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Feynman Path Integral Using Operator Integration in Banach Space

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Abstract

Feynman-Path Integral in Banach Space: In 1940, R.P. Feynman attempted to find a mathematical representation to express quantum dynamics of the general form for a double-slit experiment. His intuition on several slits with several walls in terms of Lagrangian instead of Hamiltonian resulted in a magnificent work. It was known as Feynman Path Integrals in quantum physics, and a large part of the scientific community still considers them a heuristic tool that lacks a sound mathematical definition. This paper aims to refute this prejudice, by providing an extensive and self-contained description of the mathematical theory of Feynman Path Integration, from the earlier attempts to the latest developments, as well as its applications to quantum mechanics. About a hundred years after the beginning of modern physics, it was realized that light could in fact show behavioral characteristics of both waves and particles. In 1927, Davisson and Germer demonstrated that electrons show the same dual behavior, which was later extended to atoms and molecules. We shall follow the method of integration with some modifications to construct a generalized Lebesgue-Bochner-Stieltjes (LBS) integral of the form $\int u(f, \mathrm{d}\mu)$, where u is a bilinear operator acting in the product of Banach spaces, f is a Bochner summable function, and μ is a vector-valued measure. We will demonstrate that the Feynman Path Integral is consistent and can be justified mathematically with LBS integration approach.

Keywords

Feynman Path Integral, Lebesgue-Bochner-Stieltjes Integral, Operator Integral, Particle-Wave Function, Operator Integration, Position and Momentum Operators

1. Introduction

1.1. Introduction to Young's Double Slit Experiment

In modern physics, the double-slit experiment is a demonstration that light and matter can display characteristics of both classically defined waves and particles; moreover, it displays the fundamentally probabilistic nature of quantum mechanical phenomena [1]. Thomas Young in 1801 is a demonstration of the wave behavior of visible light [2]. At that time, it was thought that light consisted of *either* waves *or* particles.

About a hundred years later, it was realized that light could in fact show behavior characteristic of *both* waves *and* particles. In 1927, Davisson and Germer demonstrated that electrons show the same behavior, which was later extended to atoms and molecules [3] [4].

Thomas Young's experiment with light was part of classical physics long before the development of quantum mechanics and the concept of wave-particle duality. He believed it demonstrated that the wave theory of light was correct, and his experiment is sometimes referred to as Young's experiment or Young's slits.

The experiment belongs to a general class of "double path" experiments, in which a wave is split into two separate waves (the wave is typically made of many photons and better referred to as a wave front, not to be confused with the wave properties of the individual photon) that later combine into a single wave. Changes in the path lengths of both waves result in a phase shift, creating an interference pattern.

A wide-ranging interview with the legendary mathematical physicist Freeman Dyson, in which he discusses his work with Richard Feynman, his attempts to build a spaceship propelled by nuclear bombs and his controversial views on climate change.

1.2. From Double Slit to Multiple Slit and Multiple Screen Wall

In 1940, Richard P. Feynman attempted to find a mathematical representation to express quantum dynamics of the general form of double-slit experiment. It was known as Feynman Path Integrals in quantum physics [5] [6].

The scientific community considers his work a heuristic tool that lacks a sound mathematical foundation.

This paper aims to refute this prejudice, by providing an extensive and self-contained description of the mathematical theory of Feynman Path Integration. We will use a new approach called Lebesgue-Bochner-Steiltjes or briefly LBS Integration Approach [7].

1.3. Characteristic of the Path Integral

We will review a traveling a particle-wave through double, triple, or multiple slits. In this experiment, the light wave may be passing through one or several walls with slits from the source labeled S to a destination object called O. Feynman Path

Integrals, suggested heuristically by Feynman in the 40 s, have become the basis of much contemporary physics.

Applications: 1) To non-relativistic quantum mechanics to quantum fields, gauge fields, gravitation, and cosmology. 2) In areas of mathematics like topology and differential geometry, algebraic geometry, infinite-dimensional analysis and geometry, and number theory. Vectors are considered in infinite dimensional Banach Spaces.

1.4. Review of STEPS in Rieman Integration

- Input: Function *f*, domain, and range of the integration:

A continuous function $f: X = [a,b] \mapsto R$ is a Riemann Integral function if there exists $A \in R$ such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition of the domain [*a*, *b*] into a finite number of intervals.

- Impose an arbitrary partition in the domain: Assume $\{\Delta_i\}, i = 1, \cdots, N$ such that $\max|\Delta_i| < \delta$ and any choice of sampling points $t_i \in \Delta_i$, where

$$P = \{a = t_0, t_1, t_2, t_3, \cdots, b = t_N\} \text{ and } \Delta = \{\Delta_i = [t_{i-1}, t_i], i = 1, \cdots, N\}$$

- The approximation of objective element: $\Delta A \cong f(t_i)|\Delta_i|$.
- Evaluate upper and lower Riemann SUM:

$$U(f, \Delta) = \sup(\Delta A \cong f(t_i)|\Delta_i|)$$

- Total estimated value:

The corresponding Riemann SUM:

$$S_R(f, \{\Delta_i\}, \{t_i\}) = \sum_{i=1}^{N} f(t_i)|\Delta_i| \tag{1.1}$$

is as close as we desire to a constant value.

- A constant limit exists:

$$\left\| \sum_{i=1}^{N} f(t_i)|\Delta_i| - A \right\| < \varepsilon \tag{1.2}$$

Or we can interpret A as the limit of the Riemann SUM when N is approaching infinity. That is,

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(t_i)|\Delta_i| = A.$$

- Integral: We define the value of A when the limit in (1.2) exists, and the Riemann Integral denoted by:

$$A = \int_a^b f(t)\,\mathrm{d}t$$

1.5. Review of STEPS in Riemann Stieltjes Integral

- Input: Functions *f* and *α*, domain, and range of the integration:

A function $f: X = [a,b] \mapsto R$ is a Riemann-Steiltjes Integral function if there exists $A \in R$ such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any partition of the domain [*a*, *b*] into a finite number of intervals. We accept a condition for function $Y = f(x)$ to be a continuous function on [*a*, *b*].

- Define upper and lower Riemann-Stieltjes SUM by: Assume $f: X \to Y$ for

$X = [a, b]$:

$$U(f,\alpha,\Delta) = \sum_{i=1}^{N} \sup\{f(t): t \in \Delta_i\} \cdot \Delta\alpha_i \quad \text{where} \quad \Delta\alpha_i = \alpha(t_i) - \alpha(t_{i-1}) \tag{1.3}$$

$$L(f,\alpha,\Delta) = \sum_{i=1}^{N} \inf\{f(t): t \in \Delta_i\} \cdot |\Delta\alpha_i|$$

Notice that when the function α is a nondecreasing function, then we can drop absolute value.

- Definition: For every, there exists a Δ-partition such that $U(f,\alpha,\Delta) - L(f,\alpha,\Delta) < \varepsilon$.
- For every $\varepsilon > 0$, $\exists$ a partition $P(\varepsilon)$ and for every choice of the point $t_k \in [t_{i-1}, t_i]$, there exists a value A such that,

$$|S(f,\alpha,\Delta) - A| = \left\| \sum_{i=1}^{N} f(t_i)|\Delta\alpha_i| - A \right\| < \varepsilon \tag{1.4}$$

- In this case, we use $\lim_{N\to\infty} \sum_{i=1}^{N} f(t_i)|\Delta\alpha_i| = A$.
- Riemann-Steiltjes integral:

$$A = \int_a^b f\mathrm{d}\alpha = \int_a^b f(t)\mathrm{d}\alpha(t) \tag{1.5}$$

Important notice: When $\alpha(t) = t$, then, the Reimenn-Steiltjes integral will reduce to just Riemann integral.

1.6. Toward the LBS Integration in Banach Space

- Path Integral can be justified in a Complete Normed Linear Space (Banach Space).

Note: Many people contributed to this theory: H. Lebesgue, S. Banah, G. Fubini, S. Saks, F. Riesz, N. Dunford, and H. Halmos [8]-[13].

- $(G, *)$ = Group G with a Binary Operation "$*$": This set with binary operation is closed, associative, with identity, and every element in this set is invertible.
- Boolean ring of binary operation on sets: Assume A and B are subsets of the power set of the abstract space X. Let us use two operations of union and intersection $\cup$ and $\cap$ of sets in a set $V = P(X)$. Define:

$$A * B = (A \cup B) \setminus (A \cap B) \tag{1.6}$$

$$A \cdot B = A \cap B \tag{1.7}$$

1.7. Semiring of Subsets and Partition of an Abstract SPACE

- Lemma: Prove the symmetric differences:

$$A * B = (A \setminus B) \cup (B \setminus A)$$

Semi-ring of subsets: Assume X is an abstract space and V is a family of subsets in X. For a set $V \in P(X)$, a triple $(V, *, \cdot)$ is said to be a semi-ring.

- V contains empty set, *i.e.* $\varnothing \in V$.
- The triple $(V, *, \cdot)$ is closed under both operations.
- Closure under symmetric differences: $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Semi-ring of disjoint sets: For every set A and B in V, there exists an integer k

and mutually disjoint sets $B_1, B_2, \cdots, B_k \in V$ such that $(A \setminus B) = \bigcup_{j=1}^{j=k} B_j$. The triple $(V, \Delta, \bigcap)$ is a semi-ring.

1.8. Measure (Volume) on a Semi-Ring $(V, \Delta, \bigcap)$

Assume a function from a semi-ring $(V, \Delta, \bigcap)$ to a Banach Space Z: $\mu : V \to Z$ satisfies the following conditions: for every countable family of disjoint sets, we partition the set A into mutually disjoint sets, then,

$$A = \bigcup_T A_t \in V \to \mu(A) = \sum_T \mu(A_t) \tag{1.8}$$

The SUM in (1.8) is convergent. For every $A \in V \Rightarrow$:

$$|\mu(A)| = \sup\left\{\sum_T |\mu(A_t)|\right\} < \infty \tag{1.9}$$

where the supremum is taken over all possible decompositions of the space. The measure (volume) is positive if it has only nonnegative values.

Norm of the positive measure: let v be a positive measure (volume) defined in a semi-ring V. Define a subspace M for all volumes $\mu : V \to Z$ such that, $|\mu(A)| \le cv(A)$ for some constant number c and all $A \in V \to \|\mu\| = \min\{c\}$. Thus:

$$\|\mu\| = \min\left\{c \in R : |\mu(A)| \le cv(A), \text{ for all } A \in V\right\} \tag{1.10}$$

In the following Sections 1.9, 1.10, and 1.11, we will describe Lebesgue-Bochner-Steiltjes approach to the integration in Banach Space.

1.9. Space of Simple Functions (Basic): Define a Space of Simple Function *S*(*Y*)

$$S(Y) = \left\{h : h = y_1 c_{A_1} + y_2 c_{A_2} + \cdots + y_k c_{A_k}\right\} \tag{1.11}$$

For all $y_i \in Y$, and $A_i \in V, (i = 1, 2, \cdots, k)$.

Vectorial Form: Assume $\vec{y} = \langle y_1, y_2, \cdots, y_k \rangle$ and $\vec{c} = \langle c_{A_1}, c_{A_2}, \cdots, c_{A_k} \rangle$ then the relation (1.11) can be described by:

$$h = \langle \vec{y}, \vec{c} \rangle = \sum_{i=1}^{i=k} y_i c_{A_i} \tag{1.12}$$

where the *characteristic function* c_{A_i} can be defined by: $c_{A_i}(t) = \begin{cases} 1 & \text{if } t \in A_i \\ 0 & \text{if } t \notin A_i \end{cases}$.

The norm of the relation (1.12): Let

$$\vec{v}(A) = \langle v(A_1), v(A_2), \cdots, v(A_k) \rangle \text{ then}$$

$$\|h\| = \langle |\vec{y}|, \vec{v}(A) \rangle = \sum_{i=1}^{i=k} |y_i| v(A_i)$$

Notice that the symbol $|y_i|$ represents the absolute value of the i-th component of y.

1.10. Generalized Lebesgue-Bochner-Stieltjes (LBS) Integral

Here is a general form of presenting integration based on a bilinear operator u

acting on h with a measure μ:

$$\begin{aligned}\int u(h,\mathrm{d}\mu) &= \sum_{i=1}^{i=k} u(y_i,\mu(A_i)) \\ &= u(y_1,\mu(A_1))+u(y_2,\mu(A_2))+u(y_3,\mu(A_3))+\cdots+u(y_k,\mu(A_k))\end{aligned} \tag{1.13}$$

Bochner Integral:

$$\int y\mathrm{d}v = (y_1 v(A_1))+(y_2 v(A_2))+(y_3 v(A_3))+\cdots+(y_k v(A_k)) \tag{1.14}$$

These two operators are well defined, that is they are independent of the choice $h \in S(Y)$ in (1.11) where $\int |h| = \|h\|$.

1.11. Basic Sequence of Simple Functions and Summability

A sequence of functions $s_n \in S(Y)$ is a BASIC if there exists a sequence $h_n \in S(Y)$ and a constant $M > 0$ such that,

$$s_n = h_1 + h_2 + \cdots + h_n \quad \text{and} \quad \|h_n\| \le 4^{-n} M \text{, for } n = 1,2,\cdots \tag{1.15}$$

The space of Summable functions: The space of summable function $L(Y)$ is the set of all functions f which,

$$L(Y) = \{ f : \exists s_n \in S(Y) - \text{Basic such that } \lim_{n\to\infty}(s_n) = f \text{ a.e} \}$$

Conclusion:

$$\int u(f,\mathrm{d}\mu) = \lim_{n\to\infty}\int u(s_n,\mathrm{d}\mu) \quad \text{and} \quad \int f\mathrm{d}v = \lim_{n\to\infty}\int s_n \mathrm{d}v \tag{1.16}$$

2. Some Characteristics of Path Integral

The objective is to justify the mathematics of multi-slit and multi-screen experiments in both classical and quantum senses, particularly the mathematical justification of Feynman Path Integral.

The original definition presented in (1.14) and demonstrated in Reference [7], did not present the precise definition and conditions for the integral operator. However, it was claimed that this integral operator is well defined, and it is independent of the choice of h. It was concluded that it is well-defined.

We will demonstrate that this originally undefined and not clarified conclusion has an amazing and powerful application in a variety of disciplines.

Notice that this work is just a mathematical justification according to theory of integration in Banach Space. No numerical computation or new experiment will be presented. This new approach of Feynman Path Integral is open for application to other parts of mathematical Physics for further research.

2.1. Justification to Have Infinite Dimensional Abstract Space for Feynman Path Integral

Quantum interference of different paths depending on their phases, and hence their classical action, will have all possible paths. The question is, what is the probability density of the amplitude in Δt = 10 Nano sec., of the infinite number of walls with infinite number of slits?

Thus, a finite dimensional Riemann-Steiltjes integral does not work for Feynman Path Integral. The infinite dimensional Bochner integral which is designed in Banach or Hilbert Space is feasible for this problem.

Dealing with some problems in Quantum Mechanics, as we are expecting in this article, like solving the Schrodinger Equation, requires certain conditions to analyze and describe the physical phenomenon. Thus, we need some rigorous mathematical work to justify the Feynman Path Integral, which was originally accepted intuitively. Many attempts were made to bridge the gap of mathematical work; thus, much research and investigation produced some significant results.

We try to show some of the foundations and principles of the Feynman Path Integral, which requires us to begin in the following mathematical language:

1) Paths are decided to be within an infinite dimensional space. As a result, the integration will be designed in Banach Space using Bochner integration [14] [15].

2) Feynman Paths satisfy the Orthogonality condition.

3) Due to the Uncertainty Principle the trajectories must be selected from the probability distribution spaces. As a result, we may select normalized functions with orthonormal paths.

4) The Lebesgue Integration theory can be used properly for Probability distribution functions.

5) Due to many impulsive responses for a variety of operators, we need to include integration theory which can include Dirac's Delta functions. Thus, we are including Dirac's Integration System.

6) In our investigation, we will demonstrate that Bogdan's approach toward Lebesgue-Bochner-Stieltjes integration will be a feasible approach for Feynman Path Integral.

7) Operator Integral is a necessary step toward the general definition of Using Lebesgue-Bochner-Stieltjes integral in Banach space (see [16] and [17]).

We are looking for some integration methods in which we can cover all these characteristic conditions. In addition to the list of 1) - 7), we need to apply operators to the natural phenomenon like Hamiltonian, position, momentum, …, and energy operators to be able to justify Feynman Path Integral.

In the last stage of Lebesgue-Bochner-Stieltjes integration, we try to use the following general formulation.

2.2. Integrals with Operators

Assume that *Y, Z, and W* are Banach Spaces. Let a be a fixed bilinear operator $u \in U : Y \times Z \to W$. Let us define a set of operators U such that,

$$U = \left\{ u : Y \times M \to Z, u\left(y, \mathrm{d}\mu \right) = y\mathrm{d}\mu \right\} \tag{2.1}$$

One example of this operator integral is the following Hamiltonian wave function:

$$u = \phi\left(t \right) \quad \text{such that} \quad \phi\left(t \right) = \mathrm{e}^{-i\hbar Ht}$$

where $t \in \mathbb{R}$, and H-Hamiltonian.

For fixed operator $u \in U$, and $\mu \in M$, we define a vector $y = \langle y_1, y_2, \cdots, y_k \rangle$:

$$\begin{aligned}\int u(y, \mathrm{d}\mu) &= \sum_{j=1}^{k} u\left(y_j, \mu\left(A_j\right)\right) \\ &= u\left(y_1, \mu\left(A_1\right)\right) + u\left(y_2, \mu\left(A_2\right)\right) + \cdots + u\left(y_k, \mu\left(A_k\right)\right)\end{aligned} \tag{2.2}$$

such that $y = y_1 \chi_{A_1} + y_2 \chi_{A_2} + \cdots + y_k \chi_{A_k}$, where $y_j \in Y$, and $A_j \in V$.

We would like to replace a wave-particle for the operator such that bra vector:

$$u = \langle a_1, a_2, \cdots, a_k \rangle \Rightarrow u|\psi\rangle = \sum_j a_j \psi_j$$

We can define the measure on a one-dimensional *x-axis*:

$$\int u\left(\psi_j, \mu\left(A_j\right)\right) = \sum_{j=1}^{k} \left(a_j \psi_j \cdot \mu\left(A_j\right)\right) \tag{2.3}$$

Now, let us assume that $u = \langle a, b \rangle$ is a vector and u is operating on two components $\left(\psi_j, \mu\left(A_j\right)\right)$. We may define the bilinear or trilinear operator $u\left(\psi_j, \mu\left(A_j\right)\right) = a\psi_j + b\mu\left(A_j\right)$.

We can use the general operator defined in (2.2) and demonstrate the integral using the Hamiltonian operator such that,

$$H\left(|\psi\rangle\right) = u(\psi)$$

where $H\left(|\psi\rangle\right) = i\hbar \frac{\partial}{\partial t}|\psi\rangle$.

$$\int u\left(|\psi\rangle, \mathrm{d}\mu\right) = \int \mathrm{e}^{-\frac{i}{\hbar}Ht} |\psi(0)\rangle \mathrm{d}\mu$$

Example of Orthogonality of paths: Verify that two sequence functions:

$$f_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad g_m(x) = \sin\left(\frac{m\pi x}{L}\right) \tag{2.4}$$

are orthogonal, that is $\langle f_n, g_m \rangle = \begin{cases} 0 & \text{for } m \neq n \\ 2\pi & \text{for } m = n \end{cases}$.

2.3. Position Operator

We let u represent a position operator on the *x-axis*. As a result, when u is applied to a wave-particle ψ with Dirac's bra-kett symbolic notation:

$u|\psi\rangle = |x\psi\rangle$, thus,

$$u\left[\psi(x)\right] = x\psi \to \langle \psi | u | \psi \rangle = \int \psi^* x \psi \mathrm{d}x \tag{2.5}$$

The symbol ψ^* is the complex conjugate (see [18] and [19]).

Eigenvalue λ and eigenvector of the position operators can be calculated:

$$x\psi(x) = \lambda\psi(x) \Rightarrow [x - \lambda]\psi(x) = 0$$

This implies that $\psi(x) = 0$ everywhere except at $x = \lambda$. This is a behavior of Dirac Delta function for eigenvector. That is,

$$\delta(x - \lambda) = \begin{cases} 0 & \text{if } x \neq \lambda \\ \infty & \text{if } x = \lambda \end{cases} \tag{2.6}$$

Let us apply the relation (2.3) for two different values $x = \lambda_i$ and $x = \lambda_j$, where (i and j are integer subscripts). The integral (2.4) can be evaluated at these points:

$$\int \psi^*(x_i)x\psi(x_j)\mathrm{d}x = \int \psi_i^*\psi_j \mathrm{d}x = \delta_{ij}$$

The relation (2.6) demonstrates the Orthogonality of the wave-particles.

In addition, the next relation proves the impulsive behavior,

$$\int_{-\infty}^{\infty} \delta(x-\lambda)\mathrm{d}x = 1. \tag{2.7}$$

2.4. Momentum Operator

It is well known that the momentum operator acting on a wave-particle ψ can be described by:

$$P = -i\hbar\frac{\partial}{\partial x} \Leftrightarrow u|\psi\rangle = p|\psi\rangle$$

For eigenvalue and eigenfunction, we can demonstrate:

$$p|\psi\rangle = -i\hbar\frac{\partial}{\partial x}|\psi\rangle \tag{2.8}$$

This differential equation can be solved by separation of variables. Thus,

$$\frac{\mathrm{d}\psi}{\psi} = \frac{i}{\hbar}p\mathrm{d}x \Rightarrow \psi(x) = A\cdot \mathrm{e}^{ikx} = A\cdot \mathrm{e}^{\frac{ipx}{\hbar}} \tag{2.9}$$

where A is a constant of proportionality. It should be normalized, and it will be in the following form:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right) \tag{2.10}$$

Assume p and p' represent momentum of the wave-particles $\psi(x)$ and $\psi(x')$ at two different points x and x'. Using Dirac's Quantum model and identity operator:

$$\langle p'|I|p\rangle = \delta(p-p') \tag{2.11}$$

$$\int_{-\infty}^{+\infty} \langle p'|x\rangle\langle x|p\rangle \mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{2\pi\hbar}\mathrm{e}^{i(k-k')x}\mathrm{d}x \tag{2.12}$$

Notice that (2.12) is a result of the product of the following using the relation (2.10) when they are acting on the wave-particle:

$$\langle p'|x\rangle = \psi^*(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{-ipx}{\hbar}\right) \quad \text{and} \tag{2.13}$$

$$\langle x|p\rangle = \psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right) \tag{2.14}$$

Combine these two relations (2.13) and (2.14):

$$\int \psi_i^*\psi_j \mathrm{d}x = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{2.15}$$

This proves the orthonormality of the paths.

3. Schrodinger Wave Equation

The Schrodinger wave equation is a fundamental equation of quantum mechanics that describes the behavior of quantum particles, such as electrons or photons. It was conceived by the Austrian physicist Erwin Schrodinger in 1925.

Derivation using wave equation: The derivation of the Schrodinger equation starts with the classical wave equation, which describes the propagation of waves. The classical wave equation is given by,

$$\frac{\partial^2}{\partial t^2}\psi - v^2\nabla^2\psi = 0 \tag{3.1}$$

where ψ is the wave function, $\frac{\partial^2}{\partial t^2}\psi$ is the second derivative of ψ with respect to time, $\nabla^2\psi$ is the Laplacian operator acting on ψ, and v is the velocity of the wave. In quantum mechanics, the wave function ψ is a complex-valued function $\psi = \mathrm{e}^{i(kx-\omega t)}$ that carries information about the particle's probability density. Schrodinger postulated that the wave function satisfies a similar equation, called the Schrodinger equation, but with a modified form:

$$H\psi = E\psi \tag{3.2}$$

where H is the Hamiltonian operator, which represents the total energy of the particle, E is the energy of the particle and ψ is the wave function.

The Hamiltonian operator H is defined as:

$$H = \frac{1}{2m}p^2 + V \tag{3.3}$$

We represent the total energy of the system, where m is the mass of the particle, p is the momentum operator, and V is the potential energy, by substituting the classical wave equation into the Schrodinger equation, we obtain:

$$\left(\frac{1}{2m}p^2 + V\right)\psi = E\psi \tag{3.4}$$

To simplify this equation, we make use of the de Broglie relation, which states that the momentum of a particle is related to its wavelength by:

$p = \hbar k$, where $\hbar = h/2\pi$ is the reduced Planck's constant and k is the wave number or constant vector [20] [21] [22].

Substituting $p = \hbar k$ into the equation and rearranging terms, we get:

$$\frac{1}{2m}(\hbar k)^2\psi = (V - E)\psi \tag{3.5}$$

This is a time-independent Schrodinger equation for a particle with fixed energy E.

To obtain the time-dependent Schrodinger equation, we introduce the concept of a time evolution operator, denoted by $U(t)$, which allows us to evolve the wave function ψ at different points in time.

Assume that a wave function is described by:

$$\psi = \mathrm{e}^{i(kx-\omega t)} \tag{3.6}$$

The time derivative of the wave function will be:

$$\frac{\partial \psi}{\partial t} = -i\omega\psi \to i\frac{\partial \psi}{\partial t} = \omega\psi \tag{3.7}$$

Next, multiply both sides of the Plank-Einestein equation $E = \hbar\omega$ by ψ $\to$:

$$E\psi = \hbar\omega\psi \tag{3.8}$$

Now, multiply both sides of (3.8) by $\frac{-i}{\hbar}$:

$$\frac{-i}{\hbar}E\psi = \hbar\omega\psi\frac{-i}{\hbar} = -i\omega\psi \Rightarrow E\psi = i\omega\psi \tag{3.9}$$

We can use the relation (3.7) to replace $i\frac{\partial \Psi}{\partial t} = \omega\psi = i\psi_t$ in (3.9) and obtain:

$$E\psi = i\hbar\psi_t \tag{3.10}$$

Now, let's take the second partial derivative of the wave function (3.6) w.r.t *x*:

$$\frac{\partial^2}{\partial x^2}\psi = -k^2\psi \Rightarrow \partial_{xx}\psi = -\left(\frac{p}{\hbar}\right)^2\psi \Rightarrow p^2\psi = -\hbar^2\psi_{xx} \tag{3.11}$$

The time evolution operator (3.8) satisfies equation (3.5), where i is the imaginary unit. By substituting the expression for H and rearranging terms, we get:

$$i\frac{\partial \Psi}{\partial t} = \frac{1}{2m}(\hbar k)^2\psi + V\psi \Rightarrow i\psi_t = \frac{1}{2m}p^2\psi + V\psi$$

Substituting (3.11) in this equation: $i\psi_t = \frac{1}{2m}\left(-\hbar^2\partial_{xx}\psi\right) + V\psi$, and writing in a proper order we will get the final form of the Schrodinger equation:

$$\frac{-\hbar^2}{2m}\left(\psi_{xx}(x,t)\right) - i\psi_t(x,t) + V(x)\psi(x,t) = 0 \tag{3.12}$$

This is the time-dependent Schrodinger equation, which describes the time evolution of a particle's wave function.

In summary, the Schrodinger wave equation is derived by applying the principles of quantum mechanics to the classical wave equation and introducing the concept of a Hamiltonian operator to represent the total energy of the particle. The resulting equation, called the Schrodinger equation, describes the behavior of the particle's wave function and allows for the calculation of its probabilities and energies.

4. Discussion

For a brief review of the Lebesgue-Bochner-Stieltjes (LBS) integration, we started to show the operator integration. This is a generalization of integration in Banach space, which was done by Bochner [23]. Authors are not aware that any prior application of this "operator integration" to sciences or engineering education was used by others.

The operator integration (LBS) is a form $\int u(f, \mathrm{d}\mu)$, where u is a bilinear operator acting in the product of Banach spaces, f is a Bochner summable function, and μ is a vector-valued measure.

This article has been intended as a mathematical justification of Feynman Path Integral using the integration theory in Banach Space. The theory of integration evolved from Riemann, Steiltjes, and Lebesgue throughout past centuries. Bochner provided the integration theory in Banach spaces. The Feynman Path Integral was originally motivated and presented heuristically.

Several characteristics of the path integral guide us to plan for the rigorous mathematical work. 1) It should work in infinite dimensional space. 2) It should be consistent with Dirac's Integral System. 3) It should be working with a variety of operator differential equations. 4) The position vector can be selected as a complex variable. 5) The nature of the quantum-level computation is required to use the Lebesgue-Stieltjes measurable space.

Much research has been justified by the integration theory based on Hilbert space or Banach space [24] [25]. Operator integration approach, called Lebesgue Bochner-Steiltjes, is used in this paper to demonstrate that Feynman Path Integral is a mathematically consistent theory. This work is an introductory development of Feynman Path Integral, and it is yet to be used in many other applications in theoretical physics.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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