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Using Affine Quantization to Analyze Non-Renormalizable Scalar Fields and the Quantization of Einstein's Gravity

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Abstract

Affine quantization is a parallel procedure to canonical quantization, which is ideally suited to deal with non-renormalizable scalar models as well as quantum gravity. The basic applications of this approach lead to the common goals of any quantization, such as Schroedinger's representation and Schroedinger's equation. Careful attention is paid toward seeking favored classical variables, which are those that should be promoted to the principal quantum operators. This effort leads toward classical variables that have a constant positive, zero, or negative curvature, which typically characterize such favored variables. This focus leans heavily toward affine variables with a constant negative curvature, which leads to a surprisingly accommodating analysis of non-renormalizable scalar models as well as Einstein's general relativity.

Keywords

Favored Variables, Affine Quantization, Non-Renormalizable Scalars, General Relativity

1. Introduction

1.1. A Brief Look at Three Quantization Procedures

Canonical quantization is traditionally used to quantize most classical theories. For a simple system, a favored pair of phase-space variables, *i.e.* p and q, for which $-\infty < p, q < \infty$, and which are also Cartesian coordinates, arising from a flat surface [1], *i.e.* a constant zero curvature surface, to become P and Q, the basic pair of quantum variables, with $[Q, P] = i\hbar \mathbb{1}$.

Another familiar approach deals with the $SU(2)$ or $SO(3)$ groups, and its favored classical variable pair arises from a spherical surface, *i.e.* a constant positive curvature surface of fixed radius determined by the Hilbert space dimension

[2], along with its basic operators S_1, S_2, and S_3, such that $[S_1, S_2] = i\hbar S_3$ with valid permutations.

A third example, which is less well known, involves affine quantization, that, in one example, involves a favored pair of phase-space variables, p and q, for $-\infty < p < \infty$ while $0 < q < \infty$, and the geometric surface is that of a constant negative curvature [3], along with the basic pair of operators $0 < Q < \infty$ and $D = (QP + PQ)/2$, which fulfills $[Q, D] = i\hbar Q$ [2] [4][1].

Favored Classical Variables

Favored phase-space coordinates promoted to quantum operators apply to all three quantization procedures. To illustrate the meaning of favored coordinates we examine an example from canonical quantization. A classical harmonic oscillator Hamiltonian, say $H(p,q) = (p^2 + q^2)/2$, in one set of coordinates, can also be described by alternative phase-space coordinates, say $\tilde{p}$ and $\tilde{q}$, as one example, where $p = \tilde{p}/\tilde{q}^2$ and $q = \tilde{q}^3/3$. It follows that $H(p,q) = \tilde{H}(\tilde{p},\tilde{q}) = (\tilde{p}^2/\tilde{q}^4 + \tilde{q}^6/9)/2$. Although the quantum operators obey $[Q,P] = i\hbar \mathbb{1} = [\tilde{Q}, \tilde{P}]$, it follows that $\hat{H}(P,Q) = (P^2 + Q^2)/2 \neq \hat{\tilde{H}}(\tilde{P},\tilde{Q}) = (\tilde{P}^2/\tilde{Q}^4 + \tilde{Q}^6/9)/2$. The spectrum of these two Hamiltonians are different despite the fact that they agree when $\hbar \to 0$; here, apart from linear transformations, one choice of phase-space variables is correct, while any other choice of phase-space variables is incorrect, and that difference may already show up at the lowest order of $\hbar \neq 0$.

It is essential to identify the favored classical variables, and only promote them to quantum operators; otherwise you risk a false quantization! We now focus on affine quantization.

1.2. The Essence of Affine Quantization

Canonical quantization is the standard approach, but it can fail to yield an acceptable quantization, such as for a classical "harmonic oscillator" with $0 < q < \infty$. This very problem is easy to quantize with affine quantization; see [2]. Coherent states for affine quantization, with positive q and Q having passed their dimensions to p (or carried by D), rendering them dimensionless for simplicity, are given by

$$|p;q\rangle \equiv e^{ipQ/\hbar} e^{-i\ln(q)D/\hbar} |b\rangle, \tag{1}$$

with $[(Q-1) + iD/b\hbar]|b\rangle = 0$. If $\mathcal{H}'(D,Q)$ denotes the quantum Hamiltonian, then, a semiclassical expression called the "weak correspondence principle" [8] is given by

$$\begin{aligned} H(p,q) &\equiv H'(pq,q) = \langle p;q|\mathcal{H}'(D,Q)|p;q\rangle \\ &= \langle b|\mathcal{H}'(D + pqQ, qQ)|b\rangle \\ &= \mathcal{H}'(pq,q) + \mathcal{O}(\hbar; p,q), \end{aligned} \tag{2}$$

[1]The affine variable Q can instead satisfy $-\infty < Q < 0$ or even $-\infty < Q \neq 0 < \infty$, a reducible operator that the program of enhanced quantization permits [5] [6]. The word "affine" has been chosen for the similarity to an affine group, especially the symbolic equality of their Lie algebras [7].

implying that when $\hbar \to 0$, leading to the standard classical limit, then $H'(pq,q) = \mathcal{H}'(pq,q)$; namely, the quantum variables have the same functional positions as the appropriate classical variables. In addition, we find that these variables lead to a constant negative curvature surface (equal to $-2/b\hbar$) as shown by the equation[2]

$$\mathrm{d}\sigma(p,q)^2 \equiv 2\hbar\left[\left\|d\,|\,p;q\rangle\right\|^2 - \langle p;q|d|p;q\rangle^2\right] = (b\hbar)^{-1} q^2\mathrm{d}p^2 + (b\hbar)q^{-2}\mathrm{d}q^2. \quad (3)$$

This latter property, *i.e.* seeing that these particular classical variables arise from a constant negative curvature renders them as favored coordinates, just like the favored variables of canonical quantization are those that are Cartesian coordinates, *i.e.* a constant zero curvature [1].

After this background, we turn attention to the Schrödinger representation and equations for affine quantization. The quantum action functional (q), with normalized Hilbert space vectors, is given by

$$A_q = \int_0^T \langle\Psi(t)|\left[i\hbar(\partial/\partial t) - \mathcal{H}'(D,Q)\right]|\Psi(t)\rangle\,\mathrm{d}t, \quad (4)$$

and variational efforts lead to a form of Schrödinger's equation

$$i\hbar\left(\partial|\Psi(t)\rangle/\partial t\right) = \mathcal{H}'(D,Q)|\Psi(t)\rangle. \quad (5)$$

Schrödinger's representation is $Q \to x$ and

$$D \to -i\frac{1}{2}\hbar\left[x(\partial/\partial x) + (\partial/\partial x)x\right] = -i\hbar\left[x(\partial/\partial x) + 1/2\right],$$

where $0 < x < \infty$ (provided $0 < Q < \infty$), and $|\Psi(t)\rangle \to \psi(x,t)$. This analysis leads to the familiar form of the Schrödinger equation

$$i\hbar\,\partial\psi(x,t)/\partial t = \mathcal{H}'\left(-i\hbar\left[x(\partial/\partial x) + 1/2\right], x\right)\psi(x,t). \quad (6)$$

There is a new feature in affine quantization, one that is not in canonical quantization, namely that

$$Dx^{-1/2} = -i\hbar\left[x(\partial/\partial x) + 1/2\right]x^{-1/2} = 0. \quad (7)$$

The analog of this relation in canonical quantization is $P\mathbb{1} = -i\hbar(\partial/\partial x)\mathbb{1} = 0$, which is self-evident, and leads to no useful relation.

The equations above, dealing with some basic properties, have their analogues in more complex systems, which are analyzed next in Section 2 regarding quantizing non-renormalizable scalar fields, followed by Section 3 regarding quantizing gravity.

2. Canonical and Affine Quantization of Non-Renormalizable Scalar Fields

2.1. Possible Results from Canonical Quantization

The conventional version of covariant scalar fields deals with the quantization of models given by the classical Hamiltonian

[2]Similar stories for canonical and spin quantizations appear in [2].

$$H_c(\pi,\varphi)=\int\left\{\frac{1}{2}\left[\pi(x)^2+(\vec{\nabla}\varphi)(x)^2+m_0^2\varphi(x)^2\right]+g_0\varphi(x)^p\right\}\mathrm{d}^s x, \tag{8}$$

where p is the (even positive integer) power of the interaction term, s is the (positive integer) number of spatial dimensions (with $n\equiv s+1$ as the number of spacetime dimensions), $m_0^2>0$ is the mass term, and $g_0\ge 0$ is the coupling constant.

Canonical quantization leads to expected results for "free models" (*i.e.* $g_0=0$) and all $n\ge 2$, while "non-free models" (*i.e.* $g_0>0$) require that $p<2n/(n-2)$. The case of $p=4=n$ was determined to "become free" by Monte Carlo studies [9], which probably would also apply to the case $p=6$ and $n=3$. The remaining models, where $p>2n/(n-2)$, are non-reormalizable and, following a perturbation expansion of g_0 there is an infinite number of different, divergent terms; or, if treated as a whole, such models collapse to "free theories" with a vanishing interaction term [10] [11].

Briefly summarized, canonical quantization leads to unacceptable results whenever $p>2n/(n-2)$. On the other hand, a classical analysis of cases where $p>2n/(n-2)$ leads to natural and expected results.

We now show how models for which $p>2n/(n-2)$ can be successfully quantized using affine quantization rather than canonical quantization.

2.2. Possible Results from Affine Quantization

The classical Hamiltonian in (8) is the same starting point, except that we require that $\varphi(x)\ne 0$ and replace the momentum field $\pi(x)$ with the affine field $\kappa(x)\equiv\pi(x)\varphi(x)$, which leads to the affine version of the classical Hamiltonian given by

$$H_c'(\kappa,\varphi)=\int\left\{\frac{1}{2}\left[\kappa^2(x)\varphi(x)^{-2}+(\vec{\nabla}\varphi)(x)^2+m_0^2\varphi(x)^2\right]+g_0\varphi(x)^p\right\}\mathrm{d}^s x, \tag{9}$$

and the parameters p, s, $m_0^2>0$, and $g_0\ge 0$ have the same meaning as before. The Poisson bracket $\{\varphi(x),\kappa(x')\}=\delta^s(x-x')\varphi(x)$, with $\varphi(x)\ne 0$ (see footnote 1), points toward the commutator $[\hat{\varphi}(x),\hat{\kappa}(x')]=i\hbar\delta^s(x-x')\hat{\varphi}(x)$, with $\hat{\varphi}(x)\ne 0$.

The Schrödinger representation is $\hat{\varphi}(x)=\varphi(x)\ne 0$ and

$$\hat{\kappa}(x)=-i\frac{1}{2}\hbar\left[\varphi(x)(\delta/\delta\varphi(x))+(\delta/\delta\varphi(x))\varphi(x)\right], \tag{10}$$

which leads to an affine Schrödinger quantization of the classical affine Hamiltonian given by

$$\mathcal{H}'(\hat{\kappa},\hat{\varphi})=\int\left\{\frac{1}{2}\left[\hat{\kappa}(x)\varphi(x)^{-2}\hat{\kappa}(x)+(\vec{\nabla}\varphi)(x)^2+m_0^2\varphi(x)^2\right]+g_0\varphi(x)^p\right\}\mathrm{d}^s x, \tag{11}$$

and which appears to be only a "formal representation and equation", since it is true that $\delta\varphi(x')/\delta\varphi(x)=\delta^s(x'-x)$, leads to ∞ if $x'=x$.

These functional derivatives are derived from regularized procedures which replace $\varphi(x)$ with a discrete basis that treats all of x as an s-dimensional lattice so $\varphi(x)\to\varphi_{\mathbf{k}}$, and the normal space $x\to\mathbf{k}a$, $\mathbf{k}\in\{\cdots,-1,0,1,2,3,\cdots\}^s$, and $a>0$ is the physical distance between rungs of the lattice. In this regularization,

$$\hat{\kappa}_{\mathbf{k}}=-i\frac{1}{2}\hbar\left[\varphi_{\mathbf{k}}\left(\partial/\partial\varphi_{\mathbf{k}}\right)+\left(\partial/\partial\varphi_{\mathbf{k}}\right)\varphi_{\mathbf{k}}\right]a^{-s}. \tag{12}$$

Additionally, a^s is a tiny physical volume, and ba^s (with $b\simeq 1$) is a tiny dimensionless volume. This expression leads to $\hat{\kappa}_{\mathbf{k}}\varphi_{\mathbf{k}}^{-1/2}=0$, which, in the limit $a\to 0$, leads to $\kappa(x)\varphi(x)^{-1/2}=0$ (see (7)).

This analysis points toward a regularized (r) quantum Hamiltonian given by

$$\begin{aligned}\mathcal{H}_r&=\frac{1}{2}\sum\nolimits_{\mathbf{k}}\hat{\kappa}_{\mathbf{k}}\left(\varphi_{\mathbf{k}}^2\right)^{-\left(1-2ba^s\right)}\hat{\kappa}_{\mathbf{k}}a^s+\frac{1}{2}\sum\nolimits_{\mathbf{k},\mathbf{k}^*}\left(\varphi_{\mathbf{k}^*}-\varphi_{\mathbf{k}}\right)^2a^{s-2}\\&\quad+\frac{1}{2}m_0^2\sum\nolimits_{\mathbf{k}}\varphi_{\mathbf{k}}^2a^s+g_0\sum\nolimits_{\mathbf{k}}\varphi_{\mathbf{k}}^pa^s-E_0,\end{aligned} \tag{13}$$

where $\mathbf{k}^*$ is one positive step forward from the site $\mathbf{k}$ for each of the s nearest lattice sites, in which the site labels may be spatially periodic. Equation (13) is the first example of a regularized Hamiltonian.

A second example of a regularized Hamiltonian is given, with $J_{\mathbf{k},\mathbf{l}}\equiv 1/(2s+1)$ for $\mathbf{l}=\mathbf{k}$ and the $2s$ nearest spacial neighbors to $\mathbf{k}$, by

$$\begin{aligned}\mathcal{H}_r'&=\frac{1}{2}\sum\nolimits_{\mathbf{k}}\hat{\kappa}_{\mathbf{k}}\left(\Sigma_{\mathbf{l}}J_{\mathbf{k},\mathbf{l}}\varphi_{\mathbf{l}}^2\right)^{-\left(1-2ba^s\right)}\hat{\kappa}_{\mathbf{k}}a^s+\frac{1}{2}\sum\nolimits_{\mathbf{k},\mathbf{k}^*}\left(\varphi_{\mathbf{k}^*}-\varphi_{\mathbf{k}}\right)^2a^{s-2}\\&\quad+\frac{1}{2}m_0^2\sum\nolimits_{\mathbf{k}}\varphi_{\mathbf{k}}^2a^s+g_0\sum\nolimits_{\mathbf{k}}\varphi_{\mathbf{k}}^pa^s-E_0'.\end{aligned} \tag{14}$$

A different kind of regularization offers a third regularized Hamiltonian operator given by

$$\begin{aligned}\mathcal{H}_r''&=-\frac{1}{2}\hbar^2a^{-2s}\sum\nolimits_{\mathbf{k}}\frac{\partial^2}{\partial\varphi_{\mathbf{k}}^2}a^s+\frac{1}{2}\sum\nolimits_{\mathbf{k},\mathbf{k}^*}\left(\varphi_{\mathbf{k}^*}-\varphi_{\mathbf{k}}\right)^2a^{s-2}\\&\quad+\frac{1}{2}m_0^2\sum\nolimits_{\mathbf{k}}\varphi_{\mathbf{k}}^2a^s+g_0\sum\nolimits_{\mathbf{k}}\varphi_{\mathbf{k}}^pa^s+\frac{1}{2}\hbar^2\sum\nolimits_{\mathbf{k}}\mathcal{F}_{\mathbf{k}}(\varphi)a^s-E_0.\end{aligned} \tag{15}$$

In this expression, the counterterm is proportional to $\hbar^2$, and specifically is chosen so that

$$\begin{aligned}\mathcal{F}_{\mathbf{k}}(\varphi)&\equiv\frac{a^{-2s}}{\Pi_{\mathbf{l}}\left[\Sigma_{\mathbf{m}}J_{\mathbf{l},\mathbf{m}}\varphi_{\mathbf{m}}^2\right]^{-\left(1-2ba^s\right)/4}}\frac{\partial^2\Pi_{\mathbf{l}}\left[\Sigma_{\mathbf{m}}J_{\mathbf{l},\mathbf{m}}\varphi_{\mathbf{m}}^2\right]^{-\left(1-2ba^s\right)/4}}{\partial\varphi_{\mathbf{k}}^2}\\&=\frac{1}{4}\left(1-2ba^s\right)^2a^{-2s}\left(\sum\nolimits_{\mathbf{l}}\frac{J_{\mathbf{l},\mathbf{k}}\varphi_{\mathbf{k}}}{\left[\Sigma_{\mathbf{m}}J_{\mathbf{l},\mathbf{m}}\varphi_{\mathbf{m}}^2\right]}\right)^2\\&\quad-\frac{1}{2}\left(1-2ba^s\right)a^{-2s}\sum\nolimits_{\mathbf{l}}\frac{J_{\mathbf{l},\mathbf{k}}}{\left[\Sigma_{\mathbf{m}}J_{\mathbf{l},\mathbf{m}}\varphi_{\mathbf{m}}^2\right]}\\&\quad+\left(1-2ba^s\right)a^{-2s}\sum\nolimits_{\mathbf{l}}\frac{J_{\mathbf{l},\mathbf{k}}^2\varphi_{\mathbf{k}}^2}{\left[\Sigma_{\mathbf{m}}J_{\mathbf{l},\mathbf{m}}\varphi_{\mathbf{m}}^2\right]^2}.\end{aligned} \tag{16}$$

2.3. Affine Coherent States for Covariant Scalar Fields

In choosing suitable coherent states we need to deal with the fact that $-\infty<\varphi(x)\neq 0<\infty$ as well as $-\infty<\hat{\varphi}(x)\neq 0<\infty$, where

$$\Pi_x\left[\left(\hat{\varphi}(x)-1\right)+i\hat{\kappa}(x)/\beta\hbar\right]|\beta\rangle=0. \tag{17}$$

The coherent states then become

$$|\pi;\varphi\rangle=\mathrm{e}^{(i/\hbar)\int\pi(x)\hat{\varphi}(x)\mathrm{d}^s x}\mathrm{e}^{-(i/\hbar)\int\ln(|\varphi(x)|)\hat{\kappa}(x)\mathrm{d}^s x}|\beta\rangle, \tag{18}$$

and the semiclassical Hamiltonian is given by

$$\begin{aligned} H(\pi,\varphi)&=\langle\pi;\varphi|\mathcal{H}(\hat{\kappa},\hat{\varphi})|\pi;\varphi\rangle\\ &=\langle\beta|\mathcal{H}\left(\hat{\kappa}(x)+\pi(x)|\varphi(x)|\hat{\varphi}(x),|\varphi(x)|\hat{\varphi}(x)\right)|\beta\rangle\\ &=\langle\beta|\mathcal{H}\left(\hat{\kappa}(x)+\pi(x)\varphi(x)|\hat{\varphi}(x)|,\varphi(x)|\hat{\varphi}(x)|\right)|\beta\rangle\\ &=\mathcal{H}(\pi,\varphi)+\mathcal{O}(\hbar;\pi,\varphi). \end{aligned} \tag{19}$$

For a suitable L it follows that

$$\begin{aligned} \mathrm{d}\sigma(\pi,\varphi)^2&=L\hbar\left[\left\|d|\pi;\varphi\rangle\right\|^2-\left|\langle\pi;\varphi|d|\pi;\varphi\rangle\right|^2\right]\\ &=\int\left\{(\beta\hbar)^{-1}\varphi(x)^2\mathrm{d}\pi(x)^2+(\beta\hbar)\varphi(x)^{-2}\mathrm{d}\varphi(x)^2\right\}\mathrm{d}^s x. \end{aligned} \tag{20}$$

The result is a constant negative curvature, namely $-2/\beta\hbar$, for each and every point *x*.

2.4. Arguments Supporting Non-Renormalizable Behavior

An important feature of many non-renormalizable models is the fact that reducing the intersection term to zero does not return the model to a free theory. This unusual feature can be illustrated on a toy model the basic Hamiltonian of which is given for $-\infty<p,q<\infty$ and $g_0\geq 0$,

$$H(p,q)=\frac{1}{2}\left(p^2+q^2\right)+g_0 q^{-4}, \tag{21}$$

which, if $g_0=0$ appears to be a free harmonic oscillator. However, that is deceptive because if that g_0 is turned on, *i.e.* $g_0>0$, and then turned off, namely $g_0\to 0$, it follows from continuity that $q=0$ is forbidden, namely $-\infty<p<\infty$ but now $-\infty<q\neq 0<\infty$; the result can be called a "pseudofree theory". That may seem to be a tiny change, but the spectrum of the free and the pseudofree quantum theories becomes markedly different. Instead of the free (*f*) theory propagator, which is given by

$$K_f(q'',T;q',0)=\sum_{n=0,1,2,3,\cdots}h_n(q'')h_n(q')\mathrm{e}^{-i(n+1/2)T/\hbar}, \tag{22}$$

where $h_n(q)$ are the Hermite functions, the pseudofree (*pf*) theory propagator is instead given by

$$K_{pf}(q'',T;q',0)=2\theta(q''q')\sum_{n=1,3,5,7,\cdots}h_n(q'')h_n(q')\mathrm{e}^{-i(n+1/2)T/\hbar}, \tag{23}$$

with $\theta(u)=1$ if $u>0$, while $\theta(u)=0$ if $u<0$. Clearly, a perturbation about the free theory leads to unlimited divergences, while a perturbation about the given pseudofree theory leads to an acceptable approach to study this example. The lesson that this toy model offers is that *domains matter*; the domain here being the set of continuous functions, $\{p(t),q(t)\}_0^T$, $T>0$, for which $\int_0^T H(p(t),q(t))\,\mathrm{d}t<\infty$.

A different example also demonstrates that the quantum theory of a nonrenormalizable model is connected to a pseudofree quantum version and not to its free quantum version. The model in question is that of an ultralocal (u) scalar field, and its affine classical Hamiltonian is given by

$$H'_u(\kappa,\varphi)=\int\left\{\frac{1}{2}\left[\kappa(x)^2\varphi(x)^{-2}+m_0^2\varphi(x)^2\right]+g_0\varphi(x)^p\right\}\mathrm{d}^s x, \tag{24}$$

which differs from (8) because the gradient term is gone. Clearly, for every example with $p>2$, the domain for the interacting version is smaller than the domain for the non-interacting version [14].

The Schrödinger representation involves $\hat{\varphi}(x)\to\varphi(x)\neq 0$ and

$$\hat{\kappa}(x)\to -i\frac{1}{2}\hbar\left[\varphi(x)\left(\delta/\delta\varphi(x)\right)+\left(\delta/\delta\varphi(x)\right)\varphi(x)\right]. \tag{25}$$

Then the regularized quantum Hamiltonian for this model is given by

$$\mathcal{H}'(\hat{\kappa},\hat{\varphi})=\sum_{\mathbf{k}}\left\{\frac{1}{2}\left[\hat{\kappa}_{\mathbf{k}}(\varphi_{\mathbf{k}})^{-2}\hat{\kappa}_{\mathbf{k}}+m_0^2\varphi_{\mathbf{k}}\right]+g_0\varphi_{\mathbf{k}}^p\right\}a^s. \tag{26}$$

With (12) as $\hat{\kappa}_{\mathbf{k}}$, then $\hat{\kappa}_{\mathbf{k}}\varphi_{\mathbf{k}}^{-1/2}=0$. Schrödinger's equation, $i\hbar\partial\psi(\varphi,t)/\partial t=\mathcal{H}'(\hat{\kappa},\hat{\varphi})\psi(\varphi,t)$, and the regularized ground state is given by

$$\psi_0(\varphi)=\mathrm{e}^{-W(\varphi)/2}\Pi_{\mathbf{k}}\left[\left(ba^s\right)^{-1/2}|\varphi_{\mathbf{k}}|^{-\left(1-2ba^s\right)/2}\right], \tag{27}$$

where $W(\varphi)$ is real.

The characteristic function, *i.e.* the Fourier transform of the normalized version of $|\psi_0(\varphi)|^2$ for this model, takes the form

$$\begin{aligned}C(f)&=\lim_{a\to 0}\Pi_{\mathbf{k}}\int\left\{\mathrm{e}^{if_{\mathbf{k}}\varphi_{\mathbf{k}}/\hbar}\mathrm{e}^{-W(\varphi_{\mathbf{k}})}\left(ba^s\right)|\varphi_{\mathbf{k}}|^{-\left(1-2ba^s\right)}\right\}\mathrm{d}\varphi_{\mathbf{k}}\\&=\lim_{a\to 0}\Pi_{\mathbf{k}}\left\{1-\left(ba^s\right)\int\left[1-\mathrm{e}^{if_{\mathbf{k}}\varphi_{\mathbf{k}}/\hbar}\right]\mathrm{e}^{-W(\varphi_{\mathbf{k}})}|\varphi_{\mathbf{k}}|^{-\left(1-2ba^s\right)}\,\mathrm{d}\varphi_{\mathbf{k}}\right\}\\&=\exp\left\{-b\int\mathrm{d}^s x\int\left[1-\mathrm{e}^{if(x)\lambda/\hbar}\right]\mathrm{e}^{-w(\lambda,\hbar)}\,\mathrm{d}\lambda/|\lambda|\right\},\end{aligned} \tag{28}$$

where $\varphi_{\mathbf{k}}\to\lambda$, and w may involve parameter renormalization as well. The result in (28), which, besides a Gaussian distribution, is the only other outcome of the Central Limit Theorem, and is called a (generalized) Poisson distribution. For this solution, as $g_0\to 0$, the factor $w(\lambda,\hbar)\to c\lambda^2$, where $c>0$, which leads to the pseudofree solution for this example.

The example of a field theory without any gradients has led to a well-defined continuum result. This result points to reasonable continuum limits for the earlier models that do have gradients, which will even soften the analysis.

2.5. Computer Studies of Non-Renormalizable Models

A Monte Carlo (MC) study, by Freedman, Smolensky, and Weingarten in 1982 [9], examined two covariant scalar fields of the φ_n^p type, where n is the spacetime dimension. This study for $p, n = 4, 3$ confirmed a proper quantization of that scalar field, and, as well, showed that a proper quantization of a $p, n = 4, 4$ model failed and instead that it led to a free theory. A MC study of a $p, n = 4, 4$ model using the regularized version shown in (15) and (16) has given a hint that such a regularization may offer a positive result. However, such studies can take considerable time and effort. A less time-consuming model of a conventional non-renormalizable model, namely $p, n = 8, 3$ has begun but not yet points to whether or not the same regularized version would be a success or a falure in overcoming its conventional non-renormalizability.

The author of this paper urges additional MC studies by others to see if any of the proposed regularized versions of non-renormalizable models presented in this paper could lead to acceptable quantizations.

3. Canonical and Affine Quantization of Einstein's Gravity

3.1. Canonical Quantization and Einstein's Gravity

Classical general relativity, as defined by Einstein, is a marvelous theory that has proven to be correct in a variety of ways. The standard phase-space variables [15], namely the spacial metric field $g_{ab}(x)$ (symmetric in *ab*) and the spacial momentum field $\pi^{cd}(x)$ (symmetric in *cd*), and where $a, b, c, d = 1, 2, 3$, prove difficult to quantize since the classical metric is strictly positive, e.g. $\mathrm{d}s^2 = g_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b > 0$. Using canonical quantization is limited to a successful result only if all the classical variables can assume arbitrary values between $-\infty$ and $+\infty$. Efforts to get around these difficulties have led to deviations from the original general relativity by adding higher powers of the scalar curvature, adding additional derivatives to the equations of motion, non-commuting spacial variables, as well as factorizing the metric field into the product of two terms, *i.e.* $g_{ab}(x) = E_a^i(x)\delta_{ij}E_b^j(x)$, with $i, j = 1, 2, 3$, and where δ_{ij} is 1 if $i = j$, or is 0 if $i \neq j$; these variables also appear with modest variations. In this case $E_a^i(x)$ obeys the rule to be between $-\infty$ and $+\infty$; but, these rules also allow some $E_a^i(x) = 0$, in which case the metric $g_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b \geq 0$, and fails to be strictly positive. Moreover, choosing (a slight variation of) $E_a^i(x)$ and a natural partner $A_i^a(x)$, which have a constant for their Poisson bracket, become candidate partners to promote to the basic pair of quantum operators. If these two classical variables were also suitable to be Cartesian coordinates, as Dirac has observed [1], then they could be favored variables. Unfortunately, the variables $E_a^i(x)$ and $A_i^a(x)$, which are primary variables in the program of "loop quantum gravity" (see, e.g. [16] [17] [18] [19]), are not suited to be a pair of Cartesian coordinates, which then implies that quantization of these two variables would lead to a false quantization [20].

Moreover, in several ways, loop quantum gravity is different than traditional (*i.e.* canonical or affine) quantization. This is because the loops and their intersection are important and play a significant role, space is also discrete, etc.

On the other hand, affine quantization is very much like canonical quantization, where space is continuous, etc. The only difference is because a chosen classical variable has a limited range of values, leading to a focus on a related variable to promote to an operator.

It is generally accepted that canonical quantization has not yet produced a satisfactory quantization of Einstein's gravity. Let's really see what affine quantization can do.

3.2. Affine Quantization and Einstein's Gravity

In this section we also start with the classical phase-space variables that are used to explore the realm of classical gravity; namely, we again introduce the metric field $g_{ab}(x)$ and the momentum field $\pi^{cd}(x)$ exactly as before. Canonical quantization chooses to promote these two fields to quantum operators, or at least it tries to do that. Affine quantization does not choose these classical variables but replaces the momentum field with the affine field $\pi^a_b(x) \equiv \pi^{ac}(x) g_{bc}(x)$, with an explicit sum on c, and retains the metric field $g_{de}(x)$ alongside the affine field[3].

The standard Poisson bracket for the metric and momentum fields is given by

$$\left\{ g_{ab}(x), \pi^{cd}(x') \right\} = \frac{1}{2}\delta^3(x,x')\left[\delta^c_a\delta^d_b + \delta^d_a\delta^c_b\right], \tag{29}$$

and the Poisson brackets for either two metric fields or two momentum fields would vanish. Instead, the set of Poisson brackets for the metric and affine fields is given by

$$\begin{aligned} &\left\{ \pi^a_b(x), \pi^c_d(x') \right\} = \frac{1}{2}\delta^3(x,x')\left[\delta^a_d \pi^c_b(x) - \delta^c_b \pi^a_d(x)\right], \\ &\left\{ g_{ab}(x), \pi^c_d(x') \right\} = \frac{1}{2}\delta^3(x,x')\left[\delta^c_a g_{bd}(x) + \delta^c_b g_{ad}(x)\right], \\ &\left\{ g_{ab}(x), g_{cd}(x') \right\} = 0. \end{aligned} \tag{30}$$

Observe that these Poisson brackets are true even if we change $g_{ab}(x)$ to $-g_{ab}(x)$, and indeed we can even restrict $\{g_{ab}(x)\} > 0$. This is not possible with the Poisson bracket for the canonical variables.

3.2.1. Affine Coherent States For Gravity

We choose the basic affine operators to build our coherent states for gravity [2]; specifcally,

$$|\pi;\eta\rangle = \mathrm{e}^{(i/\hbar)\int \pi^{ab}(x)\hat{g}_{ab}(x)\mathrm{d}^3x}\,\mathrm{e}^{-(i/\hbar)\int \eta^a_b(x)\hat{\pi}^b_a(x)\mathrm{d}^3x}|\alpha\rangle \quad \left[=|\pi;g\rangle\right]. \tag{31}$$

The fiducial vector $|\alpha\rangle$ has been chosen so that the matrix $\eta(x) \equiv \{\eta^a_b(x)\}$ enters the coherent states solely in the form given by

$$\langle\pi;\eta|\hat{g}_{ab}(x)|\pi;\eta\rangle = \left[\mathrm{e}^{\eta(x)/2}\right]^c_a \langle\alpha|\hat{g}_{cd}(x)|\alpha\rangle\left[\mathrm{e}^{\eta(x)/2}\right]^d_b \equiv g_{ab}(x), \tag{32}$$

which preserves metric positivity, *i.e.* $\{g_{ab}(x)\} > 0$. A companion relation is given by

[3]This section is partially based on [2] [4] [12] [13].

$$\langle \pi;\eta | \hat{\pi}^a_b(x) | \pi;\eta \rangle = \pi^{ac}(x) g_{cb}(x) \equiv \pi^a_b(x), \tag{33}$$

which involves the metric result from (32). These relations permit us to rename the coherent states from $|\pi;\eta\rangle$ to $|\pi;g\rangle$.

As a consequence, the inner product of two gravity coherent states is given by

$$\langle \pi'';g'' | \pi';g' \rangle = \exp\left\{ -2\int b(x) \mathrm{d}^3x \ln\left\{ \frac{\det\left\{ \frac{1}{2}\left[g''^{ab}(x) + g'^{ab}(x) \right] + i\frac{1}{2\hbar} b(x)^{-1}\left[\pi''^{ab}(x) - \pi'^{ab}(x) \right] \right\}}{\det\left[g''^{ab}(x) \right]^{1/2} \det\left[g'^{ab}(x) \right]^{1/2}} \right\} \right\}. \tag{34}$$

Here the scalar density function $b(x) > 0$ ensures the covariance of this expression.

To test whether or not we have "favored coordinates" we examine, with a suitable factor *J*, the Fubini-Study metric given by

$$\begin{aligned} \mathrm{d}\sigma(\pi,g)^2 &\equiv J\hbar\left[\left\| d|\pi;g\rangle \right\|^2 - \left| \langle \pi;g| d|\pi;g\rangle \right|^2 \right] \\ &= \int \left\{ (b(x)\hbar)^{-1} g_{ab}(x) g_{cd}(x) \mathrm{d}\pi^{bc}(x) \mathrm{d}\pi^{da}(x) \right. \\ &\quad \left. + (b(x)\hbar) g^{ab}(x) g^{cd}(x) \mathrm{d}g_{bc}(x) \mathrm{d}g_{da}(x) \right\} \mathrm{d}^3x. \end{aligned} \tag{35}$$

This metric, like the one in the previous section, represents a multiple family of constant negative curvature spaces. The product of coefficients of the differential terms is proportional to a constant rather like the previous affine metric stories. Based on the previous analysis we accept that the basic affine quantum variables have been promoted from basic affine classical variables.

The given choice of coherent states and their quantum operators therein have passed the test to involve constant negative curvature coordinates, which makes them favored affine coordinates for an affine quantization.

3.2.2. Schrödinger's Representation and Equation

Passing to operator commutations, the relations (32) and (33) point toward a promotion of the set of Poisson brackets to operator commutations given by

$$\begin{aligned} \left[\hat{\pi}^a_b(x), \hat{\pi}^c_d(x') \right] &= i\frac{1}{2}\hbar\delta^3(x,x')\left[\delta^a_d \hat{\pi}^c_b(x) - \delta^c_b \hat{\pi}^a_d(x) \right], \\ \left[\hat{g}_{ab}(x), \hat{\pi}^c_d(x') \right] &= i\frac{1}{2}\hbar\delta^3(x,x')\left[\delta^c_a \hat{g}_{bd}(x) + \delta^c_b \hat{g}_{ad}(x) \right], \\ \left[\hat{g}_{ab}(x), \hat{g}_{cd}(x') \right] &= 0. \end{aligned} \tag{36}$$

As with the Poisson brackets, these commutators are valid if we change $\hat{g}_{ab}(x)$ to $-\hat{g}_{ab}(x)$. For the metric and affine fields, we again find that we can choose the subset for which $\{\hat{g}_{ab}(x)\} > 0$.

The classical Hamiltonian for our models is given [15] by

$$H'(\pi,g) = \int \left\{ g(x)^{-1/2}\left[\pi^a_b(x)\pi^b_a(x) - \frac{1}{2}\pi^a_a(x)\pi^b_b(x) \right] + g(x)^{1/2}\, {}^{(3)}R(x) \right\} \mathrm{d}^3x, \tag{37}$$

where ${}^{(3)}R(x)$ is the 3-dimensional Ricci scalar. For the quantum operators we adopt a Schrödinger representation for the basic operators: specifically $\hat{g}_{ab}(x) = g_{ab}(x)$ and

$$\hat{\pi}_b^a(x) = -\frac{1}{2} i\hbar \left[g_{bc}(x)\left(\delta/\delta g_{ac}(x)\right) + \left(\delta/\delta g_{ac}(x)\right) g_{bc}(x) \right]. \tag{38}$$

It follows that the Schrödinger equation is given by

$$\begin{aligned} i\hbar\partial\Psi(\{g\},t)/\partial t = &\left\{ \int \left\{ \left[\hat{\pi}_b^a(x) g(x)^{-1/2} \hat{\pi}_a^b(x) - \frac{1}{2} \hat{\pi}_a^a(x) g(x)^{-1/2} \hat{\pi}_b^b(x) \right] \right.\right. \\ &\left.\left. + g(x)^{1/2}\, {}^{(3)}R(x) \right\} \mathrm{d}^3x \right\} \Psi(\{g\},t), \end{aligned} \tag{39}$$

where $\{g\}$ represents the $\{g_{ab}(x)\}$ matrix field.

Much like the scalar field of Section 2, we expect that the Schrödinger representation of eigenfunctions of the Hamiltonian operator have a "large field behavior" and a "small field behavior", and the Hamiltonian operator eigenfunctions are formally given by $\Psi(\{g\}) = W(\{g\})\left[\Pi_x g(x)^{-1/2}\right]$, where the "small field behavior" is formally obtained by the relation $\hat{\pi}_b^a F(g) = 0$, which implies that $\left[g_{bc}(\partial/\partial g_{ac}) + \frac{1}{2}\delta_b^a \right] F(g) = 0$ and this leads to

$g_{bc} g^{ac} g \,\mathrm{d}F(g)/\mathrm{d}g + \frac{1}{2}\delta_b^a F(g) = 0$, which requires that

$g\,\mathrm{d}F(g)/\mathrm{d}g + \frac{1}{2}F(g) = 0$; hence $F(g) \propto g^{-1/2}$. In summary, we observe that

$$\hat{\pi}_b^a(x) g(x)^{-1/2} = 0, \quad \hat{\pi}_b^a(x) \Pi_y g(y)^{-1/2} = 0. \tag{40}$$

We next insert a brief, but relative, comment about the Hamiltonian operator constraints.

Using (40), the factor $g(x)^{-1/2}$ can be moved to the left in the Hamiltonian density; see (39). This permits changing the Hamiltonian density, essentially by multiplying the Hamiltonian density by $g(x)^{1/2}$, and using that expression to make the result a simpler approach to fulfill the Hamiltonian operator constraints [15] to seek Hilbert space states $\Omega(\{g\})$ such that

$$\left\{ \left[\hat{\pi}_b^a(x) \hat{\pi}_a^b(x) - \frac{1}{2} \hat{\pi}_a^a(x) \hat{\pi}_b^b(x) \right] + g(x)\, {}^{(3)}R(x) \right\} \Omega(\{g\}) = 0. \tag{41}$$

As were the procedures in Section 2.2, we regularize the chosen eigenfunctions by replacing the spacial continuum by a set of $N' < \infty$ points labeled by the usual points $\mathbf{k}a$ and introduce a regularized (r) eigenfunction given by

$$\Psi_r(\{g\}) = W_r(\{g\}) \left\{ \Pi_{\mathbf{k}} \left(ba^3\right)^{1/2} \left[\Sigma_{\mathbf{l}} J_{\mathbf{k},\mathbf{l}} g_{\mathbf{l}} \right]^{-\left(1-ba^3\right)/2} \right\}, \tag{42}$$

where the factors $J_{\mathbf{k},\mathbf{l}}$ are the same factors as in Section 2.2. Because the affine variable complex in (37) is not positive definite, the quantum eigenvalues will, most likely, range over the whole real line.

Thus, $W_r(\{g\})$ will, again most likely, be positive and negative for all eigenfunctions, and we focus attention on an appropriate eigenfunction that is non-zero in the vicinity of very small values of g. Just as in the covariant scalar case, we choose the "large field behavior" of the regularized quantum Hamiltonian operator from the classical Hamiltonian, and we choose the "small field behavior" of the regularized quantum Hamiltonian, *i.e.* the term $\Pi_{\mathbf{k}}\left(ba^3\right)^{1/2}\left[\Sigma_{\mathbf{l}} J_{\mathbf{k},\mathbf{l}} g_{\mathbf{l}}\right]^{-\left(1-ba^3\right)/2}$. Based on Section 2.4, we are led to the regularized form of the quantum Hamiltonian in the Schrödinger density representation given by

$$\mathfrak{H}_r=\sum\nolimits_{\mathbf{k}}\left\{\hat{\pi}^a_{b\mathbf{k}}\mathbf{J}_{\mathbf{k}}(g)\hat{\pi}^b_{a\mathbf{k}}-\frac{1}{2}\hat{\pi}^a_{a\mathbf{k}}\mathbf{J}_{\mathbf{k}}(g)\hat{\pi}^b_{b\mathbf{k}}+g^{1/2}_{\mathbf{k}}\,{}^{(3)}R_{\mathbf{k}}\right\}a^3, \tag{43}$$

where $\mathbf{J}_{\mathbf{k}}(g)\equiv\left[\Sigma_{\mathbf{l}} J_{\mathbf{k},\mathbf{l}} g_{\mathbf{l}}\right]^{-\left(1-ba^3\right)/2}$ and

$$\hat{\pi}^a_{b\mathbf{k}}=-i\frac{1}{2}\hbar\left\{\frac{\partial}{\partial g_{ac\mathbf{k}}}g_{bc\mathbf{k}}+g_{bc\mathbf{k}}\frac{\partial}{\partial g_{ac\mathbf{k}}}\right\}a^{-3}. \tag{44}$$

We have strongly focused on making the Hamiltonian operator well defined so that, when we consider the constraints, we are ensured that the operator will result in the correct properties.

3.3. Enforcing the Constraints

The classical action functional for gravity is given [15] by

$$A=\int_0^T\int\left\{\pi^{ab}(x,t)\dot{g}_{ab}(x,t)-N^a(x,t)\pi^b_{a|b}(x,t)-N(x,t)H(x,t)\right\}\mathrm{d}^3x\mathrm{d}t, \tag{45}$$

where the Lagrange multipliers, the lapse, $N(x,t)$, and the three shifts, $N^a(x,t)$, enforce the classical Hamiltonian constraints, $H(x,t)=0$, and the classical diffeomorphism constraints, $\pi^b_{a|b}(x,t)=0$, for all x & t. Since the classical constraints are first class, the Lagrange multipliers can assume any values in the equations of motion, such as $N(x,t)=1$ and/or $N^a(x,t)=0$. However, in the quantum theory, $H(x,t)$ and $\pi^b_{a|b}(x,t)$ become operators, while $N(x,t)$ and $N^a(x,t)$ remain classical functions.

Let us focus on the regularized classical Hamiltonian constraints, $H_{\mathbf{k}}=0$, for all $\mathbf{k}$, and the three regularized classical diffeomorphism constraints, $\pi^a_{b\mathbf{k}|a}=0$, for all b and $\mathbf{k}$, where $|$ denotes a regularized covariant scalar derivative. The four regularized quantum constraints should follow the classical story as closely as possible, and so, following Dirac, we initially propose that vectors in the physical Hilbert space obey $\mathfrak{H}_{\mathbf{k}}|\Psi\rangle_{phys}=0$ for all $\mathbf{k}$ and $\hat{\pi}^a_{b\mathbf{k}|a}|\Psi\rangle_{phys}=0$ for all b and $\mathbf{k}$, for a "wide class" of non-zero Hilbert space vectors. However, that goal is not possible since, for certain $\mathbf{k}$ and $\mathbf{m}$, $\left[\mathfrak{H}_{\mathbf{k}},\mathfrak{H}_{\mathbf{m}}\right]|\Psi\rangle_{phys}\neq0$ due to quantum second-class constraints. Instead, we choose an appropriate projection operator $\mathbb{E}=\mathbb{E}\left(N'^{-1}\left[\Sigma_{\mathbf{k}}\mathfrak{H}^2_{\mathbf{k}}+\Sigma_{a,\mathbf{k}}\hat{\kappa}^{b2}_{a\mathbf{k}|b}\right]\le\delta(\hbar)^2\right)$, which is adjusted so that the constraints have the smallest, non-vanishing values. If $\langle\Psi|\Phi\rangle$ denotes the inner product in the original, kinematical Hilbert space $\mathcal{H}$, then $\langle\Psi|\mathbb{E}|\Phi\rangle$ denotes the inner product in the reduced, physical Hilbert space $\mathcal{H}_{phys}$; or symbolically stated, $\mathcal{H}_{phys}=\mathbb{E}\mathcal{H}$.

The projection operator $\mathbb{E}$ can be constructed by a suitable functional integral

[21] [22]. In the general case, choosing a set of arbitrary, self-adjoint, constraint operators, C_α, where $\alpha \in \{1,2,\cdots,A\}$, we construct a functional integral given by

$$\mathbb{E}\left(\Sigma_\alpha C_\alpha^2 \le \delta(\hbar)^2\right) = \int \mathbb{T}\mathrm{e}^{-i\int_0^T \Sigma_\alpha C_\alpha \lambda_\alpha(t)\mathrm{d}t} \mathcal{D}R(\lambda), \tag{46}$$

where $\mathbb{T}$ implies a time-ordered integral and $R(\lambda)$ is a suitable weak measure (see [21]) which is dependent only on: 1) the time $T > 0$, 2) the upper limit $\delta(\hbar)^2 \ge 0$, and 3) the number of constraints $A \le \infty$. The measure $R(\lambda)$ is completely independent of the choice of the constraint operators $\{C_\alpha\}_{\alpha=1}^N$!

A Master Constraint Operator

There is an alternative procedure to enforce the quantum constraints as well. Following Thiemann (e.g. [19]), we too can introduce a "Master Constraint Operator" to accommodate the Hamiltonian constraints that the Hamiltonian density $H(x)\Omega(\{g\})$ should vanish. Exploiting the relation (41), we introduce

$$\begin{aligned}\mathcal{M} \equiv \int &\left\{\left[\hat{\pi}_b^a(x)\hat{\pi}_a^b(x) - \frac{1}{2}\hat{\pi}_a^a(x)\hat{\pi}_b^b(x)\right] + g(x)\,{}^{(3)}R(x)\right\} g(x)^{-2} \\ &\times \left\{\left[\hat{\pi}_b^a(x)\hat{\pi}_a^b(x) - \frac{1}{2}\hat{\pi}_a^a(x)\hat{\pi}_b^b(x)\right] + g(x)\,{}^{(3)}R(x)\right\} g(x)^{1/2}\,\mathrm{d}^3x,\end{aligned} \tag{47}$$

and thus $\mathcal{M}\Omega(\{g\}) = 0$ for all vectors in the physical Hilbert space. Indeed, exploiting (40), we can simplify the last equation to become

$$\mathcal{M} = \int \left\{\left[\hat{\pi}_b^a(x)\hat{\pi}_a^b(x) - \frac{1}{2}\hat{\pi}_a^a(x)\hat{\pi}_b^b(x)\right] + g(x)\,{}^{(3)}R(x)\right\}^2 g(x)^{-3/2}\,\mathrm{d}^3x. \tag{48}$$

The other constraints for gravity are the three equations $\pi_{a|b}^b(x) = 0$. We can deal with these constraints by constructing

$$\mathcal{N}' \equiv \int \left\{\pi_{a|b}^b(x) g^{ac}(x) \pi_{c|d}^d(x)\right\} g(x)^{1/2}\,\mathrm{d}^3x. \tag{49}$$

Finally, we can include all constraints in

$$\mathcal{L} \equiv \mathcal{M} + \mathcal{N}'. \tag{50}$$

Physical Hilbert states $\Omega(\{g\})$ are those for which $\mathcal{L}\Omega(\{g\}) = 0$, while $\Omega(\{g\}) \ne 0$.

To offer an example of a few vectors that are in the physical Hilbert space, it helps to reduce the underlying spacial space to a finite level. In that case, the vector $\Omega(\{g\}) = g(x)^{-1/2}$ for which $g(x) = \det(g_{ab}(x))$, where, e.g. $g_{11}(x) = 3.2$, $g_{22}(x) = 1.7$, $g_{33}(x) = 2.4$, and $g_{12}(x) = g_{21}(x) = 0.34$; all other elements are zero. Let us call this particular example $\Omega_1(\{g\})$, namely the first example. A second example is $\Omega_2(\{g\})$, with a different set of constant values, and that type of vector can also lead to $\Omega_a(\{g\}) = 0.8\Omega_1(\{g\}) + 1.2(1+i)\Omega_2(\{g\})$, etc.

Admittedly, these are simple vectors, but nevertheless, they are vectors in the physical Hilbert space. Clearly, more vectors are needed.

4. Conclusions

If the reader can accept that an "harmonic oscillator" for which $0 < q < \infty$ cannot be quantized by canonical quantization but can be quantized by affine quantization (which is demonstrated in [2]), then it is a natural step to examine the affine quantization of non-renormalizable scalar fields and Einstein's gravity, with both not having been generally accepted as being successfully quantized by canonical quantization. Affine quantization used for these same problems offers entirely reasonable solutions, despite their complex results.

For many years the author has recognized the possibilities of affine quantization, which imitate all of the procedures of canonical quantization, but differs only by a different pair of basic quantum operators that also have their roots in appropriate classical theories; a focused lesson regarding affine quantization appears in [4]. Perhaps there are other areas of theoretical physics that could profit from exploiting the power of affine quantization.

The data that support the findings of this study are available within the article [and its supplementary material].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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