \\ &- \partial_r b\dot{r}\dot{\theta}^2 + \partial_\theta a\dot{r}^2\dot{\theta}. \end{aligned} \tag{3.22}$$

(3.19) and (3.20) are integrable due to the two Killing vectors $(\partial_t, \partial_\phi)$. The first integrals of (3.19) and (3.20) are given by

$$\dot{t} = -m\frac{w^2 - v}{vu^2} - n\frac{w}{uv}, \quad \dot{\phi} = m\frac{w}{uv} + n\frac{1}{v}, \tag{3.23}$$

where m, n are constants. Substituting (3.23) into the line element equation, we have an equation for null geodesic

$$a\dot{r}^2 + b\dot{\theta}^2 = \frac{m^2}{u^2} - \frac{(nu + mw)^2}{u^2 v}. \tag{3.24}$$

By (3.17) and (3.23), the covariant speed becomes

$$V_\mu = g_{\mu\nu}V^\nu = \left(m, -a\dot{r}, -b\dot{\theta}, -n\right). \tag{3.25}$$

V_t and V_ϕ are constants related with the Killing vectors $(\partial_t, \partial_\phi)$. According to Theorem 4, the metric can be converted into (2.19) if and only if there exists a function $T(t, r, \theta, \phi)$ such that $V_\mu = \partial_\mu T$ is a null vector. Then by (3.25), we have

$$\partial_t T = m, \quad \partial_\phi T = -n, \tag{3.26}$$

$$\partial_r T = -a\dot{r}, \quad \partial_\theta T = -b\dot{\theta}. \tag{3.27}$$

Solving (3.26) we get

$$T = mt - k\theta - n\phi - h(r,\theta), \tag{3.28}$$

where k is a constant. $k\theta$ is split from $h(r,\theta)$ for simplicity of following calculation. Substituting (3.28) into (3.27) we get

$$\dot{r} = \frac{1}{a}\partial_r h, \quad \dot{\theta} = \frac{1}{b}(k + \partial_\theta h). \tag{3.29}$$

By (3.28) we find m is the scale of time, so we set $m = 1$. Substituting (3.29) into (3.21) and (3.22), we get

$$(\partial_r h)^2 = a\left(\frac{1}{u^2} - \frac{(nu+w)^2}{u^2 v} - \frac{k^2}{b}\right), \quad \partial_\theta h = 0, \quad k \in \{0,1\}. \tag{3.30}$$

By (3.30), we find $h = h(r)$.

(3.30) includes many cases of space-time with LCS. We only discuss the case $k = 0$ in normal spherical coordinate system. By (3.30) and $k = 0$, we get

$$a = \frac{u^2 v h'(r)^2}{v - (nu+w)^2}. \tag{3.31}$$

(3.31) is a necessary condition that the metric (3.17) can be converted into (2.19) in the case $k = 0$.

Comparing the Kerr metric in the Boyer-Lindquist form with (3.17) [2] [3] [5], we obtain

$$u^2 = \frac{r^2 + \alpha^2\cos^2\theta - 2mr}{r^2 + \alpha^2\cos^2\theta}, \tag{3.32}$$

$$v = \frac{2\alpha^2 mr\left(r^2 + \alpha^2\cos^2\theta + 6mr\right)\sin^4\theta}{\left(r^2 + \alpha^2\cos^2\theta - 2mr\right)\left(r^2 + \alpha^2\cos^2\theta\right)} + \left(r^2 + \alpha^2\right)\sin^2\theta, \tag{3.33}$$

$$w = \frac{4\alpha mr\sin^2\theta}{\sqrt{\left(r^2 + \alpha^2\cos^2\theta - 2mr\right)\left(r^2 + \alpha^2\cos^2\theta\right)}}, \tag{3.34}$$

$$a = \frac{r^2 + \alpha^2\cos^2\theta}{r^2 - 2mr + \alpha^2}, \quad b = r^2 + \alpha^2\cos^2\theta, \tag{3.35}$$

where m is the mass of a star, and α is a constant proportional to the angular momentum. Substituting (3.32)-(3.35) into (3.31), we find it contradicts $\partial_\theta h = 0$, so the Kerr metric cannot be converted into (2.19). Or equivalently, we cannot construct a global light-cone coordinate system in the Kerr's space-time.

Now we transform the metric (3.17) with (3.31) into the canonical form (2.1). For (3.31), we make transformation $\tilde{r} = h(r)$, then we remove the function $h(r)$ from the metric in the new system $(t,\tilde{r},\theta,\phi)$. This process is equivalent to setting $h(r) = r$. Substituting $h = r$ and $k = 0$ into (3.28), we get the new time coordinate $T = t - n\phi - r$. By symmetry of the boundary condition, we should have

$$n = 0, \quad T = t - r. \tag{3.36}$$

By (3.29) and $k = 0$, the covariant speed V^μ defined in (3.18) becomes

$$V^\mu = \left(a^{-1}, a^{-1}, 0, \frac{w}{uv}\right), \quad a = \frac{u^2 v}{v - w^2}. \tag{3.37}$$

For general functions (u, v, w), (2.37) cannot be solved explicitly. If we set the scale function in (2.37) as $f = V^r = a^{-1}$, we can solve the new coordinate

$$z = r + Z(\tilde{t}, \theta, \Phi), \tag{3.38}$$

where Z is an arbitrary function, we set $Z = 0$ for simplicity.

$$\Phi = \phi - \int \frac{uw}{v - w^2} \mathrm{d}r. \tag{3.39}$$

Solving other equations in (2.37), we get $F = F(\tilde{t}, \theta, \Phi)$. We can choose any two independent functions

$$x = X(\tilde{t}, \theta, \Phi), \quad y = Y(\tilde{t}, \theta, \Phi), \tag{3.40}$$

as the new coordinates. In the new coordinate system $(\tilde{t}, z, x, y)$ defined by (3.36), (3.38) and (3.40), the metric (3.17) becomes $(\tilde{g}_{\mu\nu}) = J^*(g_{\alpha\beta})J^{-1}$. By calculation, we find that

$$\tilde{g}_{zz} = \tilde{g}_{zx} = \tilde{g}_{zy} = 0. \tag{3.41}$$

However, we have $\tilde{g}_{xy} \neq 0$ in general case. if

$$\frac{uw}{v - w^2} = \frac{\mathrm{d}}{\mathrm{d}r}\varphi(r), \tag{3.42}$$

where $\varphi(r)$ is any smooth function independent of θ, we can get $\tilde{g}_{xy} = 0$. In this case, we have

$$(\tilde{t}, z, x, y) = (t - r, r, \theta, \phi - \varphi), \tag{3.43}$$

and the new metric becomes canonical form.

Now we use the LCS to simplify the Einstein's field equation of an evolving star with spherical symmetry [8]. In this case the line element is equivalent to

$$\mathrm{d}s^2 = ab\mathrm{d}t^2 + 2\sqrt{b}\mathrm{d}t\mathrm{d}r - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right), \tag{3.44}$$

where (a, b) are continuous functions of (t, r) with suitable smoothness until the star becomes singular. Denote the 4-vector speed of the fluid by $U^\mu = \{U, V, 0, 0\}$ which satisfies the line element equation $g_{\mu\nu}U^\mu U^\nu = \left(abU + 2\sqrt{b}V\right)U = 1$. For the perfect fluid model, the nonzero components of $T_{\mu\nu} = (\rho + P)U_\mu U_\nu - Pg_{\mu\nu}$ are given by

$$T_{tt} = b(\rho + P)\left(a\sqrt{b}U + V\right)^2 - abP, \tag{3.45}$$

$$T_{tr} = b(\rho + P)\left(a\sqrt{b}U + V\right)U - \sqrt{b}P = T_{rt}, \tag{3.46}$$

$$T_{rr} = b(\rho + P)U^2, \quad T_{\theta\theta} = Pr^2, \quad T_{\varphi\varphi} = Pr^2\sin^2\theta, \tag{3.47}$$

where $P = P(\rho)$ is a given equation of state.

The nonzero components of Einstein tensor are given by

$$G_{tt} = -\frac{1}{r}\left(\sqrt{b}\partial_t a - ab\partial_r a\right) - \frac{1}{r^2}ab(1-a), \tag{3.48}$$

$$G_{tr} = \frac{1}{r}\sqrt{b}\partial_r a - \frac{1}{r^2}\sqrt{b}(1-a) = G_{rt}, \quad G_{rr} = -\frac{\partial_r b}{ra}, \tag{3.49}$$

$$G_{\theta\theta} = \left(\frac{a}{r}\left(\frac{\partial_r b}{2b} + \frac{\partial_r a}{a}\right) - \frac{1-a}{r^2} + \frac{\mathcal{R}}{2}\right)r^2, \quad G_{\varphi\varphi} = G_{\theta\theta}\sin^2\theta, \tag{3.50}$$

where the scalar curvature $\mathcal{R}$ depends on the second order derivatives of the metric functions (b, a). But it is not used in the following discussion, because the related equations are not independent, which can be derived from other equations.

By detailed calculations, we find only the following 3 equations are independent ones in the Einstein's equation $G_{\mu\nu} = -8\pi G T_{\mu\nu}$,

$$\partial_r b = 8\pi G r(\rho + P)b^2 U^2, \tag{3.51}$$

$$\partial_t a = 8\pi G(\rho + P)rV\sqrt{b\left(a + V^2\right)}, \tag{3.52}$$

$$\partial_r a = -4\pi G r\left((\rho - P) + (\rho + P)abU^2\right) + \frac{1-a}{r}. \tag{3.53}$$

The equations of metric become ordinary differential equations.

The above equations have still a weakness, that is, the geometrical variables (a, b) and mechanical variables (ρ, V) couple each other in a complicated manner. Besides, the physical meaning of (U, V) is unclear, which is quite different from the usual definition $\frac{\mathrm{d}r}{\mathrm{d}t}$.

To simplify the relations, we introduce the following transformation

$$U = \sqrt{\frac{1-v}{ab(1+v)}}, \quad V = \frac{\sqrt{a}v}{\sqrt{1-v^2}}, \tag{3.54}$$

where the speed $|v| < 1$ is approximately the usual definition. Define an auxiliary energy function by

$$F \equiv (\rho + P)abU^2 = (\rho + P)\frac{1-v}{1+v}. \tag{3.55}$$

For a static star, we have $F = \rho + P$. Substituting (3.54) and (3.55) into (3.51)-(3.53), we get simplified relations

$$\partial_r a = -8\pi G r\frac{\rho - Pv}{1+v} + \frac{1-a}{r}. \tag{3.56}$$

$$\partial_r b = 8\pi G r(\rho + P)\frac{b(1-v)}{a(1+v)}, \tag{3.57}$$

$$\partial_t a = 8\pi G r(\rho + P)\frac{a\sqrt{b}v}{1-v^2}, \tag{3.58}$$

Obviously, the geometrical variables (a, b) are separated from mechanical ones

(ρ, P, v) and integrable now. The solutions are given by

$$a = 1 - \frac{8\pi G}{r}\int_0^r \frac{\rho - Pv}{1+v} r^2 \mathrm{d}r \equiv a(0,r)\exp\left(8\pi G\int_0^t (\rho + P)\frac{\sqrt{b}vr}{1-v^2}\mathrm{d}t\right), \quad (3.59)$$

$$b = \exp\left(-8\pi G\int_r^R (\rho + P)\frac{(1-v)r}{(1+v)a}\mathrm{d}r\right). \quad (3.60)$$

By (3.59) and (3.60), for an evolving star, we have

$$\rho_{\text{grav}} = \frac{1}{2}(\rho - P + F) = \frac{\rho - Pv}{1+v}. \quad (3.61)$$

For any $\rho(\cdot, r) \in L^\infty([0,\infty))$, we have $a(\cdot, r) \in C^0([0,\infty))$, and it has a positive minimum $a_{\min} > 0$. $b(\cdot, r) \in C^1([0,R])$ is a monotonic increasing function of *r*. For a normal star, the variables have the following range of value,

$$0 < b \le 1, \quad 0 < a \le 1, \quad 0 \le \rho < \infty. \quad (3.62)$$

4. Conclusion

The above discussion shows that we can set up an LCS if and only if there is a null gradient field in the space-time. In such coordinate system, the metric takes a wonderful canonical form (2.19), and the Einstein's field equation becomes simpler [7] [8]. This coordinate system might be also helpful to understand the propagation of the gravitational wave. However, LCS has some limitations in application, because it holds only in average or approximate sense in usual cases. Another kind of special coordinate system with unique realistic time is given in [31]. The physical requirement for space-time with LCS is that, there is a light at some point or on some 2-dimensional surface in the space-time, and its wave front is stable enough to act as coordinates. Therefore, the feature of light-cone coordinate system can be summarized in a powerful word: Beacon the whole world by one light.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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