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Exponential Estimation of the Lyapunov Function for Delay-Coupled Neural Networks with Integral Terms

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Abstract

This paper investigates a class of coupled neural networks with delays and addresses the exponential synchronization problem using delay-compensatory impulsive control. Razumikhin-type inequalities involving some destabilizing delayed impulse gains are proposed, along with a new delay-compensatory concept demonstrating two crucial roles in system stability. Based on the constructed inequalities and the introduced delay-compensatory concept, sufficient stability and synchronization criteria for globally exponential synchronization of coupled neural networks are provided. To address the exponential synchronization problem in coupled neural networks. Utilizing delay-compensatory impulsive control and Razumikhin-type inequalities. The Lyapunov function for coupled neural networks with delays and integral terms exhibits exponential estimates.

Keywords

Delayed Impulsive Control, Hybrid System, Synchronization, Delay-Compensatory

1. Introduction

In recent years, impulsive control systems, possessing the dual characteristics of both continuous-time dynamics (piecewise continuous part of the system) and the discrete one (instantaneous jump part of the system), have been widely used for the modeling of physical evolutionary processes showing instantaneous system state changes, such as, cyber- physical systems [1] [2] [3], networked control systems [4] [5] [6], and mechanical systems [7] [8] [9] [10]. The core idea of the impulse control method is to alter system states instantaneously at some specific time so that control information can only be transmitted in some discrete time.

The stability of impulsive control systems, as the basic problem in the field of control theory, has received more and more attention, and much significant work has been presented in the literature [11] [12] [13] [14]. In particular, the exponential synchronization problem of delay-coupled neural networks with delay-compensatory impulsive control has been established. Therefore, this paper mainly investigates whether the Lyapunov function still exhibits exponential estimates when an integral term is added to the system.

2. Problem Formulation and Preliminaries

Notations. Let R and R^+ represent the set of real numbers and the set of nonnegative real numbers, respectively. Z^+ and Z_0^+ represent the set of positive integer numbers and the set of nonnegative integer numbers, respectively. R^n and $R^{n\times m}$ represent the set of n-dimensional real-valued vectors and $n \times m$ dimensional real matrices, respectively. The notation $\lambda_{\max}(\mathbf{A})$ and $\mathbf{A}^{\mathrm{T}}$ denote the largest eigenvalue and the transpose of a matrix $\mathbf{A}=(a_{ij})\in\mathbb{R}^{n\times n}$, respectively. $\mathbf{D}^+$ indicates the upper right-hand Dini derivative and $\otimes$ refers to the Kronecker product. Let $\mathbf{I}$ indicate the identity matrix with appropriate dimensions and $|\cdot|$ the absolute value of a function. $\|\mathbf{x}\|=\sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}}$ is defined for a vector $\mathbf{x}\in\mathbb{R}^n$. Moreover, we denote the norm of matrix $\mathbf{A}$ by $\|\mathbf{A}\|=\sqrt{\lambda_{\max}(\mathbf{A}^{\mathrm{T}}\mathbf{A})}$. We denote by $\mathcal{C}([t_b,t_f],\mathbb{R}^n)$ and $\mathcal{PC}([t_b,t_f],\mathbb{R}^n)$ the set of continuous and piecewise right continuous functions $\phi:[t_b,t_f]\to\mathbb{R}^n$, respectively. Denote $\|\phi(t_0)\|_\tau^{\sup}$ as $\sup_{t_0-\tau\le\theta\le t_0}\|\phi(\theta)\|$. Function expressions are sometimes simplified; for example, one functional $V(t,x(t))$ is denoted by $V(t)$.

Consider a type of delayed coupled neural network with delayed impulses and integral terms:

$$\begin{cases}\dot{x}_i(t)=Dx_i(t)+B_1f(x_i(t))+B_2g(x_i(t-\tau))\\ \qquad\qquad +s\sum_{j=1,j\ne i}^{m}a_{ij}\Gamma(x_j(t)-x_i(t))+\int_{t_0}^{t}\mathrm{e}^{-a(t-s)}g(x_i(s))\mathrm{d}s,\ t\ne t_k,\\ \text{for } (i,j) \text{ satisfying } a_{ij}>0\\ \Delta x_{ij}(t_k)=h_k(x_i(t-k-\tau))-h_k(x_j(t-k-\tau)),\\ x_i(t)=\phi_i(t),\ t\in[t_0-\tau,t_0],\end{cases}\tag{A}$$

where $x_i\in\mathbb{R}^n$ is the ith neuron state, $i=1,2,\cdots,m$. $\Delta x_{ij}(t_k)=x_i(t_k)-x_j(t_k)$, D represents a self-feedback constant matrix, $s>0$ is a coupling gain, and the inner-coupling matrix Γ is positive-definite. $f(x_i)=(f_1(x_i),\cdots,f_n(x_i))^{\mathrm{T}}$ and $g(x_i)=(g_1(x_i),\cdots,g_n(x_i))^{\mathrm{T}}$ are the activation functions on $PC([t_{k-1},t_k],\mathbb{R}^n)$, $\phi_i(t)=[\phi_{i1}(t),\cdots,\phi_{in}(t)]^{\mathrm{T}}$ on $C([t_0-\tau,t_0],\mathbb{R}^{mn})$. The impulse sequence $\{t_k\}$ needs to meet $t_k-t_{k-1}>0$, $\lim_{t\to+\infty}t_k=+\infty$, and $t_k-\tau_k>t_{k-1}$. Suppose that the solution $x(t)$ is precewise continuous, and $x(t^-)$, $x(t^+)$ exist at every time. This paper is devoted to deriving sufficient conditions for system stability under the delayed impulsive controllers $\{t_k,h_k,\tau_k\}$.

Remark 1. *Note that the ith neural network is coupled with the other ones defined by matrix A. Moreover, at the impulse time* t_k, *what changes is the difference between the states of two adjacent neural networks rather than a change in the state of a single neural network itself.*

To solve the synchronization problem of neural networks (*1*), *firstly, construct a new Razumikhin-type differential inequalities with variably delayed impulses*:

$$D+V\left(t,x(t)\right)\leq aV\left(t,x(t)\right), \tag{1}$$

whenever

$$V\left(t-\tau,x(t-\tau)\right)\leq \mathrm{e}^{\eta}V\left(t,x(t)\right),t\neq t_k,k\in\mathbb{Z}^{+}, \tag{2}$$

$$V\left(t_k,x(t_k)\right)\leq \mathrm{e}^{-d_k}V\left(\left(t_k-\tau_k\right)^{-},x\left(\left(t_k-\tau_k\right)^{-}\right)\right), \tag{3}$$

where the continuity of the function $V\left(t,x(t)\right):\mathbb{R}\times\mathbb{R}^{nm}\to\mathbb{R}^{+}$ *is broken by some points* t_k, *at which* $V\left(t_k^{-},x\left(t_k^{-}\right)\right)$ *and* $V\left(t_k^{+},x\left(t_k^{+}\right)\right)$ *exist, and suppose* $V\left(t_k^{+},x\left(t_k^{+}\right)\right)=V\left(t_k^{-},x\left(t_k^{-}\right)\right)$, $a>0$, $d_k\in\mathbb{R}$, *and* $\eta>0$ *with* $\eta=\delta^{*}\left(\tau/T^{*}+N_0\right)+\hat{\delta}_0-\lambda\tau$ *with* δ^{*}, $\hat{\delta}_0$, T^{*}, N_0, *and* λ *defined as the following definitions.*

Definition 1 [15]**.** Let N_{t,t_s} be the number of impulses that the impulse sequence $\{t_k\}$ occurs in the interval $\left(t_s,t\right]$, if

$$\frac{t-t_s}{T^{*}}-N_0\leq N_{t,t_s}\leq\frac{t-t_s}{T^{*}}-N_0$$

we say $T^{\star}>0$ is the average impulsive integral (AII), and $N_0>0$ is a chatter number.

Definition 2 (Delay-Compensatory Condition) [16]**.** If two parameters $\delta^{\star}$ and $\hat{\delta}_0$ meet

$$\delta^{\star}N_{t,t_s}-\hat{\delta}_0\leq\sum_{j=N_{t_s,t_0}+1}^{N_{t,t_0}}d_j+\lambda\tau_j\leq\delta^{\star}N_{t,t_s}+\hat{\delta}_0 \tag{4}$$

Then we say $\delta^{\star}$ is the average parameter of the delay-compensatory based condition, where $\delta^{\star}>0$, and $\hat{\delta}_0>0$ is a chatter number. Also, N_{t,t_s} is defined as above, and $\lambda=\delta^{\star}/T^{\star}-\nu$ with ν representing the decay rate corresponding to exponential stability, which will be defined later.

Suppose at least one impulse occurs in the interval $\left(t_s,t\right]$ in this paper. The newly constructed Razumikhin-type inequality differs from the one proposed by Li *et al.* (refer to reference). The inequalities in Equation (3) are associated with the variable d_k. Therefore, when $d_k<0$, it can be observed that the impulse corresponds to a destabilizing gain. Since $d_k\in\mathbb{R}$, some destabilizing gains can be encompassed within the entire set of impulses set. In addition, the instantaneous jump counterpart may be asynchronous. The main idea of AID is shown in the inequality

$$\delta^{\star}N_{t,t_s}-\hat{\delta}_0\leq\sum_{j=N_{t_s,t_0}+1}^{N_{t,t_0}}d_j+\lambda\tau_j\leq\delta^{\star}N_{t,t_s}+\hat{\delta}_0 \tag{5}$$

Inspired by this idea, a new idea of delay-compensatory is proposed. We construct a delay-compensatory condition for a system with integral terms, aiming to achieve the stabilization effect of pulse delays in unstable delayed systems and simultaneously compensate for the adverse effects caused by some destabilizing gains. In the delay-compensatory condition, the three coefficients d_k associated with impulse gains, the decay rate v corresponding to exponential stability, and impulse delays τ_k are interdependent and integrated into the stability criteria. When the decay rate v is determined, one can design some larger delays τ_k to balance the aforementioned conditions, thereby allowing for the introduction of some destabilizing gains in pulse control.

The coupled network system (a) is said to achieve Globally Exponential Synchronization (GES) if two parameters $M > 0$ and $v > 0$ meet.

$$\left\|\mathbf{x}_i(t)-\mathbf{x}_j(t)\right\| \leq M\mathrm{e}^{-v(t-t_0)}. \tag{6}$$

The following lemma will give the exponential estimate for the differential dynamics (3) and (4).

Assumption 1. There exist matrices $L_1=(l_{1ij})_{n\times n}$ and $L_2=(l_{2ij})_{n\times n}$ such that for any $u=(u_1,\cdots,u_n)^{\mathrm{T}}$ and $v=(v_1,\cdots,v_n)^{\mathrm{T}}$, $\left|f_i(u)-f_i(v)\right| \leq \sum_{j=1}^{n} l_{1ij}\left|u_j-v_j\right|$, and $\left|g_i(u)-g_i(v)\right| \leq \sum_{j=1}^{n} l_{2ij}\left|u_j-v_j\right|$, where $f_i(\cdot)$ and $g_i(\cdot)$ are given in (A).

Theorem 2.1. *Given constants $a>0$, $d_k\in\mathbb{R}$. Suppose there exist positive constants $T^*, N_0, \delta^*, \hat{\delta}_0, v$ that satisfy $\lambda > a$ and the AII condition (4) and the delay-compensatory condition (5) hold, then one exponential estimation can be obtained for inequalities (2) and (3).*

$$V(t) \leq \Gamma_k \mathrm{e}^{\lambda(t-t_0)} V_0,\ t\in[t_k,t_{k+1}),\ k\in\mathbb{Z}_0^+. \tag{7}$$

where $\lambda=\dfrac{\delta^*}{T^*}-v$, $V_0=\sup_{u\in[t_0-\tau,t_0]}V(u)$, $\Gamma_k=\mathrm{e}^{-\delta_k\Gamma_{k-1}}=\mathrm{e}^{-\sum_{j=0}^{k}\delta_j}$ *with* $\delta_k := d_k+\lambda\tau_k$, *and* $d_0=0$, $\tau_0=0$.

Proof. For convenience, construct the following auxiliary inequality:

$$\varphi(t)=\begin{cases} V(t)\mathrm{e}^{-\lambda(t-t_k)}, & \text{if } t\in[t_k,t_{k+1}), \\ V(t), & \text{if } t_0-\tau\leq t\leq t_0. \end{cases} \tag{8}$$

Thus, the proof of inequality (7) is now transformed into the following one:

$$\varphi(t) \leq \Gamma_k \mathrm{e}^{\lambda(t_k-t_0)} V_0,\ t\in[t_k,t_{k+1}). \tag{9}$$

We construct a new function $V(t)$ based on the proof technique from Lemma 1 in Reference [16]. Let the Lyapunov function in the reference be denoted as $V_{\text{original}}(t)$. Then, our new Lyapunov function is given by

$$V(t)=V_{\text{original}}(t)+\rho\int_{t_0}^{t}\mathrm{e}^{-\alpha(t-s)}g^{\mathrm{T}}(x(s))Qg(x(s))\mathrm{d}s. \tag{10}$$

According to the proof of Lemma 1, it can be concluded that $V_{\text{original}}(t)$ satisfies conditions (1), (2), and (3) [16]. Next, we prove that the newly constructed

$V(t)$ satisfies inequality (1).

$$\dot{V}(t)=\dot{V}_{\text{original}}(t)+\rho\Big(g^{\mathrm{T}}(x(s))Qg(x(s))-\alpha\int_{t_0}^{t}\mathrm{e}^{-\alpha(t-s)}g^{\mathrm{T}}(x(s))Qg(x(s))\mathrm{d}s\Big) \quad (11)$$

For the equation:

$$g^{\mathrm{T}}(x(s))Qg(x(s)) \quad (12)$$

$g(x)$ is globally Lipschitz continuous ($g(0) = 0$) Thus, we have

$$\|g(x)-g(0)\|\le L_2\|x-0\| \quad (13)$$

Namely,

$$\|g(x)\|\le L_2\|x\| \quad (14)$$

$$g^{\mathrm{T}}(x(s))Qg(x(s))=\|g(x(t))\|_Q^2 \quad (15)$$

$$\le (L_2\|x\|)_Q^2 \quad (16)$$

$$\le L_2^2\lambda_{\max}(Q)\|x(t)\|^2 \quad (17)$$

There exists c s.t

$$cP\ge \rho L_2^2\lambda_{\max}(Q)I \quad (18)$$

Then

$$X^{\mathrm{T}}(cP)X\ge \rho L_2^2\lambda_{\max}(Q)\|x\|^2 \quad (19)$$

$$\rho L_2^2\lambda_{\max}(Q)\|x\|^2\le cV_{\text{original}}(t) \quad (20)$$

Consider $\dot{V}_{\text{original}}(t)\le aV_{\text{original}}(t)$ (see Reference [16]). Then

$$\dot{V}(t)\le aV_{\text{original}}(t)+cV_{\text{original}}(t)+\rho\int_{t_0}^{t}\mathrm{e}^{-\alpha(t-s)}g^{\mathrm{T}}(x(s))Qg(x(s))\mathrm{d}s \quad (21)$$

$$=(a+c)V_{\text{original}}(t)+\rho\int_{t_0}^{t}\mathrm{e}^{-\alpha(t-s)}g^{\mathrm{T}}(x(s))Qg(x(s))\mathrm{d}s \quad (22)$$

$$\le b\Big(V_{\text{original}}(t)+\int_{t_0}^{t}\mathrm{e}^{-\alpha(t-s)}g^{\mathrm{T}}(x(s))Qg(x(s))\mathrm{d}s\Big) \quad (23)$$

$$=bV(t) \quad (24)$$

where $b=\max\{a,c\}$ Thus, it is shown that (1) holds. Thus, following a similar discussion as in Reference (Equation (10)), it can be proved that $V(t)\le\Gamma_k\mathrm{e}^{\lambda(t-t_0)}V_0$, for $t\in[t_k,t_{k+1})$, $k\in\mathbb{Z}_{0^+}$.

Proof is over. The following theorem states the globally exponential stability for $V(t,x(t))$.

Theorem 2.2. *Given constants* $a>0$, $d_k\in\mathbb{R}$. *Suppose that there exist positive constants* $T^*,N_0,\delta^*,\hat{\delta}_0,\nu$ *such that* $\lambda>a$ *and the AII condition* (4) *and the delay-compensatory condition* (5) *hold, then the system composed of* (*2*) *and* (*3*) *is globally exponentially stable,*

$$V(t)\le MV_0\mathrm{e}^{-\nu(t-t_0)},\ t\in[t_k,t_{k+1}) \quad (25)$$

where $V_0=\sup_{u\in[t_0-\tau,t_0]}V(u)$, *and* $M=\mathrm{e}^{\hat{\delta}_0+\delta^*N_0}$.

The proof follows a similar process as in Reference, which proves inequality (19). Details are omitted here.

3. Conclusion

This paper investigates the exponential estimation of the Lyapunov function for neural network systems with integral terms and pulse delays. It is demonstrated that, under certain parameter conditions, there indeed exists an exponential estimation for the Lyapunov function.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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