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On the Cauchy Problem for a 1D Euler-Alignment System in Besov Spaces

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Abstract

In this paper, we investigate a 1D pressureless Euler-alignment system with a non-local alignment term, describing a kind of self-organizing problem for flocking. As a result, by the transport equation theory and Lagrange coordinate transformation, the local well-posedness of the solutions for the 1D pressureless Euler-alignment in Besov spaces $B_{p,1}^{1+\frac{1}{p}} \times B_{p,1}^{\frac{1}{p}}$ with $1 \leq p < \infty$ is established. Next, the ill-posedness of the solutions for this model in Besov spaces $B_{p,\infty}^{s+1} \times B_{p,\infty}^{s}$ with $1 \leq p \leq \infty$ and $s > \max\left\{\frac{1}{p}, \frac{1}{2}\right\}$ is also deduced. Finally, the precise blow-up criteria of the solutions for this system is presented in Besov spaces $B_{p,1}^{1+\frac{1}{p}} \times B_{p,1}^{\frac{1}{p}}$ with $1 \leq p < \infty$.

Keywords

Euler-Alignment Equations, Local Well-Posedness, Blow-Up Criteria, Ill-Posedness

1. Introduction

The Cucker-Smale Model. The flocking behaviors are widespread in biological systems, such as the swimming of fish, the movement of wildebeest groups and the migration of birds called as self-organization biological behaviors, which have attracted much attention in [1] [2] [3] [4] [5]. Understanding the population properties of interacting systems, and how individual components function, are important questions. In biology, studying the collective behaviors of animals can better understand the structure of ecosystems and provide guidance for ecosystem management and conservation. In medicine, such as cancer cells, there is a collective arrangement of patterns in the human body, and studying this collective pattern of destructive cells can more effectively affect them and promote the understanding of disease. The flocking behavior refers to the motion of a cluster of finite particles in which the velocity of each particle is consistent with the weighted average of the velocities of its neighbors. The discrete Vicsek model in time and two-dimensional spaces is usually used to describe the flocking behaviors in [6]: the velocity angle $\theta_j(t)$ of j-th particle satisfies

$$\theta_j(t+1) = \frac{1}{|\mathcal{N}_j(t)|} \sum_{i \in \mathcal{N}_j(t)} \theta_j(i) + \eta \Delta\theta. \tag{1}$$

Here, $\mathcal{N}_j(t) = \{i : |x_i(t) - x_j(t)| \le r\}$ ($r > 0$), a random variable $\Delta\theta$ is uniformly distributed in $[-1,1]$, and a parameter $\eta > 0$ is used to measure the intensity of the noise. Furthermore, a generalization of the Vicsek model was proposed by Cucker and Smale in [7]:

$$\begin{cases} \frac{\mathrm{d}x_i}{\mathrm{d}t} = v_i, \\ \frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{1}{N} \sum_{j=1}^{N} \phi(|x_i - x_j|)(v_j - v_i), \end{cases} \tag{2}$$

where $\{x_i, v_i\}_{i \in \mathbb{N}}$ refers to the position and velocity of the agents *i*, and *N* is the total number of two groups. The nonnegative and decreased communication weight function ϕ measures the strength of the interaction between two particles, since the distance of the particles increases, the interaction usually becomes smaller. The Cucker-Smale model can be used to analyze the flocking behaviors based on the decay properties of the kernel $\phi(r)$, that is to say, if $\phi(r)$ decays weaker than r^{-1} as $r \to +\infty$, the velocities $v_i(t)$ of the agents converge to a limit velocity $\bar{v}(t)$, and the relative positions $x_i(t) - x_j(t)$ also converge to a limit position $\bar{x}_{ij}$ as $r \to +\infty$. This is what we would call the flocking behavior: all particles move with nearly identical velocities.

A Kinetic Cucker-Smale Model. When the number of particles is large, to make it easier to simulate the movement of each particle, kinetic models are often used to describe the behaviors of global flocking by the density function $f(t,x,v)$ with $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$. Furthermore, about the kinetic limit of the Cucker-Smale model (2), Ha and Tadmor established a nonlinear and non-local kinetic equation in [8]:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (Q[f]f) = 0, \tag{3}$$

there $Q[f]$ is the velocity alignment force field given by

$$Q[f](t,x,v) = \int_{\mathbb{R}^{2d}} \phi(x-y)(w-v) f(t,y,w) \mathrm{d}w\mathrm{d}y. \tag{4}$$

Here, (3)-(4) is a nonlinear and non-local kinetic version of the Cucker-Smale model. When $\phi(r)$ decays weaker than r^{-1} as $r \to +\infty$, the solutions of Equations (3)-(4) show that the global flocking behaviors in [9], where the size $S(t)$ of the support in x is uniformly bounded:

$$S(t) = \sup\{|x-y| : (x,v),(y,v') \in \mathrm{supp}(f(t,\cdot,\cdot))\},$$

and the size $V(t)$ of the support in v remains decreasing:

$$V(t)=\sup\left\{|v-v'|:(x,v),(y,v')\in\operatorname{supp}\left(f(t,\cdot,\cdot)\right)\right\}\to 0 \text{ as } t\to+\infty.$$

A 1D Euler-alignment Model. In order to deduce the standard form of the hydrodynamic limit for nonlinear kinetic equations, one can consider the mono-kinetic ansatz of the form in (3)-(4)

$$f(t,x,v)\simeq\rho(t,x)\delta_{v-u(t,x)}. \tag{5}$$

In this case, the model becomes a local alignment model which is different from the global alignment model, the particles travel locally at a single speed in which the velocities of the particles are different in space. Under the one-dimensional pressureless condition, the flocking behaviors are known as a 1D pressureless Euler-alignment system with a non-local alignment term. Submitting (5) into (3)-(4), we arrive at the pressureless one-dimensional Euler-alignment system which consists of the mass conservation equation

$$\partial_t\rho+\partial_x(\rho u)=0, \tag{6}$$

and the momentum conservation equation

$$\partial_t(\rho u)+\partial_x\left(\rho u^2\right)=\int_{\mathbb{R}}\phi(x-y)\big(u(t,y)-u(t,x)\big)\rho(t,y)\rho(t,x)\,\mathrm{d}y, \tag{7}$$

where the right side of (7) is the non-local alignment term, owing to the presence of the density ρ, the density is higher, the alignment effect between the agents becomes stronger. The 1D pressureless Euler-alignment system with a non-local alignment term simulates the population movement from irregular movement to regular movement with constant relative distance and relative velocity in one-dimensional space.

Furthermore, by submitting (6) into (7), one can get the following 1D Euler-alignment system:

$$\begin{cases}\partial_t u+u\partial_x u=\phi*(u\rho)-(\phi*\rho)u, & t>0,x\in\mathbb{R},\\ \partial_t\rho+\partial_x(\rho u)=0, & t>0,x\in\mathbb{R},\\ u(0,x)=u^0(x),\ \rho(0,x)=\rho^0(x), & t=0,x\in\mathbb{R}.\end{cases} \tag{8}$$

Many researchers showed that the characteristic of communication weight ϕ plays an important role in the regularity of the solutions for system (8): when communication weight is symmetric and uniformly bounded, Carrillo showed that system (8) exists a critical threshold of initial data in [10]: if the initial data lies above the subcritical region in the sense that $\partial_x u_0\geq-\phi*\rho_0$ for all $x\in\mathbb{R}$, there has a global classical solution for 1D Euler-alignment system (8) if the initial data lies above the supercritical region in the sense that $\partial_x u_0<-\phi*\rho_0$ for all $x\in\mathbb{R}$, the solutions can blow up in a finite time for the 1D Euler-alignment system (8).

In [11], let $G=\partial_x u+\phi*\rho$, Tan rewrote model (8) as the following equivalent equations:

$$\begin{cases} \partial_t u + u\partial_x u = \phi * (u\rho) - (\phi * \rho)u, & t > 0, x \in \mathbb{R}, \\ \partial_t \rho + u\partial_x \rho = -\partial_x u\rho, & t > 0, x \in \mathbb{R}, \\ \partial_t G + u\partial_x G = -\partial_x uG, & t > 0, x \in \mathbb{R}, \\ \partial_x u = G - \phi * \rho, & t > 0, x \in \mathbb{R}, \\ \rho(0,x) = \rho^0(x), u(0,x) = u^0(x), & t = 0, x \in \mathbb{R}, \end{cases} \tag{9}$$

where communication weight is integrable, so communication weight can be either regular or weakly singular (see [12]). If communication weight is weakly singular, that is to say, it has an integrable singularity at the origin, Tan showed that system (8) exists a critical threshold of initial data: if the initial data lies above the subcritical region in the sense that $\inf_x G_0(x) > 0$ for all $x \in \mathbb{R}$, then there exists a globally regular solution for 1D Euler-alignment system (8), if the initial data lies above the supercritical region in the sense that $\inf_x G_0(x) < 0$ for all $x \in \mathbb{R}$, then the solutions blow up in a finite time for the 1D Euler-alignment system (8) in [11].

If communication weight $\phi \in L^1(\mathbb{R})$, Tan established the local well-posedness in Sobelov spaces $H^{s+1} \times H^s$ with $s > \frac{1}{2}$ on the whole real line or the periodic domain in [11]. One natural question is: whether or not the system (8) is local well-posedness in $H^{s+1} \times H^s$ for $s = \frac{1}{2}$. The Lagrange coordinate transformation does not change the dynamic nature of the system and can make the equation easier to solve. Indeed, if there exists a small enough time $T > 0$, based on that the characteristic $y(t,\xi)$ is a homeomorphism in a small time interval $[0,T]$, we will obtain the uniqueness of the solutions. Note that $B_{2,1}^s \hookrightarrow B_{2,2}^s \approx H^s$, by compactness theory and coordinate transformation, we want to explore the local well-posedness of the Cauchy problem for the 1D Euler-alignment system (8) in Besov spaces $B_{p,1}^{1+\frac{1}{p}} \times B_{p,1}^{\frac{1}{p}}$ with $1 \le p < \infty$ (in the rest of this paper $\phi \in L^1(\mathbb{R})$ unless otherwise noted).

2. Main Results

Theorem 2.1. Suppose that $p \in [1,\infty)$ and the initial data $(u_0, \rho_0, G_0) \in \left(B_{p,1}^{\frac{1}{p}}\right)^3$. Then there exists a time $T > 0$ such that Equation (9) has a unique solution (u, ρ, G) in $\left(E_T^p\right)^3$ and $E_T^p = \mathcal{C}\left([0,T]; B_{p,1}^{\frac{1}{p}}\right) \cap \mathcal{C}^1\left([0,T]; B_{p,1}^{\frac{1}{p}-1}\right)$. Moreover, the solutions depend continuously on the initial data.

Recall that $B_{p,r}^{s+1} \times B_{p,r}^s \hookrightarrow B_{p,1}^{1+\frac{1}{p}} \times B_{p,1}^{\frac{1}{p}}$ ($1 \le p, r \le \infty$) is locally compact (see Proposition 1.3.5 in [13]), using the same argument of the proof in Theorem 2.1, we can easily get the following local well-posedness of the solutions for Equation (8) in the Besov spaces $B^{s+1}_{p,r}\times B^{s}_{p,r}$:

Assume that $1\le p$, $r\le +\infty$, $s>\max\left\{\frac{1}{p},\frac{1}{2}\right\}$ and the initial data $\left(u_0,\rho_0\right)\in B^{s+1}_{p,r}\times B^{s}_{p,r}$. Then there exists a time $T>0$ such that the Cauchy problem (8) has a unique solution $(u,\rho)\in B^{s+1}_{p,r}\times B^{s}_{p,r}$, and the map $\left(u_0,\rho_0\right)\to(u,\rho)$ is continuous from a neighborhood of $\left(u_0,\rho_0\right)$ in $B^{s+1}_{p,r}\times B^{s}_{p,r}$ into

$$\mathcal{C}\left(\left[0,T\right];B^{s'+1}_{p,r}\right)\cap \mathcal{C}^1\left(\left[0,T\right];B^{s'}_{p,r}\right)\times \mathcal{C}\left(\left[0,T\right];B^{s'}_{p,r}\right)\cap \mathcal{C}^1\left(\left[0,T\right];B^{s'-1}_{p,r}\right),$$

$s'<s$ when $r=+\infty$ whereas $s'=s$ when $r<+\infty$.

Now, another natural question is raised: whether or not the data-to-solution map of the system (8) continues in $B^{s'+1}_{p,r}\times B^{s'}_{p,r}$ for $s'=s$ and $r=+\infty$. In the following theorem, we deduce that this data-to-solution map is in ill-posedness in $B^{s+1}_{p,\infty}\times B^{s}_{p,\infty}$.

Theorem 2.2. Suppose that $1\le p\le\infty$ and $s>\max\left\{\frac{1}{p},\frac{1}{2}\right\}$, then the system (9) is ill-posedness in Besov spaces $\left(B^{s}_{p,\infty}\right)^3$. More precisely, there exists $\left(u_0,\rho_0,G_0\right)\in\left(B^{s}_{p,\infty}\right)^3$ and a positive constant δ such that the Cauchy problem for system (9) has a unique solution $(u,\rho,G)\in L^\infty\left(\left[0,T\right];\left(B^{s}_{p,\infty}\right)^3\right)$ for some $T=T\left(\left\|u_0\right\|_{B^{s}_{p,\infty}},\left\|\rho_0\right\|_{B^{s}_{p,\infty}},\left\|G_0\right\|_{B^{s}_{p,\infty}}\right)$, while

$$\liminf_{t\to 0}\left(\left\|u-u_0\right\|_{B^{s}_{p,\infty}}+\left\|\rho-\rho_0\right\|_{B^{s}_{p,\infty}}+\left\|G-G_0\right\|_{B^{s}_{p,\infty}}\right)\ge C\delta.$$

Theorem 1.3 in [11] shows that the solutions admit a finite time blow up for the 1D Euler-alignment system (8), in the sense of $\inf_x G_0(x)<0$ for all $x\in\mathbb{R}$, if and only if $\lim_{t\to T^-}G(t,\cdot)=-\infty$. Theorem 2.1 in [11] shows that the solutions of the system (8) stay smooth up to time T, in the sense of $(\rho,G)\in\mathcal{C}\left(\left[0,T\right];H^s\cap L^1_+\right)\times\mathcal{C}\left(\left[0,T\right];H^s\right)$, if and only if $\int_0^T\left(\left\|\rho(\cdot,t)\right\|_{L^\infty}+\left\|G(\cdot,t)\right\|_{L^\infty}\right)\mathrm{d}t<+\infty$. Next, we show that the solutions of the system (8) stay smooth only depending on the slope of u but not involving the components of G and ρ in the following theorem.

Theorem 2.3. Suppose that $(u,\rho,G)\in\left(\mathcal{C}\left(\left[0,T\right];B^{\frac{1}{p}}_{p,1}\right)\right)^3$ is the solution of the Cauchy problem (9) with the initial data $\left(u_0,\rho_0,G_0\right)\in\left(B^{\frac{1}{p}}_{p,1}\right)^3$. Let T is the maximal existence time of the solutions (u,ρ,G) to Equation (9). Then the solutions blow up in finite time if and only if

$$\lim_{t\to T^-}\inf_{x\in\mathbb{R}}u_x(t,\cdot)=-\infty.$$

The paper is organized as follows. In Section 2, we introduce several important results on the Littlewood-Paley decomposition, the nonhomogeneous Besov spaces and their useful properties. In Section 3, we establish the local well-posedness result in Besov spaces of the solutions for Equation (9). Moreover, we prove the blow-up criteria of the solutions to the problem (9) in Section 4. Finally, the ill-posedness result of the solutions for Equation (9) is presented in Section 5. Note that we denote a general constant $C>0$ only depending on s and $\|\phi\|_{L^1}$, since all function spaces in the following sections are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in the notation of function spaces if there is no ambiguity.

3. Preliminaries

In this section, for the convenience of readers, we introduce some facts on the Littlewood-Paley theory, which is frequently used in the following arguments. Then we introduce some properties of nonhomogeneous Besov spaces which will play a key role in proving the local well-posedness and other properties of the system (9). One may refer to [13] [14] for more details.

Proposition 3.1. (See Proposition 2.10 in [13]) Let $\mathbb{B} \doteq \left\{\xi \in \mathbb{R}^d, |\xi| \le \frac{4}{3}\right\}$ and $\mathbb{C} \doteq \left\{\xi \in \mathbb{R}^d, \frac{4}{3} \le |\xi| \le \frac{8}{3}\right\}$. There exists two radial functions $\chi \in \mathcal{C}_c^\infty(\mathbb{B})$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{C})$ such that

$$\chi(\xi) + \sum_{q\ge 0} \varphi\left(2^{-q}\xi\right) = 1, \text{ for all } \xi \in \mathbb{R}^d,$$

$$|q-q'| \ge 2 \Rightarrow \operatorname{Supp} \varphi\left(2^{-q}\cdot\right) \cap \operatorname{Supp} \varphi\left(2^{-q'}\cdot\right) = \varnothing,$$

$$q \ge 1 \Rightarrow \operatorname{Supp} \chi(\cdot) \cap \operatorname{Supp} \varphi\left(2^{-q'}\cdot\right) = \varnothing,$$

$$\frac{1}{3} \le \chi(\xi)^2 + \sum_{q\ge 0} \varphi\left(2^{-q}\xi\right)^2 \le 1, \text{ for all } \xi \in \mathbb{R}^d.$$

Moreover, let $h \doteq \mathcal{F}^{-1}\varphi$ and $\tilde{h} \doteq \mathcal{F}^{-1}\chi$. Then for all $f \in \mathcal{S}'\left(\mathbb{R}^d\right)$, the dyadic operators Δ_q and S_q can be defined as follows

$$\Delta_q f \doteq \varphi\left(2^{-q}D\right) f = 2^{qd} \int_{\mathbb{R}^d} h\left(2^q y\right) f(x-y) \mathrm{d}y, \text{ for } q \ge 0,$$

$$S_q f \doteq \chi\left(2^{-q}D\right) f = 2^{qd} \int_{\mathbb{R}^d} \tilde{h}\left(2^q y\right) f(x-y) \mathrm{d}y,$$

$$\Delta_{-1} f \doteq S_0 f \text{ and } \Delta_q f \doteq 0 \text{ for } q \le -2,$$

where $f = \sum_{q\ge 0} \Delta_q f$ in $\mathcal{S}'\left(\mathbb{R}^d\right)$, and the right-hand side is called the nonhomogeneous Littlewood-Paley decomposition of f.

Definition 3.2. (See Definition 2.68 in [13]) Let $s \in \mathbb{R}$, $1 \le p$, $r \le \infty$. The nonhomogenous Besov space $B^s_{p,r}\left(\mathbb{R}^d\right)$ is defined by

$$B^s_{p,r} \doteq \left\{ f \in \mathcal{S}'\left(\mathbb{R}^d\right); \|f\|_{B^s_{p,r}} < \infty \right\},$$

where

$$\|f\|_{B^s_{p,r}} \doteq \begin{cases} \left(\sum_{q\in\mathbb{Z}} 2^{qsr} \left\|\Delta_q f\right\|_{L_p}^r \right)^{\frac{1}{r}}, & \text{for } r < \infty, \\ \sup_{q\in\mathbb{Z}} 2^{qs} \left\|\Delta_q f\right\|_{L_p}, & \text{for } r = \infty. \end{cases}$$

If $s=\infty$, $B_{p,r}^{\infty} \doteq \bigcap_{s\in\mathbb{R}} B_{p,r}^{s}$.

Proposition 3.3. (See Corollary 2.86 in [13]) For any positive real number s and any (p,r) in $[1,\infty]^2$, the space $L^{\infty}(\mathbb{R}^d)\cap B_{p,r}^{s}(\mathbb{R}^d)$ is an algebra and a constant C exists such that

$$\|uv\|_{B_{p,r}^{s}(\mathbb{R}^d)} \le C\left(\|u\|_{L^{\infty}(\mathbb{R}^d)}\|v\|_{B_{p,r}^{s}(\mathbb{R}^d)}+\|u\|_{B_{p,r}^{s}(\mathbb{R}^d)}\|v\|_{L^{\infty}(\mathbb{R}^d)}\right).$$

If $s>\frac{d}{p}$ or $s=\frac{d}{p}$, $r=1$, then we have

$$\|uv\|_{B_{p,r}^{s}(\mathbb{R}^d)} \le C\|u\|_{B_{p,r}^{s}(\mathbb{R}^d)}\|v\|_{B_{p,r}^{s}(\mathbb{R}^d)}.$$

Proposition 3.4. Suppose that $s\in\mathbb{R}$, $1\le p,r,p_i,r_i\le\infty$ $(i=1,2)$. We have

1) (See Proposition 1.3.5 in [14]) Topological properties: $B_{p,r}^{s}$ is a Banach space which is continuously embedded in $\mathcal{S}'$.

2) (See Proposition 1.3.5 in [14]) Density: C_c^{∞} is dense in $B_{p,r}^{s}$ $\Leftrightarrow$ $1\le p,r<\infty$.

3) (See Proposition 1.3.5 in [14]) Embedding: $B_{p_1,r_1}^{s}\hookrightarrow B_{p_2,r_2}^{s-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}$, if $p_1\le p_2$ and $r_1\le r_2$. $B_{p,r_2}^{s_2}\hookrightarrow B_{p,r_1}^{s_1}$ is locally compact, if $s_1<s_2$.

4) (See Proposition 1.4.3 in [14]) Algebraic properties: for all $s>0$, $B_{p,r}^{s}\cap L^{\infty}$ is an algebra. Moreover, $B_{p,r}^{s}$ is an algebra $\Leftrightarrow$ $B_{p,r}^{s}\hookrightarrow L^{\infty}$ $\Leftrightarrow$ $s>\frac{n}{p}$ (or $s=\frac{n}{p}$ and $r=1$).

5) (See Proposition 1.3.5 in [14]) Complex interpolation:

$$\|u\|_{B_{p,r}^{\theta s_1+(1-\theta)s_2}} \le C\|u\|_{B_{p,r}^{s_1}}^{\theta}\|u\|_{B_{p,r}^{s_2}}^{1-\theta},$$

for all $u\in B_{p,r}^{s_1}\cap B_{p,r}^{s_2}$ and $\theta\in[0,1]$.

6) (See Proposition 1.3.5 in [14]) Fatou lemma: If $(u_n)_{n\in\mathbb{N}}$ is bounded in $B_{p,r}^{s}$ and $u_n\to u$ in $\mathcal{S}'$, then $u\in B_{p,r}^{s}$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ exists such that

$$\|u\|_{B_{p,r}^{s}} \le \liminf_{k\to\infty}\|u_{n_k}\|_{B_{p,r}^{s}}.$$

Lemma 3.5. (See Lemma 4.1 in [15]) The transport equation is one of the fundamental partial differential equations and appears in many mathematical problems. By virtue of the uniqueness of the transport equation, one obtains the estimates of the source term f. Let $y_0\in B_{p,1}^{\frac{1}{p}}$ with $1\le p<\infty$ and $f\in L^1\left([0,T];B_{p,1}^{\frac{1}{p}}\right)$. Define $\overline{\mathbb{N}}=\mathbb{N}\cup\infty$, for $n\in\overline{\mathbb{N}}$, denote by $y_n\in C\left([0,T];B_{p,1}^{\frac{1}{p}}\right)$ the solutions of

$$\begin{cases}\partial_t y_n + A_n(u)\partial_x y_n = f, & x\in\mathbb{R},\\ y_n(t,x)|_{t=0} = y_0(x).\end{cases}$$

Assume for some $\alpha(t)\in L^1([0,T])$, $\sup_{n\in\mathbb{N}}\|A_n(u)\|_{B_{p,1}^{1+\frac{1}{p}}}\le\alpha(t)$. If $A_n(u)$ converges in $A_\infty(u)$ in $L^1\left([0,T];B_{p,1}^{\frac{1}{p}}\right)$, then the sequence $(y_n)_{n\in\mathbb{N}}$ converges in $\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}}\right)$.

Lemma 3.6. (See Lemma 2.8 in [16]) Suppose that $(p,r)\in[1,+\infty]^2$ and $s>-\min\left\{\frac{1}{p},1-\frac{1}{p}\right\}$. Assume $f_0\in B_{p,r}^s$, $F\in L^1([0,T];B_{p,r}^s)$ and

$$\begin{cases}\partial_x v\in L^1([0,T];B_{p,r}^{s-1}), & \text{if } s>1+\frac{1}{p}\ \left(s=1+\frac{1}{p},r=1\right),\\ \partial_x v\in L^1([0,T];B_{p,r}^{s}), & \text{if } s=1+\frac{1}{p},r>1,\\ \partial_x v\in L^1\left([0,T];B_{p,\infty}^{\frac{1}{p}}\cap L^\infty\right), & \text{if } s<1+\frac{1}{p}.\end{cases}$$

If $f\in L^\infty([0,T];B_{p,r}^s)\cap\mathcal{C}([0,T];\mathcal{S}')$ solves

$$\begin{cases}\partial_t f + v\cdot\partial_x f = F, & t>0,x\in\mathbb{R},\\ f|_{t=0}=f_0.\end{cases}$$

1) Then there exists a constant C such that the following statements

$$\|f(t)\|_{B_{p,r}^s}\le e^{CV(t)}\left(\|f_0\|_{B_{p,r}^s}+\int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B_{p,r}^s}\,d\tau\right),$$

where

$$V(t)=\begin{cases}\int_0^t\|\partial_x v\|_{B_{p,r}^{\frac{1}{p}}\cap L^\infty}\,d\tau, & \text{if } s<1+\frac{1}{p},\\ \int_0^t\|\partial_x v\|_{B_{p,r}^{1+\frac{1}{p}}}\,d\tau, & \text{if } s=1+\frac{1}{p},r>1,\\ \int_0^t\|\partial_x v\|_{B_{p,r}^{s-1}}\,d\tau, & \text{if } s>1+\frac{1}{p}\left(\text{or } s=1+\frac{1}{p},r=1\right).\end{cases}$$

2) If $f=v$, then for all $s>0$, (1) holds with $V(t)=\int_0^t\|\partial_x v\|_{L^\infty}\,d\tau$.

3) If $r<\infty$, then $f\in\mathcal{C}([0,T];B_{p,r}^s)$. If $r=\infty$, then $f\in\mathcal{C}([0,T];B_{p,1}^{s'})$ for all $s'<s$.

Lemma 3.7. (See Corollary 2.86 in [13]) Assume that $1\le p$, $r\le\infty$, for $s_1\le\frac{1}{p}$, $s_2>\frac{1}{p}$, ($s_2\ge\frac{1}{p}$ if $r=1$) and $s_1+s_2>0$, the following estimates holds

$$\|fg\|_{B_{p,r}^{s_1}}\le C\|f\|_{B_{p,r}^{s_1}}\|g\|_{B_{p,r}^{s_2}},$$

where the constant C is independent of f and g.

Lemma 3.8. (See Theorem 2.100 in [13]) Let $s>0$, $1\le r\le\infty$ and $1\le p\le p_1\le\infty$. Let v be a vector field over $\mathbb{R}^d$. Define $R_j=\left[v\cdot\nabla,\Delta_j\right]f$. There exists a constant C such that

$$\left\|\left(2^{js}\left\|R_j\right\|_{L^p\left(\mathbb{R}^d\right)}\right)_j\right\|_{l^r}\le C\left(\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,r}}+\|\nabla f\|_{L^{p_2}}\|\nabla v\|_{B^{s-1}_{p_1,r}}\right),$$

where $\frac{1}{p_2}=\frac{1}{p}-\frac{1}{p_1}$. Furthermore, if $s<1$ then

$$\left\|\left(2^{js}\left\|R_j\right\|_{L^p\left(\mathbb{R}^d\right)}\right)_j\right\|_{l^r}\le C\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,r}}.$$

4. Local Well-Posedness

From the relationship $\partial_x u=G-\phi*\rho$, we can claim that

$$\|\partial_x u\|_{L^\infty}\le\|\phi\|_{L^1}\|\rho\|_{L^\infty}+\|G\|_{L^\infty},\ \|\partial_x u\|_{B^s_{p,r}}\le\|\phi\|_{L^1}\|\rho\|_{B^s_{p,r}}+\|G\|_{B^s_{p,r}}. \tag{1}$$

Firstly, Young inequality results in $\|\partial_x u\|_{L^\infty}\le\|\phi*\rho\|_{L^\infty}+\|G\|_{L^\infty}\le\|\phi\|_{L^1}\|\rho\|_{L^\infty}+\|G\|_{L^\infty}$. On the other hand, according to the definition of Besov spaces, we can get

$$\begin{aligned}\Delta_j(\phi*\rho)&=2^j\int_{\mathbb{R}}h\left(2^jy\right)(\phi*\rho)(x-y)\mathrm{d}y\\&=2^j\int_{\mathbb{R}}\int_{\mathbb{R}}h\left(2^jy\right)\phi(z)\rho(x-y-z)\mathrm{d}z\mathrm{d}y\\&=\int_{\mathbb{R}}\phi(z)\int_{\mathbb{R}}2^jh\left(2^jy\right)\rho(x-y-z)\mathrm{d}z\mathrm{d}y\\&=\phi*\Delta_j\rho.\end{aligned}$$

Applying Δ_j to $\partial_x u=G-\phi*\rho$ and taking L^p-norm to the above relationship yields

$$\left\|\Delta_j\partial_x u\right\|_{L^p}\le\left\|\Delta_j\rho*\phi\right\|_{L^p}+\left\|\Delta_jG\right\|_{L^p}\le\|\phi\|_{L^1}\left\|\Delta_j\rho\right\|_{L^p}+\left\|\Delta_jG\right\|_{L^p}. \tag{2}$$

By the definition of Besov spaces and Minkowski's inequality, for $r<\infty$, we can obtain

$$\begin{aligned}\|\partial_x u\|_{B^s_{p,r}}&=\left(\sum_{j\in\mathbb{Z}}2^{jsr}\left\|\Delta_j\partial_x u\right\|^r_{L_p}\right)^{\frac{1}{r}}\\&\le\left(\sum_{j\in\mathbb{Z}}\left(\|\phi\|_{L^1}2^{js}\left\|\Delta_j\rho\right\|_{L^p}+2^{js}\left\|\Delta_jG\right\|_{L^p}\right)^r\right)^{\frac{1}{r}}\\&\le\|\phi\|_{L^1}\left(\sum_{j\in\mathbb{Z}}2^{jsr}\left\|\Delta_j\rho\right\|^r_{L^p}\right)^{\frac{1}{r}}+\left(\sum_{j\in\mathbb{Z}}2^{jsr}\left\|\Delta_jG\right\|^r_{L^p}\right)^{\frac{1}{r}}\\&=\|\phi\|_{L^1}\|\rho\|_{B^s_{p,r}}+\|G\|_{B^s_{p,r}}.\end{aligned}$$

The case $r=\infty$ can be easily treated as above, this completes the proof of the claim (3.1).

Moreover, by Bernstein-Type Lemmas (see Lemma 2.1 in [13]), one can get

$$\|u\|_{B^{s+1}_{p,r}}+\|\rho\|_{B^s_{p,r}}\cong\|u\|_{B^s_{p,r}}+\|\partial_x u\|_{B^s_{p,r}}+\|\rho\|_{B^s_{p,r}}\cong\|u\|_{B^s_{p,r}}+\|G\|_{B^s_{p,r}}+\|\rho\|_{B^s_{p,r}}.$$

Proof. In order to prove Theorem 2.1, we proceed as the following steps.

Step 1: Existence

Firstly, we aim to construct approximate solutions to smooth solutions of some linear equations. Let $z_n(t,\cdot)=(\rho_n,G_n,u_n)$, for $z_0=(\rho_0,G_0,u_0)\triangleq(0,0,0)$, we can define the induction sequence $(z_n)_{n\in\mathbb{N}}$ by solving the following linear transport equations

$$\begin{cases}\partial_t\rho_{n+1}+u_n\partial_x\rho_{n+1}=-\partial_x u_n\rho_n,\\ \partial_t G_{n+1}+u_n\partial_x G_{n+1}=-\partial_x u_n G_n,\\ \partial_t u_{n+1}+u_n\partial_x u_{n+1}=\phi*(u_n\rho_n)-(\phi*\rho_n)u_n,\\ \rho_{n+1}(0)=S_{n+1}\rho^0,\ u_{n+1}(0)=S_{n+1}u^0.\end{cases} \tag{3}$$

By induction, we firstly assume that $z_n\in\left(L^\infty\left([0,T];B^{\frac{1}{p}}_{p,1}\right)\right)^3$ for all $T>0$. Owing to $s>\frac{1}{p}$, it implies that $B^s_{p,r}$ is an algebra. Combining Lemma 3.6 and (3), we deduce that there exists a global solution $z_{n+1}\in\left(E^p_T\right)^3$ and $E^p_T=\mathcal{C}\left([0,T];B^{\frac{1}{p}}_{p,1}\right)\cap\mathcal{C}^1\left([0,T];B^{\frac{1}{p}-1}_{p,1}\right)$. Making use of Lemma 3.6, we can obtain the following inequality

$$\begin{aligned}\|\rho_{n+1}\|_{B^{\frac{1}{p}}_{p,1}}\le{}&\exp\left\{C\int_0^t\|\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}\cap L^\infty}\mathrm{d}t'\right\}\Bigg(\|\rho_{n+1}(0)\|_{B^{\frac{1}{p}}_{p,1}}\\ &+\int_0^t\exp\left\{-C\int_0^{t'}\|\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}\cap L^\infty}\mathrm{d}t''\right\}\|\rho_n\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}}\mathrm{d}t'\Bigg),\end{aligned} \tag{4}$$

because $B^{\frac{1}{p}}_{p,1}$ is an algebra and $B^{\frac{1}{p}}_{p,1}\hookrightarrow L^\infty$, according to (1), we can obtain

$$\|\rho_n\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}}\le\|\rho_n\|_{B^{\frac{1}{p}}_{p,1}}\|\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}}\le C\left(\|\rho_n\|^2_{B^{\frac{1}{p}}_{p,1}}+\|G_n\|^2_{B^{\frac{1}{p}}_{p,1}}\right),$$

$$\|\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}\cap L^\infty}\le\|\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}}\le C\left(\|\rho_n\|_{B^{\frac{1}{p}}_{p,1}}+\|G_n\|_{B^{\frac{1}{p}}_{p,1}}\right),$$

then submittting the above inequalities into (4), we can conclude

$$\begin{aligned}\|\rho_{n+1}\|_{B^{\frac{1}{p}}_{p,1}}\le{}&\exp\left\{C\int_0^t\left(\|\rho_n\|_{B^{\frac{1}{p}}_{p,1}}+\|G_n\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'\right\}\Bigg(\|\rho_{n+1}(0)\|_{B^{\frac{1}{p}}_{p,1}}\\ &+C\int_0^t\exp\left\{-C\int_0^{t'}\left(\|\rho_n\|_{B^{\frac{1}{p}}_{p,1}}+\|G_n\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t''\right\}\left(\|\rho_n\|^2_{B^{\frac{1}{p}}_{p,1}}+\|G_n\|^2_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'\Bigg).\end{aligned} \tag{5}$$

By a similar argument as above to the component *G*, we have

$$\|G_{n+1}\|_{B^{\frac{1}{p}}_{p,1}} \le \exp\left\{C\int_0^t \left(\|\rho_n\|_{B^{\frac{1}{p}}_{p,1}} + \|G_n\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'\right\}\left(\|G_{n+1}(0)\|_{B^{\frac{1}{p}}_{p,1}} + C\int_0^t \exp\left\{-C\int_0^{t'}\left(\|\rho_n\|_{B^{\frac{1}{p}}_{p,1}} + \|G_n\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t''\right\}\left(\|\rho_n\|^2_{B^{\frac{1}{p}}_{p,1}} + \|G_n\|^2_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'\right), \tag{6}$$

and

$$\|u_{n+1}\|_{B^{\frac{1}{p}}_{p,1}} \le \exp\left\{C\int_0^t \|\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}\cap L^\infty}\mathrm{d}t'\right\}\left(\|u_{n+1}(0)\|_{B^{\frac{1}{p}}_{p,1}} + \int_0^t \exp\left\{-C\int_0^{t'}\|\partial_x u_n\|_{B^{\frac{1}{p}}_{p,1}\cap L^\infty}\mathrm{d}t''\right\}\|\phi*(u_n\rho_n)-(\phi*\rho_n)u_n\|_{B^{\frac{1}{p}}_{p,1}}\mathrm{d}t'\right), \tag{7}$$

we can see that

$$\|\phi*(u_n\rho_n)-(\phi*\rho_n)u_n\|_{B^{\frac{1}{p}}_{p,1}} \le C\left(\|u_n\|^2_{B^{\frac{1}{p}}_{p,1}} + \|\rho_n\|^2_{B^{\frac{1}{p}}_{p,1}}\right),$$

then submitting the inequality into (7), we can get

$$\|u_{n+1}\|_{B^{\frac{1}{p}}_{p,1}} \le \exp\left\{C\int_0^t\left(\|\rho_n\|_{B^{\frac{1}{p}}_{p,1}} + \|G_n\|_{B^{\frac{1}{p}}_{p,1}} + \|u_n\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'\right\}\left(\|u_{n+1}(0)\|_{B^{\frac{1}{p}}_{p,1}} + C\int_0^{t'}\exp\left\{-C\int_0^t\left(\|\rho_n\|_{B^{\frac{1}{p}}_{p,1}} + \|G_n\|_{B^{\frac{1}{p}}_{p,1}} + \|u_n\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t''\right\}\left(\|u_n\|^2_{B^{\frac{1}{p}}_{p,1}} + \|\rho_n\|^2_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'\right). \tag{8}$$

Let $Z_n(t) = \|\rho_n(t,\cdot)\|_{B^{\frac{1}{p}}_{p,1}} + \|G_n(t,\cdot)\|_{B^{\frac{1}{p}}_{p,1}} + \|u_n(t,\cdot)\|_{B^{\frac{1}{p}}_{p,1}}$, combining inequalities (5), (6), and (8), we can obtain

$$Z_{n+1}(t) \le \exp\left\{C\int_0^t Z_n \mathrm{d}t'\right\}Z^0 + C\int_0^t Z_n^2 \exp\left\{C\int_\tau^{t'} Z_n \mathrm{d}t''\right\}\mathrm{d}t', \tag{9}$$

where $Z^0 = \|\rho^0\|_{B^{\frac{1}{p}}_{p,1}} + \|G^0\|_{B^{\frac{1}{p}}_{p,1}} + \|u^0\|_{B^{\frac{1}{p}}_{p,1}}$. For fixed $T^* > 0$ such that $2Cz_0T^* < 1$ and suppose that

$$\forall t \in [0, T^*],\ Z_n(t) \le \frac{Z^0}{1-2CZ^0 t}.$$

In fact, we assume that the inequality for all $n \in \mathbb{N}$ is valid, then for $0 \le \tau < t < T^* < \frac{1}{2CZ^0}$, we can have

$$\exp\left(C\int_\tau^t Z_n \mathrm{d}t'\right) \le \exp\left(\int_\tau^t \frac{CZ^0}{1-2CZ^0 t'}\mathrm{d}t'\right) = \left(\frac{1-2CZ^0\tau}{1-2CZ^0 t}\right)^{\frac{1}{2}},$$

applying the above inequalities into (9), we can get

$$
\begin{aligned}
Z_n(t) &\le \frac{Z^0}{\sqrt{1-2CZ^0t}} + \int_0^t \frac{C\left(Z^0\right)^2\left(1-2CZ^0\tau\right)^{\frac{1}{2}}}{\left(1-2CZ^0\tau\right)^2\left(1-2CZ^0t\right)^{\frac{1}{2}}}\mathrm{d}\tau \\
&= \frac{1}{\sqrt{1-2CZ^0t}}\left(Z^0 + \int_0^t \frac{C\left(Z^0\right)^2}{\left(1-2CZ^0\tau\right)^{\frac{3}{2}}}\mathrm{d}\tau\right) \\
&= \frac{1}{\sqrt{1-2CZ^0t}}\left(Z^0 + Z^0\frac{1}{\sqrt{1-2CZ^0\tau}}\Bigg|_0^t\right) \\
&= \frac{Z^0}{1-2CZ^0t}.
\end{aligned}
$$

The above derivation implies that $(z_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\left(L^\infty\left([0,T];B_{p,1}^{\frac{1}{p}}\right)\right)^3$. Based on this, we can get that $(\partial_t z_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\left(L^\infty\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)\right)^3$. Therefore, $(z_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\left(\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}}\right)\cap\mathcal{C}^{\frac{1}{2}}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)\right)^3$.

In order to obtain a solution z of Equation (9), we make use of the compactness theory for the approximating sequence $(z_n)_{n\in\mathbb{N}}$. We take a sequence $(\varphi_j)_{j\in\mathbb{N}}$ of smooth functions with values in $[0,1]$, supported in $B(0,j+1)$ and equals to 1 on $B(0,j)$. It is easy to find that the map $z_n \mapsto \varphi_j z_n$ is compact from $\left(B_{p,1}^{\frac{1}{p}}\right)^3$ to $\left(B_{p,1}^{\frac{1}{p}-1}\right)^3$ by the virtue of Theorem 2.94 in [13]. Taking advantage of Ascoli's theorem, there exists some function z_j such that the sequence $(\varphi_j z_n)_{j\in\mathbb{N}}$ converges to z_j for any $j\in\mathbb{N}$. At the same time, according to the Cantor diagonal process, there exists a subsequence of $(z_j)_{j\in\mathbb{N}}$ such that $\varphi_j z_n$ converges to z_j in $\left(\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)\right)^3$ for any $j\in\mathbb{N}$. Owing to $\varphi_j\varphi_{j+1}=\varphi_j$, we can get $z_j=\varphi_j z_{j+1}$. Hence, there exists some function z such that the sequence $(\varphi z_n)_{n\ge1}$ tends to φz in $\left(\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)\right)^3$ for any $\varphi\in\mathcal{D}$. Then on the basis of uniform boundeness of $(z_j)_{n\ge1}$ and the Fatou property, we can obtain that z is bounded in $\left(L^\infty\left([0,T];B_{p,1}^{\frac{1}{p}}\right)\right)^3$. Taking advantage of the Fatou property yields that φz_n tends to φz in $\left(\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-\varepsilon}\right)\right)^3$ for any $\varepsilon>0$

small enough.

Furthermore, set any $\psi \in B_{p',\infty}^{1-\frac{1}{p}}$ and combining with the duality for ψ, as $n \to \infty$, we can obtain

$$\begin{aligned}&\left\langle \partial_t(\varphi u_n)-\partial_t(\varphi u),\psi\right\rangle-\left\langle(\varphi u_n)\partial_x(\varphi u_n)-(\varphi u)\partial_x(\varphi u),\psi\right\rangle\\&-\left\langle \phi*(\varphi u_n\varphi\rho_n)-(\phi*\varphi\rho_n)\varphi u_n-\phi*(\varphi u\varphi\rho)-(\phi*\varphi\rho)\varphi u,\psi\right\rangle \to 0.\end{aligned}$$

The main problem is the third term, for the sake of convenience, we treat only the term of $\left\langle \phi*(\varphi u_n\varphi\rho_n)-(\phi*\varphi\rho_n)\varphi u_n-\phi*(\varphi u\varphi\rho)-(\phi*\varphi\rho)\varphi u,\psi\right\rangle$. Therefore, we can have

$$\begin{aligned}&\left|\left\langle \phi*(\varphi u_n\varphi\rho_n)-(\phi*\varphi\rho_n)\varphi u_n-\phi*(\varphi u\varphi\rho)-(\phi*\varphi\rho)\varphi u,\psi\right\rangle\right|\\&\le\left|\left\langle \phi*(\varphi u_n\varphi\rho_n)-\phi*(\varphi u\varphi\rho),\psi\right\rangle\right|+\left|\left\langle(\phi*\varphi\rho_n)\varphi u_n-(\phi*\varphi\rho)\varphi u,\psi\right\rangle\right|.\end{aligned}\tag{10}$$

Firstly, we can estimate the first term of the inequality (10)

$$\begin{aligned}&\left|\left\langle \phi*(\varphi u_n\varphi\rho_n)-\phi*(\varphi u\varphi\rho),\psi\right\rangle\right|\\&\le\left\|\phi*(\varphi u_n\varphi\rho_n)-\phi*(\varphi u\varphi\rho)\right\|_{B_{p,1}^{\frac{1}{p}-1}}\|\psi\|_{B_{p',\infty}^{1-\frac{1}{p}}}\\&\le C\left\|\varphi u_n\varphi\rho_n-\varphi u\varphi\rho\right\|_{B_{p,1}^{\frac{1}{p}-\varepsilon}}\|\psi\|_{B_{p',\infty}^{1-\frac{1}{p}}}\\&\le C\Bigg(\|\varphi u_n\|_{L^\infty}\|\varphi\rho_n-\varphi\rho\|_{B_{p,1}^{\frac{1}{p}-\varepsilon}}+\|\varphi u_n\|_{B_{p,1}^{\frac{1}{p}-\varepsilon}}\|\varphi\rho_n-\varphi\rho\|_{L^\infty}\\&\quad+\|\varphi\rho\|_{L^\infty}\|\varphi u_n-\varphi u\|_{B_{p,1}^{\frac{1}{p}-\varepsilon}}+\|\varphi\rho\|_{B_{p,1}^{\frac{1}{p}-\varepsilon}}\|\varphi u_n-\varphi u\|_{L^\infty}\Bigg)\|\psi\|_{B_{p',\infty}^{1-\frac{1}{p}}}.\end{aligned}\tag{11}$$

For the second term of the inequality (10), taking the same approach. We have Proven that $\varphi z_n \to \varphi z$ in $\left(\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-\varepsilon}\right)\right)^3$, and φz_n is bounded in $\left(L_T^\infty\left(B_{p,1}^{\frac{1}{p}}\right)\right)^3$, then we can get that (11) tends to 0 uniformly on $[0,T]$ as $n\to\infty$. So (11) tends to 0 when $n\to\infty$. Applying the similar argument, the second term of the inequality (10) also tends to 0 as $n\to\infty$, then $\partial_t(\varphi z)\in\left(\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)\right)^3$. Hence, we deduce that z is the solution of the Equation (9), and belongs to $\left(E_T^p\right)^3$.

Step 2: Uniqueness

In this step, taking advantage of the Lagrangian coordinate, we will prove the uniqueness of the smooth solution z. Introducing a new variable $\xi\in\mathbb{R}$, and we define $y(t,\xi)$ as

$$\begin{cases}\partial_t y(t,\xi)=u(t,y(t,\xi)), & t>0, x\in\mathbb{R},\\ y(t,\xi)\big|_{t=0}=\xi.\end{cases}$$

Next, Define the new variables $U(t,\xi)=u(t,y(t,\xi))$ and $V(t,\xi)=\rho(t,y(t,\xi))$, then we can obtain $\partial_\xi U(t,\xi)=\partial_x u(t,y(t,\xi))\partial_\xi y(t,y(t,\xi))$ and $\partial_\xi V(t,\xi)=\partial_x \rho(t,y(t,\xi))\partial_\xi y(t,y(t,\xi))$. At the same time, the function $U(t,\xi)$ is a solution of

$$\partial_t U(t,\xi)=\partial_t u(t,y(t,\xi))+\partial_x u(t,y(t,\xi))\partial_t y(t,\xi)\triangleq Q(t,\xi), \tag{12}$$

where

$$Q(t,\xi)=\phi*\left(u(t,y(t,\xi))\rho(t,y(t,\xi))\right)-\left(\phi*\rho(t,y(t,\xi))\right)u(t,y(t,\xi)).$$

Moreover, $V(t,\xi)$ is a solution of

$$\begin{aligned}\partial_t V(t,\xi)&=\partial_t \rho(t,y(t,\xi))+\partial_x \rho(t,y(t,\xi))\partial_t y(t,\xi)\\&=-\rho(t,y(t,\xi))\partial_x u(t,y(t,\xi)).\end{aligned} \tag{13}$$

Taking the derivative of the Equation (12) with respect to variable ξ, we can have

$$\begin{aligned}\partial_{t\xi} U(t,\xi)&=\partial_\xi Q(t,\xi)\\&=\phi*\left(\partial_\xi U\rho\right)+\phi*\left(U\partial_\xi \rho\right)-\left(\phi*\rho\right)\partial_\xi U-\left(\phi*\partial_\xi\rho\right)U.\end{aligned}$$

Similarly, we can infer that

$$\partial_{t\xi} y(t,\xi)=\partial_\xi u(t,y(t,\xi))=\partial_\xi U(t,\xi),$$

and

$$y(t,\xi)=\xi+\int_0^t U(t,\xi)\,\mathrm{d}\tau,$$

and

$$\partial_\xi y(t,\xi)=1+\int_0^t \partial_\xi U(t,\xi)\,\mathrm{d}\tau.$$

Since the fact that (u,ρ) is $\mathcal{C}_T\left(B^{\frac{1}{p}}_{p,1}\right)\times\mathcal{C}_T\left(B^{\frac{1}{p}}_{p,1}\right)$ and the embedding $B^{\frac{1}{p}}_{p,1}\hookrightarrow W^{1,p}\cap W^{1,\infty}$, we can deduce that (u,ρ) is uniformly bounded in $\mathcal{C}_T\left(W^{1,p}\cap W^{1,\infty}\right)\times\mathcal{C}_T\left(W^{1,p}\cap W^{1,\infty}\right)$ easily. In addition, we can prove that $\partial_\xi y$ is uniformly bounded in $L^\infty_T\left(L^\infty\right)$. Moreover, based on the above discussion, for sufficiently small $T>0$, we have $\frac{1}{2}\le\partial_\xi y\le C_{u_0}$. By the continuous method and the boundedness of (u,ρ) in $\mathcal{C}_T\left(W^{1,p}\cap W^{1,\infty}\right)\times\mathcal{C}_T\left(W^{1,p}\cap W^{1,\infty}\right)$, let t is small enough, we can get

$$\left\|U(t,\xi)\right\|^p_{L_p}=\int_R\left|u(t,y(t,\xi))\right|^p\frac{1}{y_\xi}\mathrm{d}y\le\|u\|^p_{L^p}\left\|\frac{1}{y_\xi}\right\|_{L^\infty}\le2\|u\|^p_{L^p}\le C,$$

$$\left\|U(t,\xi)_\xi\right\|^p_{L_p}\le\|u_x\|^p_{L^p}\left\|y_\xi\right\|^{p-1}_{L^\infty}\le C^{p-1}_{u_0}\|u_x\|^p_{L^p}\le C,$$

$$\left\|V(t,\xi)\right\|^p_{L_p}=\int_R\left|\rho(t,y(t,\xi))\right|^p\frac{1}{y_\xi}\mathrm{d}y\le\|\rho\|^p_{L^p}\left\|\frac{1}{y_\xi}\right\|_{L^\infty}\le2\|\rho\|^p_{L^p}\le C,$$

$$\left\|V(t,\xi)_\xi\right\|^p_{L_p}=\int_R\left|\rho_x(t,y(t,\xi))\right|^p\left|y_\xi\right|^{p-1}\mathrm{d}\xi\le\|\rho_x\|^p_{L^p}\left\|y_\xi\right\|^{p-1}_{L^\infty}\le C.$$

Through the above inequalities, for any $t\in[0,T]$, we have
$U(t,\xi)\in L_T^\infty(W^{1,p}\cap W^{1,\infty})$, $V(t,\xi)\in L_T^\infty(W^{1,p}\cap W^{1,\infty})$,
$y(t,\xi)-\xi\in L_T^\infty(W^{1,p}\cap W^{1,\infty})$ and $\frac{1}{2}\le\partial_\xi y\le C_{u_0}$ satisfying (12) and (13).

Set z_1,z_2 is two solutions in $(E_T^p)^3$ of Equation (9) with the same initial data, for $i=1,2$, the function $U_j(t,\xi)=u_j(t,y_j(t,\xi))$ is a solution of

$$U_{tj}(t,\xi)=u_{jt}(t,y_j)+u_{jx}(t,y_j)y_{jt}=\phi*(u_j\rho_j)-(\phi*\rho_j)u_j=Q_j.$$

Obviously, we also have $U_j(t,y_j(t,\xi)),y_j(t,\xi)-\xi\in L_T^\infty(W^{1,p}\cap W^{1,\infty})$, and $\frac{1}{2}\le\partial_{j\xi}y\le C_{u_0}$ for sufficiently small $T>0$.

Next, we will establish the estimate $\|U_1(t,\xi)-U_2(t,\xi)\|_{W^{1,p}\cap W^{1,\infty}}$, the key step is to estimate $\|Q_1(t,\xi)-Q_2(t,\xi)\|_{W^{1,p}\cap W^{1,\infty}}$. For $Q_1(t,\xi)$ and $Q_2(t,\xi)$, we can get

$$Q_1(t,\xi)-Q_2(t,\xi)=\phi*(U_1V_1-U_2V_2)-(\phi*V_1)U_1+(\phi*V_2)U_2.$$

By Young inequality and Hölder inequality, then we shall get an estimate of $L^p\cap L^\infty$ -the norm

$$\begin{aligned}
&\|Q_1(t,\xi)-Q_2(t,\xi)\|_{L^p\cap L^\infty}\\
&\le\|\phi*(U_1V_1-U_2V_2)\|_{L^p\cap L^\infty}+\|(\phi*V_1)U_1-(\phi*V_2)U_2\|_{L^p\cap L^\infty}\\
&\le C\left(\|(U_1-U_2)V_1\|_{L^p\cap L^\infty}+\|(V_1-V_2)U_2\|_{L^p\cap L^\infty}\right)\\
&\quad+\|(\phi*V_1)(U_1-U_2)\|_{L^p\cap L^\infty}+\|(\phi*(V_1-V_2))U_2\|_{L^p\cap L^\infty}\\
&\le C\left(\|U_1-U_2\|_{L^p\cap L^\infty}\|V_1\|_{L^\infty}+\|V_1-V_2\|_{L^p\cap L^\infty}\|U_2\|_{L^\infty}\right)\\
&\quad+\|U_1-U_2\|_{L^p\cap L^\infty}\|\phi*V_1\|_{L^\infty}+\|\phi*(V_1-V_2)\|_{L^p\cap L^\infty}\|U_2\|_{L^\infty}\\
&\le C\left(\|V_1-V_2\|_{L^p\cap L^\infty}+\|U_1-U_2\|_{L^p\cap L^\infty}\right).
\end{aligned}$$

Similarly, we can have

$$\begin{aligned}
Q_{1\xi}(t,\xi)-Q_{2\xi}(t,\xi)=&\ \phi*(u_1\rho_1-u_2\rho_2)_\xi-\left((\phi*\rho_1)_\xi u_1-(\phi*\rho_2)_\xi u_2\right)\\
&-\left((\phi*\rho_1)u_{1\xi}-(\phi*\rho_2)u_{2\xi}\right),
\end{aligned}$$

then we shall get an estimate about $L^p\cap L^\infty$ -norm

$$\begin{aligned}
&\|Q_{1\xi}(t,\xi)-Q_{2\xi}(t,\xi)\|_{L^p\cap L^\infty}\\
&\le C\|(u_1\rho_1-u_2\rho_2)_\xi\|_{L^p\cap L^\infty}+\|(\phi*\rho_1)_\xi u_1-(\phi*\rho_2)_\xi u_2\|_{L^p\cap L^\infty}\\
&\quad+\|(\phi*\rho_1)u_{1\xi}-(\phi*\rho_2)u_{2\xi}\|_{L^p\cap L^\infty}\\
&\le C\left(\|V_{1\xi}-V_{2\xi}\|_{L^p\cap L^\infty}+\|V_1-V_2\|_{L^p\cap L^\infty}+\|U_{1\xi}-U_{2\xi}\|_{L^p\cap L^\infty}+\|U_1-U_2\|_{L^p\cap L^\infty}\right)\\
&\le C\left(\|V_1-V_2\|_{L^p\cap L^\infty}+\|U_1-U_2\|_{W^{1,\infty}\cap W^{1,p}}\right).
\end{aligned}$$

Similarly, applying this routine process to prove, we can also get

$$\|\partial_{x_1}u_1\rho_1-\partial_{x_2}u_2\rho_2\|_{L^p\cap L^\infty}\le C\left(\|V_1-V_2\|_{L^p\cap L^\infty}+\|U_1-U_2\|_{W^{1,\infty}\cap W^{1,p}}\right).$$

Moreover,

$$
\begin{aligned}
&\|U_1-U_2\|_{W^{1,\infty}\cap W^{1,p}}+\|y_1-y_2\|_{W^{1,\infty}\cap W^{1,p}}+\|V_1-V_2\|_{L^p\cap L^\infty}\\
&\le\|U_1(0)-U_2(0)\|_{W^{1,\infty}\cap W^{1,p}}+\|y_1(0)-y_2(0)\|_{W^{1,\infty}\cap W^{1,p}}\\
&\quad+\|V_1(0)-V_2(0)\|_{L^p\cap L^\infty}+C\int_0^T\|Q_1-Q_2\|_{W^{1,\infty}\cap W^{1,p}}\\
&\quad+\|y_1-y_2\|_{W^{1,\infty}\cap W^{1,p}}+\|\partial_{x_1}u_1\rho_1-\partial_{x_2}u_2\rho_2\|_{L^p\cap L^\infty}\,\mathrm{d}t\\
&\le\|U_1(0)-U_2(0)\|_{W^{1,\infty}\cap W^{1,p}}+\|y_1(0)-y_2(0)\|_{W^{1,\infty}\cap W^{1,p}}\\
&\quad+\|V_1(0)-V_2(0)\|_{L^p\cap L^\infty}+C\int_0^T\|U_1-U_2\|_{W^{1,\infty}\cap W^{1,p}}\\
&\quad+\|y_1-y_2\|_{W^{1,\infty}\cap W^{1,p}}+\|V_1-V_2\|_{L^p\cap L^\infty}\,\mathrm{d}t.
\end{aligned}
$$

Using the Growall inequality yields

$$
\begin{aligned}
&\|U_1-U_2\|_{W^{1,\infty}\cap W^{1,p}}+\|y_1-y_2\|_{W^{1,\infty}\cap W^{1,p}}+\|V_1-V_2\|_{L^p\cap L^\infty}\\
&\le e^{CT}\left(\|U_1(0)-U_2(0)\|_{W^{1,\infty}\cap W^{1,p}}+\|y_1(0)-y_2(0)\|_{W^{1,\infty}\cap W^{1,p}}+\|V_1(0)-V_2(0)\|_{L^p\cap L^\infty}\right)\\
&\le e^{CT}\left(\|U_1(0)-U_2(0)\|_{W^{1,\infty}\cap W^{1,p}}+\|V_1(0)-V_2(0)\|_{L^p\cap L^\infty}\right)\\
&\le e^{CT}\left(\|u_1(0)-u_2(0)\|_{B^{\frac{1}{p}}_{p,1}}+\|\rho_1(0)-\rho_2(0)\|_{B^{\frac{1}{p}}_{p,1}}\right),
\end{aligned}
$$

where owing to $y_1(0)=y_2(0)=\xi$, it follows that

$$
\begin{aligned}
\|u_1-u_2\|_{L^{p-1}}&\le C\|u_1\circ y_1-u_2\circ y_1\|_{L^{p-1}}\\
&\le C\|u_1\circ y_1-u_2\circ y_2+u_2\circ y_2-u_2\circ y_1\|_{L^{p-1}}\\
&\le C\|U_1-U_2\|_{L^{p-1}}+C\|u_{2x}\|_{L^\infty}\|y_1-y_2\|_{L^{p-1}}\\
&\le C\|u_1(0)-u_2(0)\|_{B^{\frac{1}{p}}_{p,1}}.
\end{aligned}
$$

Using similar estimate, we can obtain $\|\rho_1-\rho_2\|_{L^{p-1}}\le C\|\rho_1(0)-\rho_2(0)\|_{B^{\frac{1}{p}}_{p,1}}$.

Based on the embedding $L^{p-1}\hookrightarrow B^0_{p-1,\infty}$, we can deduce

$$
\|u_1-u_2\|_{B^0_{p-1,\infty}}\le C\|u_1-u_2\|_{L^{p-1}}\le C\|u_1(0)-u_2(0)\|_{B^{\frac{1}{p}}_{p,1}},
$$

$$
\|\rho_1-\rho_2\|_{B^0_{p-1,\infty}}\le C\|\rho_1-\rho_2\|_{L^{p-1}}\le C\|\rho_1(0)-\rho_2(0)\|_{B^{\frac{1}{p}}_{p,1}}.
$$

Therefore, if $u_1(0)=u_2(0)$ and $\rho_1(0)=\rho_2(0)$, the uniqueness of the solutions is deduced. By a similar argument as above to the component G, we can conclude that z is the unique solution of the Equation (9).

Step 3: The continuous dependence

Let (u_n,ρ_n), (u_∞,ρ_∞) is the solution with the initial data (u_{0n},ρ_{0n}), $(u_{0\infty},\rho_{0\infty})$, and the initial data $(u_{0n},\rho_{0n})\hookrightarrow(u_{0\infty},\rho_{0\infty})$ in $B^{\frac{1}{p}}_{p,1}\times B^{\frac{1}{p}}_{p,1}$. Combining Step 1 and Step 2, we can obtain that (u_n,ρ_n), (u_∞,ρ_∞) are uniformly bounded in $L^\infty_T\left(B^{\frac{1}{p}}_{p,1}\right)\times L^\infty_T\left(B^{\frac{1}{p}}_{p,1}\right)$, and

$$\begin{aligned}&\left\|u_n-u_\infty\right\|_{B^0_{p-1,\infty}}\le C\left\|u_{0n}-u_{0\infty}\right\|_{B^{\frac{1}{p}}_{p,1}},\\&\left\|\rho_n-\rho_\infty\right\|_{B^0_{p-1,\infty}}\le C\left\|\rho_{0n}-\rho_{0\infty}\right\|_{B^{\frac{1}{p}}_{p,1}}.\end{aligned}\tag{14}$$

This means that (u_n,ρ_n) tends to (u_∞,ρ_∞) in $\mathcal{C}\left([0,T],B^0_{p-1,\infty}\right)\times\mathcal{C}\left([0,T],B^0_{p-1,\infty}\right)$. Under the theorem of interpolation, for any $\varepsilon>0$, we can see that $(u_n,\rho_n)\to(u_\infty,\rho_\infty)$ in $\mathcal{C}\left([0,T],B^{\frac{1}{p}-\varepsilon}_{p,1}\right)\times\mathcal{C}\left([0,T],B^{\frac{1}{p}-\varepsilon}_{p,1}\right)$. If $\varepsilon=1$, we will have $(u_n,\rho_n)\to(u_\infty,\rho_\infty)$ in $\mathcal{C}\left([0,T],B^{\frac{1}{p}-1}_{p,1}\right)\times\mathcal{C}\left([0,T],B^{\frac{1}{p}-1}_{p,1}\right)$. Combining (14) and the above relationship, we just need to prove that $(\partial_x u_n,\partial_x\rho_n)\to(\partial_x u_\infty,\partial_x\rho_\infty)$ in $\mathcal{C}\left([0,T],B^{\frac{1}{p}-1}_{p,1}\right)\times\mathcal{C}\left([0,T],B^{\frac{1}{p}-1}_{p,1}\right)$. Set $v_{1n}=\partial_x u_n$, $v_{2n}=\partial_x\rho_n$, we split $v_{1n}=z_{1n}+w_{1n}$ and $v_{2n}=z_{2n}+w_{2n}$ with (z_{1n},w_{1n}), (z_{2n},w_{2n}) satisfying the following equations

$$\begin{cases}\partial_t z_{1n}+u_n\partial_x z_{1n}=\partial_x Q_\infty-(\partial_x u_\infty)^2,\\\partial_t z_{2n}+u_n\partial_x z_{2n}=-\partial_x(\rho_\infty\partial_x u_\infty)-\partial_x u_\infty\partial_x\rho_\infty,\\z_{1n}|_{t=0}=\partial_x u_{0\infty},\ z_{2n}|_{t=0}=\partial_x\rho_{0\infty},\end{cases}$$

and

$$\begin{cases}\partial_t w_{1n}+u_n\partial_x w_{1n}=\partial_x Q_n-\partial_x Q_\infty-(\partial_x u_n)^2+(\partial_x u_\infty)^2,\\\partial_t w_{2n}+u_n\partial_x w_{2n}=-\partial_x(\rho_n\partial_x u_n-\rho_\infty\partial_x u_\infty)-\partial_x u_n\partial_x\rho_n+\partial_x u_\infty\partial_x\rho_\infty,\\w_{1n}|_{t=0}=\partial_x u_{0n}-\partial_x u_{0\infty},\ z_{2n}|_{t=0}=\partial_x\rho_{0n}-\partial_x\rho_{0\infty}.\end{cases}$$

Taking advantage of the fact that (u_n,ρ_n), (u_∞,ρ_∞) are uniformly bounded in $L^\infty_T\left(B^{\frac{1}{p}}_{p,1}\right)\times L^\infty_T\left(B^{\frac{1}{p}}_{p,1}\right)$, we can have

$$\begin{aligned}&\left\|\partial_x Q_n-\partial_x Q_\infty\right\|_{B^{\frac{1}{p}-1}_{p,1}}\\&\le\left\|\phi*(u_n\rho_n-u_\infty\rho_\infty)-(\phi*\rho_n)u_n+(\phi*\rho_\infty)u_\infty\right\|_{B^{\frac{1}{p}}_{p,1}}\\&\le C\left(\left\|\rho_n-\rho_\infty\right\|_{B^{\frac{1}{p}}_{p,1}}+\left\|u_n-u_\infty\right\|_{B^{\frac{1}{p}}_{p,1}}\right)\\&\le C\left(\left\|u_n-u_\infty\right\|_{B^{\frac{1}{p}-1}_{p,1}}+\left\|\partial_x u_n-\partial_x u_\infty\right\|_{B^{\frac{1}{p}-1}_{p,1}}+\left\|\rho_n-\rho_\infty\right\|_{B^{\frac{1}{p}-1}_{p,1}}\right.\\&\qquad\left.+\left\|\partial_x\rho_n-\partial_x\rho_\infty\right\|_{B^{\frac{1}{p}-1}_{p,1}}\right),\end{aligned}$$

and

$$\begin{aligned}
&\left\|\partial_x\left(\rho_n\partial_x u_n-\rho_\infty\partial_x u_\infty\right)\right\|_{B_{p,1}^{\frac{1}{p}-1}}\\
&\le C\left(\left\|\rho_n-\rho_\infty\right\|_{B_{p,1}^{\frac{1}{p}}}+\left\|\partial_x u_n-\partial_x u_\infty\right\|_{B_{p,1}^{\frac{1}{p}}}\right)\\
&\le C\left(\left\|\rho_n-\rho_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|\partial_x\rho_n-\partial_x\rho_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|u_n-u_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}\right.\\
&\quad\left.+\left\|\partial_x u_n-\partial_x u_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}\right).
\end{aligned}$$

Here, owing to $G=\partial_x u+\phi*\rho$ and $G\in C\left(\left[0,T\right],B_{p,1}^{\frac{1}{p}}\right)$, we can get $\left\|\partial_x u\right\|_{B_{p,1}^{\frac{1}{p}}}\le C$. In addition, applying Lemma 3.7, let $s_1=\frac{1}{p}-1$ and $s_2=\frac{1}{p}$, we can get

$$\begin{aligned}
&\left\|\left(\partial_x u_n\right)^2-\left(\partial_x u_\infty\right)^2\right\|_{B_{p,1}^{\frac{1}{p}-1}}\\
&\le C\left\|\partial_x u_n-\partial_x u_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}\left\|\partial_x u_n+\partial_x u_\infty\right\|_{B_{p,1}^{\frac{1}{p}}}\\
&\le C\left\|\partial_x u_n-\partial_x u_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}.
\end{aligned}$$

Then for all $n\in\mathbb{N}$, using the above inequalities, we can have

$$\begin{aligned}
&\partial_t\left\|w_{1n}\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\partial_t\left\|w_{2n}\right\|_{B_{p,1}^{\frac{1}{p}-1}}\\
&\le C\left(\left\|\rho_n-\rho_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|\partial_x\rho_n-\partial_x\rho_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|u_n-u_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|\partial_x u_n-\partial_x u_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}\right)\\
&\le C\left(\left\|\rho_n-\rho_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|u_n-u_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|w_{1n}-w_{1\infty}\right\|_{B_{p,1}^{\frac{1}{p}-1}}\right.\\
&\quad\left.+\left\|z_{1n}-z_{1\infty}\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|w_{2n}-w_{2\infty}\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|z_{2n}-z_{2\infty}\right\|_{B_{p,1}^{\frac{1}{p}-1}}\right)\\
&\le C\left(\left\|\rho_n-\rho_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|u_n-u_\infty\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|w_{1n}\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|z_{1n}-z_{1\infty}\right\|_{B_{p,1}^{\frac{1}{p}-1}}\right.\\
&\quad\left.+\left\|w_{2n}\right\|_{B_{p,1}^{\frac{1}{p}-1}}+\left\|z_{2n}-z_{2\infty}\right\|_{B_{p,1}^{\frac{1}{p}-1}}\right),
\end{aligned}$$

from which it follows

$$
\begin{aligned}
&\|w_{1n}\|_{B_{p,1}^{\frac{1}{p}-1}}+\|w_{2n}\|_{B_{p,1}^{\frac{1}{p}-1}}\\
&\le Ce^{Ct}\Bigg(\|v_{1n}(0)-v_{1\infty}(0)\|_{B_{p,1}^{\frac{1}{p}-1}}+\|v_{2n}(0)-v_{2\infty}(0)\|_{B_{p,1}^{\frac{1}{p}-1}}\\
&\quad+\int_0^t e^{-Ct'}\Bigg(\|\rho_n-\rho_\infty\|_{B_{p,1}^{\frac{1}{p}-1}}+\|u_n-u_\infty\|_{B_{p,1}^{\frac{1}{p}-1}}+\|w_{1n}\|_{B_{p,1}^{\frac{1}{p}-1}}\\
&\quad+\|z_{1n}-z_{1\infty}\|_{B_{p,1}^{\frac{1}{p}-1}}+\|w_{2n}\|_{B_{p,1}^{\frac{1}{p}-1}}+\|z_{2n}-z_{2\infty}\|_{B_{p,1}^{\frac{1}{p}-1}}\Bigg)dt'\Bigg).
\end{aligned}
$$

Since we prove that $v_{1n}(0)\to v_{1\infty}(0)$ and $v_{2n}(0)\to v_{2\infty}(0)$ in $B_{p,1}^{\frac{1}{p}-1}$, $(\rho_n,u_n)\to(\rho_\infty,u_\infty)$ in $\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)\times\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)$, and according to Lemma 3.5, we can deduce that $z_{1n}(t)\to z_{1\infty}(t)$ in $\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)$, $z_{2n}(t)\to z_{2\infty}(t)$ in $\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)$. Furthermore, we obtain that $w_{1n}\to 0$ in $\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)$ and $w_{2n}\to 0$ in $\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)$. Then according to Lemma 3.6 and $w_{1\infty}=w_{2\infty}=0$, we can get that $w_{1n}\to w_{1\infty}$ in $\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)$, and $w_{2n}\to w_{2\infty}$ in $\mathcal{C}\left([0,T];B_{p,1}^{\frac{1}{p}-1}\right)$.

Finally, we can deduce

$$
\begin{aligned}
&\|v_{1n}-v_{1\infty}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}+\|v_{2n}-v_{2\infty}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}\\
&\le\|z_{1n}-z_{1\infty}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}+\|w_{1n}-w_{1\infty}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}+\|z_{2n}-z_{2\infty}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}+\|w_{2n}-w_{2\infty}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}\\
&\le\|z_{1n}-z_{1\infty}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}+\|w_{1n}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}+\|z_{2n}-z_{2\infty}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)}+\|w_{2n}\|_{L_T^\infty\left(B_{p,1}^{\frac{1}{p}-1}\right)},
\end{aligned}
$$

which indicates that $(\partial_x u_n,\partial_x\rho_n)\to(\partial_x u_\infty,\partial_x\rho_\infty)$ in $\mathcal{C}\left([0,T],B_{p,1}^{\frac{1}{p}-1}\right)\times\mathcal{C}\left([0,T],B_{p,1}^{\frac{1}{p}-1}\right)$. By a similar argument as above to the component G, we can have $\partial_x z_n\to\partial_x z_\infty$ in $\left(\mathcal{C}\left([0,T],B_{p,1}^{\frac{1}{p}-1}\right)\right)^3$. Combining Step 1 to Step 3, we complete the proof of Theorem 2.1. □

5. Ill-Posedness

In this section, we shall explore that the solutions of the Equation (9) are ill-posedness in Besov spaces $\left(B_{p,r}^{s}\right)^{3}$ with the index $s>\max\left\{\frac{1}{2},\frac{1}{p}\right\}$ and $r=\infty$. To localize the frequency region, we have to introduce smooth radial cut-off functions. Set $\hat{\psi}_i\in\mathcal{C}_0^{\infty}$, $i=1,2,3$ is an non-negative, even and real-valued function on $\mathbb{R}$ satisfying

$$\hat{\psi}_i(\xi)=\begin{cases}1, & \text{if } |\xi|\le\frac{1}{4},\\ 0, & \text{if } |\xi|\ge\frac{1}{2}.\end{cases}$$

Define the initial data

$$\begin{cases}u_0(x)=\sum_{n=0}^{\infty}2^{-ns}\psi_1(x)\cos\left(\lambda_1 2^n x\right),\\ \rho_0(x)=\sum_{n=0}^{\infty}2^{-ns}\psi_2(x)\cos\left(\lambda_2 2^n x\right),\\ G_0(x)=\sum_{n=0}^{\infty}2^{-ns}\psi_3(x)\cos\left(\lambda_3 2^n x\right),\end{cases}\tag{15}$$

there $\lambda_i\in\left[\frac{67}{48},\frac{69}{48}\right]$ and $i=1,2,3$. Then we can have

$$\Delta_j\left(\psi_i(x)\cos\left(\lambda_i 2^n x\right)\right)=\begin{cases}\psi_i(x)\cos\left(\lambda_i 2^n x\right), & \text{if } j=n,\\ 0, & \text{if } j\ne n.\end{cases}$$

So we can get

$$\begin{aligned}\|u_0\|_{B_{p,\infty}^s}&=\left\|\left(2^{js}\|\Delta_j u_0\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{l^\infty}\\ &=\left\|\left(2^{js}\left\|\Delta_j\sum_{n=0}^{\infty}2^{-ns}\psi_1(x)\cos\left(\lambda_1 2^n x\right)\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{l^\infty}\\ &=\left\|\left(2^{js}\left\|2^{-js}\psi_1(x)\cos\left(\lambda_1 2^j x\right)\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{l^\infty}\\ &\le\left\|\left(\left\|\psi_1(x)\cos\left(\lambda_1 2^j x\right)\right\|_{L^p}\right)_{j\in\mathbb{Z}}\right\|_{l^\infty}\\ &\le C.\end{aligned}$$

At the same time, we can obtain

$$\|\rho_0\|_{B_{p,\infty}^s}\le C,\ \|G_0\|_{B_{p,\infty}^s}\le C.$$

On the basis of definition of Besov spaces, we can make an estimate

$$\|\phi*u\|_{B_{p,\infty}^s}=\sup_{q\in\mathbb{Z}}2^{qs}\|\Delta_q\phi*u\|_{L^p}\le\sup_{q\in\mathbb{Z}}2^{qs}\|\phi\|_{L^1}\|\Delta_q u\|_{L^p}\le C\|u\|_{B_{p,\infty}^s},$$

where C is some positive constant.

Lemma 5.1. For the above constructed initial data (u_0,ρ_0,G_0), if some n large enough, we can obtain

$$\begin{aligned}&\left\|u_0\Delta_n\partial_x u_0\right\|_{L^p}\geq C2^{-n(s-1)},\\&\left\|u_0\Delta_n\partial_x \rho_0\right\|_{L^p}\geq C2^{-n(s-1)},\\&\left\|u_0\Delta_n\partial_x G_0\right\|_{L^p}\geq C2^{-n(s-1)}.\end{aligned}\tag{16}$$

Proof. According to the definition of (15), we can get

$$\begin{aligned}u_0\Delta_n\partial_x u_0&=u_0 2^{-ns}\partial_x\left(\psi_1(x)\cos\left(\lambda_1 2^n x\right)\right)\\&=u_0 2^{-ns}\left[\partial_x\psi_1(x)\cos\left(\lambda_1 2^n x\right)-\lambda_1 2^n\psi_1(x)\sin\left(\lambda_1 2^n x\right)\right]\\&=2^{-ns}u_0\psi_{1x}\cos\left(\lambda_1 2^n x\right)-\lambda_1 2^{-n(s-1)}u_0\psi_1(x)\sin\left(\lambda_1 2^n x\right).\end{aligned}$$

Because $u_0(x)$ is a real-valued continuous function on $\mathbb{R}$, there exists $\lambda_1>0$ such that for any $x\in B_{\lambda_1}(0)$

$$\left|u_0(x)\right|\geq\frac{1}{2}u_0(0)=\frac{1}{2}\sum_{n=0}^{\infty}2^{-ns}\psi_1(0)=\frac{2^{s-1}\psi_1(0)}{2^s-1},\tag{17}$$

by applying (17), we can obtain

$$\begin{aligned}\left\|u_0\Delta_n\partial_x u_0\right\|_{L^p}&=\left\|2^{-ns}u_0\psi_{1x}\cos\left(\lambda_1 2^n x\right)-\lambda_1 2^{-n(s-1)}u_0\psi_1(x)\sin\left(\lambda_1 2^n x\right)\right\|_{L^p}\\&\geq C\left\|2^{-n(s-1)}u_0\psi_1(x)\sin\left(\lambda_1 2^n x\right)\right\|_{L^p}-\left\|2^{-ns}u_0\psi_{1x}\cos\left(\lambda_1 2^n x\right)\right\|_{L^p}\\&\geq\left(C2^n-C_1\right)2^{-ns},\end{aligned}$$

taking a large enough n such that $C_1\leq C2^{n-1}$, then we obtain (16). □

Lemma 5.2. For the above constructed initial data $(u_0,\rho_0,G_0)\in\left(B^s_{p,r}\right)^3$, there exists $T\left(\|u_0\|_{B^s_{p,\infty}},\|\rho_0\|_{B^s_{p,\infty}},\|G_0\|_{B^s_{p,\infty}}\right)$ for $0\leq t\leq T$, we can obtain

$$\left\|u(t)-u_0\right\|_{B^{s-1}_{p,\infty}}\leq Ct,\ \left\|\rho(t)-\rho_0\right\|_{B^{s-1}_{p,\infty}}\leq Ct,\ \left\|G(t)-G_0\right\|_{B^{s-1}_{p,\infty}}\leq Ct.\tag{18}$$

Proof. Owing to $(u_0,\rho_0,G_0)\in\left(B^s_{p,\infty}\right)^3$, the system (9) has a unique solution $(u,\rho,G)\in L^\infty\left([0,T];\left(B^s_{p,\infty}\right)^3\right)$, and

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\|u(t)\|_{B^s_{p,\infty}}+\|\rho(t)\|_{B^s_{p,\infty}}+\|G(t)\|_{B^s_{p,\infty}}\right)\\&\leq C\left(\|u_0\|_{B^s_{p,\infty}}+\|\rho_0\|_{B^s_{p,\infty}}+\|G_0\|_{B^s_{p,\infty}}\right).\end{aligned}\tag{19}$$

Applying the differential mean value theorem, Lemma 3.7 and (19) $t\in[0,T]$, we can obtain

$$\begin{aligned}\|u-u_0\|_{B^{s-1}_{p,\infty}}&\leq\int_0^t\|\partial_\tau u\|_{B^{s-1}_{p,\infty}}\,\mathrm{d}\tau\\&\leq\int_0^t\|u\partial_x u\|_{B^{s-1}_{p,\infty}}+\|\phi*(u\rho)-(\phi*\rho)u\|_{B^{s-1}_{p,\infty}}\,\mathrm{d}\tau\\&\leq C\int_0^t\|u\|_{B^s_{p,\infty}}\|u_x\|_{B^{s-1}_{p,\infty}}+\|u\rho\|_{B^{s-1}_{p,\infty}}+\|\phi*\rho\|_{B^s_{p,\infty}}\|u\|_{B^{s-1}_{p,\infty}}\,\mathrm{d}\tau\\&\leq C\int_0^t\|u\|^2_{B^s_{p,\infty}}+\|u\|_{B^s_{p,\infty}}\|\rho\|_{B^{s-1}_{p,\infty}}+\|\rho\|_{B^s_{p,\infty}}\|u\|_{B^s_{p,\infty}}\,\mathrm{d}\tau\\&\leq Ct,\end{aligned}$$

and

$$\begin{aligned}\|\rho-\rho_0\|_{B^{s-1}_{p,\infty}} &\le \int_0^t \|\partial_\tau \rho\|_{B^{s-1}_{p,\infty}}\,\mathrm{d}\tau \le \int_0^t \|-\partial_x u\rho\|_{B^{s-1}_{p,\infty}} + \|-u\partial_x\rho\|_{B^{s-1}_{p,\infty}}\,\mathrm{d}\tau \\ &\le \int_0^t \|u\|_{B^{s}_{p,\infty}} \|\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|\partial_x u\|_{B^{s-1}_{p,\infty}} \|\rho\|_{B^{s}_{p,\infty}}\,\mathrm{d}\tau \\ &\le C\int_0^t \|u\|_{B^{s}_{p,\infty}} \|\rho\|_{B^{s}_{p,\infty}} + \|u\|_{B^{s}_{p,\infty}} \|\rho\|_{B^{s}_{p,\infty}}\,\mathrm{d}\tau \\ &\le Ct.\end{aligned}$$

By a similar argument as above to the component G, $\|G(t)-G_0\|_{B^{s-1}_{p,\infty}} \le Ct$. Therefore, we complete the proof of Lemma 5.2. □

Lemma 5.3. According to the assumption of Theorem 2.2 for the above-constructed initial data (u_0,ρ_0,G_0), $0\le t\le T$, there $T\left(\|u_0\|_{B^{s}_{p,\infty}}, \|\rho_0\|_{B^{s}_{p,\infty}}, \|G_0\|_{B^{s}_{p,\infty}}\right)$, we can obtain

$$\begin{aligned}&\|u(t)-u_0-tw_0\|_{B^{s-2}_{p,\infty}} \le C\left(t^2+t\right),\\ &\|\rho(t)-\rho_0-tv_0\|_{B^{s-2}_{p,\infty}} \le C\left(t^2+t\right), \\ &\|G(t)-G_0-tm_0\|_{B^{s-2}_{p,\infty}} \le C\left(t^2+t\right),\end{aligned} \tag{20}$$

there $w_0=-u_0\partial_x u_0+\phi*(u_0\rho_0)-(\phi*\rho_0)u_0$, $v_0=-u_0\partial_x\rho_0-\partial_x u_0\rho_0$, and $m_0=-u_0\partial_x G_0-\partial_x u_0 G_0$.

Proof. Define

$$\tilde{u}=u(t)-u_0-tw_0,\ \tilde{\rho}=\rho(t)-\rho_0-tv_0,\ \tilde{G}=G(t)-G_0-tm_0.$$

Applying the differential mean value theorem, $t\in[0,T]$, we can obtain

$$\begin{aligned}\|\tilde{\rho}\|_{B^{s-2}_{p,\infty}} &\le \int_0^t \|\partial_\tau\rho-v_0\|_{B^{s-2}_{p,\infty}}\,\mathrm{d}\tau \\ &\le \int_0^t \|u\partial_x\rho-u_0\partial_x\rho_0\|_{B^{s-2}_{p,\infty}}\,\mathrm{d}\tau + \int_0^t \|\rho\partial_x u-\rho_0\partial_x u_0\|_{B^{s-2}_{p,\infty}}\,\mathrm{d}\tau,\end{aligned} \tag{21}$$

and

$$\begin{aligned}\|\tilde{u}\|_{B^{s-2}_{p,\infty}} &\le \int_0^t \|\partial_\tau u-w_0\|_{B^{s-2}_{p,\infty}}\,\mathrm{d}\tau \\ &\le \int_0^t \|u\partial_x u-u_0\partial_x u_0\|_{B^{s-2}_{p,\infty}}\,\mathrm{d}\tau + \int_0^t \|\phi*(u\rho)-\phi*(u_0\rho_0)\|_{B^{s-2}_{p,\infty}} \\ &\quad + \|(\phi*\rho)u-(\phi*\rho_0)u_0\|_{B^{s-2}_{p,\infty}}\,\mathrm{d}\tau.\end{aligned} \tag{22}$$

Using proposition 3.4, Lemma 3.7, and (19), we can deduce

$$\begin{aligned}&\|u\partial_x\rho-u_0\partial_x\rho_0\|_{B^{s-2}_{p,\infty}} \\ &\le \|(u-u_0)\partial_x\rho+u_0\partial_x(\rho-\rho_0)\|_{B^{s-1}_{p,\infty}} \\ &\le \|u-u_0\|_{B^{s}_{p,\infty}} \|\partial_x\rho\|_{B^{s-1}_{p,\infty}} + \|\partial_x(\rho-\rho_0)\|_{B^{s-1}_{p,\infty}} \|u_0\|_{B^{s}_{p,\infty}} \\ &\le \|u-u_0\|_{B^{s}_{p,\infty}} \|\rho\|_{B^{s}_{p,\infty}} + \|\rho-\rho_0\|_{B^{s}_{p,\infty}} \|u_0\|_{B^{s}_{p,\infty}} \\ &\le C.\end{aligned} \tag{23}$$

Taking advantage of proposition 3.4, Lemma 3.7, Lemma 5.2, and (19), we can obtain

$$\begin{aligned}
&\left\|\rho\partial_x u-\rho_0\partial_x u_0\right\|_{B^{s-2}_{p,\infty}}\\
&\le\left\|\left(\rho-\rho_0\right)\partial_x u+\rho_0\partial_x\left(u-u_0\right)\right\|_{B^{s-1}_{p,\infty}}\\
&\le\left\|\rho-\rho_0\right\|_{B^{s-1}_{p,\infty}}\left\|\partial_x u\right\|_{B^{s}_{p,\infty}}+\left\|\partial_x\left(u-u_0\right)\right\|_{B^{s-1}_{p,\infty}}\left\|\rho_0\right\|_{B^{s}_{p,\infty}}\\
&\le C\left\|\rho-\rho_0\right\|_{B^{s-1}_{p,\infty}}+\left\|u-u_0\right\|_{B^{s}_{p,\infty}}\left\|\rho_0\right\|_{B^{s}_{p,\infty}}\\
&\le C\left(\tau+1\right),
\end{aligned}\tag{24}$$

and

$$\begin{aligned}
&\left\|u\partial_x u-u_0\partial_x u_0\right\|_{B^{s-2}_{p,\infty}}\\
&\le\left\|\left(u-u_0\right)\partial_x u+u_0\partial_x\left(u-u_0\right)\right\|_{B^{s-1}_{p,\infty}}\\
&\le\left\|u-u_0\right\|_{B^{s-1}_{p,\infty}}\left\|\partial_x u\right\|_{B^{s}_{p,\infty}}+\left\|u-u_0\right\|_{B^{s}_{p,\infty}}\left\|u_0\right\|_{B^{s}_{p,\infty}}\\
&\le C\left(\tau+1\right).
\end{aligned}\tag{25}$$

Applying proposition 3.4, Lemma 3.7, and (19), we can get

$$\begin{aligned}
&\left\|\phi*\left(u\rho-u_0\rho_0\right)\right\|_{B^{s-2}_{p,\infty}}\\
&\le C\left\|u\rho-u_0\rho_0\right\|_{B^{s-1}_{p,\infty}}\\
&\le C\left\|\left(u-u_0\right)\rho+\left(\rho-\rho_0\right)u_0\right\|_{B^{s-1}_{p,\infty}}\\
&\le C\left\|u-u_0\right\|_{B^{s-1}_{p,\infty}}\left\|\rho\right\|_{B^{s}_{p,\infty}}+\left\|\rho-\rho_0\right\|_{B^{s-1}_{p,\infty}}\left\|u_0\right\|_{B^{s}_{p,\infty}}\\
&\le C\tau,
\end{aligned}\tag{26}$$

and

$$\begin{aligned}
&\left\|\left(\phi*\rho\right)u-\left(\phi*\rho_0\right)u_0\right\|_{B^{s-2}_{p,\infty}}\\
&\le\left\|\left(\phi*\rho\right)u-\left(\phi*\rho\right)u_0+\left(\phi*\rho\right)u_0-\left(\phi*\rho_0\right)u_0\right\|_{B^{s-1}_{p,\infty}}\\
&\le\left\|u-u_0\right\|_{B^{s-1}_{p,\infty}}\left\|\phi*\rho\right\|_{B^{s}_{p,\infty}}+\left\|\phi*\left(\rho-\rho_0\right)\right\|_{B^{s-1}_{p,\infty}}\left\|u_0\right\|_{B^{s}_{p,\infty}}\\
&\le C\tau.
\end{aligned}\tag{27}$$

Taking (23)-(24) into (21) and (25)-(27) into (22), we can deduce

$$\left\|\tilde{\rho}\right\|_{B^{s-2}_{p,\infty}}\le C\int_0^t\left(\tau+1\right)\mathrm{d}\tau\le C\left(t^2+t\right),$$

$$\left\|\tilde{u}\right\|_{B^{s-2}_{p,\infty}}\le C\int_0^t\left(\tau+1\right)\mathrm{d}\tau\le C\left(t^2+t\right),$$

similarly,

$$\left\|\tilde{G}\right\|_{B^{s-2}_{p,\infty}}\le C\int_0^t\left(\tau+1\right)\mathrm{d}\tau\le C\left(t^2+t\right).$$

Thus, we complete the proof of Lemma 5.3. Next, we prove Theorem 8. □

Proof. On the basis of the definition of the Besov norm, we can obtain

$$\begin{aligned}
&\left\|\rho-\rho_0\right\|_{B^{s}_{p,\infty}}\\
&\ge 2^{ns}\left\|\Delta_n\left(\rho-\rho_0\right)\right\|_{L^p}=2^{ns}\left\|\Delta_n\left(\tilde{\rho}+tv_0\right)\right\|_{L^p}\\
&\ge t2^{ns}\left\|\Delta_n v_0\right\|_{L^p}-2^{ns}\left\|\Delta_n\tilde{\rho}\right\|_{L^p}\\
&\ge t2^{ns}\left\|\Delta_n\left(u_0\partial_x\rho_0\right)\right\|_{L^p}-Ct2^{ns}\left\|\Delta_n\left(\partial_x u_0\rho_0\right)\right\|_{L^p}-2^{ns}\left\|\Delta_n\tilde{\rho}\right\|_{L^p}\\
&\ge t2^{ns}\left\|\Delta_n\left(u_0\partial_x\rho_0\right)\right\|_{L^p}-Ct2^{n}\left\|\partial_x\rho_0 u_0\right\|_{B^{s-1}_{p,\infty}}-2^{2n}\left\|\tilde{\rho}\right\|_{B^{s-2}_{p,\infty}}.
\end{aligned}\tag{28}$$

Making the following analysis

$$\left\|\partial_x\rho_0 u_0\right\|_{B^{s-1}_{p,\infty}}\le\left\|\partial_x\rho_0\right\|_{B^{s-1}_{p,\infty}}\left\|u_0\right\|_{B^{s}_{p,\infty}}\le\left\|u_0\right\|_{B^{s}_{p,\infty}}\left\|\rho_0\right\|_{B^{s}_{p,\infty}}\le C,$$

and

$$\begin{aligned}\Delta_n\left(u_0\partial_x\rho_0\right)&=\Delta_n\left(u_0\partial_x\rho_0\right)-u_0\Delta_n\left(\partial_x\rho_0\right)+u_0\Delta_n\left(\partial_x\rho_0\right)\\&=\left[\Delta_n,u_0\right]\partial_x\rho_0+u_0\Delta_n\left(\partial_x\rho_0\right).\end{aligned}$$

Applying Lemma 3.8, we can obtain

$$\left\|2^{ns}\left\|\left[\Delta_n,u_0\right]\partial_x\rho_0\right\|_{L^p}\right\|_{l^\infty}\le C\left(\left\|\partial_x u_0\right\|_{L^\infty}\left\|\rho_0\right\|_{B^{s}_{p,\infty}}+\left\|\partial_x\rho_0\right\|_{L^\infty}\left\|u_0\right\|_{B^{s}_{p,\infty}}\right)\le C.$$

Then using above inequalities into (28), we can get

$$\begin{aligned}\left\|\rho-\rho_0\right\|_{B^{s}_{p,\infty}}&\ge t2^{ns}\left\|u_0\Delta_n\left(\partial_x\rho_0\right)\right\|_{L^p}-C_1t2^n-2^{2n}\left\|\tilde{\rho}\right\|_{B^{s-2}_{p,\infty}}\\&\ge C_2t2^{ns}2^{-n(s-1)}-C_1t2^n-C_3\left(t^2+t\right)2^{2n}\\&=C_2t2^n-C_1t2^n-C_3\left(t^2+t\right)2^{2n},\\&\ge C_2t2^n-C_1t2^n-C_3t^2 2^{2n}.\end{aligned}$$

Here, we take that C_3 is a enough large number. If taking that C_1 and C_2 satisfy $C_2>2C_1$, then we can have

$$\left\|\rho-\rho_0\right\|_{B^{s}_{p,\infty}}\ge\frac{C_2t2^n}{2}-C_3t^2 2^{2n}.$$

When n is a fixed number, let $t\to 0$, choosing $t2^n\approx\delta<\frac{C_2}{4C_3}$ yields

$$\left\|\rho-\rho_0\right\|_{B^{s}_{p,\infty}}\ge\frac{C_2\delta}{2}-C_3\delta^2=\frac{C_2\delta}{4},$$

similarly, we can deduce

$$\begin{aligned}\left\|u-u_0\right\|_{B^{s}_{p,\infty}}&\ge\frac{C_4\delta}{4},\\\left\|G-G_0\right\|_{B^{s}_{p,\infty}}&\ge\frac{C_5\delta}{4}.\end{aligned}$$

This completes the proof of Theorem 2.2. □

6. Blow-Up Criteria

In this section, we will present blow-up criteria for Equation (9). We first need to establish support from the following Lemma.

Lemma 6.1. Suppose that $(u,\rho,G)\in\left(\mathcal{C}\left(\left[0,T\right];B^{\frac{1}{p}}_{p,1}\right)\right)^3$ is the solution of the Cauchy problem (9) with the initial data $\left(\rho_0,u_0,G_0\right)\in\left(B^{\frac{1}{p}}_{p,1}\right)^3$. Then the component ρ satisfies the following expression

$$\rho(t,x)=\rho_0\left(\zeta^{-1}(t,x)\right)\mathrm{e}^{-\int_0^t u_x(s,x)\mathrm{d}s},\ t\in\left[0,T\right).$$

Proof. Considering the following initial value problem

$$\begin{cases} \zeta_t = u\left(t,\zeta\left(t,x\right)\right), & t\in\left[0,T\right), x\in\mathbb{R}, \\ \zeta\left(0,x\right)=x, & t=0, x\in\mathbb{R}. \end{cases}$$

Set $T>0$ is the maximal existence time. Owing to $\zeta_{tx}=u_{\zeta}\left(t,\zeta\left(t,x\right)\right)\zeta_x\left(t,x\right)$ and $\zeta_x\left(0,x\right)=1$, this implies

$$\zeta_x\left(t,x\right)=\exp\left(\int_0^t u_{\zeta}\left(s,\zeta\left(s,x\right)\right)\mathrm{d}s\right).$$

According to the second formula for (9), we can get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\left(t,\zeta\left(t,x\right)\right)\zeta_x\right) \\ &=\rho_t\left(t,\zeta\left(t,x\right)\right)\zeta_x+\rho_{\zeta}\left(t,\zeta\left(t,x\right)\right)\zeta_t\zeta_x+\rho\left(t,\zeta\left(t,x\right)\right)\zeta_{xt} \\ &=\zeta_x\left(\rho_t\left(t,\zeta\left(t,x\right)\right)+\rho_{\zeta}\left(t,\zeta\left(t,x\right)\right)u\left(t,\zeta\left(t,x\right)\right)+\rho\left(t,\zeta\left(t,x\right)\right)u_{\zeta}\left(t,\zeta\left(t,x\right)\right)\right) \\ &=0, \end{aligned}$$

thus, $\rho\left(t,\zeta\left(t,x\right)\right)\zeta_x$ is independent on *t*, we can obtain

$$\rho\left(t,\zeta\left(t,x\right)\right)\zeta_x=\rho_0\left(x\right).$$

Consequently, we prove the Lemma 6.1. □

Theorem 6.2. Set the initial value $\left(u_0,\rho_0,G_0\right)\in\left(\mathcal{C}\left(\left[0,T\right];B_{p,1}^{\frac{1}{p}}\right)\right)^3$ and $\phi\in L^1$.

Suppose that the maximal existence time of the Cauchy problem (9) solutions is $T>0$. If $M>0$ satisfies that

$$\left\|u_x\right\|_{L^\infty}\le M,\ t\in\left[0,T\right),$$

then the solution $z\left(x,t\right)$ of the Cauchy problem (9) does not blow up in $\left[0,T\right)$.

Proof. According to Lemma 6.1, we can establish the following estimate

$$\begin{aligned} \left\|\rho\left(t,x\right)\right\|_{L^\infty} &=\left\|\rho_0\left(\zeta^{-1}\left(t,x\right)\right)\mathrm{e}^{-\int_0^t u_x(s,x)\mathrm{d}s}\right\|_{L^\infty} \\ &\le\left\|\rho_0\left(\zeta^{-1}\left(t,x\right)\right)\right\|_{L^\infty}\mathrm{e}^{\int_0^t\left\|u_x(s,x)\right\|_{L^\infty}\mathrm{d}s} \\ &\le\left\|\rho_0\right\|_{B_{p,1}^{\frac{1}{p}}}\mathrm{e}^{\int_0^t\left\|u_x(s,x)\right\|_{L^\infty}\mathrm{d}s}, \end{aligned}$$

this is because the embedding $B_{p,1}^{\frac{1}{p}}\hookrightarrow W^{1,p}\cap W^{1,\infty}$. If $M>0$ satisfies that $\left\|u_x\right\|_{L^\infty}\le\left\|u_x\right\|_{B_{p,1}^{\frac{1}{p}}}\le M$, $t\in\left[0,T\right)$, then we can have

$$\left\|\rho\left(t,x\right)\right\|_{L^\infty}\le\left\|\rho_0\right\|_{B_{p,1}^{\frac{1}{p}}}\mathrm{e}^{MT}, \tag{29}$$

similarly,

$$\left\|G\left(t,x\right)\right\|_{L^\infty}\le\left\|G_0\right\|_{B_{p,1}^{\frac{1}{p}}}\mathrm{e}^{MT}. \tag{30}$$

According to the result of local well-posedness, applying Δ_j to (9) yields

$$\begin{cases} \Delta_j u_t + u\Delta_j u_x = \Delta_j\left[\phi*(u\rho)-(\phi*\rho)u\right]+\left[u\partial_x,\Delta_j\right]u, \\ \Delta_j \rho_t + u\Delta_j \rho_x = -\Delta_j\left(u_x\rho\right)+\left[u\partial_x,\Delta_j\right]\rho, \\ \Delta_j G_t + u\Delta_j G_x = -\Delta_j\left(u_x G\right)+\left[u\partial_x,\Delta_j\right]G, \\ \Delta_j\rho(x,0)=\Delta_j\rho_0(x),\ \Delta_j u(x,0)=\Delta_j u_0(x). \end{cases}$$

Multiplying both sides of the above equation by $sgn(\Delta_j u)|\Delta_j u|^{p-1}$, $sgn(\Delta_j \rho)|\Delta_j \rho|^{p-1}$, and $sgn(\Delta_j G)|\Delta_j G|^{p-1}$ and integrating over $\mathbb{R}$ respectively, we can obtain

$$\begin{aligned} \|\Delta_j u\|_{L^p} \le \|\Delta_j u_0\|_{L^p} &+ \int_0^t \Big(\|\Delta_j[\phi*(u\rho)-(\phi*\rho)u]\|_{L^p} \\ &+ \|[u\partial_x,\Delta_j]u\|_{L^p} + \|u_x\|_{L^\infty}\|\Delta_j u\|_{L^p}\Big)\mathrm{d}t', \end{aligned}$$

and

$$\|\Delta_j\rho\|_{L^p} \le \|\Delta_j\rho_0\|_{L^p} + \int_0^t\Big(\|\Delta_j(u_x\rho)\|_{L^p} + \|[u\partial_x,\Delta_j]\rho\|_{L^p} + \|u_x\|_{L^\infty}\|\Delta_j\rho\|_{L^p}\Big)\mathrm{d}t',$$

multiplying both sides of the above inequality by $2^{\frac{j}{p}}$ and taking l^1-norm respectively, we can get

$$\begin{aligned} \|u\|_{B^{\frac{1}{p}}_{p,1}} \le \|u_0\|_{B^{\frac{1}{p}}_{p,1}} &+ \int_0^t\Bigg(\|\phi*(u\rho)-(\phi*\rho)u\|_{B^{\frac{1}{p}}_{p,1}} \\ &+ \left\|2^{\frac{j}{p}}\|[u\partial_x,\Delta_j]u\|_{L^p}\right\|_{l^1} + \|u_x\|_{L^\infty}\|u\|_{B^{\frac{1}{p}}_{p,1}}\Bigg)\mathrm{d}t', \end{aligned}$$

and

$$\|\rho\|_{B^{\frac{1}{p}}_{p,1}} \le \|\rho_0\|_{B^{\frac{1}{p}}_{p,1}} + \int_0^t\left(\|u_x\rho\|_{B^{\frac{1}{p}}_{p,1}} + \left\|2^{\frac{j}{p}}\|[u\partial_x,\Delta_j]\rho\|_{L^p}\right\|_{l^1} + \|u_x\|_{L^\infty}\|\rho\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'.$$

On the basis of Lemma 3.8, we can deduce

$$\left\|2^{\frac{j}{p}}\|[u\partial_x,\Delta_j]u\|_{L^p}\right\|_{l^1} \le C\|u_x\|_{L^\infty}\|u\|_{B^{\frac{1}{p}}_{p,1}},$$

and

$$\left\|2^{\frac{j}{p}}\|[u\partial_x,\Delta_j]\rho\|_{L^p}\right\|_{l^1} \le C\|u_x\|_{L^\infty}\|\rho\|_{B^{\frac{1}{p}}_{p,1}},$$

by the estimate (1) and Lemma 3.4, we can obtain

$$\|u\|_{B^{\frac{1}{p}}_{p,1}} \le \|u_0\|_{B^{\frac{1}{p}}_{p,1}} + \int_0^t C\left(\|u\|_{B^{\frac{1}{p}}_{p,1}}\|\rho\|_{B^{\frac{1}{p}}_{p,1}} + \|u_x\|_{L^\infty}\|u\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t',$$

and

$$\|\rho\|_{B^{\frac{1}{p}}_{p,1}} \le \|\rho_0\|_{B^{\frac{1}{p}}_{p,1}} + \int_0^t C\left(\|u_x\|_{B^{\frac{1}{p}}_{p,1}}\|\rho\|_{B^{\frac{1}{p}}_{p,1}} + \|u_x\|_{L^\infty}\|\rho\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'.$$

Similarly, by the same deduction, we can deduce

$$\|G\|_{B^{\frac{1}{p}}_{p,1}} \le \|G_0\|_{B^{\frac{1}{p}}_{p,1}} + \int_0^t C\left(\|u_x\|_{B^{\frac{1}{p}}_{p,1}}\|G\|_{B^{\frac{1}{p}}_{p,1}} + \|u_x\|_{L^\infty}\|G\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t',$$

from the above inequalities, we can get

$$\begin{aligned}\|u\|_{B^{\frac{1}{p}}_{p,1}} + \|\rho\|_{B^{\frac{1}{p}}_{p,1}} + \|G\|_{B^{\frac{1}{p}}_{p,1}} \le& \left(\|u_0\|_{B^{\frac{1}{p}}_{p,1}} + \|\rho_0\|_{B^{\frac{1}{p}}_{p,1}} + \|G_0\|_{B^{\frac{1}{p}}_{p,1}}\right)\\ &+ C\int_0^t \|u_x\|_{B^{\frac{1}{p}}_{p,1}}\left(\|u\|_{B^{\frac{1}{p}}_{p,1}} + \|\rho\|_{B^{\frac{1}{p}}_{p,1}} + \|G\|_{B^{\frac{1}{p}}_{p,1}}\right)\mathrm{d}t'.\end{aligned}$$

Applying Gronwall's inequality, we can obtain

$$\begin{aligned}&\|u\|_{B^{\frac{1}{p}}_{p,1}} + \|\rho\|_{B^{\frac{1}{p}}_{p,1}} + \|G\|_{B^{\frac{1}{p}}_{p,1}}\\ &\le \left(\|u_0\|_{B^{\frac{1}{p}}_{p,1}} + \|\rho_0\|_{B^{\frac{1}{p}}_{p,1}} + \|G_0\|_{B^{\frac{1}{p}}_{p,1}}\right)\exp\left(C\int_0^t \|u_x\|_{B^{\frac{1}{p}}_{p,1}}\mathrm{d}t'\right)\\ &\le \left(\|u_0\|_{B^{\frac{1}{p}}_{p,1}} + \|\rho_0\|_{B^{\frac{1}{p}}_{p,1}} + \|G_0\|_{B^{\frac{1}{p}}_{p,1}}\right)\exp(CMt),\end{aligned}$$

so we complete the proof. □

Then we shall establish the proof of Theorem 2.3.

Proof. Set $(u,\rho,G)\in\left(\mathcal{C}\left([0,T];B^{\frac{1}{p}}_{p,1}\right)\right)^3$ is the solution of the Cauchy problem (9) with the initial data $(u_0,\rho_0,G_0)\in\left(B^{\frac{1}{p}}_{p,1}\right)^3$, and the maximal existence time of the Cauchy problem (9) solutions is $T>0$. We suppose that the solution of Equation (9) blows up in finite time *T*, and there exists a constant $M>0$ such that

$$u_x(t,x)\ge -M,\ \forall(t,x)\in[0,T)\times\mathbb{R}. \tag{31}$$

Applying to (29) and (30), then we can get

$$\begin{aligned}\|u_x(t,x)\|_{L^\infty} &\le \|\phi\|_{L^1}\|\rho\|_{L^\infty} + \|G\|_{L^\infty}\\ &\le C\|\rho\|_{L^\infty} + \|G\|_{L^\infty}\\ &\le C\|\rho_0\|_{B^{\frac{1}{p}}_{p,1}}\mathrm{e}^{MT} + \|G_0\|_{B^{\frac{1}{p}}_{p,1}}\mathrm{e}^{MT}.\end{aligned} \tag{32}$$

By (31) and (32), we can obtain a contradiction with Theorem 6.2. On the basis of Theorem 6.2 and Sobolev embedding theorem, if

$$\lim_{t\to T^-}\inf_{x\in\mathbb{R}} u_x(t,x) = -\infty,$$

then the solution will blow up in a finite time.

This completes the proof of the Theorem 2.3. □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Sneyd, J., Theraula, G., Bonabeau, E., Deneubourg, J.-L. and Franks, N.R. (2001) Self-Organization in Biological Systems. Princeton University Press, Princeton. https://doi.org/10.1515/9780691212920

[2] Seeley, T.D. (2002) When Is Self-Organization Used in Biological Systems? *The Biological Bulletin*, **202**, 314-318. https://doi.org/10.2307/1543484

[3] Wedlich-Söldner, R. and Betz, T. (2018) Self-Organization: The Fundament of Cell Biology. *Philosophical Transactions of the Royal Society B: Biological Sciences*, **373**, Article 20170103. https://doi.org/10.1098/rstb.2017.0103

[4] Huth, A. and Wissel, C. (1992) The Simulation of the Movement of Fish Schools. *Journal of Theoretical Biology*, **156**, 365-385. https://doi.org/10.1016/S0022-5193(05)80681-2

[5] Deisboeck, T.S. and Couzin, I.D (2009) Collective Behavior in Cancer Cell Populations. *Bioessays*, **31**, 190-197. https://doi.org/10.1002/bies.200800084

[6] Do, T., Kiselev, A., Ryzhik, L. and Tan, C.H. (2018) Global Regularity for the Fractional Euler Alignment System. *Archive for Rational Mechanics and Analysis*, **228**, 1-37. https://doi.org/10.1007/s00205-017-1184-2

[7] Cucker, F. and Smale, S. (2007) Emergent Behavior in Flocks. *IEEE Transactions on Automatic Control*, **52**, 852-862. https://doi.org/10.1109/TAC.2007.895842

[8] Ha, S.-Y. and Tadmor, E. (2008) From Particle to Kinetic and Hydrodynamic Descriptions of Flocking.

[9] Carrillo, J.A., Fornasier, M., Rosado, J. and Toscani, G. (2010) Asymptotic Flocking Dynamics for the Kinetic Cucker-Smale Model. *SIAM Journal on Mathematical Analysis*, **42**, 218-236. https://doi.org/10.1137/090757290

[10] Carrillo, J.A., Choi, Y.-P., Tadmor, E. and Tan, C.H. (2016) Critical Thresholds in 1D Euler Equations with Non-Local Forces. *Mathematical Models and Methods in Applied Sciences*, **26**, 185-206. https://doi.org/10.1142/S0218202516500068

[11] Tan, C.H. (2020) On the Euler-Alignment System with Weakly Singular Communication Weights. *Nonlinearity*, **33**, Article 1907. https://doi.org/10.1088/1361-6544/ab6c39

[12] Leslie, T.M and Tan, C.H. (2023) Sticky Particle Cucker-Smale Dynamics and the Entropic Selection Principle for the 1D Euler-Alignment System. *Communications in Partial Differential Equations*, **48**, 753-791. https://doi.org/10.1080/03605302.2023.2202720

[13] Bahouri, H., Chemin, J.-Y. and Danchin, R. (2011) Fourier Analysis and Nonlinear Partial Differential Equations. *Springer Science & Business Media*, **343**. https://doi.org/10.1007/978-3-642-16830-7

[14] Danchin, R. (2005) Fourier Analysis Methods for PDEs. *Lecture Notes*, **14**, 1.

[15] Li, J.L. and Yin, Z.Y. (2016) Well-Posedness and Global Existence for a Generalized Degasperis-Procesi Equation. *Nonlinear Analysis*: *Real World Applications*, **28**, 72-90. https://doi.org/10.1016/j.nonrwa.2015.09.003

[16] Li, J.L. and Yin, Z.Y. (2016) Remarks on the Well-Posedness of Camassa-Holm Type Equations in Besov Spaces. *Journal of Differential Equations*, **261**, 6125-6143. https://doi.org/10.1016/j.jde.2016.08.031