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Global Well-Posedness of the Fractional Tropical Climate Model

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Abstract

In this paper, we consider the Cauchy problem of 3-dimensional tropical climate model. This model reflects the interaction and coupling among the barotropic mode u, the first baroclinic mode v of the velocity and the temperature θ. The systems with fractional dissipation studied here may arise in the modeling of geophysical circumstances. Mathematically these systems allow simultaneous examination of a family of systems with various levels of regularization. The aim here is the global strong solution with the least dissipation. By energy estimate and delicate analysis, we prove the existence of global solution under three different cases: first, with the help of damping terms, the global strong solution of the system with $\Lambda^{2\alpha}u$, $\Lambda^{2\beta}v$ and $\Lambda^{2\gamma}\theta$ for $\frac{3}{4}<\alpha=\gamma\le 1$, $\frac{5}{4}\le\beta$; and second, the global strong solution of the system for $1\le\alpha=\beta=\gamma\le\frac{5}{4}$ with damping terms; finally, the global strong solution of the system for $\frac{5}{4}\le\alpha=\beta=\gamma$ without any damping terms, which improve the known existence theory for this system.

Keywords

Tropical Climate Model, Fractional Diffusion, Global Existence

1. Introduction

In this paper, we consider the following tropical climate model with fractional diffusion and nonlinear damping terms

$$\partial_t u+(u\cdot\nabla)u+\mu\Lambda^{2\alpha}u+\nabla p+\operatorname{div}(v\otimes v)+\xi_1|u|^{\rho_1-1}u=0, \tag{1.1}$$

$$\partial_t v + (u \cdot \nabla) v + \nu \Lambda^{2\beta} v + \nabla \theta + (v \cdot \nabla) u + \xi_2 |v|^{\rho_2 - 1} v = 0, \tag{1.2}$$

$$\partial_t \theta + (u \cdot \nabla) \theta + \eta \Lambda^{2\gamma} \theta + \operatorname{div} v + \xi_3 |\theta|^{\rho_3 - 1} \theta = 0, \tag{1.3}$$

$$\operatorname{div} u = 0, \tag{1.4}$$

$$(u, v, \theta)(x, 0) = (u_0, v_0, \theta_0), \tag{1.5}$$

where $x \in \mathbb{R}^3$, the vector fields $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively. The scalar functions $p(x,t)$ and $\theta(x,t)$ represent the pressure and the temperature, respectively. $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\mu \geq 0$, $\nu \geq 0$, $\eta \geq 0$, $\xi_1 \geq 0$, $\xi_2 \geq 0$, $\xi_3 \geq 0$, $\rho_1 \geq 1$, $\rho_2 \geq 1$, $\rho_3 \geq 1$ are real parameters. For $s \in \mathbb{R}$, the fractional Laplacian operator $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ is defined through the Fourier transform

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi).$$

The tropical climate model (1.1) - (1.5) was originally introduced by Frierson-Majda-Pauluis in [1] without any dissipation terms ($\mu = \nu = \eta = 0$) in order to perform a Galerkin truncation to the hydrostatic Boussinesq equations. For more background about the tropical climate model, we refer to [2]. If the effect of temperature is ignored, the system is similar in form to the generalized MHD equation with divergence free condition both on u and v.

Firstly, we recall some global existence results for the tropical climate model without any damping terms. Ye [3] obtained global regularity for a class of 2-dimensional tropical climate model with $\alpha > 0$, $\beta = \gamma = 1$. Li and Titi [4] established the 2-dimensional global well-posedness of strong solution for the system with $\alpha = \beta = 1$, $\eta = 0$ by introducing a combined quantity called pseudo baroclinic velocity. Wan [5] proved the global well-posedness of the classical solutions to the climate model with the dissipation of the first baroclinic model of the velocity and some damping terms ($\mu = 0$, $\nu > 0$, $\eta = 0$, $\beta = 1$). Dong *et al.* [6] investigated the case when $\mu = 0$, $\beta > 1$, $\beta + \gamma > \frac{3}{2}$ or $\frac{3}{2} < \beta \leq 2$, $\mu = \eta = 0$ and obtained global regularity of 2-dimensional tropical climate models in $H^s, s > 2$. Zhu [7] established the regularity for the tropical climate model with $\mu > 0$, $\nu = 0$, $\eta = 0$ and $\alpha \geq \frac{5}{2}$ in $H^s, s = 3$. In [8] the authors analyzed the d-dimensional ($d = 2,3$) tropical climate model with only $\nu \neq 0$. By choosing a class of special initial data (u_0, v_0, θ_0) whose $H^s(\mathbb{R}^d)$ norm can be arbitrarily large and obtained the global smooth solution of d-dimensional tropical climate model. Yu, Li and Yin establish the global regularity for the system with $\mu > 0$, $\nu = 0$, $\eta = 0$ and $\alpha = \beta = \frac{5}{4}$ in [9]. Niu and Wang [10] dealt with the global well-posedness and large-time behavior of the 2D tropical climate model with small initial data for $\alpha = \gamma = 1$, $\beta = 0$ and $\mu > 0$, $\nu = 0$, $\eta = 0$.

Next, we will give some global existence results for the system. The global existence and uniqueness of a strong solution is established provided $\rho_1 \ge 4$ with $\alpha = \beta = \gamma = 1$, $\xi_1 > 0$, $\xi_2 = \xi_3 = 0$ by Yuan and Chen [11]. Yuan and Zhang [12] proved the global regularity assuming that one of the following three condition holds true: 1) $\rho_1, \rho_2 \ge 4$, 2) $\frac{7}{2} \le \rho_1 < 4$, $\rho_2 \ge \frac{5\alpha+7}{2\alpha}$, $\rho_3 \ge \frac{7}{2\alpha-5}$, 3) $3 < \rho_1 \le \frac{7}{2}$, $\rho_2, \rho_3 \ge \frac{7}{2\alpha-5}$ with $\alpha = \beta = \gamma = 1$. Berti, Bisconti and Catania in [13] provided a regularity criterion to obtain the smoothness of the solutions with $\alpha = \beta = \gamma = 1$, $\xi_3 = 0$, $3 \le \rho_1, \rho_2 < 4$, and $(u_0, v_0, \theta_0) \in H^s$, $\frac{3}{2} < s \le 2$.

Since the specific values of μ, ν, η do not play a special role in our discussions, for the sake of simplicity, we set $\mu = \nu = \eta = \xi_1 = \xi_2 = \xi_3 = 1$ in the rest of the paper.

We have proved the local existence of the fractional tropical climate model for $\alpha > \frac{1}{2}$ in H^s, $s > \max\left\{\frac{n}{2} + 1 - 2\alpha, 0\right\}$. When $n = 3$, for our global existence in this paper, because of the shortcoming of the damping terms, we can verify that $\alpha > \frac{3}{4}$ to ensure $\left\{\frac{5}{2} - 2\alpha, 0\right\} < s \le 1$. The local existence for this paper can be established by the procedure of the local existence of the fractional tropical climate model for $\alpha > \frac{1}{2}$ in H^s, $s > \max\left\{\frac{n}{2} + 1 - 2\alpha, 0\right\}$ and Lemma 2.3, thus it is omitted here.

It should be noted that all the above mentioned works for the system require the restriction that at least one of the α, β, γ must greater than or equal to 1. A natural question is that what would happen if they were all less than 1. In this paper, the focus of our work is to discuss the global exitence when all α, β, γ are less than $\frac{5}{4}$. Also, we improved the previous global solution when α, β, γ greater than $\frac{5}{4}$, which is meaningful.

Our main results are stated as follows:

Theorem 1.1. *Assume* $(u_0, v_0, \theta_0) \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$ *If*

$$\frac{3}{4} < \alpha = \gamma \le 1, \frac{5}{4} \le \beta, \frac{4}{2\alpha-1} \le \rho = \rho_1 = \rho_2 = \rho_3,$$

then, for any $T > 0$, the system (1.1) - (1.5) has a global strong solution (u, v, θ) such that

$$u, \theta \in L^\infty\left(0, T; H^1\left(\mathbb{R}^3\right)\right) \cap L^2\left(0, T; H^{1+\alpha}\left(\mathbb{R}^3\right)\right) \cap L^{\rho+1}\left(0, T; H^{\rho+1}\left(\mathbb{R}^3\right)\right),$$

$$v \in L^\infty\left(0, T; H^1\left(\mathbb{R}^3\right)\right) \cap L^2\left(0, T; H^{1+\beta}\left(\mathbb{R}^3\right)\right) \cap L^{\rho+1}\left(0, T; H^{\rho+1}\left(\mathbb{R}^3\right)\right).$$

Theorem 1.2. *Assume* $(u_0, v_0, \theta_0) \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} u_0 = 0$ *If*

$$1 \le \alpha = \beta = \gamma \le \frac{5}{4}, \frac{4}{2\alpha-1} \le \rho = \rho_1 = \rho_2 = \rho_3,$$

then, for any $T>0$, the system (1.1) - (1.5) has a global strong solution (u,v,θ) such that

$$u,v,\theta \in L^{\infty}\left(0,T;H^{1}\left(\mathbb{R}^{3}\right)\right)\cap L^{2}\left(0,T;H^{1+\alpha}\left(\mathbb{R}^{3}\right)\right)\cap L^{\rho+1}\left(0,T;H^{\rho+1}\left(\mathbb{R}^{3}\right)\right),$$

Theorem 1.3. *Assume* $\xi_1=\xi_2=\xi_3=0$ *and* $(u_0,v_0,\theta_0)\in H^3\left(\mathbb{R}^3\right)$ *with* $\operatorname{div} u_0=0$. *If*

$$\frac{5}{4}\le \alpha=\beta=\gamma,$$

then, for any $T>0$, the system (1.1) - (1.5) has a global strong solution (u,v,θ) such that

$$u,v,\theta\in L^{\infty}\left(0,T;H^{3}\left(\mathbb{R}^{3}\right)\right)\cap L^{2}\left(0,T;H^{3+\alpha}\left(\mathbb{R}^{3}\right)\right).$$

Throughout the whole paper, we use $\|\cdot\|_{L^p}$ to denote the $L^p\left(\mathbb{R}^n\right)$ norm. $\dot{H}^s\left(\mathbb{R}^n\right)$ and $H^s\left(\mathbb{R}^n\right)$ denote the homogeneous Sobolev space with the norm $\|u\|_{\dot{H}^s}=\left\|\Lambda^s u\right\|_{L^2}$ and nonhomogeneous Sobolev space with the norm $\|u\|_{H^s}\triangleq\left(\|u\|_{L^2}^2+\left\|\Lambda^s u\right\|_{L^2}^2\right)^{\frac{1}{2}}$, respectively. C denotes a generic positive constant, and it may be different from line to line.

We find that when α,β,γ are relatively large, with the help of dissipative terms, the global existence is relatively easy to obtain. But when all α,β,γ are relatively small, the global existence is not easy to obtain, and it needs to be controlled by damping terms.

2. Preliminaries

We state the Gagliardo-Nirenberg inequality in Lemma 2.1 and the Kato-Ponce type inequality in Lemma 2.2.

Lemma 2.1. ([14]) *Let u belongs to* L^q *in* $\mathbb{R}^n$ *and its derivatives of order m,* $D^m u$, *belong to* L^r, $1\le q,r\le\infty$. *For the derivative* $D^j u$, $0\le j<m$, *the following inequality holds*

$$\left\|D^j u\right\|_{L^p}\le C\left\|D^m u\right\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}, \tag{2.1}$$

where

$$\frac{1}{p}=\frac{j}{n}+\alpha\left(\frac{1}{r}-\frac{m}{n}\right)+(1-\alpha)\frac{1}{q}, \tag{2.2}$$

for all α in the interval

$$\frac{j}{m}\le\alpha\le 1. \tag{2.3}$$

Lemma 2.2 ([15]) *Let* $s>0$, $1<p<\infty$, *then we have*

$$\left\|\Lambda^s(fg)\right\|_{L^p}\le C\left(\|f\|_{L^{p_1}}\left\|\Lambda^s g\right\|_{L^{q_1}}+\|g\|_{L^{p_2}}\left\|\Lambda^s f\right\|_{L^{q_2}}\right) \tag{2.4}$$

with $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{p_2}+\frac{1}{q_2}$.

Lemma 2.3 ([16]) *For* $0\le s\le 1$, $\beta\ge 0$ *and* $\Lambda^{2s}u\in L^{\beta+1}$, *we have*

$$\left\langle \Lambda^s \left(|u|^{\beta-1} u \right), \Lambda^s u \right\rangle \geq C(\beta) \int \left(\Lambda^s |u|^{\frac{\beta+1}{2}} \right)^2 \mathrm{d}x,$$

where

$$C(\beta) = \begin{cases} \dfrac{2}{\beta+1}, & \beta \geq 1, \\ \dfrac{2\beta}{\beta+1}, & 0 < \beta < 1. \end{cases}$$

3. Proof of the Theorem 1.1

Proof. Multiplying (1.1), (1.2), (1.3) respectively by u, v, θ, after integrating by parts and taking the divergence free property into account, we have

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2 \right) + \left\| \Lambda^\alpha u \right\|_{L^2}^2 + \left\| \Lambda^\beta v \right\|_{L^2}^2 + \left\| \Lambda^\alpha \theta \right\|_{L^2}^2 \\ &+ \|u\|_{L^{\rho+1}}^{\rho+1} + \|v\|_{L^{\rho+1}}^{\rho+1} + \|\theta\|_{L^{\rho+1}}^{\rho+1} = 0. \end{aligned} \tag{3.5}$$

Next, applying the operator ∇ to (1.1), (1.2), (1.3) and taking the $L^2(\mathbb{R}^3)$ inner product to the resultants with $(\nabla u, \nabla v, \nabla \theta)$, after integrating by parts, we have

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) + \left\| \Lambda^{1+\alpha} u \right\|_{L^2}^2 + \left\| \Lambda^{1+\beta} v \right\|_{L^2}^2 + \left\| \Lambda^{1+\alpha} \theta \right\|_{L^2}^2 \\ &+ \left\| |u|^{\frac{\rho-1}{2}} \nabla u \right\|_{L^2}^2 + \left\| |v|^{\frac{\rho-1}{2}} \nabla v \right\|_{L^2}^2 + \left\| |\theta|^{\frac{\rho-1}{2}} \nabla \theta \right\|_{L^2}^2 \\ &+ \frac{4(\rho-1)}{(\rho+1)^2} \left(\left\| \nabla |u|^{\frac{\rho+1}{2}} \right\|_{L^2}^2 + \left\| \nabla |v|^{\frac{\rho+1}{2}} \right\|_{L^2}^2 + \left\| \nabla |\theta|^{\frac{\rho+1}{2}} \right\|_{L^2}^2 \right) \\ &= \left\langle (u \cdot \nabla) u, \Delta u \right\rangle + \left\langle \operatorname{div}(v \otimes v), \Delta u \right\rangle + \left\langle (u \cdot \nabla) v, \Delta v \right\rangle \\ &\quad + \left\langle \nabla \theta, \Delta v \right\rangle + \left\langle (v \cdot \nabla) u, \Delta v \right\rangle + \left\langle (u \cdot \nabla) \theta, \Delta \theta \right\rangle + \left\langle \operatorname{div} v, \Delta \theta \right\rangle \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned} \tag{3.6}$$

Integration by parts implies

$$I_4 + I_7 = 0. \tag{3.7}$$

Because of the divergence-free condition of *u*, the estimates for I_1, I_2, I_3 and I_6 are similar, and we take the detailed estimate for I_3 as an example.

For I_3, when $\frac{3}{4} < \alpha \leq 1$, using Kato-Ponce type inequality and Young's inequality, we can get

$$\begin{aligned} |I_3| &= \left| \left\langle (u \cdot \nabla) v, \Delta v \right\rangle \right| = \left| \left\langle \Lambda^{1-\alpha} \left[(u \cdot \nabla) v \right], \Lambda^{1+\alpha} v \right\rangle \right| \leq C \left\| \Lambda^{2-\alpha} (u \otimes v) \right\|_{L^2} \left\| \Lambda^{1+\alpha} v \right\|_{L^2} \\ &\leq C \left(\left\| \Lambda^{2-\alpha} u \right\|_{L^{\frac{2(\rho+1)}{\rho-1}}} \|v\|_{L^{\rho+1}} + \left\| \Lambda^{2-\alpha} v \right\|_{L^{\frac{2(\rho+1)}{\rho-1}}} \|u\|_{L^{\rho+1}} \right) \left\| \Lambda^{1+\alpha} v \right\|_{L^2} \\ &\leq C \|\nabla u\|_{L^2}^{1-\lambda_1} \left\| \Lambda^{1+\alpha} u \right\|^{\lambda_1} \|v\|_{L^{\rho+1}} \left\| \Lambda^{1+\alpha} v \right\|_{L^2} + C \|\nabla v\|_{L^2}^{1-\lambda_1} \left\| \Lambda^{1+\alpha} v \right\|^{\lambda_1} \|u\|_{L^{\rho+1}} \left\| \Lambda^{1+\alpha} v \right\|_{L^2} \\ &\leq \frac{1}{8} \left\| \Lambda^{1+\alpha} u \right\|_{L^2}^2 + \frac{1}{16} \left\| \Lambda^{1+\alpha} v \right\|_{L^2}^2 + C \|v\|_{L^{\rho+1}}^{\frac{2}{1-\lambda_1}} \|\nabla u\|_{L^2}^2 + \frac{1}{16} \left\| \Lambda^{1+\alpha} v \right\|_{L^2}^2 + C \|u\|_{L^{\rho+1}}^{\frac{2}{1-\lambda_1}} \|\nabla v\|_{L^2}^2 \\ &\leq \frac{1}{8} \left\| \Lambda^{1+\alpha} u \right\|_{L^2}^2 + \frac{1}{8} \left\| \Lambda^{1+\alpha} v \right\|_{L^2}^2 + C \|v\|_{L^{\rho+1}}^{\frac{2}{1-\lambda_1}} \|\nabla u\|_{L^2}^2 + C \|u\|_{L^{\rho+1}}^{\frac{2}{1-\lambda_1}} \|\nabla v\|_{L^2}^2, \end{aligned}$$

here we have used the following Gagliardo-Nirenberg inequality

$$\left\|\Lambda^{2-\alpha}u\right\|_{L^{\frac{2(\rho+1)}{\rho-1}}} \le C\left\|\nabla u\right\|_{L^2}^{1-\lambda_1}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^{\lambda_1},\ \lambda_1=\frac{\rho+4}{\alpha(\rho+1)}-1. \tag{3.8}$$

Note that $\lambda_1\in\left[\frac{1-\alpha}{\alpha},1\right]$ implies $\frac{1}{2}\le\alpha\le1$ and $\rho\ge\frac{4-2\alpha}{2\alpha-1}$. Letting $\frac{2}{1-\lambda_1}\le\rho+1$, which means $\rho\ge\frac{4}{2\alpha-1}$.

Therefore, we need $\alpha>\frac{3}{4}$ and $\rho\ge\frac{4}{2\alpha-1}$. Then, we get

$$\left|I_3\right|\le\frac{1}{8}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2+\frac{1}{8}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+C\left\|v\right\|_{L^{\rho+1}}^{\rho+1}\left\|\nabla u\right\|_{L^2}^2+C\left\|u\right\|_{L^{\rho+1}}^{\rho+1}\left\|\nabla v\right\|_{L^2}^2. \tag{3.9}$$

Similarly, for the terms I_1, I_2 and I_6, we can obtain the following estimates

$$\begin{aligned}\left|I_1\right|&=\left|\left\langle(u\cdot\nabla)u,\Delta u\right\rangle\right|\\&\le C\left\|\Lambda^{2-\alpha}u\right\|_{L^{\frac{2(\rho+1)}{\rho-1}}}\left\|u\right\|_{L^{\rho+1}}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}\\&\le\frac{1}{8}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2+C\left\|u\right\|_{L^{\rho+1}}^{\rho+1}\left\|\nabla u\right\|_{L^2}^2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\left|I_2\right|&=\left|\left\langle\operatorname{div}(v\otimes v),\Delta u\right\rangle\right|\\&\le C\left\|\Lambda^{2-\alpha}v\right\|_{L^{\frac{2(\rho+1)}{\rho-1}}}\left\|v\right\|_{L^{\rho+1}}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}\\&\le\frac{1}{8}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2+\frac{1}{8}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+C\left\|v\right\|_{L^{\rho+1}}^{\rho+1}\left\|\nabla v\right\|_{L^2}^2,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\left|I_6\right|&=\left|\left\langle(u\cdot\nabla)\theta,\Delta\theta\right\rangle\right|\\&\le C\left\|\Lambda^{2-\alpha}\theta\right\|_{L^{\frac{2(\rho+1)}{\rho-1}}}\left\|\theta\right\|_{L^{\rho+1}}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}\\&\le\frac{1}{8}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2+\frac{1}{8}\left\|\Lambda^{1+\alpha}\theta\right\|_{L^2}^2+C\left\|\theta\right\|_{L^{\rho+1}}^{\rho+1}\left\|\nabla u\right\|_{L^2}^2+C\left\|u\right\|_{L^{\rho+1}}^{\rho+1}\left\|\nabla\theta\right\|_{L^2}^2.\end{aligned} \tag{3.12}$$

It remains to estimate the term I_5. However, this term can not be treated as above due to the non-divergence free property of *v*, so we estimate I_5 as follows

$$\begin{aligned}\left|I_5\right|&=\left|\left\langle(v\cdot\nabla)u,\Delta v\right\rangle\right|\\&\le\left|\left\langle\operatorname{div}(v\otimes u),\Delta u\right\rangle\right|+\left|\left\langle(\nabla\cdot v)u,\Delta v\right\rangle\right|\\&=I_{51}+I_{52}.\end{aligned}$$

For I_{51}, similarly to 3.9, we obtain

$$\begin{aligned}I_{51}&=\left|\left\langle\Lambda^{1+\alpha}(v\otimes u),\Lambda^{2-\alpha}u\right\rangle\right|\\&\le\frac{1}{8}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2+\frac{1}{16}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+C\left\|v\right\|_{L^{\rho+1}}^{\rho+1}\left\|\nabla u\right\|_{L^2}^2+C\left\|u\right\|_{L^{\rho+1}}^{\rho+1}\left\|\nabla u\right\|_{L^2}^2,\end{aligned}$$

and for I_{52}, we have

$$
\begin{aligned}
I_{52} &\le C\|u\|_{L^2}\|\nabla v\|_{L^{12}}\|\Delta v\|_{L^{\frac{12}{5}}} \le C\|u\|_{L^2}\left\|\Lambda^{1+\frac{5}{4}}v\right\|_{L^2}\|\nabla v\|_{L^2}^{1-\lambda_2}\left\|\Lambda^{1+\beta}v\right\|_{L^2}^{\lambda_2} \\
&\le C\|u\|_{L^2}\left\|\Lambda^{1+\beta}v\right\|_{L^2}\|\nabla v\|_{L^2}^{1-\lambda_2}\left\|\Lambda^{1+\beta}v\right\|_{L^2}^{\lambda_2} \\
&\le \frac{1}{16}\left\|\Lambda^{1+\beta}v\right\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2,
\end{aligned}
$$

where

$$
\|\Delta v\|_{L^{\frac{12}{5}}} \le C\|\nabla v\|_{L^2}^{1-\lambda_2}\left\|\Lambda^{1+\beta}v\right\|_{L^2}^{\lambda_2},
$$

with $\lambda_2 = \frac{5}{4\beta}$ and $\lambda_2 \in \left[\frac{1}{\beta},1\right]$ which implies that $\beta \ge \frac{5}{4}$.

Therefore, we have

$$
|I_5| \le \frac{1}{8}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2 + \frac{1}{8}\left\|\Lambda^{1+\beta}v\right\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 + C\left(\|u\|_{L^{\rho+1}}^{\rho+1} + \|v\|_{L^{\rho+1}}^{\rho+1}\right)\|\nabla u\|_{L^2}^2. \quad (3.13)
$$

Taking the above estimations (3.9) - (3.13) into (3.6), we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) + \left\|\Lambda^{1+\alpha}u^R\right\|_{L^2}^2 + \left\|\Lambda^{1+\beta}v^R\right\|_{L^2}^2 + \left\|\Lambda^{1+\alpha}\theta^R\right\|_{L^2}^2 \\
&+\left\||u|^{\frac{\rho-1}{2}}\nabla u\right\|_{L^2}^2 + \left\||v|^{\frac{\rho-1}{2}}\nabla v\right\|_{L^2}^2 + \left\||\theta|^{\frac{\rho-1}{2}}\nabla\theta\right\|_{L^2}^2 \\
&+\frac{4(\rho-1)}{(\rho+1)^2}\left(\left\|\nabla|u|^{\frac{\rho+2}{2}}\right\|_{L^2}^2 + \left\|\nabla|v|^{\frac{\rho+2}{2}}\right\|_{L^2}^2 + \left\|\nabla|\theta|^{\frac{\rho+2}{2}}\right\|_{L^2}^2\right) \\
&\le C\left(\|u\|_{L^{\rho+1}}^{\rho+1} + \|v\|_{L^{\rho+1}}^{\rho+1} + \|\theta\|_{L^{\rho+1}}^{\rho+1} + 1\right)\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right).
\end{aligned} \quad (3.14)
$$

This completes the proof of the Theorem 1.1 by Gronwall's inequality and energy estimate (3.5). □

4. Proof of the Theorem 1.2

Proof. Multiplying (1.1), (1.2), (1.3) respectively by u, v, θ, after integrating by parts and taking the divergence free property into account, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2\right) + \left\|\Lambda^{\alpha}u\right\|_{L^2}^2 + \left\|\Lambda^{\alpha}v\right\|_{L^2}^2 + \left\|\Lambda^{\alpha}\theta\right\|_{L^2}^2 \\
&+\|u\|_{L^{\rho+1}}^{\rho+1} + \|v\|_{L^{\rho+1}}^{\rho+1} + \|\theta\|_{L^{\rho+1}}^{\rho+1} = 0.
\end{aligned} \quad (4.15)
$$

Next, applying the operator ∇ to (1.1), (1.2), (1.3) and taking the $L^2(\mathbb{R}^3)$ inner product to the resultants with $(\nabla u, \nabla v, \nabla\theta)$, after integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\right) + \left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2 + \left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2 + \left\|\Lambda^{1+\alpha}\theta\right\|_{L^2}^2 \\
&+\left\||u|^{\frac{\rho-1}{2}}\nabla u\right\|_{L^2}^2 + \left\||v|^{\frac{\rho-1}{2}}\nabla v\right\|_{L^2}^2 + \left\||\theta|^{\frac{\rho-1}{2}}\nabla\theta\right\|_{L^2}^2 \\
&+\frac{4(\rho-1)}{(\rho+1)^2}\left(\left\|\nabla|u|^{\frac{\rho+2}{2}}\right\|_{L^2}^2 + \left\|\nabla|v|^{\frac{\rho+2}{2}}\right\|_{L^2}^2 + \left\|\nabla|\theta|^{\frac{\rho+2}{2}}\right\|_{L^2}^2\right) \\
&= \langle (u\cdot\nabla)u, \Delta u\rangle + \langle \operatorname{div}(v\otimes v), \Delta u\rangle + \langle (u\cdot\nabla)v, \Delta v\rangle \\
&\quad + \langle \nabla\theta, \Delta v\rangle + \langle (v\cdot\nabla)u, \Delta v\rangle + \langle (u\cdot\nabla)\theta, \Delta\theta\rangle + \langle \operatorname{div} v, \Delta\theta\rangle \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned} \quad (4.16)
$$

Integration by parts implies

$$I_4 + I_7 = 0. \tag{4.17}$$

We take the detailed estimate for I_3 as an example.

For I_3, when $1 \le \alpha \le \frac{5}{4}$, using Kato-Ponce type inequality, we can get

$$\begin{aligned} |I_3| &= \left|\left\langle (u\cdot\nabla) v, \Delta v\right\rangle\right| \\ &\le \|u\|_{L^{\rho+1}} \|\nabla v\|_{L^{\frac{2(\rho+1)}{\rho-1}}} \|\Delta v\|_{L^2} \\ &\le C\|u\|_{L^{\rho+1}} \|\nabla v\|_{L^2}^{1-\lambda_3} \|\Lambda^{1+\alpha} v\|_{L^2}^{\lambda_3} \|\nabla v\|_{L^2}^{1-\lambda_4} \|\Lambda^{1+\alpha} v\|_{L^2}^{\lambda_4} \\ &\le C\|u\|_{L^{\rho+1}} \|\nabla v\|_{L^2}^{2-\lambda_3-\lambda_4} \|\Lambda^{1+\alpha} v\|_{L^2}^{\lambda_3+\lambda_4} \\ &\le \frac{1}{16}\|\Lambda^{1+\alpha} v\|_{L^2}^2 + C\|u\|_{L^{\rho+1}}^{\frac{2\alpha}{2\alpha-1-\frac{3}{\rho+1}}} \|\nabla v\|_{L^2}^2, \end{aligned}$$

with $\lambda_3 = \frac{3}{\alpha(\rho+1)}$, $\lambda_4 = \frac{1}{\alpha}$. Here, we can yeild $\frac{2\alpha}{2\alpha - 1 - \frac{3}{\rho+1}} \le \rho + 1$.

Then, we have

$$|I_3| \le \frac{1}{16}\|\Lambda^{1+\alpha} v\|_{L^2}^2 + C\|u\|_{L^{\rho+1}}^{\rho+1} \|\nabla v\|_{L^2}^2. \tag{4.18}$$

Due to the non-divergence free condition we have used, so we can obtain other terms using the same way. Then we have

$$\begin{aligned} |I_1| &= \left|\left\langle (u\cdot\nabla) u, \Delta u\right\rangle\right| \\ &\le \frac{1}{16}\|\Lambda^{1+\alpha} u\|_{L^2}^2 + C\|u\|_{L^{\rho+1}}^{\rho+1} \|\nabla u\|_{L^2}^2, \end{aligned} \tag{4.19}$$

$$\begin{aligned} |I_2| &= \left|\left\langle \operatorname{div}(v\otimes v), \Delta u\right\rangle\right| \\ &\le C\|v\|_{L^{\rho+1}} \|\nabla v\|_{L^{\frac{2(\rho+1)}{\rho-1}}} \|\Delta u\|_{L^2} \\ &\le \frac{1}{16}\|\Lambda^{1+\alpha} u\|_{L^2}^2 + \frac{1}{16}\|\Lambda^{1+\alpha} v\|_{L^2}^2 + C\|v\|_{L^{\rho+1}}^{\rho+1}\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right), \end{aligned} \tag{4.20}$$

$$\begin{aligned} |I_5| &\le C\|v\|_{L^{\rho+1}} \|\nabla v\|_{L^{\frac{2(\rho+1)}{\rho-1}}} \|\Delta u\|_{L^2} \\ &\le \frac{1}{16}\|\Lambda^{1+\alpha} u\|_{L^2}^2 + \frac{1}{16}\|\Lambda^{1+\alpha} v\|_{L^2}^2 + C\|v\|_{L^{\rho+1}}^{\rho+1}\left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right), \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} |I_6| &= \left|\left\langle (u\cdot\nabla)\theta, \Delta\theta\right\rangle\right| \\ &\le \frac{1}{16}\|\Lambda^{1+\alpha}\theta\|_{L^2}^2 + C\|u\|_{L^{\rho+1}}^{\rho+1} \|\nabla\theta\|_{L^2}^2. \end{aligned} \tag{4.22}$$

Taking the above estimations (4.17) - (4.22) into (4.16), we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla \theta\|_{L^2}^2\right)+\left\|\Lambda^{1+\alpha}u^R\right\|_{L^2}^2+\left\|\Lambda^{1+\alpha}v^R\right\|_{L^2}^2+\left\|\Lambda^{1+\alpha}\theta^R\right\|_{L^2}^2$$
$$+\left\||u|^{\frac{\rho-1}{2}}\nabla u\right\|_{L^2}^2+\left\||v|^{\frac{\rho-1}{2}}\nabla v\right\|_{L^2}^2+\left\||\theta|^{\frac{\rho-1}{2}}\nabla \theta\right\|_{L^2}^2$$
$$+\frac{4(\rho-1)}{(\rho+1)^2}\left(\left\|\nabla|u|^{\frac{\rho+2}{2}}\right\|_{L^2}^2+\left\|\nabla|v|^{\frac{\rho+2}{2}}\right\|_{L^2}^2+\left\|\nabla|\theta|^{\frac{\rho+2}{2}}\right\|_{L^2}^2\right)$$
$$\le C\left(\|u\|^{\rho+1}+\|v\|_{L^{\rho+1}}^{\rho+1}\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla \theta\|_{L^2}^2\right).$$

This completes the proof of the Theorem 1.2 by Gronwall's inequality and energy estimate (4.15). ☐

5. Proof of the Theorem 1.3

Proof. Multiplying (1.1), (1.2), (1.3) respectively by u,v,θ, after integrating by parts and taking the divergence free property into account, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2\right)+\left\|\Lambda^{\alpha}u\right\|_{L^2}^2+\left\|\Lambda^{\alpha}v\right\|_{L^2}^2+\left\|\Lambda^{\alpha}\theta\right\|_{L^2}^2=0. \tag{5.23}$$

For the H^1 *-estimates*:

Next, applying the operator ∇ to (1.1), (1.2), (1.3) and taking the $L^2(\mathbb{R}^3)$ inner product to the resultants with $(\nabla u,\nabla v,\nabla\theta)$, after integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla \theta\|_{L^2}^2\right)+\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2+\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+\left\|\Lambda^{1+\alpha}\theta\right\|_{L^2}^2\\&=\langle(u\cdot\nabla)u,\Delta u\rangle+\langle\operatorname{div}(v\otimes v),\Delta u\rangle+\langle(u\cdot\nabla)v,\Delta v\rangle\\&\quad+\langle\nabla\theta,\Delta v\rangle+\langle(v\cdot\nabla)u,\Delta v\rangle+\langle(u\cdot\nabla)\theta,\Delta\theta\rangle+\langle\operatorname{div}v,\Delta\theta\rangle\\&=I_1+I_2+I_3+I_4+I_5+I_6+I_7.\end{aligned} \tag{5.24}$$

Integration by parts implies

$$I_4+I_7=0. \tag{5.25}$$

We also consider I_3 first.

$$\begin{aligned}|I_3|&=\left|\langle(u\cdot\nabla)v,\Delta v\rangle\right|\le C\|u\|_{L^6}\|\nabla v\|_{L^3}\|\Delta v\|_{L^2}\\&\le C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}^{1-\lambda_5}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^{\lambda_5}\|\nabla v\|_{L^2}^{1-\lambda_6}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^{\lambda_6}\\&\le\frac{1}{16}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+C\|\nabla u\|_{L^2}^{\frac{2}{2-\lambda_5-\lambda_6}}\|\nabla v\|_{L^2}^2\\&\le\frac{1}{16}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+C\|u\|_{L^2}^{(1-\lambda_7)\frac{2}{2-\lambda_5-\lambda_6}}\left\|\Lambda^{\alpha}u\right\|_{L^2}^{\lambda_7\frac{2}{2-\lambda_5-\lambda_6}}\|\nabla v\|_{L^2}^2\\&\le\frac{1}{16}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+C\left(\left\|\Lambda^{\alpha}u\right\|_{L^2}^2+1\right)\|\nabla v\|_{L^2}^2,\end{aligned} \tag{5.26}$$

here, we need $\alpha\ge\frac{5}{4}$ and have used the following inequalities

$$\|\nabla v\|_{L^3}\le C\|\nabla v\|_{L^2}^{1-\lambda_5}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^{\lambda_5},\ \lambda_5=\frac{1}{2\alpha},$$

$$\|\Delta v\|_{L^2}\le C\|\nabla v\|_{L^2}^{1-\lambda_6}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^{\lambda_6},\ \lambda_6=\frac{1}{\alpha},$$

$$\|\nabla u\|_{L^2} \le C\|u\|_{L^2}^{1-\lambda_7}\|\Lambda^\alpha u\|_{L^2}^{\lambda_7},\ \lambda_7=\frac{1}{\alpha}.$$

Similarly, we can obtain I_5 as follows

$$\begin{aligned}
|I_5| &= \left|\langle (v\cdot\nabla)u,\Delta v\rangle\right| \le C\|v\|_{L^6}\|\nabla u\|_{L^3}\|\Delta v\|_{L^2}\\
&\le C\|\nabla v\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\Lambda^{1+\alpha}u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla v\|_{L^2}^{1-\frac{1}{\alpha}}\|\Lambda^{1+\alpha}v\|_{L^2}^{\frac{1}{\alpha}}\\
&\le \frac{1}{16}\|\Lambda^{1+\alpha}u\|_{L^2}^2+\frac{1}{16}\|\Lambda^{1+\alpha}v\|_{L^2}^2+C\|\nabla v\|_{L^2}^{\frac{2}{2-\frac{1}{2\alpha}-\frac{1}{\alpha}}}\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2\right)\\
&\le \frac{1}{16}\|\Lambda^{1+\alpha}u\|_{L^2}^2+\frac{1}{16}\|\Lambda^{1+\alpha}v\|_{L^2}^2\\
&\quad +C\|v\|_{L^2}^{\left(1-\frac{1}{\alpha}\right)\frac{2}{2-\frac{1}{2\alpha}-\frac{1}{\alpha}}}\|\Lambda^\alpha v\|_{L^2}^{\frac{2}{\alpha\left(2-\frac{1}{2\alpha}-\frac{1}{\alpha}\right)}}\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2\right)\\
&\le \frac{1}{16}\|\Lambda^{1+\alpha}u\|_{L^2}^2+\frac{1}{16}\|\Lambda^{1+\alpha}v\|_{L^2}^2+C\left(\|\Lambda^\alpha v\|_{L^2}^2+1\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2\right).
\end{aligned}\tag{5.27}$$

Though above estimate for I_3 and I_5, we know that we don't need to use the divergence free condition. Therefore, we can obtain

$$\begin{aligned}
|I_1| &= \left|\langle (u\cdot\nabla)u,\Delta u\rangle\right|\\
&\le \frac{1}{16}\|\Lambda^{1+\alpha}u\|_{L^2}^2+C\left(\|\Lambda^\alpha u\|_{L^2}^2+1\right)\|\nabla u\|_{L^2}^2,
\end{aligned}\tag{5.28}$$

$$\begin{aligned}
|I_2| &= \left|\langle \operatorname{div}(v\otimes v),\Delta u\rangle\right|\\
&= \left|\langle (v\cdot\nabla)v,\Delta u\rangle+\langle (\nabla\cdot v)v,\Delta u\rangle\right|\\
&\le \frac{1}{16}\|\Lambda^{1+\alpha}u\|_{L^2}^2+\frac{1}{16}\|\Lambda^{1+\alpha}v\|_{L^2}^2+C\left(\|\Lambda^\alpha v\|_{L^2}^2+1\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2\right),
\end{aligned}\tag{5.29}$$

$$\begin{aligned}
|I_6| &= \left|\langle (u\cdot\nabla)\theta,\Delta\theta\rangle\right|\\
&\le \frac{1}{16}\|\Lambda^{1+\alpha}\theta\|_{L^2}^2+C\left(\|\Lambda^\alpha u\|_{L^2}^2+1\right)\|\nabla\theta\|_{L^2}^2.
\end{aligned}\tag{5.30}$$

Taking the above estimations (5.25) - (5.38) into (5.24), we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)+\|\Lambda^{1+\alpha}u\|_{L^2}^2+\|\Lambda^{1+\alpha}v\|_{L^2}^2+\|\Lambda^{1+\alpha}\theta\|_{L^2}^2\\
&\le C\left(\|\Lambda^\alpha u\|_{L^2}^2+\|\Lambda^\alpha v\|_{L^2}^2+1\right)\left(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right).
\end{aligned}\tag{5.31}$$

For the H^2 *-estimates*:

Taking Δ to (1.1), (1.2), (1.3), multiplying (1.1), (1.2), (1.3) by Δu, Δv, $\Delta\theta$, after integrating by parts and taking the divergence free property into account, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta u\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\|\Delta\theta\|_{L^2}^2\right)+\|\Lambda^{2+\alpha}u\|_{L^2}^2+\|\Lambda^{2+\alpha}v\|_{L^2}^2+\|\Lambda^{2+\alpha}\theta\|_{L^2}^2\\
&= -\langle \Delta((u\cdot\nabla)u),\Delta u\rangle-\langle \Delta\operatorname{div}(v\otimes v),\Delta u\rangle-\langle \Delta((u\cdot\nabla)v),\Delta v\rangle\\
&\quad -\langle \Delta\nabla\theta,\Delta v\rangle-\langle \Delta((v\cdot\nabla)u),\Delta v\rangle-\langle \Delta((u\cdot\nabla)\theta),\Delta\theta\rangle-\langle \Delta\operatorname{div} v,\Delta\theta\rangle\\
&= II_1+II_2+II_3+II_4+II_5+II_6+II_7.
\end{aligned}\tag{5.32}$$

Integration by parts implies

$$II_4 + II_7 = 0. \tag{5.33}$$

We consider II_5 first.

$$\begin{aligned}|II_5| &= \left|\left\langle \Delta\left((v\cdot\nabla)u\right), \Delta v\right\rangle\right| \\ &\le C\|v\|_{L^\infty}\|\Delta u\|_{L^2}\|\nabla\Delta v\|_{L^2} + C\|\nabla v\|_{L^6}\|\nabla u\|_{L^3}\|\nabla\Delta v\|_{L^2} \\ &\le C\left(\left\|\Lambda^{1+\alpha}v\right\|_{L^2} + \|v\|_{L^2}\right)\|\Delta u\|_{L^2}\left\|\Lambda^{2+\alpha}v\right\|_{L^2}^{\frac{1}{\alpha}}\|\Delta v\|_{L^2}^{1-\frac{1}{\alpha}} \\ &\quad + C\|\Delta v\|_{L^2}\|u\|_{L^2}^{1-\frac{3}{2(1+\alpha)}}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^{\frac{3}{2(1+\alpha)}}\left\|\Lambda^{2+\alpha}v\right\|_{L^2}^{\frac{1}{\alpha}}\|\Delta v\|_{L^2}^{1-\frac{1}{\alpha}} \\ &\le \frac{1}{16}\left\|\Lambda^{2+\alpha}v\right\|_{L^2}^2 + C\|\Delta v\|_{L^2}^2 + C\left(\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2 + \|v\|_{L^2}^2\right)\|\Delta u\|_{L^2}^2 \\ &\quad + \frac{1}{16}\left\|\Lambda^{2+\alpha}v\right\|_{L^2}^2 + C\|\Delta v\|_{L^2}^2 + C\left(\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2 + 1\right)\|\Delta v\|_{L^2}^2 \\ &\le \frac{1}{8}\left\|\Lambda^{2+\alpha}v\right\|_{L^2}^2 + C\left(\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2 + \|v\|_{L^2}^2\right)\|\Delta u\|_{L^2}^2 + C\left(\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2 + 1\right)\|\Delta v\|_{L^2}^2,\end{aligned} \tag{5.34}$$

here, we have used the following inequalities

$$\|\nabla\Delta v\|_{L^2} \le C\|\Delta v\|_{L^2}^{1-\frac{1}{\alpha}}\left\|\Lambda^{2+\alpha}v\right\|_{L^2}^{\frac{1}{\alpha}},$$

$$\|\nabla u\|_{L^3} \le C\|u\|_{L^2}^{1-\frac{3}{2(1+\alpha)}}\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^{\frac{3}{2(1+\alpha)}}.$$

By the Kato-Ponce, Hölder, Gagliardo-Nirenberg and Young inequalities, we have

$$\begin{aligned}|II_2| &= \left|\left\langle \Delta\mathrm{div}(v\otimes v), \Delta u\right\rangle\right| \\ &= \left|\left\langle \Delta\left((v\cdot\nabla)v\right), \Delta u\right\rangle + \left\langle \Delta\left((\nabla\cdot v)v\right), \Delta u\right\rangle\right| \\ &\le \frac{1}{8}\left\|\Lambda^{2+\alpha}u\right\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2 + C\left(\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2 + \|v\|_{L^2}^2 + 1\right)\|\Delta v\|_{L^2}^2,\end{aligned} \tag{5.35}$$

$$\begin{aligned}|II_3| &= \left|\left\langle \Delta\left((u\cdot\nabla)v\right), \Delta v\right\rangle\right| \\ &\le \frac{1}{8}\left\|\Lambda^{2+\alpha}v\right\|_{L^2}^2 + C\left(\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2 + \|u\|_{L^2}^2 + 1\right)\|\Delta v\|_{L^2}^2,\end{aligned} \tag{5.36}$$

$$\begin{aligned}|II_1| &= \left|\left\langle (u\cdot\nabla)u, \Delta u\right\rangle\right| \\ &\le \frac{1}{8}\left\|\Lambda^{2+\alpha}u\right\|_{L^2}^2 + C\left(\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2 + \|u\|_{L^2}^2 + 1\right)\|\Delta u\|_{L^2}^2,\end{aligned} \tag{5.37}$$

$$\begin{aligned}|II_3| &= \left|\left\langle (u\cdot\nabla)\theta, \Delta \theta\right\rangle\right| \\ &\le \frac{1}{8}\left\|\Lambda^{2+\alpha}\theta\right\|_{L^2}^2 + C\left(\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2 + \|u\|_{L^2}^2 + 1\right)\|\Delta \theta\|_{L^2}^2.\end{aligned} \tag{5.38}$$

Taking the above estimations (5.25) - (5.38) into (5.24), we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\right) + \left\|\Lambda^{2+\alpha}u\right\|_{L^2}^2 + \left\|\Lambda^{2+\alpha}v\right\|_{L^2}^2 + \left\|\Lambda^{2+\alpha}\theta\right\|_{L^2}^2 \\ &\le C\left(\left\|\Lambda^{1+\alpha}u\right\|_{L^2}^2 + \|u\|_{L^2}^2 + \left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2 + \|v\|_{L^2}^2 + 1\right)\left(\|\Delta u\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\right).\end{aligned} \tag{5.39}$$

For the H^3 *-estimates*:

Taking $\nabla\Delta$ to (1.1), (1.2), (1.3), multiplying (1.1), (1.2), (1.3) by $\nabla\Delta u$, $\nabla\Delta v$, $\nabla\Delta\theta$, after integrating by parts and taking the divergence free property into account, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Lambda^3 u\|_{L^2}^2+\|\Lambda^3 v\|_{L^2}^2+\|\Lambda^3\theta\|_{L^2}^2\right)+\|\Lambda^{3+\alpha}u\|_{L^2}^2+\|\Lambda^{3+\alpha}v\|_{L^2}^2+\|\Lambda^{3+\alpha}\theta\|_{L^2}^2\\&=-\langle\nabla\Delta((u\cdot\nabla)u),\nabla\Delta u\rangle-\langle\nabla\Delta\mathrm{div}(v\otimes v),\nabla\Delta u\rangle-\langle\nabla\Delta((u\cdot\nabla)v),\nabla\Delta v\rangle\\&\quad-\langle\nabla\Delta\nabla\theta,\nabla\Delta v\rangle-\langle\nabla\Delta((v\cdot\nabla)u),\nabla\Delta v\rangle\\&\quad-\langle\nabla\Delta((u\cdot\nabla)\theta),\nabla\Delta\theta\rangle-\langle\nabla\Delta\mathrm{div}v,\nabla\Delta\theta\rangle\\&=III_1+III_2+III_3+III_4+III_5+III_6+III_7.\end{aligned}\tag{5.40}$$

Integration by parts implies

$$III_4+III_7=0.\tag{5.41}$$

Now, we estimate the terms in the right side of (5.40) one by one

$$\begin{aligned}|III_1|&=\left|\int_{\mathbb{R}^3}\nabla\Delta((u\cdot\nabla)u)\cdot\nabla\Delta u\mathrm{d}x\right|\\&=\int_{\mathbb{R}^3}\left(\nabla\Delta((u\cdot\nabla)u)-u\cdot\nabla(\nabla\Delta u)\right)\cdot\nabla\Delta u\mathrm{d}x\\&\le C\|\nabla u\|_{L^\infty}\|\Lambda^3 u\|_{L^2}^2\\&\le C\left(\|\Lambda^{2+\alpha}u\|_{L^2}^2+1\right)\|\Lambda^3 u\|_{L^2}^2,\end{aligned}\tag{5.42}$$

$$\begin{aligned}|III_3|&=\left|\int_{\mathbb{R}^3}\nabla\Delta(u\cdot\nabla v)\cdot\nabla\Delta v\mathrm{d}x\right|\\&=\left|\int_{\mathbb{R}^3}\left(\nabla\Delta(u\cdot\nabla v)-u\cdot\nabla(\nabla\Delta v)\right)\cdot\nabla\Delta v\mathrm{d}x\right|\\&\le C\|\nabla u\|_{L^\infty}\|\Lambda^3 v\|_{L^2}^2+C\|\nabla v\|_{L^3}\|\Lambda^3 u\|_{L^6}\|\Lambda^3 u\|_{L^2}\\&\le C\|\nabla u\|_{L^\infty}\|\Lambda^3 v\|_{L^2}^2+C\|\nabla v\|_{L^3}\|\Lambda^4 u\|_{L^2}\|\Lambda^3 u\|_{L^2}\\&\le C\left(\|\Lambda^{2+\alpha}u\|_{L^2}^2+1\right)\|\Lambda^3 v\|_{L^2}^2\\&\quad+C\|v\|_{L^2}^{1-\frac{3}{2(1+\alpha)}}\|\Lambda^{1+\alpha}v\|_{L^2}^{\frac{3}{2(1+\alpha)}}\|\Lambda^{3+\alpha}u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^3 u\|_{L^2}^{1-\frac{1}{\alpha}}\|\Lambda^3 u\|_{L^2}\\&\le C\left(\|\Lambda^{2+\alpha}u\|_{L^2}^2+1\right)\|\Lambda^3 v\|_{L^2}^2+\frac{1}{16}\|\Lambda^{3+\alpha}u\|_{L^2}^2+C\|\Lambda^3 u\|_{L^2}^2\\&\quad+C\left(\|\Lambda^{1+\alpha}v\|_{L^2}^2+1\right)\|\Lambda^3 u\|_{L^2}^2\\&\le\frac{1}{16}\|\Lambda^{3+\alpha}u\|_{L^2}^2+C\left(\|\Lambda^{1+\alpha}v\|_{L^2}^2+1\right)\|\Lambda^3 u\|_{L^2}^2+C\left(\|\Lambda^{2+\alpha}u\|_{L^2}^2+1\right)\|\Lambda^3 v\|_{L^2}^2.\end{aligned}\tag{5.43}$$

Similarity,

$$\begin{aligned}|III_6|&=\left|\int_{\mathbb{R}^3}\nabla\Delta(u\cdot\nabla\theta)\cdot\nabla\Delta\theta\mathrm{d}x\right|\\&\le\frac{1}{16}\|\Lambda^{3+\alpha}u\|_{L^2}^2+C\left(\|\Lambda^{1+\alpha}\theta\|_{L^2}^2+1\right)\|\Lambda^3 u\|_{L^2}^2+C\left(\|\Lambda^{2+\alpha}u\|_{L^2}^2+1\right)\|\Lambda^3\theta\|_{L^2}^2.\end{aligned}\tag{5.44}$$

The most difficulty are the following two terms.

$$
\begin{aligned}
\left|III_2+III_5\right| &= \left|\int_{\mathbb{R}^3}\nabla\Delta\left(\operatorname{div}\left(v\otimes v\right)\right)\cdot\nabla\Delta u\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\Delta\left(\left(v\cdot\nabla\right)u\right)\cdot\nabla\Delta v\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^3}\nabla\Delta\left(\left(\nabla\cdot v\right)v\right)\cdot\nabla\Delta u\mathrm{d}x+\int_{\mathbb{R}^3}\nabla\Delta\left(\left(v\cdot\nabla\right)v\right)\cdot\nabla\Delta u\mathrm{d}x\right. \\
&\quad \left.+\int_{\mathbb{R}^3}\nabla\Delta\left(\left(v\cdot\nabla\right)u\right)\cdot\nabla\Delta v\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}^3}\nabla\Delta\left(\left(\nabla\cdot v\right)v\right)\cdot\nabla\Delta u\mathrm{d}x+\int_{\mathbb{R}^3}\left(\nabla\Delta\left(\left(v\cdot\nabla\right)v\right)-v\cdot\nabla\left(\nabla\Delta v\right)\right)\cdot\nabla\Delta u\mathrm{d}x\right. \\
&\quad \left.+\int_{\mathbb{R}^3}\left(\nabla\Delta\left(\left(v\cdot\nabla\right)u\right)-v\cdot\nabla\left(\nabla\Delta u\right)\right)\cdot\nabla\Delta v\mathrm{d}x+\int_{\mathbb{R}^3}\operatorname{div} v\left(\nabla\Delta u\right)\cdot\nabla\Delta v\mathrm{d}x\right| \\
&\le C\left\|\nabla v\right\|_{L^3}\left\|\Lambda^3 u\right\|_{L^6}\left\|\Lambda^3 v\right\|_{L^2}+C\left\|\nabla u\right\|_{L^\infty}\left\|\Lambda^3 v\right\|_{L^2}^2 \\
&\quad +C\left\|v\right\|_{L^\infty}\left\|\Lambda^3 v\right\|_{L^2}\left\|\Lambda^4 u\right\|_{L^2}+C\left\|\nabla v\right\|_{L^3}\left\|\Delta v\right\|_{L^6}\left\|\Lambda^4 u\right\|_{L^2} \\
&= E_1+E_2+E_3+E_4.
\end{aligned}
$$

For E_2, we can estimate it as follows

$$
\begin{aligned}
E_2 &= C\left\|\nabla u\right\|_{L^\infty}\left\|\Lambda^3 v\right\|_{L^2}^2 \\
&\le C\left(\left\|\Lambda^{2+\alpha}u\right\|_{L^2}^2+1\right)\left\|\Lambda^3 v\right\|_{L^2}^2.
\end{aligned}
$$

For E_1, we have

$$
\begin{aligned}
E_1 &= C\left\|\nabla v\right\|_{L^3}\left\|\Lambda^3 u\right\|_{L^6}\left\|\Lambda^3 v\right\|_{L^2} \\
&\le C\left\|\nabla v\right\|_{L^3}\left\|\Lambda^4 u\right\|_{L^2}\left\|\Lambda^3 v\right\|_{L^2} \\
&\le C\left\|v\right\|_{L^2}^{1-\frac{3}{2(1+\alpha)}}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^{\frac{3}{2(1+\alpha)}}\left\|\Lambda^{3+\alpha}u\right\|_{L^2}^{\frac{1}{\alpha}}\left\|\Lambda^3 u\right\|_{L^2}^{1-\frac{1}{\alpha}}\left\|\Lambda^3 v\right\|_{L^2} \\
&\le \frac{1}{16}\left\|\Lambda^{3+\alpha}u\right\|_{L^2}^2+C\left\|\Lambda^3 u\right\|_{L^2}^2+C\left(\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+1\right)\left\|\Lambda^3 v\right\|_{L^2}^2.
\end{aligned}
$$

Similarity,

$$
\begin{aligned}
E_3 &= C\left\|v\right\|_{L^\infty}\left\|\Lambda^3 v\right\|_{L^2}\left\|\Lambda^4 u\right\|_{L^2} \\
&\le C\left\|v\right\|_{L^2}^{1-\frac{3}{2(1+\alpha)}}\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^{\frac{3}{2(1+\alpha)}}\left\|\Lambda^3 v\right\|_{L^2}\left\|\Lambda^{3+\alpha}u\right\|_{L^2}^{\frac{1}{\alpha}}\left\|\Lambda^3 u\right\|_{L^2}^{1-\frac{1}{\alpha}} \\
&\le \frac{1}{16}\left\|\Lambda^{3+\alpha}u\right\|_{L^2}^2+C\left\|\Lambda^3 u\right\|_{L^2}^2+C\left(\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+1\right)\left\|\Lambda^3 v\right\|_{L^2}^2,
\end{aligned}
$$

$$
E_4 \le \frac{1}{16}\left\|\Lambda^{3+\alpha}u\right\|_{L^2}^2+C\left\|\Lambda^3 u\right\|_{L^2}^2+C\left(\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+1\right)\left\|\Lambda^3 v\right\|_{L^2}^2.
$$

Therefore, we can get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Lambda^3 u\right\|_{L^2}^2+\left\|\Lambda^3 v\right\|_{L^2}^2+\left\|\Lambda^3\theta\right\|_{L^2}^2\right)+\left\|\Lambda^{3+\alpha}u\right\|_{L^2}^2+\left\|\Lambda^{3+\alpha}v\right\|_{L^2}^2+\left\|\Lambda^{3+\alpha}\theta\right\|_{L^2}^2 \\
&\le C\left(\left\|\Lambda^{2+\alpha}u\right\|_{L^2}^2+\left\|\Lambda^{1+\alpha}v\right\|_{L^2}^2+\left\|\Lambda^{1+\alpha}\theta\right\|_{L^2}^2+1\right)\left(\left\|\Lambda^3 u\right\|_{L^2}^2+\left\|\Lambda^3 v\right\|_{L^2}^2+\left\|\Lambda^3\theta\right\|_{L^2}^2\right).
\end{aligned}
$$

This completes the proof of the Theorem 1.3 by Gronwall's inequality and energy estimate (5.23). □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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