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Finite-Time Synchronization of Fractional-Order Chaotic Systems with Different Structures under Stochastic Disturbances

Weiqiu Pan, Tianzeng Li*

School of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, China
Email: 2396269767@qq.com, *litianzeng27@163.com

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Abstract

This paper studies the finite-time synchronization of fractional-order chaotic systems with different structures under parameter disturbance and external disturbance. We put forward a fractional-order controller that can achieve the finite-time synchronization of any-order fractional-order chaotic systems under stochastic disturbances. This controller has good robustness and anti-interference performance. With the concept of the finite-time stability theory given, some judgment criterions for the synchronization of fractional-order chaotic systems are proved. This method can not only make the error systems have a faster convergence rate but also can be implemented in engineering easily. The numerical simulations of two specific examples demonstrate the effectiveness of the method. At the same time, the synchronised time of finite-time synchronization is shorter and faster than the complete synchronization and the time can be adjusted according to the parameters in the controller.

Keywords

Fractional Order, Chaos, Finite-Time Synchronization, Stochastic Disturbance

1. Introduction

Although the development history of fractional and integer calculus is not much different, it turns out that fractional calculus is of great significance to express the model we are studying. This is because it not only has the memory function but also more accurately describes the fractional order dynamic models. Of course, fractional calculus has many other advantages [1] [2] [3]. It's because of these benefits that the theories of fractional calculus are widely used in physics [4], engineering [2], chemistry [5] and other subjects. In recent decades, these fields including information processing [6], secure communication [7], thermal systems [8], robot control and problems [9] have been discovered that many physical processes exhibit fractional order dynamic behavior.

Chaos has achieved tremendous and far-reaching development in many fields after discovering the first chaotic attractor by Lorenz. At present, the research on chaos control and synchronization has spread across many disciplines. In the past 20 years, many methods of chaos control have appeared, such as OGY control [10], drive-response control [11], pulse control [12], sliding mode control [13], active control [14], Lyapunov direct method [15], etc. Taking into account above methods, some scholars have considered the synchronization problem for a class of fractional-order chaotic systems [16] [17] [18] while others only concentrate on the specific chaotic systems [19] [20] [21]. And many scholars have solved the synchronization problem of chaotic systems of any order [22] [23].

Currently, the main research direction is to make the system reach synchronization in infinite time, that is, to consider the asymptotically stable of systems. However, facing the actual situation, we may need a specified time, that is, finite-time synchronization. It not only has a faster convergence rate but also stronger robustness [24]. Keyong S *et al.* used the finite-time synchronization theories to define a nonlinear controller which realizes that the synchronization of four-dimensional fractional order hyperchaotic system [25]. Lingdong Z *et al.* adopted the finite-time stability theory to accomplish the synchronization of hyperchaotic Lorenz systems [26]. In literature [27], a robust non-singular terminal sliding mode controller is proposed for synchronizing two different input nonlinear uncertain chaotic systems. However, these results are aimed at specific chaotic systems. What we need more is a controller that can control all fractional-order chaotic systems. What's more, in real life, chaotic systems do not exist in society in isolation. Of course, there are some disturbances. For example, the signal transmission caused by random fluctuations is a disturbed process. Disturbances can be divided into parameter disturbance [28], external disturbance [29] and internal disturbance [30]. Their presence will make the system unstable and difficult to control. Therefore, some scholars began to consider the impact of disturbance. Literatures [31] [32] [33] have solved the synchronization of fractional-order chaotic systems under random disturbance, but they only considered one of the above three disturbances. If they can consider multiple disturbances, it has more practical significance. Although literatures [34] [35] [36] consider the case of multiple disturbances, the object of their research is the integer order chaotic system.

In response to this situation, we are going to consider the synchronization of fractional-order chaotic systems under two kinds of disturbances, namely parameter disturbances and external disturbances, where these two disturbances are random. The controller we designed has good robustness and anti-interference performance and this method we derived can make the error systems have a faster convergence rate. Many figures provided by the numerical simulations guarantee our theoretical analysis. Meanwhile, conclusions that the synchronised time of finite-time synchronization is shorter and faster than the complete synchronization and the time can be adjusted according to the parameters in the controller are established. The composition of this article is shown below. In chapter 2, we present the definitions, lemmas and stability theories that need to be used. In chapter 3, synchronization conditions are presented. In chapter 4, the numerical simulations proved that our method is very effective. In chapter 5, we have a sum up for this paper.

2. Related Theories of Fractional Order System

2.1. Definitions and Lemmas of Fractional Derivative

Next, we will introduce the Riemann-Liouville (R-L) derivative and the Caputo derivative. When the order α is a negative real number and a positive integer, they are equivalent. The R-L definition is more suitable for theoretical analysis and can simplify the calculation of fractional order derivatives. The definition of Caputa is more suitable for modern engineering and makes the Laplace transformation more concise.

Definition 1. [1] The mathematical expression of Caputo derivative with order α is given as

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{t} (t-\upsilon)^{m-\alpha-1} f^{(m)}(\upsilon)\,\mathrm{d}\upsilon, \tag{1} $$

where $m-1<\alpha<m, m\in\mathbb{Z}^{+}$.

Definition 2. [1] The mathematical expression of Riemann-Liouville derivative with order α is given as

$$ {}_{a}^{RL}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^{m}}{\mathrm{d}t^{m}} \int_{a}^{t} (t-\upsilon)^{m-\alpha-1} f(\upsilon)\,\mathrm{d}\upsilon, \tag{2} $$

where $m-1<\alpha<m, m\in\mathbb{Z}^{+}$.

Lemma 1. [22] When $\boldsymbol{x}(t)\in\mathbb{R}^{n}$ has continuous first derivative, then

$$ {}_{a}D_{t}^{\alpha}\left(\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(t)Q\boldsymbol{x}(t)\right) \le \boldsymbol{x}^{\mathrm{T}}Q_{a}D_{t}^{\alpha}\boldsymbol{x}(t), \tag{3} $$

where $\alpha\in(0,1)$ and Q is an arbitrary n order positive definite matrix.

2.2. Stability Theories of Fractional Order System

Considering that most of the things around us are nonlinear, we write the fractional order nonlinear system to be:

$$ {}_{0}D_{t}^{\alpha}\boldsymbol{x}(t) = \boldsymbol{f}\left(t,\boldsymbol{x}(t)\right), \tag{4} $$

where $\alpha \in (0,1)$, $\boldsymbol{f} = (f_1, f_2, \cdots, f_n)^{\mathrm{T}}$, $\boldsymbol{x}(t) \in \mathbb{R}^n$. And $f : [t_0, \infty] \times \Omega \to \mathbb{R}^n$ meets Lipschitz conditions; the initial value is $x(t_0) = x_0, t_0 \geq 0$. The equilibrium point $\boldsymbol{x}^*$ of (4) can be obtained from $\boldsymbol{f}(\boldsymbol{x}^*) = 0$.

Theorem 1. [37] Suppose that $\mathbb{D} \in \mathbb{R}^n$ is a domain that contains the origin. If there is a locally bounded Lyapunov function $V(t, \boldsymbol{x}(t)) : [t_0, \infty] \times \mathbb{D} \to \mathbb{R}$ which meets the local Lipschitz condition about x adapting to

$$\eta_1\left(\|\boldsymbol{x}(t)\|^a\right) \leq V(t, \boldsymbol{x}(t)) \leq \eta_2\left(\|\boldsymbol{x}(t)\|^{ab}\right),$$

$$ {}_0D_t^\alpha V(t, \boldsymbol{x}(t)) \leq -\eta_3\left(\|\boldsymbol{x}(t)\|^{ab}\right), \tag{5}$$

where $\alpha \in (0,1)$, $a > 0$, $b > 0$, $\eta_i (i = 1,2,3) > 0$, then the system (4) is called Mittag-Leffler stable.

Theorem 2. [37] Suppose that $\mathbb{D} \in \mathbb{R}^n$ is a domain that contains the origin. If there is a locally bounded Lyapunov function $V(t, \boldsymbol{x}(t)) : [t_0, \infty] \times \mathbb{D} \to \mathbb{R}$ which meets the local Lipschitz condition about x adapting to

$$1)\ \eta_1\left(\|\boldsymbol{x}(t)\|^a\right) \leq V(t, \boldsymbol{x}(t)) \leq \eta_2\left(\|\boldsymbol{x}(t)\|^{ab}\right),$$

$$2)\ kV^{1/\beta}(t, \boldsymbol{x}(t)) \leq \eta_3\left(\|\boldsymbol{x}(t)\|^{ab}\right), \tag{6}$$

$$3)\ {}_0D_t^\alpha V(t, \boldsymbol{x}(t)) \leq -\eta_3\left(\|\boldsymbol{x}(t)\|^{ab}\right),$$

where $\alpha \in (0,1)$, $a > 0$, $b > 0$, $k > 0$, $\beta > 1$, $\eta_i (i = 1,2,3) > 0$, the system (4) is called finite-time stable. The stabilization time of the system (4) has the following form

$$T \leq \left(\frac{\beta(\alpha+1)}{k(\beta-1)} V^{(\beta-1)/\beta}(0, x_0)\right)^{1/\alpha}. \tag{7}$$

Corollary 1. It follows from Lemma 1 and Theorem 2 that the system (4) must first satisfy the criterions of the Mittag-Leffler stability. From the conditions (2) and (3) in Theorem 2, we get

$$ {}_0D_t^\alpha V(t, \boldsymbol{x}(t)) \leq -kV^{1/\beta}(t, \boldsymbol{x}(t)). \tag{8}$$

Hence, the system (4) is called finite-time stable if it satisfies (8) and the criterions of the Mittag-Leffler stability.

3. Sufficients Condition for Finite-Time Synchronization

In this chapter, we believe that a small disturbance can make a great change in the orbit of the chaotic system. Therefore, it is reasonable to treat them as bounded. This will also make our theory easier to understand. With the help of Lyapulov function, we obtain our conclusions successfully.

The fractional-order drive-response system with parametric disturbance and external disturbance is demonstrated as follows. The drive system is:

$$ {}_0D_t^\alpha \boldsymbol{x}(t) = (A + \Delta A)\boldsymbol{x}(t) + \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{d}_1(\boldsymbol{x}, t). \tag{9}$$

The response system is:

$$ {}_0D_t^a \boldsymbol{y}(t) = (B+\Delta B)\boldsymbol{y}(t) + \boldsymbol{g}(\boldsymbol{y}(t)) + \boldsymbol{d}_2(\boldsymbol{x},t) + \boldsymbol{u}(t), \tag{10} $$

where A and B are the parameter matrices of the systems; ΔA and ΔB are the parameter interference matrices; $\boldsymbol{f}(\boldsymbol{x}(t))$ and $\boldsymbol{g}(\boldsymbol{y}(t))$ are the nonlinear vectors; $\boldsymbol{d}_1(\boldsymbol{x},t)$ and $\boldsymbol{d}_2(\boldsymbol{x},t)$ are external disturbances.

We suppose the state error among the driving system with response system as $\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{x}(t)$. Subtract (9) from (10) to get the error system:

$$ {}_0D_t^a \boldsymbol{e}(t) = C\boldsymbol{y}(t) - D\boldsymbol{x}(t) + \boldsymbol{g}(\boldsymbol{y}(t)) - \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{d}(\boldsymbol{x},t) + \boldsymbol{u}(t), \tag{11} $$

where $C = B + \Delta B$, $D = A + \Delta A$ and $\boldsymbol{d}(\boldsymbol{x},t) = \boldsymbol{d}_2(\boldsymbol{x},t) - \boldsymbol{d}_1(\boldsymbol{x},t)$.

For the sake of ensuring our conclusion more realistic, we need to make the following assumptions.

Assumption 1. For any $\boldsymbol{x}(t) \in \mathbb{R}^n$, the nonlinear function $\boldsymbol{g}(\boldsymbol{x}(t))$ and $\boldsymbol{f}(\boldsymbol{x}(t))$ are continuous and smooth. That is, there is a constant M adapting to

$$ \left\| \boldsymbol{f}(\boldsymbol{x}(t)) - \boldsymbol{g}(\boldsymbol{x}(t)) \right\| \le M, \tag{12} $$

where $M > 0$, $\|\cdot\|$ represents the 2-norm of matrix.

Assumption 2. For any $\boldsymbol{x}(t), \boldsymbol{y}(t) \in \mathbb{R}^n$, the nonlinear function $\boldsymbol{g}(\cdot)$ meets the Lipschitz condition, namely

$$ \left\| \boldsymbol{g}(\boldsymbol{y}(t)) - \boldsymbol{g}(\boldsymbol{x}(t)) \right\| \le L \|\boldsymbol{y} - \boldsymbol{x}\|. \tag{13} $$

Remark 1. Regardless of whether the nonlinear functions $\boldsymbol{g}(\cdot)$ and $\boldsymbol{f}(\cdot)$ are about $\boldsymbol{x}$ or $\boldsymbol{y}$, they are bounded because the system state variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are bounded.

Assumption 3. The parameter matrices A and B, the parameter interference matrices ΔA and ΔB, the external disturbances $\boldsymbol{d}_1(\boldsymbol{x},t)$ and $\boldsymbol{d}_2(\boldsymbol{x},t)$ all have a bounded norm, namely

$$ \begin{gathered} \|\Delta A\| + \|\Delta B\| \le p, \\ \|A\| + \|B\| \le q, \\ \left\| \boldsymbol{d}_2(\boldsymbol{x},t) - \boldsymbol{d}_1(\boldsymbol{x},t) \right\| \le l, \end{gathered} \tag{14} $$

where $p > 0$, $q > 0$, $l > 0$.

Theorem 3. When the assumptions 1 - 3 are all satisfied, the systems (9) and (10) are stable for a finite time with the following controller:

$$ \boldsymbol{u}(t) = (A+\Delta A)\boldsymbol{y}(t) - (B+\Delta B)\boldsymbol{x}(t) - \left[m_1\boldsymbol{e} + (m_2+1)\frac{\boldsymbol{e}}{\|\boldsymbol{e}\|} \right], \tag{15} $$

where $m_1 \ge L + p + q$, $m_2 \ge l + M$ and $\|\boldsymbol{e}\| \ne 0$.

Proof. Take the Lyapulov function as

$$ V(t, \boldsymbol{x}(t)) = \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}. \tag{16} $$

The fractional order derivative of the Lyapulov function is

$$
\begin{aligned}
&{}_0D_t^{\alpha}V\left(t,\boldsymbol{x}(t)\right) \le \boldsymbol{e}^{\mathrm{T}}\, {}_0D_t^{\alpha}\boldsymbol{e} \\
&= \boldsymbol{e}^{\mathrm{T}}\left[\left(B+\Delta B\right)\boldsymbol{y}(t) - \left(A+\Delta A\right)\boldsymbol{x}(t) + \boldsymbol{g}\left(\boldsymbol{y}(t)\right) - \boldsymbol{f}\left(\boldsymbol{x}(t)\right)\right. \\
&\quad \left. + \boldsymbol{d}_2(\boldsymbol{x},t) - \boldsymbol{d}_1(\boldsymbol{x},t) + \boldsymbol{u}(t)\right] \\
&= \boldsymbol{e}^{\mathrm{T}}\left[\left(B\boldsymbol{y}(t) - B\boldsymbol{x}(t)\right) + \left(A\boldsymbol{y}(t) - A\boldsymbol{x}(t)\right) + \left(\Delta B\boldsymbol{y}(t) - \Delta B\boldsymbol{x}(t)\right)\right. \\
&\quad \left. + \left(\Delta A\boldsymbol{y}(t) - \Delta A\boldsymbol{x}(t)\right)\right] + \left(\boldsymbol{g}\left(\boldsymbol{y}(t)\right) - \boldsymbol{g}\left(\boldsymbol{x}(t)\right)\right) + \left(\boldsymbol{g}\left(\boldsymbol{x}(t)\right) - \boldsymbol{f}\left(\boldsymbol{x}(t)\right)\right) \\
&\quad + \boldsymbol{d}_2(\boldsymbol{x},t) - \boldsymbol{d}_1(\boldsymbol{x},t) - \left[m_1\boldsymbol{e} + (m_2+1)\frac{\boldsymbol{e}}{\|\boldsymbol{e}\|}\right] \\
&\le \|\boldsymbol{e}\|^2\left(\|B\| + \|A\| + \|\Delta B\| + \|\Delta A\| + L\right) + \|\boldsymbol{e}\|M + l\|\boldsymbol{e}\| \\
&\quad - \|\boldsymbol{e}\| \cdot \left(\left\| m_1\boldsymbol{e} + (m_2+1)\frac{\boldsymbol{e}}{\|\boldsymbol{e}\|}\right\|\right) \\
&= \|\boldsymbol{e}\|^2\left(\|B\| + \|A\| + \|\Delta B\| + \|\Delta A\| + L\right) + \|\boldsymbol{e}\|M + l\|\boldsymbol{e}\| - m_1\|\boldsymbol{e}\|^2 - (m_2+1)\|\boldsymbol{e}\| \quad (17) \\
&\le \|\boldsymbol{e}\|^2(q+p+L) + \|\boldsymbol{e}\|M + l\|\boldsymbol{e}\| - \|\boldsymbol{e}\|^2(L+q+p) - \|\boldsymbol{e}\|(l+M+1) \\
&= -\|\boldsymbol{e}\|.
\end{aligned}
$$

Finally, ${}_0D_t^{\alpha}V\left(t,\boldsymbol{x}(t)\right) < 0$ is obtained. Therefore, we must be able to find α_3 to make ${}_0D_t^{\alpha}V\left(t,\boldsymbol{x}(t)\right) \le -\alpha_3\|x\|^2$ hold. It follows from Lemma 2 that the system (4) is Mittag-Leffler stable. Then, because the formula ${}_0D_t^{\alpha}V\left(t,\boldsymbol{x}(t)\right) \le -\|\boldsymbol{e}\| = -\left(2V\left(t,\boldsymbol{x}(t)\right)\right)^{1/2} = -\sqrt{2}V\left(t,\boldsymbol{x}(t)\right)^{1/2}$ is held, we can obtain that the systems (9) and (10) achieve finite-time synchronization according to Theorem 1. And

$$
T \le \left(\frac{2(\alpha+1)}{\sqrt{2}}V^{1/2}(0,x_0)\right)^{1/\alpha}, \tag{18}
$$

where $\alpha \in (0,1)$.

4. Numerical Simulation

We exemplify a pair of three-dimensional and four-dimensional fractional chaotic systems to affirm the availability of our method. The results indicate that the error variables of the system quickly stabilize in a finite time, and the synchronization time is able to adjust via m_1, m_2. Among them, the influence of m_1 on the synchronization time is small, and we can ignore it.

Example 1. Let the fractional order Chen chaotic system [38] under stochastic disturbances be the drive system

$$
\begin{aligned}
{}_0D_t^{\alpha}x_1 &= (a+\Delta a)(x_2 - x_1) + d_{11}, \\
{}_0D_t^{\alpha}x_2 &= (c+\Delta c - a - \Delta a)x_1 - x_1x_3 + (c+\Delta c)x_2 + d_{12}, \\
{}_0D_t^{\alpha}x_3 &= x_2x_1 - (b+\Delta b)x_3 + d_{13},
\end{aligned} \tag{19}
$$

where $\alpha \in (0,1)$, $a = 35, b = 3, c = 28$ and $\boldsymbol{d}_1 = (d_{11}, d_{12}, d_{13})$.

Let the fractional order Lorenz chaotic system [39] under stochastic disturbances and controller be the response system

$$\begin{aligned}
{}_0D_t^{\alpha} y_1 &= \left(a_1 + \Delta a_1\right)\left(y_2 - y_1\right) + d_{21} + u_1, \\
{}_0D_t^{\alpha} y_2 &= y_1\left(b_1 + \Delta b_1 - y_3\right) - y_2 + d_{22} + u_2, \\
{}_0D_t^{\alpha} y_3 &= y_2 y_1 - \left(c_1 + \Delta c_1\right) y_3 + d_{23} + u_3,
\end{aligned} \tag{20}$$

where $\alpha \in (0,1)$, $a_1 = 10, b_1 = 28, c_1 = 8/3$ and $\boldsymbol{d}_2 = \left(d_{21}, d_{22}, d_{23}\right)$.

When we select the initial value as $(1,-2,-2)$ and the order $\alpha = 0.995$, the driving system (19) appears chaotic attractors which are presented in **Figure 1**. For the response system (20), if we take the initial value $(0,1,-1)$ and the order $\alpha = 0.995$, it appears chaotic attractors which are presented in **Figure 2**. We realize that the trajectories about the state variables of fractional order Chen and Lorenz systems are not synchronized with time without any control. Next, we will verify the effectiveness of our controller.

We use the MATLAB software to obtain the state variable trajectory of the systems (19) and (20) obtaining the following results

$$-21 \le x_1 \le 22, \quad -24 \le x_2 \le 25, \quad 7 \le x_3 \le 38,$$

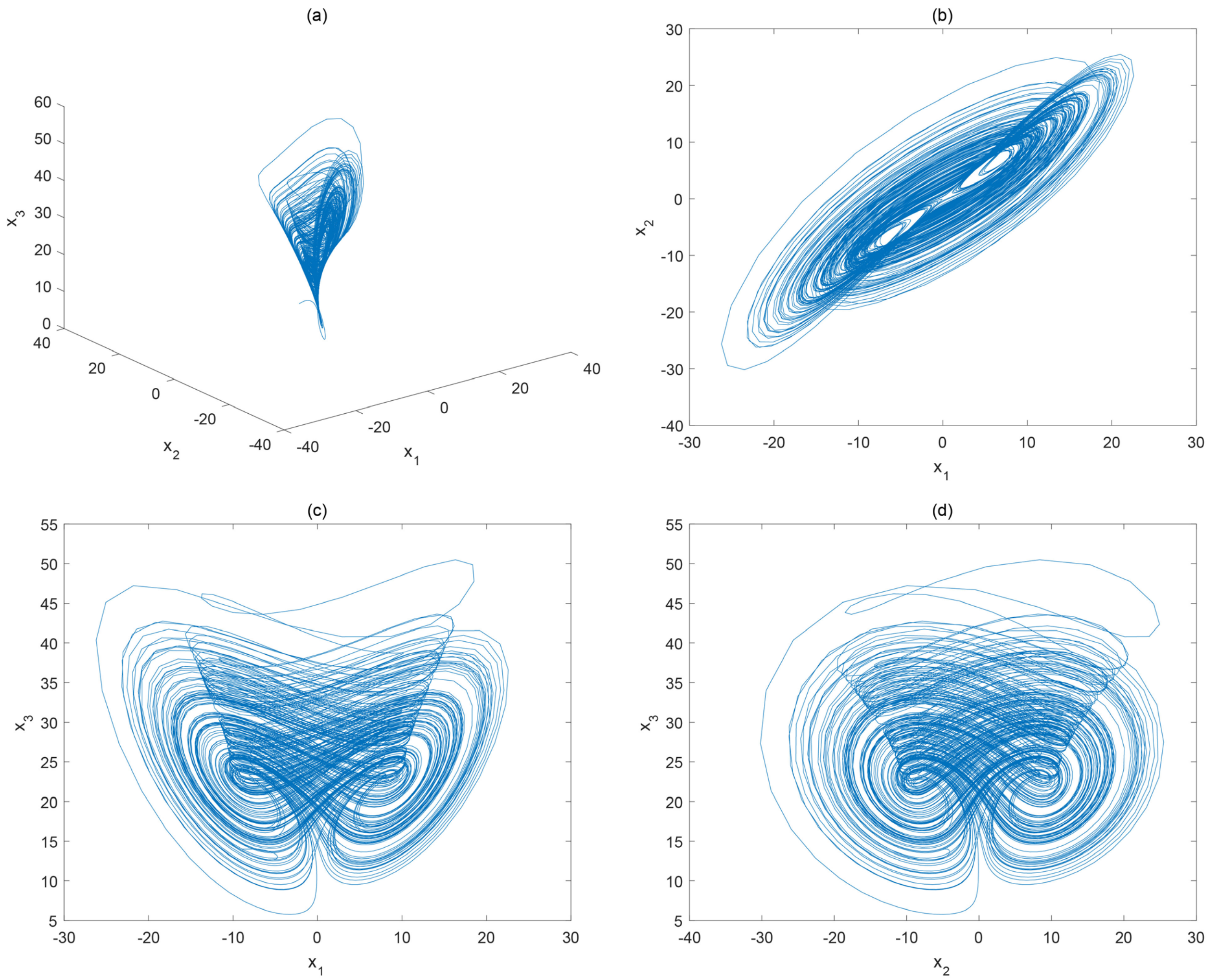

Figure 1. The attractors with respect to fractional order Chen chaotic system choosing $\alpha = 0.995$ and $a = 35, b = 3, c = 28$ show in sub-pictures (a)-(d) respectively which are $x_1 - x_2 - x_3$, $x_1 - x_2$, $x_1 - x_3$ and $x_2 - x_3$.

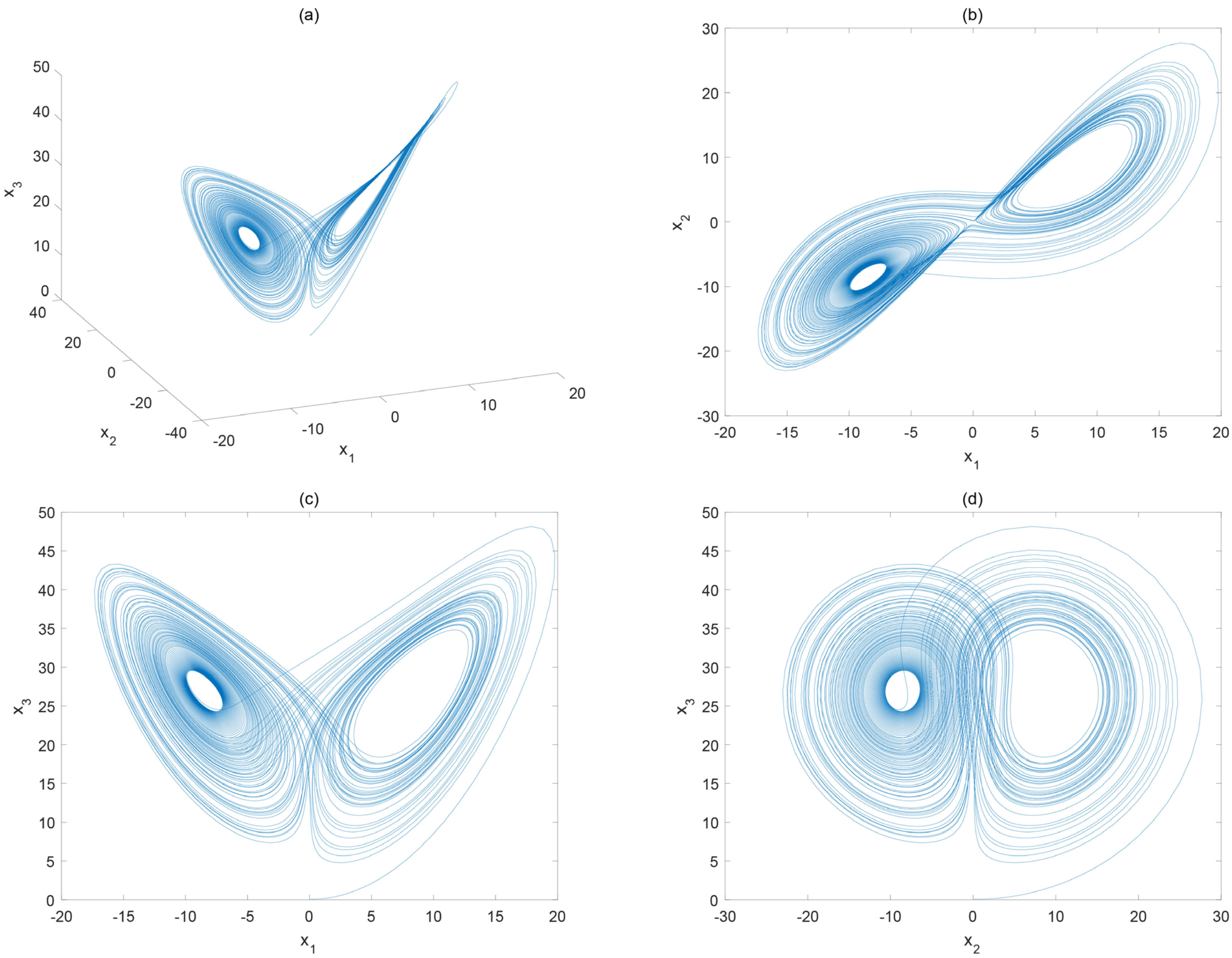

Figure 2. The attractors with respect to fractional order Lorenz chaotic system choosing $\alpha = 0.995$ and $a_1 = 10, b_1 = 28, c_1 = 8/3$ show in sub-pictures (a)-(d) respectively which are $x_1 - x_2 - x_3$, $x_1 - x_2$, $x_1 - x_3$ and $x_2 - x_3$.

$$-11 \le y_1 \le 20,\quad -13 \le y_2 \le 28,\quad 0 \le y_3 \le 49.$$

According to (19) and (20), we get the following matrices. The parameter matrices are

$$A = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix},\quad B = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}. \tag{21}$$

The parameter interference matrices are

$$\begin{aligned} \Delta A &= diag\left(-0.3\sin t, 0.2\sin t, 0.15\sin(3t)\right), \\ \Delta B &= diag\left(-0.1\sin t, 0.2\sin t, 0.3\sin(3t)\right). \end{aligned} \tag{22}$$

The nonlinear vectors of systems are

$$\boldsymbol{f}(\boldsymbol{x}(t)) = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_2x_1 \end{pmatrix},\quad \boldsymbol{g}(\boldsymbol{y}(t)) = \begin{pmatrix} 0 \\ -y_1y_3 \\ y_2y_1 \end{pmatrix}. \tag{23}$$

The external disturbances are

$$\boldsymbol{d}_1(\boldsymbol{x},t)=\begin{pmatrix}0.05x_2\sin t\\0.1x_3\sin(3t)\\0.01x_1\cos(2t)\end{pmatrix},\quad \boldsymbol{d}_2(\boldsymbol{x},t)=\begin{pmatrix}0.6y_2\sin t\\0.9y_3\sin(3t)\\0.8y_1\cos(2t)\end{pmatrix}. \tag{24}$$

It follows from the assumptions 1 - 3 and Theorem 3 that the following calculation results are obtained.

$$\|A\|<58,\ \|B\|<32,\ \|\Delta A\|<0.4,\ \|\Delta B\|<0.4.$$

$$\|\boldsymbol{d}_2(\boldsymbol{x},t)-\boldsymbol{d}_1(\boldsymbol{x},t)\|<46,\ \|\boldsymbol{f}(\boldsymbol{x}(t))-\boldsymbol{g}(\boldsymbol{x}(t))\|=0,$$

$$\|\boldsymbol{g}(\boldsymbol{y}(t))-\boldsymbol{g}(\boldsymbol{x}(t))\|\le 63\|\boldsymbol{y}-\boldsymbol{x}\|.$$

Hence, it follows from Assumption 3 and Theorem 3 that we get

$$L=63,\ p=0.8,\ q=90,\ l=47,\ M=0,\ m_1\ge 153.8,\ m_2\ge 47.$$

We get the controller as

$$\boldsymbol{u}(t)=\begin{pmatrix}(-35-0.3\sin t)y_1+35y_2+(10+0.1\sin t)x_1-10x_2-k_1\\-7y_1+(28+0.2\sin t)y_2-28x_1-(0.2\sin t-1)x_2-k_2\\(-3+0.15\sin(3t))y_3-(0.3\sin(3t)-8/3)x_3-k_3\end{pmatrix}. \tag{25}$$

where $k_i(i=1,2,3)=m_1(y_i-x_i)-m_2(y_i-x_i)/|y_i-x_i|$.

Under the control of the above controller, the synchronised time satisfies $T\le 4.21$ seconds. Numerical simulations show that the (19) and (20) achieve the finite-time synchronization revealing in **Figures 3-5**. Comparing **Figures 3-5**, it's worth noting that the synchronization time gradually decreases as the

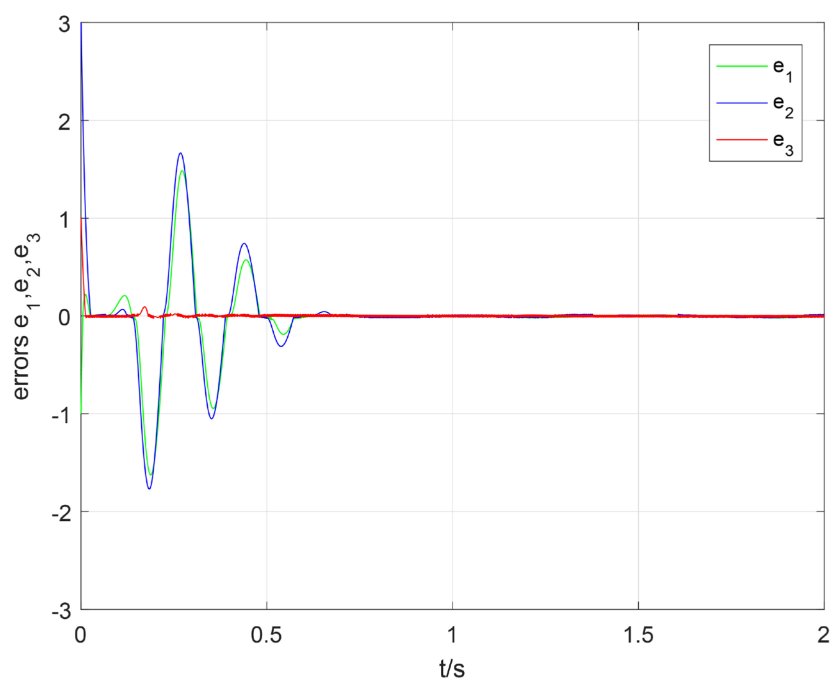

Figure 3. The finite-time synchronization errors e_1,e_2,e_3 change with time t when $m_1=154, m_2=55$. The synchronization time is 0.73 second.

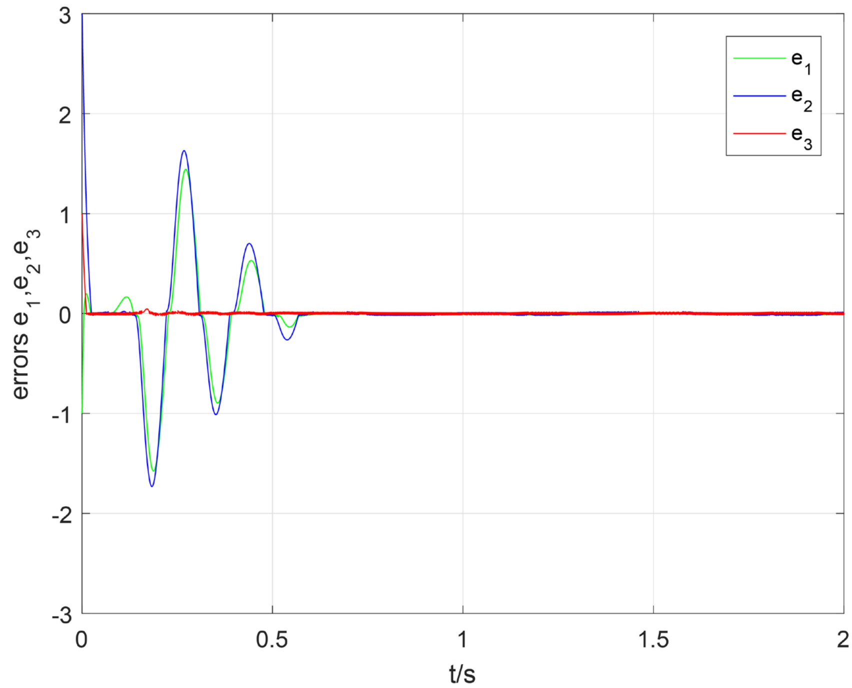

Figure 4. The finite-time synchronization errors e_1, e_2, e_3 change with time t when $m_1 = 154, m_2 = 64$. The synchronization time is 0.58 second.

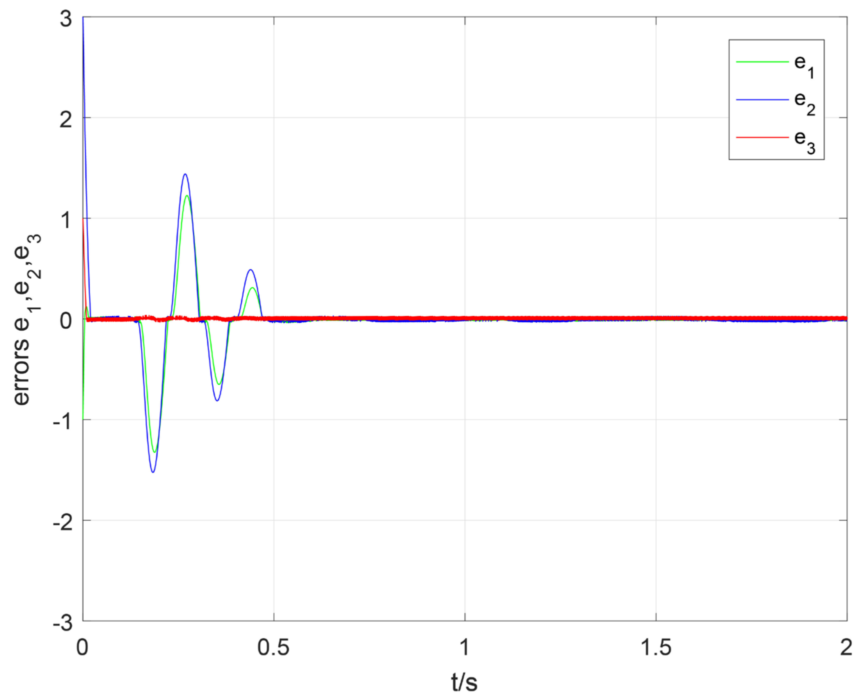

Figure 5. The finite-time synchronization errors e_1, e_2, e_3 change with time t when $m_1 = 154, m_2 = 100$. The synchronization time is 0.46 second.

value of m_2 increases. We can also say that the synchronised time of finite-time synchronization is shorter and faster than the complete synchronization in infinite time, and it can be adjusted according to the m_2.

Example 2. Let the fractional order hyperchaotic Lorenz system [38] under stochastic disturbances be the driving system

$$\begin{aligned} {}_0D_t^\alpha x_1 &= (a+\Delta a)(x_2 - x_1) + x_4 + d_{11}, \\ {}_0D_t^\alpha x_2 &= (c+\Delta c)x_1 - x_1x_3 - x_2 + d_{12}, \\ {}_0D_t^\alpha x_3 &= x_2x_1 - (b+\Delta b)x_3 + d_{13}, \\ {}_0D_t^\alpha x_4 &= -x_2x_3 + (d+\Delta d)x_4 + d_{14}. \end{aligned} \tag{26}$$

where $\alpha \in (0,1)$, $a = 10, b = 8/3, c = -28, d = -1$ and $\boldsymbol{d}_1 = (d_{11}, d_{12}, d_{13}, d_{14})$.

Let the fractional order hyperchaotic Liu chaotic system [40] under stochastic disturbances and controller be the response system

$$\begin{aligned} {}_0D_t^\alpha y_1 &= (a_1+\Delta a_1)(y_2 - y_1) + d_{21} + u_1, \\ {}_0D_t^\alpha y_2 &= -y_1y_3 + (b_1+\Delta b_1)y_1 + y_4 + d_{22} + u_2, \\ {}_0D_t^\alpha y_3 &= (m+\Delta m)y_1^2 - (c_1+\Delta c_1)y_3 + d_{23} + u_3, \\ {}_0D_t^\alpha y_4 &= -(d_1+\Delta d_1)y_1 + d_{24} + u_4. \end{aligned} \tag{27}$$

where $\alpha \in (0,1)$, $a_1 = 10, b_1 = 40, c_1 = 2.5, d_1 = 10, m = 4$ and $\boldsymbol{d}_2 = (d_{21}, d_{22}, d_{23}, d_{24})$.

When we select the initial value as $(11,2,1,-1)$ and the order $\alpha = 0.99$, the driving system (26) appears chaotic attractors which are presented in **Figure 6**. For the response system (27), if we take the initial value $(3,-4,2,1)$ and the order $\alpha = 0.99$, it appears chaotic attractors which are presented in **Figure 7**. We realize that the trajectories about the state variables of fractional order hyperchaotic Lorenz and Liu systems are not synchronized with time without any control. Next, we will verify the effectiveness of our controller.

We use the MATLAB software to obtain the state variable trajectory of the systems (26) and (27) obtaining the following results

$$-20 \le x_1 \le 25, \quad -25 \le x_2 \le 28, \quad 0 \le x_3 \le 47, \quad -124 \le x_4 \le 180,$$

$$-15 \le y_1 \le 20, \quad -35 \le y_2 \le 30, \quad 0 \le y_3 \le 110, \quad -40 \le y_4 \le 40.$$

According to (26) and (27), we get the following matrices. The parameter matrices are

$$A = \begin{pmatrix} -10 & 10 & 0 & 1 \\ -28 & -1 & 0 & 0 \\ 0 & 0 & -8/3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -10 & 10 & 0 & 0 \\ 40 & 0 & 0 & 1 \\ 0 & 0 & -2.5 & 0 \\ -10 & 0 & 0 & 0 \end{pmatrix}, \tag{28}$$

The parameter interference matrices are

$$\begin{aligned} \Delta A &= diag\left(-0.1\sin t, 0.2\sin(2t), 0.3\sin(3t), 0.4\sin(4t)\right), \\ \Delta B &= diag\left(-0.1\cos t, 0.2\cos(2t), 0.3\cos(3t), 0.4\cos(4t)\right). \end{aligned} \tag{29}$$

The nonlinear vectors of systems are

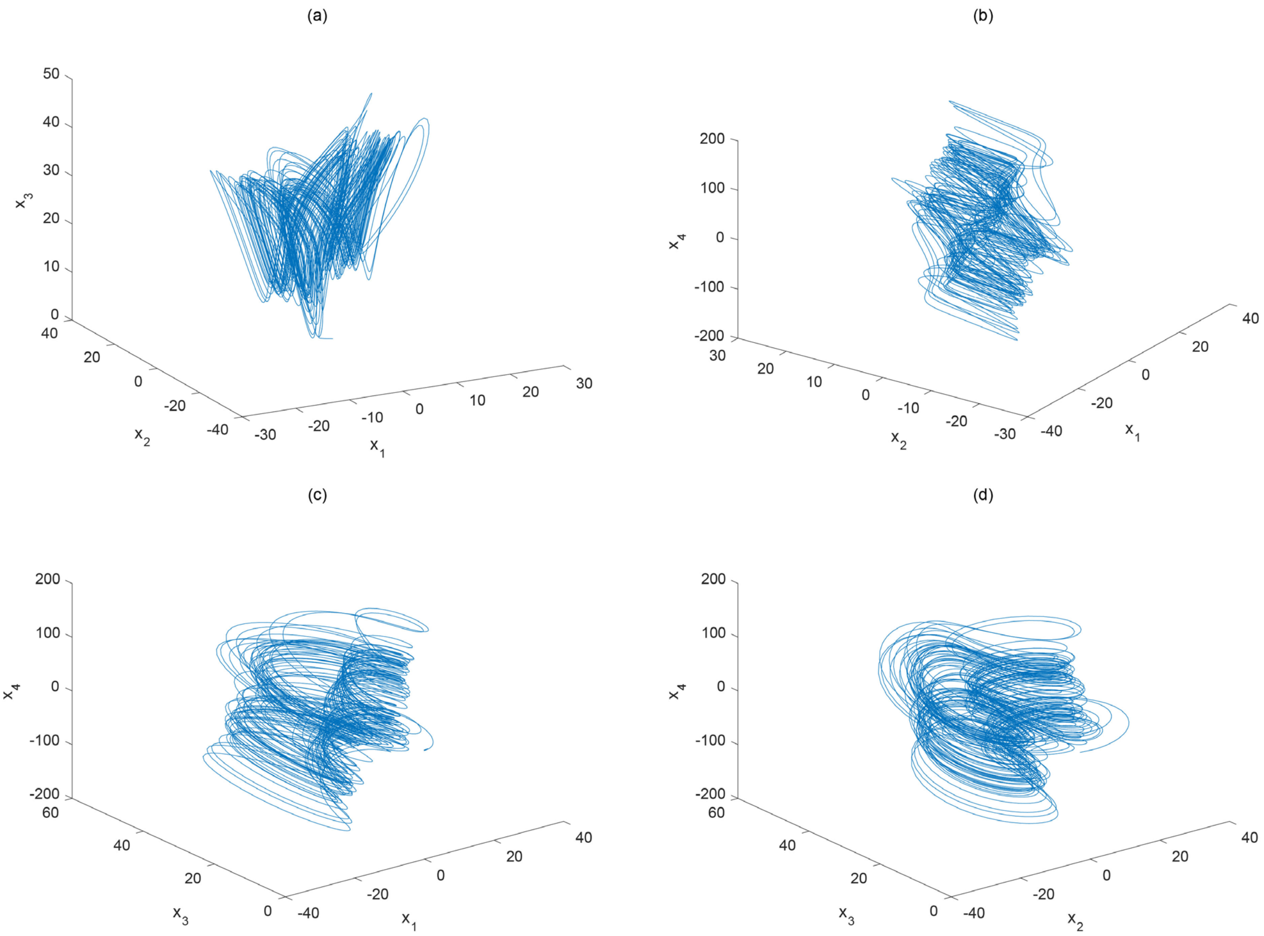

Figure 6. The attractors with respect to fractional order hyper-chaotic Lorenz system choosing $\alpha = 0.99$ and $a = 10, b = 8/3, c = 28, d = -1$ show in sub-pictures (a)-(d) respectively which are $x_1 - x_2 - x_3$, $x_1 - x_2 - x_4$, $x_1 - x_3 - x_4$ and $x_2 - x_3 - x_4$.

$$\boldsymbol{f}\left(\boldsymbol{x}(t)\right) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_2 x_1 \\ -x_2 x_3 \end{pmatrix}, \quad \boldsymbol{g}\left(\boldsymbol{y}(t)\right) = \begin{pmatrix} 0 \\ -y_1 y_3 \\ 4y_1^2 \\ 0 \end{pmatrix}. \tag{30}$$

The external disturbances are

$$\boldsymbol{d}_1\left(\boldsymbol{x}, t\right) = \begin{pmatrix} -0.1\cos t \\ -0.1\cos t \\ -0.1\cos t \\ -0.1\cos t \end{pmatrix}, \quad \boldsymbol{d}_2\left(\boldsymbol{x}, t\right) = \begin{pmatrix} 0.1\cos t \\ 0.1\cos t \\ 0.1\cos t \\ 0.1\cos t \end{pmatrix}. \tag{31}$$

It follows from the assumptions 1 - 3 and Theorem 3 that the following calculation results are obtained.

$$\|A\| < 32, \quad \|B\| < 44, \quad \|\Delta A\| < 0.55, \quad \|\Delta B\| < 0.55.$$

$$\left\|\boldsymbol{d}_2\left(\boldsymbol{x}, t\right) - \boldsymbol{d}_1\left(\boldsymbol{x}, t\right)\right\| < 0.4, \quad \left\|f\left(\boldsymbol{x}(t)\right) - g\left(\boldsymbol{x}(t)\right)\right\| < 3004,$$

$$\left\|g\left(\boldsymbol{y}(t)\right) - g\left(\boldsymbol{x}(t)\right)\right\| \le 213\|\boldsymbol{y} - \boldsymbol{x}\|.$$

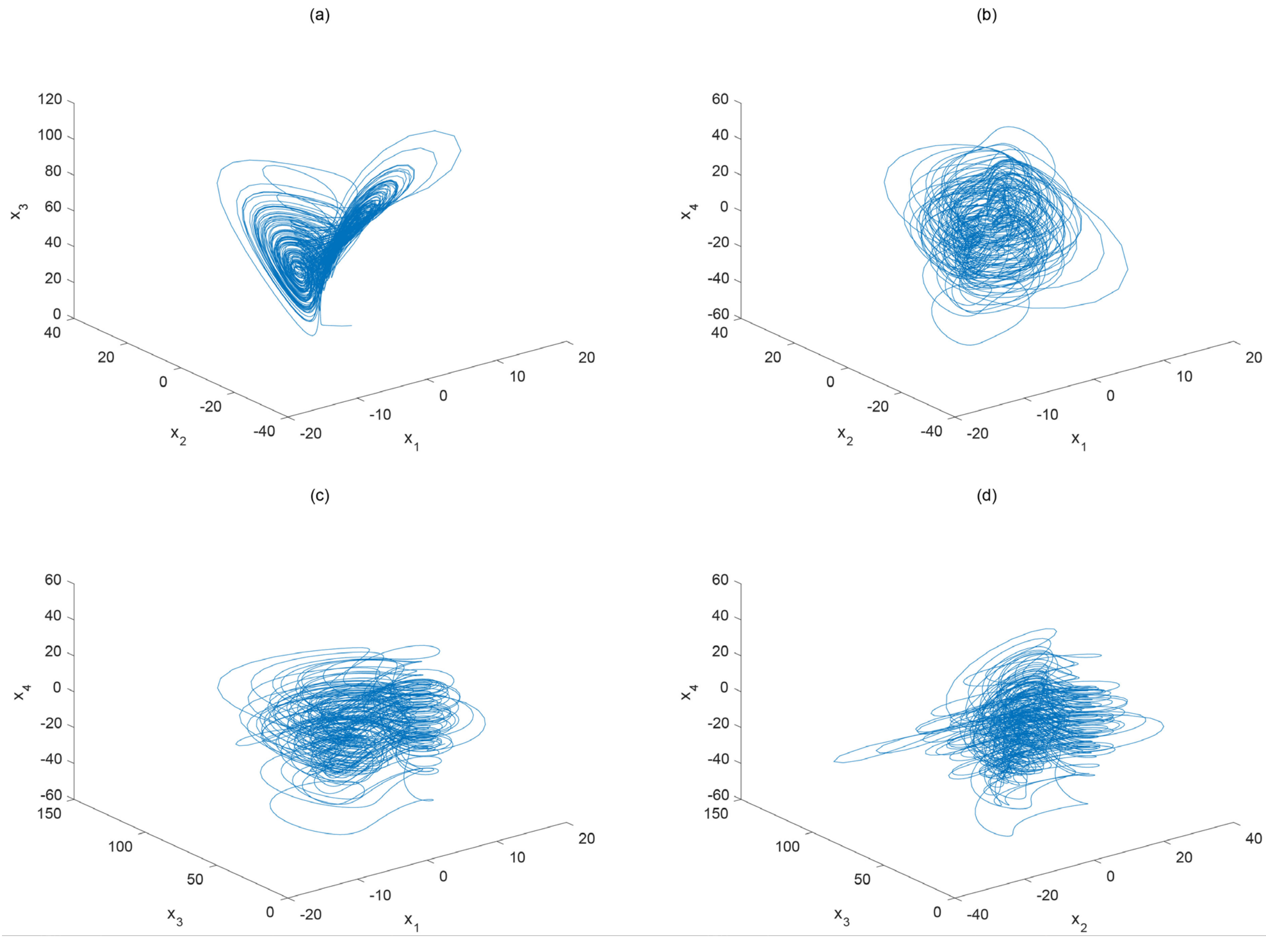

Figure 7. The attractors with respect to fractional order hyperchaotic Liu system choosing $\alpha = 0.99$ and $a_1 = 10, b_1 = 40, c_1 = 2.5, m = 4, d_1 = 10$ show in sub-pictures (a)-(d) respectively which are $x_1 - x_2 - x_3$, $x_1 - x_2 - x_4$, $x_1 - x_3 - x_4$ and $x_2 - x_3 - x_4$.

Hence, it follows from Assumption 3 and Theorem 3 that we get

$$L = 213,\quad p = 1.2,\quad q = 76,\quad l = 0.4,\quad M = 3004,\quad m_1 \geq 290.2,\quad m_2 \geq 3004.4.$$

We get the controller as

$$\boldsymbol{u}(t) = \begin{pmatrix} (-10 - 0.1\sin t)\, y_1 + 10 y_2 + y_4 + (10 + 0.1\cos t)\, x_1 - 10 x_2 - k_1 \\ -28 y_1 + (-1 + 0.2\sin(2t))\, y_2 - 40 x_1 - (0.2\cos(2t))\, x_2 - x_4 - k_2 \\ (-8/3 + 0.3\sin(3t))\, y_3 - (-2.5 + 0.3\cos(3t))\, x_3 - k_3 \\ (-1 + 0.4\sin(4t))\, y_4 + 10 x_1 - (0.4\cos(4t))\, x_4 - k_4 \end{pmatrix}. \quad (32)$$

where $k_i (i = 1,2,3,4) = m_1 (y_i - x_i) - m_2 (y_i - x_i)/|y_i - x_i|$.

Under the control of above controller, the synchronised time satisfies $T \leq 6.47$ seconds. Numerical simulations show that the (26) and (27) achieve the finite-time synchronization revealing in **Figures 8-10**. Comparing **Figures 8-10**, it's worth noting that the synchronization time gradually decreases as the value of m_2 increases. We can also say that the synchronised time of finite-time synchronization is shorter and faster than the complete synchronization in infinite time, and it can be adjusted according to the m_2.

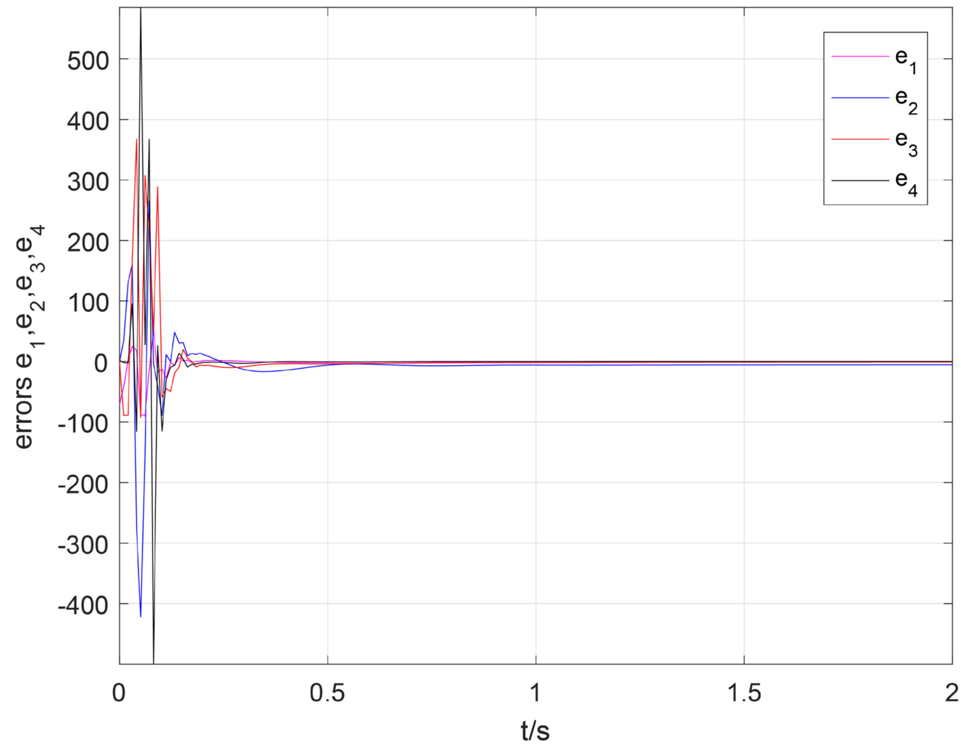

Figure 8. The finite-time synchronization errors e_1, e_2, e_3, e_4 change with time t when $m_1 = 291, m_2 = 3005$. The synchronization time is 0.50 second.

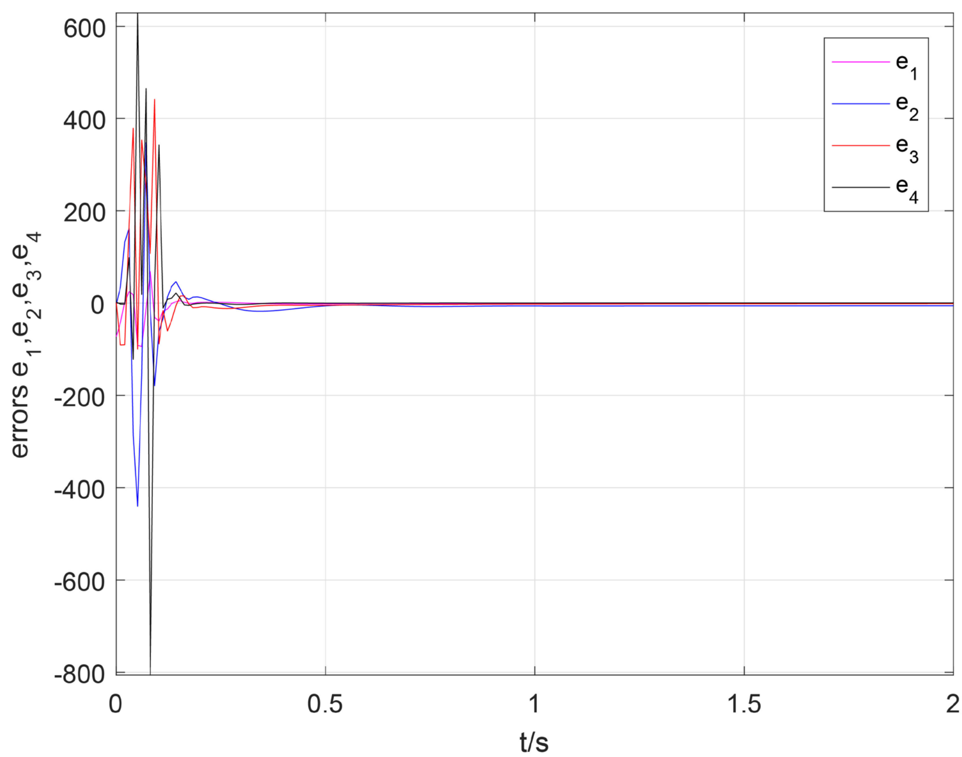

Figure 9. The finite-time synchronization errors e_1, e_2, e_3, e_4 change with time t when $m_1 = 291, m_2 = 3025$. The synchronization time is 0.45 second.

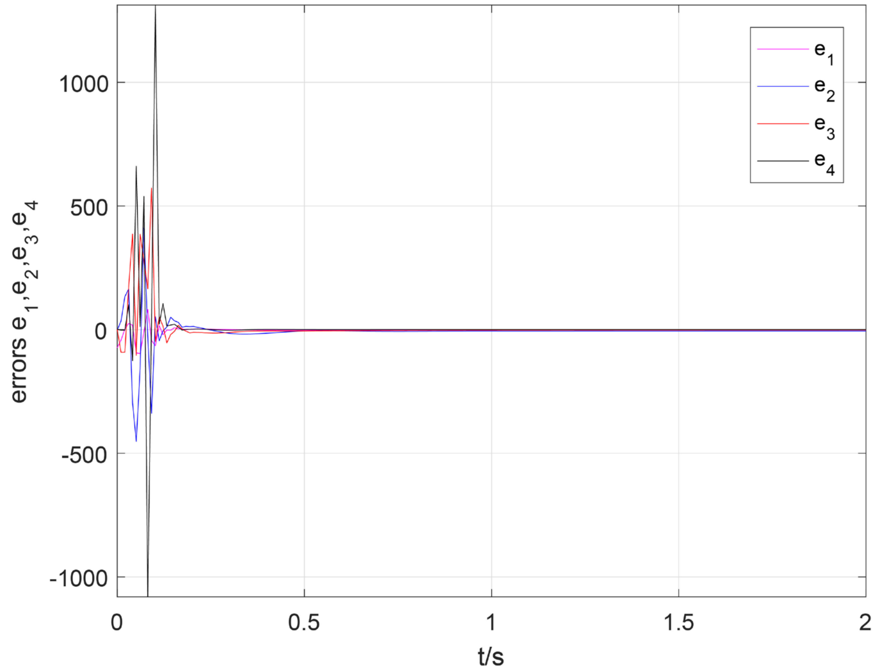

Figure 10. The finite-time synchronization errors e_1, e_2, e_3, e_4 change with time t when $m_1 = 291, m_2 = 3065$. The synchronization time is 0.27 second.

5. Conclusion

In this article, we have studied the finite-time synchronization of fractional-order chaotic systems with different structures under parameter disturbance and external disturbance. With the help of theory of fractional order calculus and the finite-time Lyapunov principle, we put forward a new fractional-order controller which can synchronize any-order fractional-order chaotic system under stochastic disturbances. From the numerical simulation results, it can be seen that the error variables of the systems quickly converges to the equilibrium point in a finite time and the synchronization time gradually decreases with the increase of m_2. Compared with complete synchronization in infinite time, the synchronised time of finite-time synchronization is shorter and faster and the time can be adjusted according to the parameter value in controller. Therefore, this controller is effective and has strong robustness. Next, we will study the time-delay systems under stochastic disturbances, and the system parameters are unknown.

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Authors Contributions

WQ PAN proposed the main the idea and prepared the manuscript initially. TZ LI gave the numerical simulation of this paper.

Availability of Data and Material

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

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