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Space Topologies and Their Dual Space Topologies for Conventional Functional Space Topologies

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Abstract

In this paper, we have studied the topology of some classical functional spaces. Among these spaces, there are standard spaces, spaces that can be metrizable and others that cannot be metrizable. But they are all topological vector spaces and it is in this context that we have chosen to present this work. We are interested in the topology of its spaces and in the topologies of their dual spaces. The first part, we presented the fundamental topological properties of topological vector spaces. The second part, we studied Frechet spaces and particularly the space $S(\mathbb{R}^n)$ of functions of class C^∞ on $\mathbb{R}^n$ which are as well as all their rapidly decreasing partial derivatives. We have also studied its dual $S'(\mathbb{R}^n)$ the space of tempered distributions. The last part aims to define a topological structure on an increasing union of Frechet spaces called inductive limit of Frechet spaces. We study in particular the space $D(\Omega)$ of functions of class C^∞ with compact supports on Ω as well as its dual $D'(\Omega)$ the space distributions over the open set Ω.

Keywords

Linear Forms, Dual Spaces, Frechet Spaces, Partial Derivatives, Distributions, Topological Structure, Weak Topology, Strong Topology

1. Introduction

On some diagrams, only a specific collection of nodes or edges (arcs) represent a structure. A topology on a collection of nodes and edges of a non-directed graph and a topology on a set of nodes and arcs of a directed graph are defined in [1]. In addition, Dr. Alqahtani. M highlighted some of the topological characteristics of these spaces, and we examine some of the connections between them and the graphs. We further demonstrate that this topology meets the requirement of being Alexandrov.

In the group of pretopological spaces with *p*-continuous maps, we present the T_0-reflection as arrows. Following that, we'll look at a few of this category's separation axioms, [2] [3]. On the other hand, [4] if *A* is open (respectively closed) and (respectively $A \setminus A^{\circ}$ is a finite array, then the subset *A* of the topological space X is known as an F-open (respectively F-closed) set. He examined the key aspects of these definitions in the study and illustrated and F-closed groups relate to other kinds, including regularly open, regularly closed, closed, and open groups in topological spaces. Moreover, utilizing the ideas of *F*-open and *F*-closed groups, [5] I will explain several topological features for the group, including internal *F*, closure *F*, derivative *F*, etc. Eventually, I'll discuss *F*-Continuous, *F*-compact, and other concepts and ideas that are relevant.

We now provide our lesson on some classical function spaces' topology. There are standard spaces among them, as well as areas that can be measured and some that cannot. Yet, given that they are all topological vector spaces. We have decided to display this work in this setting. Both the topology of its dual spaces and the topology of those spaces are of interest to us. Fundamental topological features of topological spaces comprised our first section. Part 2, He investigated functions of class C^{∞} over $\mathbb{R}^n$ in Frechite spaces, particularly the $S(\mathbb{R}^n)$ space. In addition to all of its quickly deteriorating partial derivatives, it exists. Also, we looked at the diluted distributions' Double $S'(\mathbb{R}^n)$ area. The final section seeks to specify the topology. The inductive limit of Frechet spaces is based on an increasing union of Frechet spaces. We focus on the area $D(\Omega)$ of the class C^{∞} functions on the plus $D'(\Omega)$ space distributions over the open group. In addition to all of its quickly deteriorating partial derivatives, it exists. Also, we looked at the diluted distributions' Double $S'(\mathbb{R}^n)$ area.

2. Continuous Linear Forms and Dual Space

2.1. Standard Vector Spaces

Proposition 2.1. *E is a vector space on* $\mathbb{R}$ *or* $\mathbb{C}$.

1) *It is* $a,b \in E$. *We call the end segment a and b the subset* $[ab]$ *of E defined by*

$$[a,b] = \{x = a + tb, t \in [0,1]\}.$$

2) *Let, Ω is a subset of E. They say that,*

a) *Ω is convex if for all* $a,b \in \Omega$, *we have* $[a,b] \in \Omega$.

b) *Ω is equilibrium, if for all* $a \in \Omega$ *and all* $\lambda \in \mathbb{K}$, *then*

$$|\lambda| \le 1 \to \lambda a \in \Omega .$$

c) *Ω is absorbent, if for all* $a \in E$, *it exists* $\alpha > 0$ *such that,*

$$\forall \lambda \in \mathbb{K}, |\lambda| \le \alpha \to \lambda a \in \Omega .$$

Remark 2.2.

- *A balanced or absorbing subset necessarily contains 0.*
- *Vector sub-spaces are convex and balanced.*
- *Only E is an absorbing vector subspace.*

Definition 2.3.

1) *A norm on E is a map* $\| \|: E \to [0, +\infty[$ *verifying,*

a) $\forall x \in E$, $\|x\| = 0 \Leftrightarrow x = 0$,

b) $\forall x \in E$ *and* $\forall \lambda \in \mathbb{K}$, $\|\lambda x\| = |\lambda| \|x\|$,

c) $\forall x, y \in E$, $\|x + y\| \leq \|x\| + \|y\|$.

2) *We say that the pair* $(E, \| \|)$ *is a norm vector space if* $\| \|$ *is a norm on E.*

$(E, \| \|)$ is a norm vector space on $\mathbb{K}$. We call an open ball with center $a \in E$ and radius $r > 0$ the set,

$$B_r = \{x \in E, \|x - a\|_E < r\} = a + B_r(0).$$

Any ball of E is convex and the balls with center 0 are balanced and absorbent.

A norm vector space admits a metric space structure. For the topology associated with this metric the balls with center 0 form a fundamental system of neighborhood of 0.

Two norms $\| \|_1$ and $\| \|_2$ over E are said to be equivalent, if there exist two strictly positive real numbers α and β such that,

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1, \ \forall x \in E.$$

Two equivalent norms $\| \|_1, \| \|_2$ on E define the same topology.

Proposition 2.4. *Let* $(E, \| \|_E)$ *be a norm vector space.*

1) *The map* $\begin{array}{ccc} E \times E & \to & E \\ (x, y) & \to & x + y \end{array}$ *is continuous.*

2) *The map* $\begin{array}{ccc} \mathbb{K} \times E & \to & E \\ (\lambda, x) & \to & \lambda x \end{array}$ *is continuous.*

3) *Let* $a \in E$, *then*

a) *The translation by the vector a defined by* $T_a : x \to a + x$ *is a homeomorphism on E,*

b) *The multiplication by a scalar defined by* $\lambda \to \lambda x$ *is continued on* $\mathbb{K}$.

Example 2.5.

- $\forall x = (x_1, \cdots, x_n) \in \mathbb{K}^n$,

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}, \ p \in [1, \infty[\ \text{ and } \ \|x\|_\infty = \sup_{1 \leq k \leq n} |x|.$$

$\| \|_p$ *and* $\| \|_\infty$ *are two norms on* $\mathbb{K}$. *They are both equivalent.*

- *For* $1 \leq p < \infty$,

$$l^p(\mathbb{K}) = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}, \sum_{n=0}^{\infty} |x_n|^p < \infty \right\}$$

$l^p(\mathbb{K})$ *is a a vector space on norm by*

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$$

- *On* $C([0,1],\mathbb{R})$, *we define the following norms,*

$$\forall f \in C([0,1],\mathbb{R}),\ \|f\|_\infty = \sup_{x\in[0,1]} |f(x)|,$$

$$\forall f \in C([0,1],\mathbb{R}),\ \|f\|_p = \left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}},\ 1 \le p < \infty.$$

- *More generally*

- *Let* (X,τ,ν) *be a measure space and* $1 \le p \le \infty$. *The spaces* $L^p(X,\tau,\nu), \|\ \|_p$ *are norm vector spaces.*
- *For* $k \in \mathbb{N} \cup \infty$ *and Ω an open set of* $\mathbb{R}^n$, *we note,*

✧ $C^k(\Omega)$ *the space of functions* $f:\Omega \to \mathbb{C}$ *of class* C^k *on Ω.*

✧ $C_p^k(\Omega)$ *the space of functions* $f \in C^k(\Omega)$ *shach that f and all its partial derivatives up to order k are bounded on Ω.*

✧ $C_c^k(\Omega)$ *the space of functions* $f \in C^k(\Omega)$ *has compact support included in Ω.*

✧ $C_0^k(\Omega)$ *the space of functions* $f \in C^k(\Omega)$ *shach that f and all its partial derivatives up to order k tend to 0 at infinity.*

The map $\|\ \|_\infty : f \to \sup_{x\in\Omega} |f(x)|$ *is a norm on the spaces* $C_b^k(\Omega, C_c^k(\Omega))$ *and* $C_0^k(\mathbb{R}^n)$.

Definition 2.6. *Let* $(E,\|\ \|)$ *be a norm vector space and* $k \subset E$. *We say that K is a compact if for any covering of K by a family of open sets of E, we can extract a finite subcovering. More precisely, let I be any set* $(\Omega_\alpha)_{\alpha\in I}$ *a family of open sets of E such that* $K \subset \bigcup_{\alpha\in I} \Omega_\alpha$, *then there is* $\alpha_1,\cdots,\alpha_n \in I$ *such that* $K \subset \bigcup_{1\le i\le n} \Omega_{\alpha_i}$.

Remark 2.7. *It is easy to verify that*

- *The finished sets are compacts.*
- *If* $(a_n)_{n\ge0} \subset E$ *converge to* $a \in E$. *Then* $\{a_n, n\in\mathbb{N}\} \cup \{a\}$ *is a compact.*
- *If K is a compact, then K is a closed bound of E.*
- *If K is a compact of E and* $M > 0$, *then* $\{\lambda K, |\lambda| \le M\}$ *is compact.*

Definition 2.8. *Let* $(E,\|\ \|)$ *be a norm vector space. We say that K locally compact if 0 admits a compact neighborhood.*

Proposition 2.9. $(E,\|\ \|)$ *is locally compact if and only if the closed unit ball* $\bar{B}_1(0)$ *is compact.*

Definition 2.10. *Let* $(E,\|\ \|)$ *be a norm vector space.*

1) *Let* $(a_n)_{n\ge0}$ *be a sequence in E. We say that* $(a_n)_{n\ge0}$ *is a cauchy sequence if,*

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N},\ n \ge m \ge n_0 \Rightarrow \|x_n - x_m\| \le \varepsilon.$$

2) *We say that* $(E,\|\ \|)$ *is complete if every cauchy sequence in E is convergent. In this case, we say that E is a Bannach space.*

2.2. Continuous Linear Forms and Dual Space

$(E,\|\ \|)$ denotes a norm vector space.

We call linear form on E any linear application defined on E and takes its values in the body $\mathbb{K}$.

Proposition 2.11. *Let l be a linear form on E. Then the following properties are equivalent*

1) *l is continuous on E.*

2) *l is uniformly continuous on.*

3) *l is continuous in 0.*

4) *Le noyau de l est un sous-espace vectoriel ferme de E.*

5) *There is a constant* $M > 0$, *such that*

$$\forall x \in E, \left| l(x) \right| \le M \|x\|.$$

Let l be a continuous linear form on E. Then l is bounded on the closed unit ball of E and we set:

$$N(l) = \sup_{\|x\| \le 1} \left| l(x) \right|.$$

Remark 2.12.

1) *There are linear shapes that are not continuous. For example, consider the vector space* $l : E \to \mathbb{K}$ *be defined by* $l(P) = P'(1)$. *The map l is a linear form and we have: for all* $n \in \mathbb{N}^*$

$$\left\| X^n \right\| = 1 \quad and \quad \left| l\left(X^n\right) \right| = n.$$

So l is not continuous.

2) *On the other hand, if E is not reduced to* $\{0\}$, *then there exist nonzero continuous linear forms on E. The existence of such linear forms is ensured by the following Hahn-Banach theorem.*

Theorem 2.13. *Hahn-Banach theorem*

Let E be a norm vector space and F a vector subspace of E not reduced to $\{0\}$. *If l is a continuous linear form on F, then there exists* $\tilde{l}$ *a continuous linear form on E such that*

1) $\tilde{l}_{|F} = l$,

2) $N(\tilde{l}) = N(l)$,

Let $a \in E$ *be nonzero. Then there exists l a continuous linear form on E such that* $l(a) = \|a\|$ *et* $N(l) = 1$.

Corollary 2.14. *Let E be a norm vector space and* $a \in E$ *a nonzero vector. Then there exists l a continuous linear form on E which satisfies* $l(a) = 1$.

Proof. Let $a \in E$ be a nonzero vector. Let l be the linear form defined on $F = \mathbb{K}a$ by $f(\lambda a) = \lambda$. So we have:

$$\left| f(\lambda) \right| = |\lambda| = \frac{1}{\|a\|} \|\lambda a\|.$$

So l is continuous on F and $N(l) = \frac{1}{\|a\|}$. The Hahn-Banach theorem ensures the existence of a continuous linear form $\tilde{l}$ on E which verifies

$$\tilde{l}(a) = 1 \text{ and } N(\tilde{l}) = \frac{1}{\|a\|}.$$

□

Proposition 2.15. *Let E be a norm vector space. So E' is a space vectoril on* $\mathbb{K}$*, the map* $l \mapsto \|l\|_{E'} = N(l)$ *is a norm on E' etr* $(E', \| \|_{E'})$ *is a Banach space.*

We note E" the dual of E' also calls the bidual space of E. The space E" endowed with norm

$$\|\phi\|_{E''} = \sup_{\|l\|_{E'} \le 1} |\phi(l)|,$$

is a Banach space.

Let $a \in E$*. The map* $J(a): E \to \mathbb{K}$ *defined by* $J(a)(l) = l(a)$*, is a linear form on E' which verifies*

$$\sup_{\|l\|_{E'} \le 1} |J(a)(l)| = \sup_{\|l\|_{E'} \le 1} |l(a)(l)| = \|a\|_E.$$

We deduce that the map $J: E \to E''$ *is an isometric bijection of E in E".*

In general, the map J is not surjective and particularly it is not surjective if E is not a Banach space. But the map J makes it possible to prove that any norm vector space E can be injected isometrically into a Banach space. Therefore it is dense in the Banach space $\overline{J(E)}$.

Definition 2.16. *We say that E is a reflexive space if the isometric injection J from E into E" is surjective. We are going to define a topology on E weak topology site. Let*

$$\tau = \left\{ l^{-1}(\Omega), l \in E' \text{ and } \Omega \text{ open from } \mathbb{K} \right\}.$$

Definition 2.17. *We call weak topology on E and we denote by* $\sigma(E, E')$ *the topology on E generated by* τ.

Remark 2.18.

1) The topology $\sigma(E, E')$ *is less fine than the topology of the standard space E.*

2) Let $a \in E$*. We obtain a fundamental system of neighborhoods of a for the topology* $\sigma(E, E')$ *by considering the open sets of the form*

$$V(a, \varepsilon, l_1, \cdots, l_p) = \bigcap_{1 \le i \le p} l_i^{-1}\left(D_{\mathbb{K}}\left(l_i(a), \varepsilon\right)\right),$$

where $\varepsilon > 0$, $p \in \mathbb{N}^*$, $l_1, \cdots, l_p \in E'$.

3) We assume that E is not of finite dimension. Let $V(0, \varepsilon, l_1, \cdots, l_p)$ *an open neighborhood of 0 for the weak topology. There exists a nonzero vector in E, such that*

$$l_1(a) = \cdots = l_p(a) = 0.$$

So the line $\mathbb{K}a$ *is included in* $V(0, \varepsilon, l_1, \cdots, l_p)$*. This proves that the balls* $B_r(0)$ *are not open for the weak topology.*

Remark 2.19. *Let E be a norm vector space. therefore*

1) *The topology* $\sigma(E, E')$ *is separated.*

2) *The map* $(x, y) \mapsto x + y$ *of* $E \times E$ *in E and the map* $(\lambda, x) \mapsto \lambda x$ *of* $\mathbb{K} \times E$ *in E are continuous when E is endowed with the weak topology.*

Proof. Let *a* and *b* be two distinct vectors in *E*. By the corollary (2.14) there exists $l \in E'$ such that $l(a-b)=1$. So the disks $D_{\mathbb{K}}\left(l(a),\frac{1}{4}\right)$ $D_{\mathbb{K}}\left(l(b),\frac{1}{4}\right)$ are disjoint.

We deduce that $l^{-1}D_{\mathbb{K}}\left(l(a),\frac{1}{4}\right)$ and $l^{-1}D_{\mathbb{K}}\left(l(b),\frac{1}{4}\right)$ are indeed two disjoint open sets and which contain respectively *a* and *b*. □

Definition 2.20. *Let (x_n) be a sequence in E and $x \in E$.*

1) *We say that (x_n) converges (strongly) to x, if the sequence $(\|x_n - x\|)$ converges to 0.*

2) *We say that (x_n) converges weakly to x, if the sequence (x_n) converges to x for the topology $\sigma(E,E')$.*

Since the weak topology is less fine than the topology defined by the norm, then we have the following result:

Proposition 2.21. *Let (x_n) be a sequence in E and $x \in E$.*

1) The sequence (x_n) converges weakly to *x*, if and only if for all $l \in E'$ the sequence $(l(x_n))$ converges to $l(x)$.

2) If (x_n) converges strongly to *x*, then (x_n) converges weakly to *x*.

2.3. Finite Dimensional Vector Spaces

The characteristic properties of finite-dimensional vector spaces follow from the following properties of $\mathbb{R}$,

- $\mathbb{R}$ endowed with the absolute value is a complete metric space.
- The compacts of $(\mathbb{R},\| \|)$ are the closed bounded subsets.

Theorem 2.22. *A finite dimensional norm vector space, $(E,\| \|)$. Therefore,*

- *E is compact.*
- *All the norms on E are equivalent.*

Corollary 2.23. *E a finite dimensional norm vector space.*

1) *If E has finite dimension n, then it is homeomorphic to $\mathbb{K}^n$.*

2) *Any linear map defined on a finite-dimensional norm vector space with values in a norm vector space is continuous.*

3) *Let F be a vector subspace of E. If F is of finite dimension then it is closed in E.*

Proof.

1) Let $(E,\| \|_E)$ be a finite dimensional vector space and let $(e_1,\cdots,e_n)$ be a basis of *E*. We consider the map $\phi : E \to \mathbb{K}^n$ defined by

$$\phi\left(\sum_{1\le j\le n} x_j e_j\right) = (x_1,\cdots,x_n).$$

The map ϕ is an isomorphism of *E* on $\mathbb{K}^n$. We endow $\mathbb{K}^n$ with the norm *N* defined by

$$N(x_1,\cdots,x_n) = \left\|\sum_{1\le j\le n} x_j e_j\right\|_E$$

Thus the application φ verifies, for any vector $u \in E$,

$$N(\phi(u)) = \|u\|_E.$$

We deduce that φ is a surjective isometry between the spaces $(E, \|\ \|_E)$ and $(\mathbb{K}, N)$ and therefore it is a homeomorphism.

2) Let $(E, \|\ \|_E)$, $(F, \|\ \|_F)$ be two norm vector spacesa, and let $L: E \to F$ be a linear map. We assume that E is finite dimensional and let $(e_1, \cdots, e_n)$ be a basis of E. We consider on E the norm $\|\ \|_\infty$ defined by

$$\left\| \sum_{1 \le j \le n} x_j e_j \right\|_\infty = \max_{1 \le j \le n} |x_j|.$$

Let

$$M = \sum_{1 \le j \le n} x \left\| L(e_j) \right\|_F,$$

therefore

$$\forall u \in E,\ \|L(u)\|_F = \sum_{1 \le j \le n} |x_j| \left\| L(e_j) \right\|_F \le M \|u\|_\infty.$$

3) If F is a finite dimensional vector subspace, then F endowed with the norm of E is complete. So it is firm in E. □

Proposition 2.24. *Let E be a finite dimensional vector space n, then*

1) *E' be a finite dimensional vector space n.*

2) *The* $\sigma(E, E')$ *topology coincides with the usual topology.*

Proof.

1) Let $(e_1, \cdots, e_n)$ be a basis of E. We consider the linear forms $(e_1^\star, \cdots, e_n^\star)$ defined on E by

$$e_j^\star(e_k) = \sigma_j^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases}$$

It is easy to check that if l is a linear form on E, then

$$l = \sum_{1 \le j \le n} l(e_j) e_j^\star.$$

So $(e_1^\star, \cdots, e_n^\star)$ is a basis of E', called dual basis of $(e_1, \cdots, e_n)$, and hence E' has dimension n.

2) To show this result, it suffices to prove that for all $r > 0$, the ball $B_r(0)$ is a neighborhood of 0 for the topology $\sigma(E, E')$. Let $(e_1, \cdots, e_n)$ be a basis of E, satisfying $\|e_1\| = \cdots = \|e_n\| = 1$. We denote $(e_1^\star, \cdots, e_n^\star)$ its dual basis. For all $x \in E$, we have

$$x = e_1^\star(x) e_1 + \cdots + e_n^\star(x) e_n.$$

If $x \in W(0, e_1^\star, \cdots, e_n^\star, r/n)$ then

$$\|x\| \le \sum_{i=1}^n |e_i^\star(x)| < n \frac{r}{n} = r,$$

which implies $x \in B_E(a, r)$, so

$$V(0, r/n, e_1^\star, \cdots, e_n^\star) \subset B_E(0, r).$$

□

2.4. Hilbert Space

Let H be a vector space over $\mathbb{K}$.

Definition 2.25.

1) We call sesquilinear form on H any map $b: H \times H \to \mathbb{K}$ *satisfying for all* $x, y, z \in H$ *and* $\lambda \in \mathbb{K}$,

$$b(x+\lambda y, z) = b(x,z) + \lambda b(y,z) \text{ and } b(x+y+\lambda z) = b(x,y) + \bar{\lambda} b(x,z).$$

2) A Hermitian form b on H is a ferifying sesquilinear form

$$\forall x, y \in H,\ b(x,y) = \overline{b(y,x)}.$$

3) A Hermitian form b on H is said to be positive if for all $x \in H$ *we have,*

$$b(x,x) \geq 0.$$

4) A Hermitian form b on H is said to be positive define, if it is positive and if for all $x \in H$ *we have,*

$$b(x,x) = 0 \Rightarrow x = 0.$$

In this case, we say that b is a scalar product (or hermitian product) on H. Such a product is generally denoted by (x/y) *instead of* $b(x,y)$. *When* $\mathbb{K} = \mathbb{R}$, *a sesquilinear form is a bilinear form and a Hermitian form is a symmetric bilinear form.*

Theorem 2.26. *Let b be a positive Hermitian form on H Then for all* $x, y \in H$ *we have,*

1) $|b(x,y)| \leq \sqrt{b(x,x)}\sqrt{b(y,y)}$ *(Schwarz's inequality). If moreover b is positive definite, then the equality holds if and only if x and y are collinear.*

2) Minkowski's Inequality,

$$\sqrt{b(x+y,x+y)} \leq \sqrt{b(x,x)} + \sqrt{b(y,y)}.$$

3) If moreover b is positive definite,

$$x \to \|x\| = \sqrt{b(x,x)},$$

is a norm on H.

Definition 2.27. *A pre-Hilbertian space is a vector space H over* $\mathbb{K}$ *endowed with a scalar product* $(./.)$. *It is implied standard by* $\|.\| = \sqrt{(./.)}$.

A Hilbert space is a complete pre-Hilbert space.

Definition 2.28. *Let* $(H_1, (./.)_1)$ *and* $(H_2, (./.)_2)$ *be two Hilbert spaces. They are said to be isomorphic if there exists an isomorphism of vector spaces* $u: H_1 \to H_2$ *which satisfies*

$$\forall x, y \in H_1, (u(x) | u(y))_2 = (x | y)_1.$$

Example 2.29

- *The space* $\mathbb{C}^n$ *is a Hilbert space for the Hermitian product*

$$(x | y) = \sum_{k=1}^{n} x_k \bar{y}_k,$$

defined for all $x = (x_1, \cdots, x_n)$ *and* $y = (y_1, \cdots, y_n)$ *in* $\mathbb{C}^n$.

- *Every finite-dimensional pre-Hilbert space is a Hilbert space.*
- *The set* $l^2(\mathbb{K})$ *is a Hilbert space for the Hermitian product*

$$(x \mid y) = \sum_{k=0}^{\infty} x_k \overline{y}_k,$$

defined for all $x = (x_n)_{n\in\mathbb{N}}$ *and* $y = (y_n)_{n\in\mathbb{N}}$ *in* $l^2(\mathbb{K})$.

- *Let* (X, τ, ν) *be a measure space. The vector space* $L^2(X, \tau, \nu)$ *is a Hilbert space for the scalar product* $(.|.)$ *defined for all* f, g *by*

$$(f \mid g) = \int_X f(x)\overline{g}(x)\mathrm{d}\mu(x).$$

1) *The space* $l^2(\mathbb{K})$ *is a special case for* $X = \mathbb{N}$, $M = p(\mathbb{N})$ *and* μ *the measure of the cardinal.*

2) *Let Ω be an open set of* $\mathbb{R}^n$ *and* ω *a positive measurable function on Ω for the Lebesgue measure dx. Then* $L^2(\Omega, \omega(x)\mathrm{d}x)$ *into a Hilbert space.*

Let H be a Hilbert space. For all fixed a in H, the map from H to $\mathbb{K}$

$$l_a : x \to (x \mid a),$$

is a continuous linear form because $|(x \mid a)| \le \|x\|\|a\|$ *and we thus obtain all the continuous linear forms on H.*

Theorem 2.30. (Riesz representation theorem)

1) *Let l be a continuous linear form on a Hilbert space H. Then there exists a unique element* $a \in H$ *such that* $l(x) = (x \mid a)$ *for all* $x \in H$ *and we have*

$$\|l\|_{H'} = \|a\|.$$

2) *The map* ϕ *of H in H' which has each a in H associates* l_a *is bijective antilinear. In particular* $(H', \|\ \|)$ *is a Hilbert space and* ϕ *is an ant-isopmorphism of Hilbert space. So the canonical injection J of H into its bidual H" given by*

$$J(a)(l_b) = l_b(a) = (a \mid b), \forall\, a, b \in H.$$

is surjective and verifies

$$\|J(a)\|_{H''} = \|a\|, \forall a, b \in H.$$

So H" is a Hilbert space and the injection J is an isomorphism of Hilbert spaces, we deduce the following result.

Corollary 2.31 *Hilbert spaces are reflexive spaces.*

2.5. Bannach Spaces

Banach spaces are complete vector spaces.

Example 2.32.

- *Finite dimensional spaces and Hilbert spaces are Bannach spaces.*
- *For* $1 \le p \le \infty$, *the spaces* $(l^p(\mathbb{K}), \|\ \|_p)$ *are Bannach spaces.*
- *More generally, if* (X, τ, ν) *a measure space and* $1 \le p \le \infty$, *then the spaces* $(l^p(X, \tau, \nu), \|\ \|_p)$ *are Bannach spaces.*
- *Let* $k \in \mathbb{N}$, *Ω be an open set of* $\mathbb{R}^n$, *and* $f \in C_b^k(\Omega)$. *We set*

$$\|f\|_{\infty,k} = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_\infty,$$

where $\alpha \in \mathbb{N}^k$ *is a multi-index of length* $|\alpha|$ *and* $\partial^\alpha f$ *is the partial derivative of f of order* α. *Then the map* $f \to \|f\|_{\infty,k}$ *is a norm on* $C_b^k(\Omega)$ *and we have*

- $C_b^k(\Omega), \|f\|_{\infty,k}$ *is Bannach space.*
- $C_b^k(\Omega), \|f\|_{\infty,k}$ *is a norm vector subspace of* $C_b^k(\Omega), \|f\|_{\infty,k}$. *We denote* $C_0^k(\Omega)$ *its adherence which is a Bannach space.*
- *The space* $C_0^k(\mathbb{R}^n)$ *corresponds to the spaces of functions f of class* C^k *on* $\mathbb{R}^n$ *which tend to 0 a l 'infinity as well as their derivatives of order* α *such as* $|\alpha| \le k$.

• *If E is a norm vector space, then E' and E" are Bannach spaces. If E is dense without a Bannach space F, then* $E' = F'$ (*we restrict the elements of F' to E*).

Corollary 2.33. *Let E be a Bannach space. If* $(l_n)_{n\ge 0}$ *is a sequence in E' such that, for all* $x \in E$, *the sequence* $(l_n(x))_{n\ge 0}$ *converges to* $l(x)$, *then* $l \in E'$ *and*

$$\|l\|_{E'} \le \liminf \|l_n\|_{E'}$$

In general, a Bannach space is not necessarily reductive as shown by the following proposition.

Proposition 2.34. *We consider the Bannach spaces* $(l^\infty, \|\ \|_\infty)$, $(l^1, \|\ \|_1)$ *and* $c_0 = \{x = (x_n)_{n\ge 0} \in l^\infty(\mathbb{K}), \lim_\infty x_n = 0\}$. So we have $(l^1)' = l^\infty$, $c_0' = l^1$, $(l^\infty)' \ne l^1$.

Proof. We will start by showing that $(l^1)' = l^\infty$.

We consider, for all $m \in \mathbb{N}$, the sequence $S_m = (\sigma_{n,m})_{m\ge 0}$ where $\sigma_{n,m}$ denotes the symbol by Kroneker. So S_m is the sequence of which all the terms are zero except the term of order $m+1$ which is worth 1. Let $a = (a_n)_{n\ge 0} \in l^\infty$. We denote by l_a the linear form on l^1 defined by

$$l_a(x) = \sum_{n\ge 0} a_n x_n, \ \forall x = (x_n)_{n\ge 0} \in l^\infty.$$

Then l_a is a continuous linear form on l^1 and it is easy to show that

$$\|l_a\|_{(l^1)'} = \|a\|_\infty.$$

Conversely, let $l \in (l^1)'$. Then the sequence $a = (l(S_n))_{n\ge 0}$ is bounded and we have $l_a = l$ on C_c the space of sequences which are zero from a certain rank. But C_c is dense in l^1 which implies that $l_a = l$ and $\|l_a\|_{(l^1)'} = \|a\|_\infty$. So the map $a \to l_a$ is a one-to-one isometry from l^∞ to $(l_1)'$.

Let's show that $c_0' = l^1$.

Let $a = (a_n)_{n\ge 0} \in l^1$ and l_a be the linear form on c_0 defined by

$$l_a(x) = \sum_{n\ge 0} a_n x_n, \forall x = (x_n)_{n\ge 0} \in c_0.$$

So $l_a(x) \le \|a\|_1 \|x\|_\infty$ and hence l_a is a continuous linear form on c_0 and $\|l_a\|_{(c_0)'} \le \|a\|_1$.

To show that $\| l_a \|_{c_0'} = \| a \|_1$, we consider the sequence $(X_k)_{k\ge 0}$ defined for $k \ge 0$ by $X_k = (x_{k,n})_{n\ge 0}$ or

$$x_{k,n} = \begin{cases} \dfrac{\overline{a}_n}{|a_n|} & \text{if } n \le k, a_n \ne 0 \\ 0 & \text{if } n > k \end{cases}$$

Then the sequence $(X_k)_{k\ge 0}$ belongs to c_0 and verifies

$$\|X\|_\infty = 1 \text{ and } l_0(X_k) = \sum_{1\, leqj \le k} |a_j|,$$

which implies by passing to the limit that $\|l_a\|_{c_0'} = \|a\|_1$. So the map $a \to l_a$ is an isommetry of $l^1(\mathbb{K})$ in c_0'. For surjectivity, we consider, for all $m \in \mathbb{N}$, the sequence $S_m = (\delta_{n,m})_{m\ge 0}$. Let $l \in c_0'$. We consider the sequence $b = (b_k)_{k\ge 0}$ defined by $b_k = l(S_k)$ and $(l_n)_{n\ge 0}$ the sequence in c_0' defined by $l_n = l_{a_n}$ or

$$a_n = \sum_{0\le j\le n} l(S_j) S_j = (b_0, b_1, \cdots, b_n, 0, 0, \cdots),$$

therefore

$$\|l_n\|_{c_0} = \|a_n\|_1 = \sum_{i=0}^{n} |b_i|.$$

Let $x = (x_k)_{k\ge 0} \in c_0$. We consider the sequence $(X_n)_{n\ge 0}$ in c_0 defined by

$$X_n = \sum_{0\ge j\ge n} x_k S_k = (x_0, x_1, \cdots, x_n, 0, \cdots).$$

So we have

$$\|X_n - x\|_\infty = \sup_{k>n} |x_k|,$$

therefore the sequence (X_n) converges to x in c_0. On the other hand, we have $l_n(x) = l(X_n)$. So

$$\lim_{n\to+\infty} l_n(x) = \lim_{n\to+\infty} l(X_n) = l(x).$$

So the series $\sum_{j=0}^{\infty} b_j x_j$ is convergent and equals $l(x)$. To conclude, it suffices to show that the sequence $b \in l^1$. We consider the sequence $(B_k)_{k\ge 0}$ defined for $k \ge 0$ for $B_k = (b_{k,n})_{n\ge 0}$ where

$$b_{k,n} = \begin{cases} \dfrac{\overline{b}_n}{|b_n|} & \text{if } n \le k, a_n \ne 0 \\ 0 & \text{if } n > k \end{cases}$$

The sequence $(B_k)_{k\ge 0}$ is in c_0 and $\|b_k\|_\infty \le 1$. So, if $m \ge 0$, then we have $(b_{k,n})_{n\ge 0}$ or

$$\sum_{j=1}^{n} |b_j| = |l_n(B_n)| = |l(B_n)| \le \|l\| c_0'.$$

We deduce that $b \in l^1$ and $l_b = l$.

Let us show that $(l^\infty)' \ne l^1$.

We consider E the space of convergent sequences in $\mathbb{K}$. So E is a vector subspace of l^∞ which contains c_0. Let l be the linear form on E defined by

$$l\left((x_n)_{n\ge 0}\right) = \lim_{n\to\infty} x_n$$

Then l is continuous and $\|l\|_{E'} = 1$.

According to the Hahn-Banach theorem (2.13), there exists a linear form $\tilde{l} \in c_0$ is zero. There is no $a \in l^1$ such that $\tilde{l} = l_a$. □

Theorem 2.35. *Let E be a Banach space. So we have,*

- E is reflexive if and only if the closed unit ball is compact for the weak topology $\sigma(E, E')$.
- E is reflexive if and only if E' is reflexive.

We deduce from this theorem that the closed subspaces of a reflexive Banach space are reflexive.

2.6. Dual of L^p Spaces

The triple (X, τ, μ) denotes a measurable space and $p, q \in [1, \infty]$ two conjugate exponents.

Definition 2.36.

- μ *is said to be a finite measure if* $\mu(X) < \infty$.
- μ *is said to be a* σ*-finite measure if there exists a sequence* $(X_n)_{n \geq 0}$ *in* τ *such that*

$$X = \bigcup_{n \geq 0} X_n \quad \text{and} \quad \mu(X_n) < \infty, \forall\, n \geq 0.$$

- *We say that* μ *is a semi-finite measure if for all* $A \in \tau$ *with* $\mu(A) = +\infty$, *then there exists* $B \in \tau$ *included in* A *and such that* $0 < \mu(B) < \infty$. *For* $g \in L^p(X)$, *we denote* l_g *the linear form on* $L^q(X)$ *defined by,*

$$l_g : f \to \int fg\mathrm{d}\mu.$$

Lemma 2.37. *Suppose that* p *and* q *are two conjugate exponents and* $1 \leq q \leq \infty$.

If $g \in L^p(X)$, *then*

$$\|g\|_q = \sup\left\{\left|l_g(f)\right|, f \in L^p(X) \text{ and } \|f\|_p = 1\right\}.$$

Moreover, if μ *is semi-finite then the previous equality is true for* $q = \infty$.

We deduce that the linear map $g \to l_g$ is an isometry of L^q in $(L^p)'$. More precisely, we have the following theorem,

Theorem 2.38. *Suppose p and q are two conjugate exponents if* $1 < 1 < \infty$. *then the linear map* $g \to l_g$ *is an isometric isomorphism from* L^q *to* $(L^p)'$. *the result remains true if* $p = 1$ *and* μ *is* σ*-finite.*

Corollary 2.39. *If* $1 < p < \infty$ *then* L^p *is a reflexive Banach space.*

For the case $p = \infty$, *the map* $g \to l_g$ *may not be surjective.*

In this case the spaces L^1 *and* L^∞ *are not reductive.*

2.7. Dual of Space $C_0(\Omega)$

Let (X, τ) a space measures a signed measure on (X, τ) is a mapping $\nu : \tau \to [-\infty, +\infty[$ or $]-\infty, +\infty]$ such that

1) $\nu(\varnothing) = 0$
2) $\forall (X_n)_{n \geq 0} \in \tau$ the series $\sum_{n \geq 0} |\nu(X_n)|$ is convergent and

$$\nu\left(\bigcup_{n\geq 0} X_n\right) = \sum_{n\geq 0} \nu\left(X_n\right).$$

If ν is a signed measure, then there is a unique pair $\left(\nu^+, \nu^-\right)$ of positive measures such that $\nu = \nu^+ - \nu^-$. We call total variation of ν the positive measure $|\nu| = \nu^+ + \nu^-$ and we define the spaces,

$$L^p\left(X,\tau,\nu\right) = L^p\left(X,\tau,\nu^+\right) \cap L^p\left(X,\tau,\nu^-\right) = L^p\left(X,\tau,|\nu|\right).$$

If ν is a complex measure, then its real part ν_r and its imaginary part ν_i are two signed measures and $\nu = \nu_r + i\nu_i$ and we define the spaces,

$$L^p\left(X,\tau,\nu\right) = L^p\left(X,\tau,\nu_r\right) \cap L^p\left(X,\tau,\nu_i\right) = L^p\left(X,\tau,|\nu|\right).$$

where $|\nu|$ is a positive measure, called the total change of ν and it is defined by

$$|\nu|(E) = \sup\left\{\left|\int_E f \mathrm{d}\nu\right| = \left|\int_E f \mathrm{d}\nu_r\right| + i\left|\int_E f \mathrm{d}\nu_i\right|, \ |f| \leq 1\right\}.$$

In the following, we move to the case $\left(X,\tau\right) = \left(\Omega, B(\Omega)\right)$ where Ω is an open set of $\mathbb{R}^d$ and $B(\Omega)$ is the Borelian tribe.

A (borelian) measure signed ν on $B(\Omega)$ is said to be of Radon if it is finite on compact sets. Let ν be a complex (borelian) measure on B, then ν_r and ν_i are Radon measures. Similarly, its total variation $|\nu|$ is a positive Radon measure.

We denote by $M(\Omega)$ the set of complex measures on $B(\Omega)$, and we define for $\nu \in M(\Omega)$ the map

$$\|\nu\| = |\nu|(\Omega).$$

Proposition 2.40. $M(\Omega)$ *is a vector space over* $\mathbb{K}$ *and the map*

$$\nu \to \|\nu\|$$

is a norm over $M(\Omega)$.

$C_0(\Omega)$ designates the complete for the norm $\|\ \|_\infty$ of the space of continuous functions with compact support in Ω. The space $C_0\left(\mathbb{R}^d\right)$ is the set of continuous functions which tend to 0 at infinity.

Theorem 2.41. (Riesz representation theorem) *Let* Ω *be an open set of* $\mathbb{R}^d$. *For* $\nu \in M(\Omega)$ *and* $f \in C_0(\Omega)$, *let*

$$l_\nu(f) = \int f \mathrm{d}\nu.$$

Then the application

$$\nu \to l_\nu$$

is an isometric isomorphism from $M(\Omega)$ *to* $\left(C_0(\Omega)\right)'$

Corollary 2.42. *If* K *is a compact of* $\mathbb{R}^d$, *then* $\left(C(K)\right)'$ *is isometrically isomorphic to* $M(K)$.

3. Introduction to Topological Vector Spaces

Definition 3.1. *A topological vector space (EVT) E is a vector space over* $\mathbb{K}$ *endowed with a topology* τ *such as the maps*

$$\begin{array}{ccc} E\times E & \to & E \\ (x,y) & \to & x+y, \end{array}$$

and

$$\begin{array}{ccc} \mathbb{K}\times E & \to & E \\ (\lambda,y) & \to & \lambda x, \end{array}$$

are continues.

Proposition 3.2. *Let* (E,τ) *be a topological vector space,* $a\in E$ *and* $\lambda\in\mathbb{K}$ *nonzero, then*

1) *The translation by the vector a defined by*

$$t_a : x \to x+a,$$

is a homeomorphism from E to E.

2) *The* λ *ratio scaling defined by*

$$h_\lambda : x \to \lambda x,$$

is a continuous isomorphism on E as well as its inverse.

Proof. The maps l_a and h_λ are invertible and their inverses are given by $t_a^{-1} = t_{-a}$ and $g_\lambda^{-1} = g_{\frac{1}{\lambda}}$. The continuity of the map and the multiplication by a scalar imply respectively the continuity of t_a and h_λ and likewise the continuity of their inverses.

From this proposition, we deduce that the topology τ is inveriant under translation. More precisely, if $A\subset E$ is an open set, then its translates $a+A$, with $a\in E$ are open sets of *E*. Consequently, the topology τ is completely determined by the given basis of neighborhoods of any point of *E*, in particular at 0. Thus in a topological vector space (E,τ) the term neighborhood base always means a neighborhood base of 0.

Proposition 3.3. *Let E be a topological vector space and U an open neighborhood of 0 in E*

1) The open set U is an absorbing subset of E.

2) There exists $V\subset U$ *an equilibrium open neighborhood of 0. Moreover, for all* $t\geq 1$ *we have* $V\subset tV$.

Proof. Let U be an open neighborhood of 0 in *E*. Let $\alpha>0$ be a real number. We denote

$$D_{\mathbb{K}}(0,\alpha) = \{\lambda\in\mathbb{K}, |\lambda|<\alpha\}.$$

1) Let $a\in E$. The map $f_a:\mathbb{K}\to E$ defined by $f_a(\lambda)=\lambda a$ is continuous. So the reciprocal image of U by f_a is an open neighborhood of 0 in $\mathbb{K}$. So there exists $\alpha_a>0$ such that

$$D_{\mathbb{K}}(0,\alpha_a)\subset f_a^{-1}(U),$$

so $f_a(D_{\mathbb{K}}(0,\alpha_a))\subset U$ which proves that

$$\forall\lambda\in\mathbb{K}, |\lambda|<\alpha_a \Rightarrow f_a(\lambda)=\lambda a\in U,$$

therefore U is absorbent.

2) Let $F:\mathbb{K}\times E\to E$ be the map defined, for all $(\lambda,x)\in\mathbb{K}\times E$, by

$$F(\lambda,x)=\lambda x.$$

By definition of the topological vector space, the map F is continuous so $F^{-1}(U)$ is an open neighborhood of $(0,0)$ in $\mathbb{K}\times E$. Therefore, there is $\alpha>0$ and W an open neighborhood of 0 in E such that

$$D_{\mathbb{K}}(0,\alpha)\times W\subset F^{-1}(U).$$

Therefore

$$V=F(D_{\mathbb{K}}(0,\alpha)\times W)=D_{\mathbb{K}}(0,\alpha)W=\bigcup_{\lambda\in D_{\mathbb{K}}(0,\alpha)}\lambda\cdot W$$

is an open neighborhood of 0 included in U and it is equilibrium. Let $x\in V$ and $t\geq 1$ be a real. Since V is balanced then $\frac{1}{t}x\in V$ and since $x=t\left(\frac{1}{t}x\right)$ then $x\in tV$. or $V\subset tV$.

Definition 3.4. *A topological space is said to be separate if for all distinct points x and y of E there exists a neighborhood* U_x *of x and a neighborhood* V_y *of y such that* $U_x\cap V_y=\emptyset$

Proposition 3.5. *A topological vector space is separate if and only if the singleton* $\{0\}$ *is closed.*

Proof. Let E be a topological vector space.

If E is separated then it is easy to see that $E-\{0\}$ is an open and therefore $\{0\}$ is closed. Conversely, suppose that $\{0\}$ is closed.

Let $\Omega=\{(x,y)\in E\times E, x\neq y\}$ and $f:E\times E\to E$ defined by

$$f(x,y)=x-y,\quad\forall x,y\in E.$$

The map f is continuous and we have $f^{-1}=\Omega^c$. So Ω is an open set of E. Let $a,b\in E$ be distinct, hence $(a,b)\in\Omega$. Since Ω is an open, then there exists V_a an open neighborhood of a and V_b an open neighborhood of b such that $V_a\times V_b\subset\Omega$. This implies that $V_a\cap V_b=\emptyset$ and that E is a separate topological vector space. □

In the following, all topological vector spaces will be assumed to be separate

Definition 3.6. *Let E be a topological vector space.*

1) *Let* $A\subset E$. *We say that A is bounded, if for every neighborhood V of 0, there exists* $\alpha>0$ *such that*

$$\forall\beta\geq\alpha,\ A\subset\beta V.$$

2) *Let* $(x_n)_{n\geq 0}$ *be a sequence in E. We say that* $(x_n)_{n\geq 0}$ *is a Cauchy sequence if for every Ω a neighborhood of 0 in E, there exists an integer* n_0 *such that*

$$m\geq n\geq n_0\Rightarrow(x_m-x_n)\in\Omega.$$

3) *We say that E is complete if any cauchy sequence in E converges in E.*

Example 3.7.

1) *Any finite union of bounded sets of topological vector spaces TVS is bounded.*

2) *Any finite subset of a topological vector space is bounded.*

3) *Any Cauchy sequence is bounded.*

Proof.

1) Let E be a TVS and $A_1,\cdots,A_n \subset E$ bound subsets. Let U be an open neighborhood of 0. Then, for all $1\le j\le n$, there exists $\alpha_j \ge 0$ such that $\beta \ge \alpha_j$ implies that $A_j \subset \beta U$. Let $\alpha = \max_{1\ge j\ge n} \alpha_j$. SO

$$\forall \beta \ge \alpha, \bigcup_{1\le j\le n} A_j \subset \beta U,$$

this proves that $\bigcup_{1\le j\le n} A_j$ is bounded.

2) According to the proposition 3.3, any neighborhood of 0 is absorbing. so every singleton is bounded. We conclude with item 1. of 3.3.

3) Let $(x_n)_{n\ge 0}$ be a Cauchy sequence in *E*. Let *U* be a neighborhood of 0 in *E*. By proposition 3.3, there exists $V\subset U$ an equilibrium open neighborhood of 0 which satisfies $V \subset tV$ for all $t\ge 1$. There exists an integer n_0 such that for any integer $n\ge n_0$, we have $x_n \in x_{n_0} + V$. Since the singleton $\{x_{n_0}\}$ is bounded, then there exists $\alpha > 0$ such that $x_{n_0} \in \beta V$, for all $\beta \ge \alpha$. This implies that the whole

$$\{x_n, n\ge n_0\} \subset (1+\beta)V.$$

We deduce that $\{x_n, n\ge 0\}$ is bounded because it is a union finish of ensembleornes. □

Definition 3.8. *Let E be a TVS and* $K\subset E$. *We say that K is said to be a compact if from any cover of K by open sets we can extract a finite subcover.*

Proposition 3.9. *Let E be a TVS and* $K\subset E$. *We say that K is said to be a compact if from any cover of K by open sets we can extract a finite subcover.*

Proof. Let *K* be a compact of a TVS *E*. To begin, we will show that *K* is firm.

Let $y\in\Omega = K^c$. Then for all $x\in K$, since E is separated, there exists V_x, open neighborhood of *x* in *E* and $U_{x,y}$ an open neighborhood of *y* in *E* such that $V_x \cap U_{x,y} = \varnothing$. So $K \subset U_{x\in K} V_x$, since K is compact, then there exists $x_1,\cdots,x_n \in K$ such that

$$K \subset \bigcup_{i=1}^n V_{x_i}.$$

The set $U = \bigcap_{i=1}^n V_{x_i,y}$ is an open neighborhood of *y* which verifies,

$$(U\cap K) \subset U \cap \bigcup_{i=1}^n V_{x_i} \subset \bigcup_{i=1}^n \left(V_{x_i} \cap U_{x_i,y}\right) = \varnothing.$$

The open set *U* is included in Ω and contains *y* so Ω is an open set and therefore *K* is a closed set. Show that *K* is bounded, let *U* be an open neighborhood of 0 in *E*. By proposition 3.3, there exists $V\subset U$ an equilibrium open neighborhood of 0 which satisfies $V\subset tV$ for all $t\ge 1$. Therefore, for any integer $n\ge 1$ nV is an open and balanced neighborhood of 0. Moreover the sequence $(nV)_{n\ge 1}$ is increasing and it satisfies

$$K \subset \bigcup_{n\ge 1} nV = E.$$

□

So there exists an integer $n_0 \geq 1$ such that $K \subset n_0 V$ and consequently, for all $t \geq n_0$ we have

$$K \subset n_0 V \subset \frac{t}{n_0} n_0 V = tV \subset tU.$$

which shows the result.

Definition 3.10. *Let E be a TVC. We say that,*

1) *E is locally convex if E admits a basis of convex neighborhoods.*

2) *E is locally bounded if 0 admits a bounded neighborhood.*

3) *E is locally compact if 0 admits a compact neighborhood.*

4) *E is metrisable if there is a distance on CE which defines the topology of E.*

Example 3.11.

1) *Norm vector spaces are topological vector spaces. Moreover, they are E is metrisable if there is a distance on E which defines the topology of E.*

a) *locally convex if E admits a basis of convex neighborhoods.*

b) *E is locally convex, because the balls are convex.*

c) *locally bounded, because* $B_1(0)$ *is a bounded neighborhood of 0.*

d) *E is locally compact only when they are of finite dimension.*

e) *metrizable*.

2) *The norm vector spaces endowed with the weak topology* $\sigma(E, E')$ *are topological vector spaces.*

Topology Defined by a Family of Semi-Norms

Definition 3.12. *Let E be a vector space. A semi-norm on E is a map* $p : E \to [0, +\infty[$ *verifying*

1) $p(x+y) \leq p(x) + p(y)$, $\forall x, y \in E$.

2) $p(\lambda x) \leq |\lambda| p(x)$, $\forall x \in E$ *and* $\lambda \in \mathbb{K}$.

In the following E **denotes a vector space over** $\mathbb{K}$. Let *p* be a semi-norm on *E*, $a \in E$ and $r > 0$.

We call *p*-ball with center *a* and radius *r*, the set,

$$B_p(a, r) = \{x \in E, p(x-a) < r\}.$$

More generally, let $P = (p_j)_{j \in J}$ be a family of semi-norms on *E*.

It is said to be separant if it verifies,

$$\forall x \in E - \{0\}, \exists j \in J, p_j(x) \neq 0.$$

Let $a \in E$. We call *P*-ball with center *a* any set of the form,

$$W(a, p_{j_1}, dots, p_{j_k}, r) = \bigcap_{1 \leq i \leq k} B_{p_{j_i}}(a, r),$$

where $j_1, \cdots, j_k \in J$ and $r > 0$ a real number. We denote by T_P the topology on E generated by the *P*-balls. The topology T_P is inveriant under translation and the *P*-balls with center 0 form a fundamental system of neighborhood of 0.

Proposition 3.13. *Let* $P = (p_j)_{j \in J}$ *be a family of separating semi-norms on E. Then E endowed with the topology* T_P *is a locally convex and separate to-*

pological vector species. also for a sequence $(x_n)_{n\geq 0}$ *converges to x in E, if and only if for all* $j \in J$, *we have,*

$$\lim_{n\to+\infty} p_j(x - x_n) = 0.$$

Proof. Let $\varphi:(x,y)\to x+y$ defined over $E\times E$ and has values in E and Ω a neighborhood of 0 in E. Then there is $W(0, p_j,\cdots,p_{j_k},r)\subset\Omega$. inclusion,

$$W(0, p_j,\cdots,p_{j_k},r/2)\times W(0, p_j,\cdots,p_{j_k},r/2) = W(0, p_j,\cdots,p_{j_k},r)\subset\Omega,$$

Let $\varphi^{-1}(\Omega)$ is a neighborhood of $(0,0)$ IN $E\times E$ and hence φ is continuous.

Let $\phi:(\lambda,x)\to\lambda x$ defined over $\mathbb{K}\times E$ and has values in E and Ω a neighborhood of 0 in E. Then there is $W(0, p_j,\cdots,p_{j_k},r)\subset\Omega$. inclusion,

$$D_{\mathbb{K}}(0,1)\times W(0, p_j,\cdots,p_{j_k},r)\subset\varphi^{-1}(\Omega),$$

implies that $\phi^{-1}(\Omega)$ is a neighborhood of $(0,0)$ in $\mathbb{K}\times E$ and therefore ϕ is continuous. The continuity of the maps φ and ϕ implies that (E,T_P) is a topological vector space.

To show that this topology is separated, it suffices to show that {0} is a closed one. Let $a\in E$ be nonzero, then there exists $p_j\in P$ a semi-norm which satisfies $p_j(a)\neq 0$. The P- ball $W(a,p_j,r) = B_{p_j}(a,r)$ is a neighborhood of a which does not contain 0, so {0} is firm and the topological vector space E,T_P is separate. Finally, (E,T_P) is locally convex because the P-balls are convex. □

Proposition 3.14. *Let* (E,T_P) *be a topological vector space defined by a family of semi-norms.*

1) *Balls with center 0 are balanced sets.*

2) *A set* $A\subset E$ *is bounded if and only if*

$$\forall\, j\in J,\ \sup_{x\in A} p_j(x) < \infty.$$

E **denotes a topological vector space.**

Proposition 3.15. *Let l be a nonzero linear form on E. Then we have the following equivalences,*

- *l is continued on E.*
- *l is bounded on a neighborhood of 0.*
- *l is continuous at 0.*
- *The core of l is firm.*
- *The kernel of l is not dense in E.*

Definition 3.16. E *is the set of continuous linear forms on E,* E *is a vector space on* $\mathbb{K}$, *called dual (topological) space of E.*

Example 3.17. *Let* (X,μ,τ) *be a measure space. Let* $0<p<1$ *and* $L^p(X)$ *be the space of classes of functions measurable on X such that,*

$$\Delta_p(f) = \int_X |f|^p\, \mathrm{d}\mu < \infty.$$

The application Δ_p *verifies*

$$\Delta_p(f+g) \leq \Delta_p(f) + \Delta_p(g),$$

which implies that $L^p(X)$ *is a vector space over* $\mathbb{K}$ *and the map*

$$d_p(f,g)=\Delta_p(f-g),$$

defines a translation invariant distance on $L^p(X)$. *Then* $L^p(X)$ *endowed with this distance is a locally bounded topological vector space and does not contain any nontrivial convex open set. Moreover, on* $L^p(X)$ *the only continuous linear form is the zero linear form, so* $(L^p(X))'=\{0\}$.

The following Hahn-Banach theorem ensures that $LE'\neq\{0\}$, if E is locally convex not reduced to {0}.

Theorem 3.18. (Hahn-Banach) *Let E be a locally convex topological vector space. Then, for all nonzero* $a\in E$, *there exists* $l\in E'$ *satisfying* $l(a)=1$.

We are provided with two topologies defined by semi-norms,

Strong topology on E'**,** τ_b

Let J_b be the set of bounded subsets of E. For all $B\in J_b$, we consider the semi-norm q_B defined by

$$q_B(f)=\sup_{x\in B}|f(x)|.$$

We denote by τ_b the topology on E' defined by the family of semi-norms $(q_B)_{B\in J_b}$. The space E' endowed with the topology τ_b is a topological vector space.

Weak topology on E'**,** τ_f

For all $x\in E$, we consider the semi-norm p_x defined by,

$$p_x(f)=|f(x)|.$$

We denote by τ the topology on E' defined by the family of semi-norms $(p_x)_{x\in E}$. The space E' endowed with the topology τ_f is a topological vector space.

It is clear that the topology τ_f is less fine than the topology τ_b.

Proposition 3.19. *The topological vector spaces* (E',τ_f) *and* (E',τ_b) *are locally convex and separate.*

Proof. It suffices to show that the families of semi-norms $(p_x)_{x\in E}$ and $(q_B)_{B\in J_b}$ are separating.

4. Frechet Space

4.1. Definition and Properties of Frechet Spaces

Proposition 4.1. *Let E be a vector space and* $(p_n)_{n\geq 0}$ *a sequence of separating semi-norms on E. So we have,*

- *The application*

$$d(x,y)=\sum_{n\geq 0}\frac{1}{2^{n+1}}\frac{p_n(x-y)}{1+p_n(x-y)},$$

is a distance on E.

- *For all* $n\in\mathbb{N}$, *the map* $p_n:(E,d)\to[0,\infty[$ *is continuous.*
- *The topology on E defined by the family of semi-norms* $(p_n)_{n\geq 0}$ *coincides*

with the metric topology (E,d).

Proof.

- The continuity of a semi-norm p_n follows from the inequality

$$p_n(x-y) \le \frac{2^{n+1} d(x,y)}{1-2^{n+1} d(x,y)}$$

valid for all $x, y \in E$ such that $d(x,y) < \frac{1}{2^{n+1}}$.

- Let $x_2 \in E$ and Ω be a neighborhood of x_0 in (E,d). Then there exists $r > 0$ such that

$$B_d(x_0, r) = \{x \in E, d(x, x_0) < r\} \subset \Omega .$$

let $\rho = \frac{r}{2}$ and m a natural integer satisfying

$$\sum_{n \ge m+1}^{+\infty} \frac{1}{2^{n+1}} < \rho,$$

therefore

$$\sum_{n \ge m+1}^{+\infty} \frac{1}{2^{n+1}} \frac{p_n(x-x_0)}{1+p_n(x-x_0)} < \rho, \ \forall x \in E.$$

Then if

$$x \in \bigcap_{n=0}^{m} B_{p_n}(x_0, \rho),$$

we have

$$\sum_{n=0}^{m} \frac{1}{2^{n+1}} \frac{p_n(x-x_0)}{1+p_n(x-x_0)} < \sum_{n=0}^{m} \frac{1}{2^{n+1}} \rho < \rho,$$

so $d(x, x_0) < r$. As a result. $\bigcap_{0 \le n \le m} B_{p_n} \subset \Omega$. This proves that Ω is a neighborhood of x_0 for the topology defined by the semi-norms. The reciprocal is deduced from the continuity of the p_n.

Definition 4.2. *See* [6].

A Frechet space is a complete topological vector space whose topology is defined by a sequence of separate semi-norms.

Example 4.3. (Examples of Frechet spaces)

- *Finite dimensional vector spaces, Hilbert spaces and Bannach spaces are Frechet spaces.*
- $C^k(\Omega)$ *he space of functions of class* C^k *on an open set of* $\mathbb{R}^n$. *Let Ω be an open set of* $\mathbb{R}^d$ *and* $k \in \mathbb{N} \cup \{\infty\}$. *We consider* $(K_n)_{n \ge 0}$ *an increasing sequence of compact sets such that*

$$\forall\, n \ge 0, \ K_n \subset K_{n+1}^{\circ} \ \text{and} \ \Omega = \bigcup_{n \ge 0} K_n .$$

We denote by p_n *the family of semi-norms in* $C^k(\Omega)$ *defined by,*

$$p_n(f) = \sup_{x \in K_n, |\alpha| \le k} \left| \left(\partial^{\alpha} f\right)(x) \right|,$$

if k is finite, else

$$p_n(f) = \sup_{x\in K_n, |\alpha|\le n} \left|\left(\partial^\alpha f\right)(x)\right|.$$

The space $C^k(\Omega)$ *equipped with the topology* τ *defined by the sequence of semi-norms* (p_n) *is a Frechet space. Moreover, this topology does not depend on the choice of the sequence of compacts* $(K_n)_{n\ge 0}$.

A sequence (f_n) *convergent to f in* $C^k(\Omega)$ *if and only if, for any multi-index* α *such that* $|\alpha| \le k$, *the sequences* $(\partial^\alpha f_n)$ *converges uniformly on any compact from Ω to* $\partial^\alpha f$.

- $C_K^k(\Omega)$ *the subspace of* $C^k(\Omega)$ *of functions with support included in a compact K.*

Let Ω be an open set of $\mathbb{R}^d$ *and* $K \subset \Omega$ *a compact. We consider the space*

$$C_K^k(\Omega) = \left\{ f \in C^k(\Omega),\ f(x) \equiv 0 \text{ in } K^c \right\}.$$

The space $C_K^k(\Omega)$ *endowed with the induced topology is closed in* $C^k(\Omega)$ *and it is therefore a Frechet space. Indeed, let* $a \in \Omega$ *and* $l_a : f \to f(a)$ *be a linear form on* $C^k(\Omega)$. *There exists* $j \ge 0$ *such that* aK_j. *So*

$$|l_a(f)| \le p_j(f),$$

this proves that l_a *is continuous, so its kernel is a farm and we have,*

$$C_K^k(\Omega) = \bigcap\nolimits_{x\notin K} Ker(l_x).$$

If $k \ne \infty$, *then* $\left(C_K^k(\Omega), \|\ \|_k\right)$ *is a Banach space or*

$$\|f\|_k = \sup_{|\alpha|\le k} \left\|\partial^\alpha f\right\|_\infty.$$

When $k = \infty$, *we also denote* $D_K(\Omega)$ *the space* $C_K^\infty(\Omega)$.

- $L_{loc}^p(\Omega)$ *the space of functions locally* L^p.

Let Ω be an open set of $\mathbb{R}^d$ *and* $(\Omega, \Im(\Omega), dx)$ *the measure space where* $\Im(\Omega)$ *is the tribe of Lebesgue and dx is the Lebesgue measure.*

Let $p \in [1, +\infty]$. *A function f measurable on Ω is said locally* L^p *if for any compact* $K \subset \Omega$ *the function* $f \in L^p(K)$. *We denote by* $L_{loc}^p(\Omega)$ *the set of these functions. We consider* $(K_n)_{n\ge 0}$ *an increasing sequence of compacts satisfying the properties* (1). *For all* $n \ge 0$, *let* p_n *be the semi-norm in* $L_{loc}^p(\Omega)$ *defined by*

$$p_n(f) = \left\| f 1_{K_n} \right\|_p,$$

or 1_{K_n} *designates the characteristic function of* K_n. *By considering this sequence of semi-norms we endow* $L_{loc}^p(\Omega)$ *with a Frechet space structure.*

- $S(\mathbb{R}^d)$ *is the Schwartz space. A function* $f : \mathbb{R}^d \to \mathbb{C}$ *is said to have rapid decrease if for all* $\alpha \in \mathbb{N}^d$ *the function*

$$x \to x^\alpha f(x),$$

is bounded on $\mathbb{R}^d$. *We denote by* $S(\mathbb{R}^d)$ *the space of functions f of class* C^∞ *such that for* $\beta \in \mathbb{N}^d$, *the function* $\partial^\beta f$ *is fast decreasing over* $\mathbb{R}^d$. *This space is also called Schwartz space.*

We endow $S(\mathbb{R}^d)$ *with a sequence of semi-norms* $(p_{k,n})_{k\geq 0,n\geq 0}$ *defined by,*

$$p_{k,n}(f)=\sup_{|\alpha|\leq k}\sup_{x\in\mathbb{R}^d}\left[\left(1+\|x\|^2\right)^n\left|\left(\partial^\alpha f\right)(x)\right|\right].$$

Proposition 4.4. *The space* $S(\mathbb{R}^d)$ *endowed with the sequence of semi-norms* $(p_{k,n})$ *is a Frechet space.*

Proof. Let (f_m) be a cauchy sequence in $S(\mathbb{R}^d)$. So, for all $\alpha\in\mathbb{N}^d$, $(\partial^\alpha f_m)$ converges uniformly on $\mathbb{R}^d$ to a function $\partial^\alpha g$ with g is of class C^∞. It remains to show that the functions $\partial^\alpha g$ are rapidly decreasing and that the sequence (f_m) converges to g for the topology of $S(\mathbb{R}^d)$. Let $\varepsilon>0$ and $k,n\in\mathbb{N}$. There exists $m_0\in\mathbb{N}$ such that

$$p\geq m\geq m_0\Rightarrow p_{k,n}\left(f_p-f_m\right)\leq\varepsilon.$$

then, $\forall x\in\mathbb{R}^d$, $\forall\alpha\in\mathbb{N}^d/|\alpha|\leq k$ we have

$$\left(1+\|x\|^2\right)^n\left|\left(\partial^\alpha f_p\right)(x)-\partial^\alpha f_m(x)\right|\leq\varepsilon.$$

We fix x and let p tend to infinity, then we get,

$$\left(1+\|x\|^2\right)^n\left|\left(\partial^\alpha g\right)(x)-\partial^\alpha f_m(x)\right|\leq\varepsilon.$$

Therefore

$$\left(1+\|x\|^2\right)^n\left|\left(\partial^\alpha g\right)(x)\right|\leq\left(1+\|x\|^2\right)^n\left|\left(\partial^\alpha f_{m_0}\right)(x)\right|+\varepsilon.$$

So $\partial^\alpha g$ is rapidly decreasing and verified,

$$m\geq m_0\Rightarrow p_{k,n}\left(g-f_m\right)\leq\varepsilon.$$

This then proves that the Cauchy sequence (f_m) is convergent in $S(\mathbb{R}^d)$ and therefore $S(\mathbb{R}^d)$ is a Frechet space.

Proposition 4.5.

1) *The space* $S(\mathbb{R}^d)$ *is a subspace of the following topological spaces* $\left(L^p(\mathbb{R}^d),dx\right)$, $\left(L^p_{loc}(\mathbb{R}^d),dx\right)$ *and* $C^k(\mathbb{R}^d)$. *Moreover, the injection of* $S(\mathbb{R}^d)$ *into each of the preceding spaces is continued.*

2) *Let K be a compact of* $\mathbb{R}^d$. *The canonical injection of* $D_K(\mathbb{R}^d)$ *into* $S(\mathbb{R}^d)$ *is continuous.*

4.2. Continuous Linear Forms and Dual Space (See [7])

Proposition 4.6. *Let E be a Frechet space. Then its dual E is complete for the strong topology* τ_b.

Definition 4.7. *Let E be a Frechet space. The strong bidual E'' of E is the strong dual of the strong dual E of E. We say that E is reflexive if the canonical injection of E into E'' is an isomorphism of E onto E''.*

Theorem 4.8. *Let E be a Frechet space. For E' to be reflexive, it suffices that every weakly closed and bounded set in E be weakly compact.*

In particular, if the bounded firm sets of a Frechet space E are compact, then E is reflexive. In this case, we say that E is a Montel space. So the only norm vector spaces that are Montel spaces are the finite dimensional spaces.

Theorem 4.9. *Let K be a compact of* $\mathbb{R}^d$. *In* $D_K(\mathbb{R}^d)$, *any closed and bounded subset is a compact set. So* $D_K(\mathbb{R}^d)$ *is a Montel space.*

The most notable example of a Frechet space is the space $S(\mathbb{R}^d)$.

Proposition 4.10. *The space* $S(\mathbb{R}^d)$ *is a reflexive Frechet space.*

Definition 4.11. *We call tempered distribution the linear forms belonging to* $S'(\mathbb{R}^d)$ *the dual of* $S(\mathbb{R}^d)$.

Let T be a linear form on $S(\mathbb{R}^d)$. Then T is a tempered distribution if there exists $C>0$ and a pair of positive integers (k,n) such that for all $f\in S(\mathbb{R}^d)$, we have

$$|T(f)|=|\langle T,f\rangle|\le Cp_{k,n}(f)=\sup_{|\alpha|\le k}\sup_{x\in\mathbb{R}^d}\left[\left(1+\|x\|^2\right)^n\left|\left(\partial^\alpha f\right)(x)\right|\right].$$

Since $S(\mathbb{R}^d)$ is metrisable, a linear form T is a tempered distribution if it is sequentially continuous. So T is continuous if for any sequence $f_n\to 0$ in $S(\mathbb{R}^d)$, the sequence

$$|T(f_n)|=|\langle T,f_n\rangle|\to 0.$$

Let g be a measurable function on $\mathbb{R}^d$ such that for all $f\in S(\mathbb{R}^d)$, the function fg belongs to $L^1(\mathbb{R}^d)$. We denote by T_g the linear form on $S(\mathbb{R}^d)$, defined by

$$T_g(f)=\langle T_g,f\rangle=\int_{\mathbb{R}^d}g(x)f(x)\mathrm{d}x.$$

We then have,

Proposition 4.12. *If* $p\in[1,+\infty]$, *then the map* $g\to T_g$ *is a continuous injection of* $\left(L^p(\mathbb{R}^d),\|\ \|_p\right)$ *in* $S'(\mathbb{R}^d)$ *endowed with its weak topology.*

5. Inductive Limit Spaces of Frechet Spaces (See [7])

Definition 5.1. *Let E be a vector space over* $\mathbb{K}$. *We suppose that there exists a strictly increasing sequence* $(E_n)_{n\in\mathbb{N}}$ *of vector subspaces of E satisfying*

- *For all* $n\in\mathbb{N}$, E_n *is a Frechet space.*
- *For all* $n\in\mathbb{N}$, *the restriction of the topology of* E_{n+1} *to its subspace* E_n *coincides with the initial topology of* E_n.
- $E=\bigcup_{n\ge 0}E_n$.

Under these conditions, we endow E with a topology τ *as follows*

Let $\Omega\subset E$. *We say that Ω is an open set of E, if and only if, for all* $n\in\mathbb{N}$, $\Omega\cap E_n$ *is an open set of* E_n. *We thus define a topology on E, called inductive limit of Frechet spaces and we denote the space* L.F

Proposition 5.2. *Let* $E=\bigcup_{n\ge 0}E_n$ *be an inductive limiting space of Frechet spaces. So we have,*

1) *E is a separate topological vector space.*

2) *For a set* $A\subset E$ *to be bounded, it is necessary and sufficient that there exists an integer n such that* $A\subset E_n$ *and that A is bounded in* E_n.

3) *A sequence* $(x_m)_{m\ge 0}$ *is convergent in E, if and only if there exists an integer n such that* $(x_m)_{m\ge 0}$ *is a convergent sequence in* E_n.

4) *E is complete but it is not metrisable.*

Example 5.3. *Let* Ω *be an open set of* $\mathbb{R}^d$ *and* $k \in \mathbb{N} \cup \{\infty\}$.

The space $C_c^k(\Omega)$ *is the set of functions* $f : \Omega \to \mathbb{C}$ *which are of class* C^k *and have compact support.*

We consider $(K_n)_{n \geq 0}$ *an increasing sequence of compact sets such that*

$$\forall n \geq 0, \ K_n \subset K_{n+1}^{\circ} \text{ and } \Omega = \bigcup_{n \geq 0} K_n .$$

The spaces $C_{K_n}^k(\Omega)$ *are Frechet spaces and we have,*

$$C_c^k(\Omega) = \bigcup_{n \geq 0} C_{K_n}^k(\Omega).$$

We thus provide $C_c^k(\Omega)$ *with an inductive limit space structure of Frechet spaces which does not depend on the choice of the sequence of compact sets* (K_n). *When* $k = \infty$, *we denote* $D(\Omega)$ *The space* $C_c^{\infty}(\Omega)$.

5.1. Continuous Linear Forms and Dual Space

Theorem 5.4. *Let* $E = \bigcup_{n \geq 0} E_n$ *be a space* L.F .

1) *Let T be a linear form on E. Then T is continuous, if and only if the restriction of T to* E_n *is continuous for all* $n \in \mathbb{N}$.

2) *Let T be a linear form on E. Then T is continuous, if and only if it is sequentially continuous.*

Proposition 5.5. *Let E be a space* L.F . *Then its dual E' is complete for the strong topology* τ_b.

Definition 5.6. *Let E be a space* L.F . *The strong bidual E" of E is the strong dual of the strong dual E of E. We say that E is reflexive if the canonical injection of E into E" is an isomorphism of E onto E".*

Theorem 5.7. *If E is a space* L.F , *for E to be reflexive, it is necessary and sufficient that every weakly closed and bounded set in E be weakly compact.*

We deduce that, if $E = \bigcup_{n \geq 0} E_n$ *is a space* L.F *such that, for all* $n \in \mathbb{N}$, E_n *is reflexive, then E is reflexive.*

5.2. The Space of Distributions $D'(\Omega)$ See [8]

Definition 5.8. *Let* Ω *be an open set of* $\mathbb{R}$. *A distribution T on* Ω *is a continuous linear form on* $D(\Omega)$. *We note* $D'(\Omega)$ *the dual of* $D(\Omega)$.

Let T be a linear form on $D(\Omega)$. *Then T is a distribution if and only if, for any compact K the sequence of compacts satisfying the property* (1)

Therefore, for any compact $K \subset \Omega$, *there exists* $C > 0$ *and an integer* $n \geq 0$ *such that, for any* $f \in D_K(\Omega)$, *we have,*

$$p_n(f) \leq C \sup_{|\alpha| \leq k} \left\| \partial^{\alpha} f \right\|_{\infty}.$$

Let g be a measurable function on Ω such that for all $f \in D(\Omega)$, the function fg belongs to $L^1(\Omega)$. We denote by T_g the linear form on $D(\Omega)$ defined by,

$$T_g(f) = \langle T_g, f \rangle = \int_{\Omega} g(x) f(x) \mathrm{d}x.$$

Then we have,

Proposition 5.9.

1) If $p \in [1, +\infty]$, then the map $g \to T_g$ is a continuous injection of $\left(L^p_{loc}(\Omega), \| \|_p\right)$ daNS $D'(\Omega)$ endowed with its weak topology.

2) $S'(\mathbb{R}^d) \subset D'(\mathbb{R}^d)$ and this injection is continuous.

Definition 5.10. *Let* $T \in D'(\Omega)$.

1) *Let F be a farm of Ω). We say that the support of T is included in F if for all* $f \in D(\Omega)$ *the condition* $\text{support}(f) \cap F = \emptyset$ *implies* $\langle T, f \; range = 0$.

2) *We say that T has compact support if its support is included in a compact. We denote by* $\xi'(\Omega)$ *the space of distributions* $T \in D'(\Omega)$ *which have compact support.*

3) *We say that T is of finite order if there exists an integer* $k \geq 0$ *such that for every compact* $K \subset \Omega$, *there exists a constant* $C > 0$ *such that, for every* $f \in D_K(\Omega 0$, *we have,*

$$p_n(f) \leq C \sup_{|\alpha| \leq k} \left\| \partial^\alpha f \right\|_\infty .$$

If T is of finite order, we call order of T the smallest integer k which satisfies the preceding inequality. We denote $D'^k(\Omega)$ *the subspace of distributions of order less than or equal to k.*

If T is not of finite order, we say that it is of infinite order.

Example 5.11. *Let Ω be an open set of* $\mathbb{R}^d$ *and* $a \in \Omega$.

- *Let* $\alpha \in \mathbb{N}^d$, *then the map* $f \to \left\langle \sigma_a^{(\alpha)}, f \right\rangle = \left(\partial^\alpha f\right)(a)$ *is a distribution with support* $\{a\}$. *Moreover,* $\sigma_a^{(\alpha)}$ *is of finite order equal to* $|\alpha|$.
- *Let* $T \in D'(\mathbb{R})$ *be defined by*

$$\langle T, f \rangle = \sum_{n \geq 0} \left\langle \sigma_n^{(n)}, f \right\rangle = \sum_{n \geq 0} f^{(n)}(n),$$

Then, T is a distribution of infinite order and inclusive support $\mathbb{N}$.

6. Conclusions

This work examines the topology of classical functional spaces, such as standard spaces, metrizable spaces, and those that cannot be metrizable. It presents the fundamental topological properties of topological vector spaces, as well as the space $S(\mathbb{R}^n)$ of functions of class C^∞ on $\mathbb{R}^n$ and its rapidly decreasing partial derivatives. It also defines a topological structure on an increasing union of Frechet spaces, called the inductive limit of Frechet spaces, and studies the space $D(\Omega)$ of functions of class C^∞ with compact supports on Ω.

Let $T \in D'(\Omega)$, let $T \in D'(\Omega)$ be a distribution with support included in a compact *K*. Let K_1 be a compact satisfying $K \subset K_1^\circ \subset K_1 \subset \Omega$ and $\varphi \in D(\Omega)$ such that $\varphi(x) = 1$ for all $x \in K_1$. Then, for all $f \in D(\Omega)$, we have,

$$\langle T, f \rangle = \langle T, \varphi f \rangle,$$

this is what gives that any distribution with compact support is of finite order. Let $k \in \mathbb{N}$ and Ω be an open set of $\mathbb{R}^d$. Then the dual of $C_c^k(\Omega)$ is $D'^k(\Omega)$

the space of distributions of order less than or equal to k.

Let $k \in \mathbb{N} \cup \{\infty\}$ and an open set Ω an open set of $\mathbb{R}^d$. Then the dual of $C^k(\Omega)$ is $D'^k(\Omega) \cap \xi'(\Omega)$, the space of distributions of order less than or equal to k and has compact support. Finally, for E the space of continuous and rapidly decreasing functions $f : \mathbb{R}^d \to \mathbb{C}$. The space E endowed with the family of semi-norms (see [9] [10])

$$p_n(f) = \left\| \left(1 + \|x\|^2\right)^n f \right\|_\infty$$

is a Frechet space. Its dual E' is equal to $S'\left(\mathbb{R}^d\right) \cap D'^0\left(\mathbb{R}^d\right)$, the space of tempered distributions of order 0. We say that these distributions are temperate measures.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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