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The Dual of the Two-Variable Exponent Amalgam Spaces $\left(L^{q()},l^{p()}\right)(\Omega)$

Sambourou Massinanke, Sékou Coulibaly, Mamadou Traore

Departement d'Etudes et de Recherches de Mathématiques et d'Informatique, Université des Sciences, des Techniques et des Technologies de Bamako, Bamako, Mali
Email: smass74@yahoo.com, csekou91@yahoo.fr, mamadoutraor664@yahoo.com

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Abstract

Wiener amalgam spaces are a class of function spaces where the function's local and global behavior can be easily distinguished. These spaces are extensively used in Harmonic analysis that originated in the work of Wiener. In this paper: we first introduce a two-variable exponent amalgam space $\left(L^{q()},l^{p()}\right)(\Omega)$. Secondly, we investigate some basic properties of these spaces, and finally, we study their dual.

Keywords

Amalgam Spaces, Variable Exponent Lebesgue Spaces, Dual of a Vector Space

1. Introduction

In recent years, function spaces with variable exponents have been intensively studied by an important number of authors. The generalized Lebesgue spaces $L^{p(x)}$ (or variable exponent Lebesgue spaces) appeared in literature for the first time already in an article by Orlicz [1], but the advanced development started with the paper [2] of Kovacik and Rakosnik in 1991. A survey of the history of the field with a bibliography of more than a hundred titles published up to 2004 can be found in [3]. To illustrate the importance of Wiener amalgams, let us mention one specific example, which today plays a central role in the theory of time-frequency analysis. This is the space $W\left(\mathcal{F}L^1,L^1\right)$ consisting of functions that are locally the Fourier transform of an L^1 function and have a global behavior L^1.

The motivation to study such function spaces comes from applications to fluid dynamics [4] [5], image processing [6], PDE (Partial Differential Equation) and the calculus of variation [7] [8].

In the early 1980s, in a series of articles, Feichtinger provides the most general definition of Wiener Amalgam (WA) [9] [10] [11].

For an introduction to WA on the real line and for some historical notes, we refer to [12].

In mathematical domain, Wiener amalgams proved to be a very useful tool, for instance in time-frequency analysis [13] (e.g. the Balian-Low theorem [12]) and sampling theory. Our interest in those spaces arose from the Wiener Amalgams of the spaces with constant exponents [14].

F. Holland began his systematic study in 1975 [15]. Since, he has been widely studied by [16] [17] [18].

Only some papers treat the Wiener amalgam with one variable exponent [19] [20] [21].

It seems that Wiener amalgams with two or more variable exponents have not yet been considered in full generality. In this work, we define a two-variable exponent amalgam space $\left(L^{q()}, l^{p()}\right)(\Omega)$ and give some properties and study their dual.

Some properties of variable exponent amalgam space can be derived in the same way as for usual amalgams $\left(L^q, l^p\right)$, where q, p are constant, while others are very complicate.

The following definitions and results on the amalgam spaces $\left(L^q, l^p\right)$ with constant exponents can be found in [15] [17] [22] [23] [24] [25] [26].

1) Classical Wiener Amalgam space $\left(L^q, l^p\right)$ with constant exponents

We give d as a fixed positive integer and $\mathbb{R}^d$ as the d-dimensional Euclidean space equipped with its Lebesgue measure dx.

For $1 \le p, q \le \infty$, the amalgam of L^q and l^p is the space $\left(L^q, l^p\right)$ defined by:

$$\left(L^q, l^p\right)\left(\mathbb{R}^d\right) = \left\{ f \in L^1_{loc}\left(\mathbb{R}^d\right) : {}_1\|f\|_{q,p} < \infty \right\},$$

where for $r > 0$

$${}_r\|f\|_{q,p} = \begin{cases} \left[\sum_{k \in \mathbb{Z}^d} \left\| f \chi_{I_k^r} \right\|_q^p \right]^{\frac{1}{p}} & \text{if } p < \infty \\ \sup_{k \in \mathbb{Z}^d} \left\| f \chi_{I_k^r} \right\|_q & \text{if } p = \infty \end{cases} \tag{1}$$

with $I_k^r = \prod_{j=1}^{d} \left[k_j \times r, \left(k_j + 1\right) \times r \right[$ and $k = \left(k_j\right)_{1 \le j \le d} \in \mathbb{Z}^d$.

The map $f \mapsto \|f\|_p$ denotes the usual norm on Lebesgue space $L^p\left(\mathbb{R}^d\right)$ on $\mathbb{R}^d$ while χ_E stands for the characteristic function of the subset E of $\mathbb{R}^d$.

2) Some basic facts about amalgam spaces $\left(L^q, l^p\right)\left(\mathbb{R}^d\right)$ with constant exponents

Let $1 \le p, q \le \infty$. Amalgam spaces $\left(L^q, l^p\right)\left(\mathbb{R}^d\right)$ are defined in (1).

Here are the well-known results properties (see, for example, [15] [17] [22] [23] [24] [25] [26]):

- For $0 < r < \infty$, $f \mapsto {}_r\|f\|_{q,p}$ is a norm on $(L^q, l^p)(\mathbb{R}^d)$ equivalent to ${}_1\|f\|_{q,p}$ (the equivalence constants depend only on *r*).

With respect to these norms, the amalgam spaces $(L^q, l^p)(\mathbb{R}^d)$ are Banach spaces.

- The spaces are strictly increasing with the global exponent p and (strictly) decreasing with a growing local exponent q more precisely:

*

$$ {}_1\|f\|_{q,p} \leq {}_1\|f\|_{q,p_1} \text{ if } 1 \leq p_1 < p \tag{2} $$

that is:

*

$$ 1 \leq p_1 < p \Rightarrow (L^q, l^{p_1})(\mathbb{R}^d) \subset (L^q, l^p)(\mathbb{R}^d) $$

*

$$ {}_1\|f\|_{q,p} \leq {}_1\|f\|_{q_1,p} \text{ if } q < q_1 \tag{3} $$

that is:

$$ q < q_1 \Rightarrow (L^{q_1}, l^p)(\mathbb{R}^d) \subset (L^q, l^p)(\mathbb{R}^d) $$

- For $0 < r < \infty$, Holder's inequality is fulfilled:

$$ \|fg\|_1 \leq {}_r\|f\|_{q,p} \times {}_r\|g\|_{q',p'}, \quad f, g \in L^1_{loc}(\mathbb{R}^d) \tag{4} $$

where q', p' are conjugate exponents of q, p that is $\frac{1}{q} + \frac{1}{q'} = \frac{1}{p} + \frac{1}{p'} = 1$.

3) Duality of the wiener amalgam spaces $(L^q, l^p)(\mathbb{R}^d)$ with constant exponents

- When $1 \leq q, p < \infty$, $(L^q, l^p)(\mathbb{R}^d)$ is isometrically isomorphic to the dual $(L^q, l^p)^*(\mathbb{R}^d)$ of $(L^q, l^p)(\mathbb{R}^d)$ in the sense that for any element T of $(L^q, l^p)^*(\mathbb{R}^d)$, there is an unique element $\varphi(T)$ of $(L^{q'}, l^{p'})(\mathbb{R}^d)$ such that:

$$ \langle T, f \rangle = \int_{\mathbb{R}^d} f(x)\varphi(T)(x)\mathrm{d}x, \; f \in (L^q, l^p)(\mathbb{R}^d) \text{ and furthermore} $$

$$ {}_1\|\varphi(T)\|_{q',p'} = \|T\|. $$

We recall that $\|T\| := \sup\left\{ |\langle T, f \rangle| : f \in L^q_{loc}(\mathbb{R}^d), {}_1\|f\|_{q,p} \leq 1 \right\}$.

- If $1 \leq q, p < \infty$, then there exist real numbers A and B such that:

$$ A \times {}_r\|f\|_{q,p} \leq r^{\frac{-d}{p}} \left\{ \int_{\mathbb{R}^d} \left[\int_{J_x^r} |f(y)|^q \mathrm{d}y \right]^{\frac{p}{q}} \mathrm{d}x \right\}^{\frac{1}{p}} \leq B \times {}_r\|f\|_{q,p} $$

for $f \in L^1_{loc}(\mathbb{R}^d)$, $r > 0$ and $J_x^r = \prod_{j=1}^{d} \left] x_j - \frac{r}{2}, x_j + \frac{r}{2} \right[$, $x = (x_j)_{1 \leq j \leq d} \in \mathbb{R}^d$.

4) Denseness of some subsets in amalgam spaces $(L^q, l^p)(\Omega)$ with constant exponents

4.1) We define $S = S(\Omega)$ to be the collection of all simple functions, that is, functions whose range is finite: $s \in S(\Omega)$ if:

$$s(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$$

where the numbers a_j are distinct and the sets $E_j \subset \Omega$ are pairwise disjoint.

4.2) Let Ω be an open non void set. Suppose that $1 \le q, p < \infty$, then $C_c(\Omega)$ and $S(\Omega)$ are dense in $(L^q, l^p)(\Omega)$.

5) Constant Lebesgue sequence spaces

a) For any real sequence $(x_k)_{k\in\mathbb{Z}}$,

$$\left\|(x_k)_{k\in\mathbb{Z}}\right\|_{l^p} = \begin{cases} \left[\sum_{k\in\mathbb{Z}} |x_k|^p\right]^{\frac{1}{p}} & \text{if } p < \infty \\ \sup_{k\in\mathbb{Z}} |x_k| & \text{if } p = \infty \end{cases} \tag{5}$$

b)

$$l^p = \left\{(x_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : \left\|(x)_{k\in\mathbb{Z}}\right\|_{l^p} < \infty\right\} \tag{6}$$

c) $c_0 = \left\{(x_k)_{k\in\mathbb{Z}} \in l^\infty : \lim_{k\to\infty} x_k = 0\right\}$

Therefore, we get the following proposition.

Proposition 1.

Let $1 \le p \le \infty$.

a) Endowed with the two usual operations, l^p is a real vector space and the mapping $(x_k)_{k\in\mathbb{Z}} \mapsto \left\|(x_k)_{k\in\mathbb{Z}}\right\|_{l^p}$ makes it a Banach space:

b) Holder's inequality: If $1 \le r, q, p \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then

$$\left\|(x_k y_k)_{k\in\mathbb{Z}}\right\|_{l^r} \le \left\|(x_k)_{k\in\mathbb{Z}}\right\|_{l^p} \left\|(y_k)_{k\in\mathbb{Z}}\right\|_{l^q} \tag{7}$$

c) Suppose that $1 \le p < \infty$. Then the topological dual of l^p is isomorphically isometric to $l^{p'}$ and the duality bracket is defined as follows:

$$\left\langle (x_k)_{k\in\mathbb{Z}}, (y_k)_{k\in\mathbb{Z}} \right\rangle = \sum_{k\in\mathbb{Z}} x_k y_k, (x_k)_{k\in\mathbb{Z}} \in l^{p'}, (y_k)_{k\in\mathbb{Z}} \in l^p$$

Furthermore, the following result is well-known.

Proposition 2.

a) c_0 is a closed sub vector space of l^∞ whose topological dual is l^1

b) For any $(x_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$:

$$1 \le p \le \tilde{p} \le \infty \Rightarrow \left\|(x_k)_{k\in\mathbb{Z}}\right\|_{l^{\tilde{p}}} \le \left\|(x_k)_{k\in\mathbb{Z}}\right\|_{l^p} \tag{8}$$

therefore l^p is continuously embedded in $l^{\tilde{p}}$.

Given a normed vector space V, we denote by V^* the normed vector space of bounded linear functionals $V \to \mathbb{R}$ endowed with the usual operator norm. We wish to study $\left[\left(L^{q()}, l^{p()}\right)(\Omega)\right]^*$. The study is motivated by norm conjugate inequality (Theorem 22). More precisely, for $g \in \left(L^{q'()}, l^{p'()}\right)(\Omega)$, we define the integral operator associated to g to be the operator $T_g : \left(L^{q()}, l^{p()}\right)(\Omega) \to \mathbb{R}$ given by $T_g(f) = \int_\Omega f(x) g(x) \mathrm{d}x$ Hölder's inequality ensures that T_g is a well-defined operator and that it is bounded. The linearity of the integral implies the linearity of T_g whence $T_g \in \left[\left(L^{q()}, l^{p()}\right)(\Omega)\right]^*$.

We have thus defined an operator $T:\left(L^{q'()}, l^{p'()}\right)(\Omega) \to \left[\left(L^{q()}, l^{p()}\right)(\Omega)\right]^*$. Again using the linearity of the integral, we find that T is linear. What's more, by Proposition 20, we have the identity:

$$\left\|T_g\right\| \equiv \sup\left\{\left|T_g(f)\right| : f \in \left(L^{q()}, l^{p()}\right)(\Omega), {}_r\|f\|_{q(),p(),\Omega} \leq 1\right\} \approx {}_r\|g\|_{q'(),p'(),\Omega}.$$

From this, it follows that T is a bijection, bounded, linear operator from $\left(L^{q'()}, l^{p'()}\right)(\Omega)$ into $\left[\left(L^{q()}, l^{p()}\right)(\Omega)\right]^*$, it actually turns out that this operator is an isomorphism.

The paper is divided into four sections. Section 2 includes fundamental notations and definitions, which will be used in the subsequent sections. Section 3 contains auxiliary results and properties. Section 4 deals with the dual of $\left(L^{q()}, l^{p()}\right)(\Omega)$.

Throughout the paper, the constants are independent of the main parameters involved, with values that may differ from line to line.

2. Definitions and Notations

- d will be a fixed positive integer, Ω a non void subset of $\mathbb{R}^d$, for any subset E of $\mathbb{R}^d$ the d-dimensional euclidean space $\mathbb{R}^d$ is equipped with its Lebesgue measure $E \mapsto |E|$ and χ_E will be the characteristic function of E, for any $x \in \mathbb{R}^d$, $|x|$ will be the usual euclidean norm of x.

* $p(\,), q(\,), r(\,), \cdots$ in general indicate that $p, q, r, \cdots$ are functions used as norm indexes ($\|\cdot\|_{p()}, \|\cdot\|_{q()}, \|\cdot\|_{r()}, \cdots$).

* $f(\cdot), g(\cdot), h(\cdot), \cdots$ in general mean that $f, g, h, \cdots$ are functions which are applied on the elements $x, y, z, \cdots$ of $\mathbb{R}^d$, the dots between the brace refer to these elements.

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p(\,):\Omega \to [1,\infty]$. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(\,)$.

Given $p(\,)$ and a set $E \subset \Omega$, let:

$$p_-(E) = \operatorname{essinf}_{x\in E} p(x), \quad p_+(x) = \operatorname{esssup}_{x\in E} p(x).$$

We simply write:

$$p_- = p_-(\Omega) \text{ and } p_+ = p_+(\Omega).$$

As in the case for the classical Lebesgue spaces, we will encounter different behaviors depending on whether:

$$p(x) = 1, \quad 1 < p(x) < \infty, \quad p(x) = \infty.$$

Therefore, we define three canonical subsets of Ω:

$$\Omega_\infty^{p()} = \Omega_\infty = \{x \in \Omega : p(x) = \infty\}$$

$$\Omega_1^{p()} = \Omega_1 = \{x \in \Omega : p(x) = 1\}$$

$$\Omega_*^{p()} = \Omega_* = \{x \in \Omega : 1 < p(x) < \infty\}$$

Below, the value of certain constants will depend on whether these sets have positive measure; if they do we will use the fact that, for instance,

$$\left\|\chi_{\Omega_1^{p(\cdot)}}\right\|_\infty = 1$$

Given $p(\ )$, we define the conjugate exponent $p'(\ )$ by:

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1,\quad x\in\Omega$$

with the convention $\frac{1}{\infty}=0$.

Since $p(\ )$ is a function, the notation $p'(\ )$ can be mistaken for the derivative of $p(\ )$, but we will never use the symbol << '>> in this sense.

The notation p' will always denote the conjugate of a constant exponent. The operation of taking the supremum/infimum of an exponent does not commute with forming the conjugate exponent. In fact, a straightforward computation shows that:

$$\left(p'(\ )\right)_+ = (p_-)',\quad \left(p'(\ )\right)_- = (p_+)'$$

For simplicity, we will omit one set of parentheses and write the left-hand side of each equality as:

$$p'(\ )_+ = (p_-)',\quad p'(\ )_- = (p_+)'$$

We will always avoid ambiguous expressions such as p'_+.

A function $r(\ ):\Omega\to\mathbb{R}$ is *locally log-Holder continuous* and denotes this by $r(\ )\in LH_0(\Omega)$, if there exists a constant C_0, such that:

$$\forall x,y\in\Omega, |x-y|\le\frac{1}{2}:\quad |r(x)-r(y)|\le\frac{C_0}{-\log(|x-y|)}$$

We say that $r(\ )$ is *log-Holder continuous at infinity* and denote this by $r(\ )\in LH_\infty(\Omega)$, if there exist C_∞ and $r_\infty = r(\infty)$ such that

$$\forall x\in\Omega:\quad |r(x)-r_\infty|\le\frac{C_\infty}{\log(e+|x|)}$$

If $r(\ )$ is log-Holder continuous locally and at infinity, we will denote this by writing $r(\ )\in LH(\Omega)=LH_0(\Omega)\cap LH_\infty(\Omega)$.

If there is no confusion about the domain we will sometimes write: LH_0, LH_∞ or LH.

- Let $L^0(\Omega,dx)=L^0(\Omega)$ be the vector space of equivalence modulo *dx*-always everywhere equality of real-valued measurable functions on Ω.
- For any $q(\ )\in\mathcal{P}(\Omega)$ and a Lebesgue measurable function *f*, we denote:

$$\rho_{L^{q()}(\Omega)}(f)=\rho_{q(),\Omega}(f)=\rho_{q()}(f)=\rho(f)=\int_{\Omega\setminus\Omega_\infty^{q()}}|f(x)|^{q(x)}\,\mathrm{d}x+\|f\|_{L^\infty\left(\Omega_\infty^{q()}\right)} \quad (9)$$

where

$$\|f\|_{L^\infty\left(\Omega_\infty^{q()}\right)}=\inf\left\{\varepsilon>0:|f(x)|\le\varepsilon\ a.e.\ x\in\Omega_\infty^{q()}\right\}$$

We define:

$$\|f\|_{L^{q()}(\Omega)} = \|f\|_{q(),\Omega} = \|f\|_{q()} = \inf\left\{\lambda > 0 : \rho_{q()}\left(\frac{f}{\lambda}\right) \le 1\right\} \tag{10}$$

$$L^{q()}(\Omega) = \left\{ f \in L^0(\Omega) : \|f\|_{L^{q()}(\Omega)} < \infty \right\} \tag{11}$$

- If f is unbounded on $\Omega_\infty^{q()}$ or $f(\,)^{q()} \notin L^1\left(\Omega^{q()} \setminus \Omega_\infty\right)$, we define $\rho_{q()}(f) = \infty$
- If $\left|\Omega_\infty^{q()}\right| = 0$ in particular when $q_+ < \infty$, we let $\|f\|_{L^\infty\left(\Omega_\infty^{q()}\right)} = 0$.
- If $\left|\Omega \setminus \Omega_\infty^{q()}\right| = 0$ then $\rho_{q()}(f) = \|f\|_{L^\infty\left(\Omega_\infty^{q()}\right)}$.

Let I be a non void countable set, $\mathcal{P}(I)$ be the set of all Lebesgue measurable functions $p(\,): I \to [1,\infty]$.

- For any $p(\,) \in \mathcal{P}(I)$ and $\{a_k\}_{k\in I} \in \mathbb{R}^I$, we define the modular $\rho_{l^{p()}(I)}$ by:

$$\rho_{l^{p()}(I)}\left(\{a_k\}_{k\in I}\right) = \rho_{p()}\left(\{a_k\}_{k\in I}\right) = \rho\left(\{a_k\}_{k\in I}\right) = \sum_{k\in I\setminus I_\infty^{p()}} |a_k|^{p(k)} + \sup_{k\in I_\infty^{p()}} |a_k| \tag{12}$$

or

$$\rho_{l^{p()}(I)}\left(\{a_k\}_{k\in I}\right) = \left\|\left\{|a_k|^{p(k)}\right\}_{k\in I}\right\|_{l^1\left(I\setminus I_\infty^{p()}\right)} + \left\|\left\{|a_k|\right\}_{k\in I}\right\|_{l^\infty\left(I_\infty^{p()}\right)}$$

- If $\left\{|a_k|^{p(k)}\right\}_{k\in I} \notin l^1\left(I \setminus I_\infty^{p()}\right)$ or $\left\{|a_k|\right\}_{k\in I}$ is unbounded on $I_\infty^{p()}$, we define $\rho_{p()}\left(\{a_k\}_{k\in I}\right) = \infty$.
- If $I_\infty^{p()} = \varnothing$, in particular when $p_+ < \infty$, we let $\sup_{k\in I_\infty^{p()}} |a_k| = 0$ therefore $\rho_{l^{p()}(I)}\left(\{a_k\}_{k\in I}\right) = \sum_{k\in I} |a_k|^{p(k)}$.
- If $I \setminus I_\infty^{q()} = \varnothing$ then $\rho_{p()}\left(\{a_k\}_{k\in I}\right) = \sup_{k\in I_\infty^{p()}} |a_k|$.

Definition 3.

Let I be a non-void countable set, $\mathcal{P}(I)$ be the set of all functions $p(): I \to [1,\infty]$.

For any $p(\,) \in \mathcal{P}(I)$, we define the variable sequence spaces $l^{p()}(I)$ by:

$$l^{p()}(I) = \left\{ \{a_k\}_{k\in I} \in \mathbb{R}^I : \left\|\{a_k\}_{k\in I}\right\|_{l^{p()}(I)} < \infty \right\} \tag{13}$$

where

$$\left\|\{a_k\}_{k\in I}\right\|_{l^{p()}(I)} = \inf\left\{\lambda > 0 : \rho_{l^{p()}(I)}\left(\frac{\{a_k\}_{k\in I}}{\lambda}\right) \le 1\right\} \tag{14}$$

$$\rho_{l^{p()}(I)}\left(\{a_k\}_{k\in I}\right) = \sum_{k\in I\setminus I_\infty^{p()}} |a_k|^{p(k)} + \sup_{k\in I_\infty^{p()}} |a_k|. \tag{15}$$

Then, for any $p(\,) \in \mathcal{P}(I)$, $\forall \lambda > 0$:

$$\rho_{l^{p()}(I)}\left(\frac{\{a_k\}_{k\in I}}{\lambda}\right) = \sum_{k\in I\setminus I_\infty^{p()}} \left(\frac{|a_k|}{\lambda}\right)^{p(k)} + \sup_{k\in I_\infty^{p()}} \frac{|a_k|}{\lambda}. \tag{16}$$

We define on $l^{p()}(I)$ some operations as follows:

For any $\{a_k\}_{k\in I} \in l^{p()}(I)$, $\{b_k\}_{k\in I} \in l^{p()}(I)$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}\setminus\{0\}$:

$\{a_k\}_{k\in I} + \{b_k\}_{k\in I} = \{a_k + b_k\}_{k\in I}$; $\alpha \cdot \{a_k\}_{k\in I} = \{\alpha \cdot a_k\}_{k\in I}$;

$\{a_k\}_{k\in I} \cdot \{b_k\}_{k\in I} = \{a_k \cdot b_k\}_{k\in I}$; $\dfrac{\{a_k\}_{k\in I}}{\beta} = \left\{\dfrac{a_k}{\beta}\right\}_{k\in I}$.

We also define the absolute value of any element $\{a_k\}_{k\in I}$ of $l^{p()}(I)$ by:

$$\left|\{a_k\}_{k\in I}\right| = \{|a_k|\}_{k\in I}$$

the *s*-power of $\{a_k\}_{k\in I}$ of $l^{p()}(I)$ (with $1 \le s < \infty$) is defined by:

$$\left|\{a_k\}_{k\in I}\right|^s = \left\{|a_k|^s\right\}_{k\in I}$$

Remark that:

- If $p() < \infty$,

then $\rho_{p(),I}\left(\{a_k\}_{k\in I}\right) = \sum_{k\in I\setminus I_\infty^{p()}} |a_k|^{p(k)} = \sum_{k\in I} |a_k|^{p(k)}$.

- If $p() = \infty$,

then $\rho_{p(),I}\left(\{a_k\}_{k\in I}\right) = \sup_{k\in I} |a_k|$.

Properties 4.

1) Let's prove that:

$$1 \le p() < \tilde{p}() \le \infty \Rightarrow \left\|\{a_k\}_{k\in I}\right\|_{l^{p()}(I)} \le \left\|\{a_k\}_{k\in I}\right\|_{l^{p()}(I)}. \tag{17}$$

Remark that (17) generalizes (8).

- Case 1: $p() < \tilde{p}() < \infty$

$p() < \tilde{p}() < \infty \Rightarrow I_\infty^{p()} = I_\infty^{\tilde{p}()} = \varnothing$

Let's prove that: $p() < \tilde{p}() < \infty$ a.e. on I

$$\begin{aligned}&\Rightarrow l^{p()}(I) = \left\{\{a_k\}_{k\in I} \in (\mathbb{R})^I : \inf\left\{\lambda > 0 : \sum_{k\in I}\left(\frac{|a_k|}{\lambda}\right)^{p(k)} \le 1\right\} < \infty\right\} \\ &\subset l^{\tilde{p}()}(I) = \left\{\{a_k\}_{k\in I} \in (\mathbb{R})^I : \inf\left\{\lambda > 0 : \sum_{k\in I}\left(\frac{|a_k|}{\lambda}\right)^{\tilde{p}(k)} \le 1\right\} < \infty\right\}.\end{aligned}$$

If $(a_k)_{k\in I}$ belongs to the left-hand side set, then:

$$\inf\left\{\lambda > 0 : \sum_k \left(\frac{|a_k|}{\lambda}\right)^{p(k)} \le 1\right\} < \infty \Rightarrow \exists\, 0 < \lambda_0 < \infty :$$

$$\sum_{k\in I}\left(\frac{|a_k|}{\lambda_0}\right)^{p(k)} \le 1 \tag{18}$$

this implies that $\forall k \in I : \left(\dfrac{|a_k|}{\lambda_0}\right)^{p(k)} \le 1 \Rightarrow$

$$\forall k \in I : \frac{|a_k|}{\lambda_0} \le 1. \tag{19}$$

Since $\forall k\in I:1\le p(k)<\tilde{p}(k)<\infty\Rightarrow\forall k\in I:\tilde{p}(k)-p(k)>0$, this inequality with (19) $\Rightarrow\left(\frac{|a_k|}{\lambda_0}\right)^{\tilde{p}(k)-p(k)}\le 1$ then $\left(\frac{|a_k|}{\lambda_0}\right)^{\tilde{p}(k)}\le\left(\frac{|a_k|}{\lambda_0}\right)^{p(k)}$, therefore $\sum_{k\in I}\left(\frac{|a_k|}{\lambda_0}\right)^{\tilde{p}(k)}\le\sum_{k\in I}\left(\frac{|a_k|}{\lambda_0}\right)^{p(k)}\overset{(18)}{\le}1$, this implies that $(a_k)_{k\in I}\in l^{\tilde{p}()}(I)$ then $l^{p()}(I)\subset l^{\tilde{p}()}(I)$.

- Case 2: $p()<\infty$; $\tilde{p}()=\infty$.

$$p()<\infty\Rightarrow I_\infty^{p()}=\varnothing \quad\text{and}\quad \tilde{p}()=\infty\Rightarrow I_\infty^{\tilde{p}()}=I$$

In this case: $\rho_{l^{p()}(I)}\left(\{a_k\}_{k\in I}\right)=\sum_{k\in I\setminus I_\infty^{p()}}|a_k|^{p(k)}+\sup_{k\in I_\infty^{p()}}|a_k|=\sum_{k\in I}|a_k|^{p(k)}$

$$\rho_{l^{\tilde{p}()}(I)}\left(\{a_k\}_{k\in I}\right)=\sum_{k\in I\setminus I_\infty^{\tilde{p}()}}|a_k|^{p(k)}+\sup_{k\in I_\infty^{\tilde{p}()}}|a_k|=\sup_{k\in I}|a_k|$$

$$l^{p()}(I)=\left\{\{a_k\}_{k\in I}\in(\mathbb{R})^I:\inf\left\{\lambda>0:\sum_{k\in I}\left(\frac{|a_k|}{\lambda}\right)^{p(k)}\le 1\right\}<\infty\right\},$$

$$l^{\tilde{p}()}(I)=\left\{\{a_k\}_{k\in I}\in(\mathbb{R})^I:\inf\left\{\lambda>0:\sup_{k\in I}\frac{|a_k|}{\lambda}\le 1\right\}<\infty\right\}.$$

Let's compare $\left\{\lambda>0:\sum_{k\in I}\left(\frac{|a_k|}{\lambda}\right)^{p(k)}\le 1\right\}$ and $\left\{\lambda>0:\sup_{k\in I}\frac{|a_k|}{\lambda}\le 1\right\}$

Take λ_0 in the first (left-hand side) set, then $\sum_{k\in I}\left(\frac{|a_k|}{\lambda_0}\right)^{p(k)}\le 1$, then $\forall k\in I:\left(\frac{|a_k|}{\lambda_0}\right)^{p(k)}\le 1\Rightarrow\forall k\in I:\frac{|a_k|}{\lambda_0}\le 1\Rightarrow\sup_{k\in I}\frac{|a_k|}{\lambda_0}\le 1$ this implies that λ_0 belongs to the right-hand side set, therefore:

We have:

$$\left\{\lambda>0:\sum_{k\in I}\left(\frac{|a_k|}{\lambda}\right)^{p(k)}\le 1\right\}\subset\left\{\lambda>0:\sup_{k\in I}\frac{|a_k|}{\lambda}\le 1\right\}$$

$$\Rightarrow\inf\left\{\lambda>0:\sup_{k\in I}\frac{|a_k|}{\lambda}\le 1\right\}\le\inf\left\{\lambda>0:\sum_{k\in I}\left(\frac{|a_k|}{\lambda}\right)^{p(k)}\le 1\right\}$$

$$\Rightarrow l^{p()}(I)\subset l^{\tilde{p}()}(I)$$

2) Given a non-void countable set I and $p()\in\mathcal{P}(I)$ such that $I_\infty^{p()}=\varnothing$, then for all s such that $\frac{1}{p_-}\le s<\infty$, we have:

$$\left\|\left\{|a_k|^s\right\}_{k\in I}\right\|_{l^{p()}(I)}=\left\|\{a_k\}_{k\in I}\right\|_{l^{sp()}(I)}^s.$$

To prove this, let $\mu=\lambda^{\frac{1}{s}}$

$$\left\|\left\{|a_k|^s\right\}_{k\in I}\right\|_{l^{p()}(I)} = \inf\left\{\lambda>0:\sum_{k\in I}\left(\frac{|a_k|^s}{\lambda}\right)^{p(k)}\le 1\right\}$$
$$= \inf\left\{\mu^s>0:\sum_{k\in I}\left(\frac{|a_k|^s}{\mu^s}\right)^{p(k)}\le 1\right\}$$
$$= \inf\left\{\mu^s>0:\sum_{k\in I}\left(\frac{|a_k|}{\mu}\right)^{sp(k)}\le 1\right\}$$
$$= \left\|\left\{a_k\right\}_{k\in I}\right\|^s_{l^{sp()}(I)}$$

3) When $p()=p$, $1\le p\le\infty$, the definition (13) is equivalent to the classical norm of l^p seen in (6), let's prove it:

For $p()=p<\infty$,

then $I_\infty^{p()}=\varnothing$ and $\rho_{p(),I}\left(\left\{a_k\right\}_{k\in I}\right)=\sum_{k\in I}|a_k|^{p(k)}$,

$$\left\|\left\{a_k\right\}_{k\in I}\right\|_{l^{p()}(I)} = \inf\left\{\lambda>0:\sum_{k\in I}\left(\frac{|a_k|}{\lambda}\right)^{p(k)}\le 1\right\} = \inf\left\{\lambda>0:\sum_{k\in I}\left(\frac{|a_k|}{\lambda}\right)^{p}\le 1\right\}$$
$$= \inf\left\{\lambda>0:\lambda\ge\left(\sum_{k\in I}|a_k|^p\right)^{\frac{1}{p}}\right\} = \left(\sum_{k\in I}|a_k|^p\right)^{\frac{1}{p}} = \left\|\left\{a_k\right\}_{k\in I}\right\|_{l^p(I)}.$$

For $p()=p=\infty$,

then

$I_\infty^{p()}=I$ and $\rho_{l^{p()}(I)}\left(\left\{a_k\right\}_{k\in I}\right)=\sup_{k\in I}|a_k|$, therefore

$$\left\|\left\{a_k\right\}_{k\in I}\right\|_{l^{p()}(I)} = \inf\left\{\lambda>0:\sup_{k\in I}\frac{|a_k|}{\lambda}\le 1\right\} = \inf\left\{\lambda>0:\lambda\ge\sup_{k\in I}|a_k|\right\}$$
$$= \sup_{k\in I}|a_k| = \left\|\left\{a_k\right\}_{k\in I}\right\|_{l^{+\infty}(I)}.$$

4) Given a countable and non-void set I and $p()\in\mathcal{P}(I)$, for all:

$$\left\{a_k\right\}_{k\in I}\in l^{p()}(I) \text{ and } \left\{b_k\right\}_{k\in I}\in l^{p'()}(I), \text{ if } \left\{a_kb_k\right\}_{k\in I}\in l^1(I),$$

then

$$\sum_{k\in I}a_kb_k\le K_{p()}\left\|\left\{a_k\right\}_{k\in I}\right\|_{l^{p()}(I)}\times\left\|\left\{b_k\right\}_{k\in I}\right\|_{l^{p'()}(I)}. \quad (20)$$

This inequality can be generalized in the following way:

5) Given a countable and non-void set I and $q(),r()\in\mathcal{P}(I)$, define $p()$ by:

$$\frac{1}{p()}=\frac{1}{q()}+\frac{1}{r()} \text{ on } I$$

Then, there exists a constant K such that for all:

$$\left\{a_k\right\}_{k\in I}\in l^{q()}(I), \left\{b_k\right\}_{k\in I}\in l^{r()}(I): \left\{a_kb_k\right\}_{k\in I}\in l^{p()}(I) \text{ and}$$

$$\left\|\{a_k b_k\}_{k\in I}\right\|_{l^{p()}(I)} \le K_{p()}\left\|\{a_k\}_{k\in I}\right\|_{l^{q()}(I)} \times \left\|\{b_k\}_{k\in I}\right\|_{l^{r()}(I)}. \tag{21}$$

In fact, we have the following result.

Proposition 5.

Given a non void countable set I and $p(\,)\in\mathcal{P}(I)$.

a) Suppose that $\{b_i\}_{i\in I}\in l^{p'()}(I)$. Then:

•) $a\mapsto T_b(a)=\sum_{i\in I}a_i b_i$ is a continuous linear functional on $l^{p()}(I)$ and

$$\|T_b\|\equiv\sup\left\{\left|\sum_{i\in I}a_i b_i\right|: a=\{a_i\}_{i\in I}\in l^{p()}(I),\|a\|_{l^{p()}(I)}\le 1\right\}=\|b\|_{l^{p'()}(I)}$$

••) If $p'(\,)<\infty$ on I, then

$$\|T_b\|=\max\left\{\left|\sum_{i\in I}a_i b_i\right|: a=\{a_i\}_{i\in I}\in l^{p()}(I),\|a\|_{l^{p()}(I)}=1\right\}.$$

b) Suppose that $b=\{b_i\}_{i\in I}\in\mathbb{C}^I$ such that

$$N_{p'()}(b)=\sup\left\{\left|\sum_{i\in I}a_i b_i\right|: a=\{a_i\}_{i\in I}\in\mathbb{C}^I, Card\left(\{i\in I: a_i\ne 0\}\right)<\infty,\|a\|_{l^{p()}(I)}=1\right\}<\infty.$$

Then,

$$b\in l^{p'()}(I)\text{ and }\|b\|_{l^{p'()}(I)}=N_{p'()}(b)$$

c) Suppose that $p(\,)\in\mathcal{P}(I)$, $p(\,)<\infty$ on I and T belongs to the dual $\left(l^{p()}(I)\right)^*$ of $l^{p()}(I)$.

Then,

there exists $b=(b_i)_{i\in I}\in l^{p'()}(I)$ such that:

$$\forall a=\{a_i\}_{i\in I}\in l^{p()}(I): T(a)=\sum_{i\in I}a_i b_i$$

For the proof of (20) and (21) and Proposition 5, consult Theorem 2.26 and Corollary 2.28 of [27] take account of the fact that $\left(l^{p()},\|\cdot\|_{l^{p()}}\right)$ is in fact the Lebesgue space $L^{p()}(E,\mathcal{A},\mu)$ where $E=\mathbb{Z}$, $\mathcal{A}=\{X: X\subset E\}=2^X$ the power of the set X, μ is defined as: $\forall k\in\mathbb{Z}:\mu\{k\}=1$, μ is a counting measure.

In this work, we will need the following lemma called Norm-modular unit ball property.

Lemma 6. (*Norm-modular unit ball property*)

Let Ω be a non void set $q(\,)\in\mathcal{P}(\Omega)$, suppose that $q_+<\infty$. For any sequence $\{f_n\}\subset L^{q()}(\Omega)$ and $f\in L^{q()}(\Omega)$, $\|f-f_n\|_{q(),\Omega}\to 0$ if only if $\rho_{L^{q()}(\Omega)}(f-f_n)\to 0$.

Remark that the discrete version of this lemma is also valid.

Historic of the definition

Recall that: For $1\le p,q\le\infty$, $r>0$,

$$I_k^r=\prod_{j=1}^{d}\left[k_j\times r,(k_j+1)\times r\right[,\text{ with }k=(k_j)_{1\le j\le d}\in\mathbb{Z}^d.$$

The amalgam of L^q and l^p is the space (L^q,l^p) defined (see (1)) by:

$$\left(L^q,l^p\right)\left(\mathbb{R}^d\right)=\left\{f\in L^1_{loc}\left(\mathbb{R}^d\right):{}_1\|f\|_{q,p}<\infty\right\},$$

where

$$_r\|f\|_{q,p}=\begin{cases}\left[\sum_{k\in\mathbb{Z}^d}\left\|f\chi_{I_k^r}\right\|_q^p\right]^{\frac{1}{p}} & \text{if } p<\infty\\ \sup_{k\in\mathbb{Z}^d}\left\|f\chi_{I_k^r}\right\|_q & \text{if } p=\infty\end{cases}$$

Suppose that q is a function $q(\,)$ and p a constant and taking account of (10), we get:

$$_r\|f\|_{q(),p}=\begin{cases}\left[\sum_{k\in\mathbb{Z}^d}\left\|f\chi_{I_k^r}\right\|_{q()}^p\right]^{\frac{1}{p}} & \text{if } p<\infty\\ \sup_{k\in\mathbb{Z}^d}\left\|f\chi_{I_k^r}\right\|_{q()} & \text{if } p=\infty\end{cases}\tag{22}$$

$\left\{\left\|f\chi_{I_k^r}\right\|_{q()}\right\}_{k\in\mathbb{Z}^d}$ is real sequence indexed by a countable set $\mathbb{Z}^d$.

Suppose that q and p are both functions $q(\,)$ and $p(\,)$ and taking account of (14), (22) becomes:

$$_r\|f\|_{q(),p()}=\left\|\left\{\left\|f\chi_{I_k^r}\right\|_{q()}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}\left(\mathbb{Z}^d\right)}$$

which may be rewritten with more information under the form:

$$_r\|f\|_{q(),p(),\Omega}=\left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}\left(\mathbb{Z}^d\right)}\tag{23}$$

where

$$L^{q()}(\Omega)=\left\{f\in L^0(\Omega):\|f\|_{L^{q()}(\Omega)}<\infty\right\}$$

$$\|f\|_{L^{q()}(\Omega)}=\inf\left\{\lambda>0:\rho_{L^{q()}(\Omega)}\left(\frac{f}{\lambda}\right)\le 1\right\}$$

$$\rho_{L^{q()}(\Omega)}(f)=\int_{\Omega\setminus\Omega_\infty^{q()}}|f(x)|^{q(x)}\,\mathrm{d}x+\|f\|_{L^\infty\left(\Omega_\infty^{q()}\right)}$$

$$l^{p()}\left(\mathbb{Z}^d\right)=\left\{\{a_k\}_{k\in I}\in(\mathbb{R})^{\mathbb{Z}^d}:\left\|\{a_k\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}\left(\mathbb{Z}^d\right)}<\infty\right\}$$

$$\left\|\{a_k\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}\left(\mathbb{Z}^d\right)}=\inf\left\{\lambda>0:\rho_{l^{p()}\left(\mathbb{Z}^d\right)}\left(\frac{\{a_k\}_{k\in\mathbb{Z}^d}}{\lambda}\right)\le 1\right\}$$

$$\rho_{l^{p()}\left(\mathbb{Z}^d\right)}\left(\{a_k\}_{k\in\mathbb{Z}^d}\right)=\sum_{k\in\mathbb{Z}^d\setminus\left(\mathbb{Z}^d\right)_\infty^{p()}}|a_k|^{p(k)}+\sup_{k\in\left(\mathbb{Z}^d\right)_\infty^{p()}}|a_k|$$

Definition 7.

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, for any $q(\,)\in\mathcal{P}(\Omega)$, $p(\,)\in\mathcal{P}\left(\mathbb{Z}^d\right)$, let

$f \in L_{loc}^{q()}(\Omega)$, $I_k^r = \prod_{j=1}^{d} \left[k_j \times r, (k_j+1) \times r \right[$, with $k = (k_j)_{1\le j\le d} \in \mathbb{Z}^d$, $0 < r < \infty$, for any Lebesgue measurable function *f* we define the non negative real number:

$$ {}_r\|f\|_{q(),p(),\Omega} = \left\| \left\{ \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} = \left\| \left\{ F(k,r) \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \tag{24} $$

where

$$ F(k,r) = \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)}, \; k \in \mathbb{Z}^d, \; 0 < r < \infty. $$

If we take $r = 1$ in (24), we get:

$$ {}_1\|f\|_{q(),p(),\Omega} = \left\| \left\{ F(k,1) \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} = \left\| \left\{ \left\| f\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}. \tag{25} $$

We define the two-variable exponential amalgam spaces $\left(L^{q()}, l^{p()}\right)(\Omega)$ by:

$$ \left(L^{q()}, l^{p()}\right)(\Omega) = \left\{ f \in L_{loc}^{q()}(\Omega) : {}_1\|f\|_{q(),p(),\Omega} < \infty \right\} \tag{26} $$

If there is no confusion:

${}_r\|f\|_{q(),p(),\mathbb{R}^d}$, $\left(L^{q()}, l^{p()}\right)(\mathbb{R}^d)$ will be smply ${}_r\|f\|_{q(),p()}$, $\left(L^{q()}, l^{p()}\right)$.

Explanation

To compute ${}_r\|f\|_{q(),p(),\Omega} = \left\| \left\{ \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$, we first calculate:

$$ \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} = \inf \left\{ \lambda > 0 : \rho_{L^{q()}(\Omega)}\left(\frac{f\chi_{I_k^r}}{\lambda} \right) \le 1 \right\}, $$

(where $\rho_{L^{q()}(\Omega)}\left(\frac{f\chi_{I_k^r}}{\lambda} \right) = \int_{\Omega\setminus\Omega_\infty^{q()}} \left| \left(\frac{f\chi_{I_k^r}(y)}{\lambda} \right) \right|^{q(y)} \mathrm{d}y + \left\| \frac{f\chi_{I_k^r}}{\lambda} \right\|_{L^\infty(\Omega_\infty^{q()})}$), this result depends at least on *k* and *r*, we denote it by $\beta(k,r)$, after that we consider $\left\{ \beta(k,r) \right\}_{k\in\mathbb{Z}^d}$ and determine:

$$ \left\| \left\{ \beta(k,r) \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} = \inf \left\{ \lambda > 0 : \rho_{l^{p()}(\mathbb{Z}^d)}\left(\frac{\left\{ \beta(k,r) \right\}_{k\in\mathbb{Z}^d}}{\lambda} \right) \le 1 \right\}. $$

where

$$ \rho_{l^{p()}(\mathbb{Z}^d)}\left(\frac{\left\{ \beta(k,r) \right\}_{k\in\mathbb{Z}^d}}{\lambda} \right) = \sum_{k\in\mathbb{Z}^d\setminus(\mathbb{Z}^d)_\infty^{p()}} \left(\frac{|\beta(k,r)|}{\lambda} \right)^{p(k)} + \left\| \left\{ \frac{|\beta(k,r)|}{\lambda} \right\}_{k\in\mathbb{Z}^d} \right\|_{L^{+\infty}\left((\mathbb{Z}^d)_\infty^{p()}\right)}, $$

this result depends at least on *r*.

Finally, the result of the calculation of ${}_r\|f\|_{q(),p()}$ depends at least on *r*.

In other hand,

$$ f \in \left(L^{q()}, l^{p()}\right)(\Omega) \Leftrightarrow $$

$$\begin{cases} \left\| f\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} < \infty, & \text{for any cube } I_k^1 \subset \Omega, \text{ and} \\ \left\| \left\{ \left\| f\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} < \infty \end{cases} \tag{27}$$

- A sequence $(f_n)_n$ of $\left(\left(L^{q()}, l^{p()}\right)(\Omega), {}_1\|\cdot\|_{q(),p()} \right)$ is said to converge in norm to *f*, we note:

$$f_n \to f, \text{ if } {}_1\|f - f_n\|_{q(),p(),\Omega} \to 0 \text{ when } n \to \infty.$$

- For two functions (eventually constants) $q(\,), p(\,)$ on Ω such that $0 < p(\,) < q(\,) < \infty$, we define:

$$L^{p()}(\Omega) + L^{q()}(\Omega) = \left\{ f = g + h : g \in L^{p()}(\Omega), h \in L^{q()}(\Omega) \right\}.$$

This is a Banach space with the norm:

$$\|f\|_{L^{p()}(\Omega)+L^{q()}(\Omega)} = \inf\left\{ \|g\|_{L^{p()}(\Omega)} + \|h\|_{L^{q()}(\Omega)} : f = g + h, g \in L^{p()}(\Omega), h \in L^{q()}(\Omega) \right\}.$$

3. Properties

In this section, we will use a method to show that $\left(\left(L^{q()}, l^{p()}\right)(\Omega), {}_1\|\cdot\|_{q(),p(),\Omega} \right)$ is a Banach space either Ω is bounded or unbounded.

Proposition 8.

Let Ω be a set such that $\mathbb{Z}^d \subset \Omega$, given $q(\,) \in \mathcal{P}(\Omega)$, $p(\,) \in \mathcal{P}(\mathbb{Z}^d)$ and $f \in L^{q()}_{loc}(\Omega)$.

1) Then $\left(L^{q()}, l^{p()}\right)(\Omega)$ is a vector space and

$${}_1\|f\|_{q(),p(),\Omega} < \infty \Rightarrow f < \infty.$$

2) The function $f \mapsto {}_1\|f\|_{q(),p(),\Omega}$ defines a norm on $\left(L^{q()}, l^{p()}\right)(\Omega)$.

Proof.

1) $0 \in \left(L^{q()}, l^{p()}\right)(\Omega) \subset L^{q()}_{loc}(\Omega)$ which is a vector space, then it will suffice to show that for all $\alpha, \beta \in \mathbb{R}$ not both 0; and $f, g \in \left(L^{q()}, l^{p()}\right)(\Omega)$:

$$\alpha f + \beta g \in \left(L^{q()}, l^{p()}\right)(\Omega)$$

From triangle inequality of $\|\cdot\|_{L^{q()}(\Omega)}$, we have:

$$\left\| (\alpha f + \beta g)\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \le \left\| (\alpha f)\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} + \left\| (\beta g)\chi_{I_k^1} \right\|_{L^{q()}(\Omega)}$$

$\|\cdot\|_{l^{p()}(\mathbb{Z}^d)}$ is order preserving, then:

$$\left\| \left\{ \left\| (\alpha f + \beta g)\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$
$$\le \left\| \left\{ \left\| (\alpha f)\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} + \left\| (\beta g)\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}.$$

From triangle inequality of $\|\cdot\|_{l^{p()}(\mathbb{Z}^d)}$:

$$\left\|\left\{\left\|(\alpha f+\beta g)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq\left\|\left\{\left\|(\alpha f)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}+\left\|\left\{\left\|(\beta g)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}.$$

From the homogeneity of $\|\cdot\|_{L^{q()}(\Omega)}$ and $\|\cdot\|_{l^{p()}(\mathbb{Z}^d)}$, we have:

$$\left\|\left\{\left\|(\alpha f+\beta g)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq|\alpha|\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}+|\beta|\left\|\left\{\left\|g\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

that is

$$_1\|\alpha f+\beta g\|_{q(),p(),\Omega}\leq|\alpha|_1\|f\|_{q(),p(),\Omega}+|\beta|_1\|g\|_{q(),p(),\Omega} \tag{28}$$

It is obvious that $_1\|f\|_{q(),p(),\Omega}<\infty\Rightarrow f<\infty$.

2) Let $f\in\left(L^{q()},l^{p()}\right)(\Omega)$.

It is easy to see that:

$$_1\|f\|_{q(),p(),\Omega}\geq 0\text{, let }_1\|f\|_{q(),p(),\Omega}=0.$$

$$_1\|f\|_{q(),p(),\Omega}=0\Rightarrow\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}=0$$

$$\Rightarrow\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}=\{0\}_{k\in\mathbb{Z}^d}$$

$$\Rightarrow\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}=0,\quad k\in\mathbb{Z}^d$$

$$\Rightarrow f\chi_{I_k^1}=0,\ k\in\mathbb{Z}^d$$

$$\Rightarrow f=0$$

Homogeneity of $_1\|\cdot\|_{q(),p(),\Omega}$:

Let $f\in\left(L^{q()},l^{p()}\right)(\Omega)$, and $\alpha\in\mathbb{R}$, by homogeneity of $\|\cdot\|_{L^{q()}(\Omega)}$ and $\|\cdot\|_{l^{p()}(\mathbb{Z}^d)}$, we have:

$$_1\|\alpha f\|_{q(),p(),\Omega}=\left\|\left\{\left\|(\alpha f)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$=\left\||\alpha|\left\{\left\|(f)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$=|\alpha|\left\|\left\{\left\|(f)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$=|\alpha|\times{}_1\|f\|_{q(),p(),\Omega}.$$

Triangle inequality of $_1\|\cdot\|_{q(),p(),\Omega}$:

In (28), if we take $\alpha=\beta=1$, we will get:

$$_1\|f+g\|_{q(),p(),\Omega} \le {}_1\|f\|_{q(),p(),\Omega} + {}_1\|g\|_{q(),p(),\Omega}.$$

□

We will need the following lemmas.

Lemma 9. [28]

Let $(X,\mathcal{A},\mu)$ be a measure space such that $\mu(X)<\infty$.

Then, $L^q(X,\mathcal{A},\mu)\subset L^p(X,\mathcal{A},\mu)$ that is $\|f\|_p \le C(q,p)\times\|f\|_q$ for any $1\le p\le q\le\infty$.

Where

$$C(q,p)=\begin{cases}[\mu(X)]^{\frac{1}{p}-\frac{1}{q}} & \text{if } q<\infty\\ [\mu(X)]^{\frac{1}{p}} & \text{if } q=\infty\end{cases}$$

Lemma 10. (*Monotone Convergence*)

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, given $q()\in\mathcal{P}(\Omega)$, $p()\in\mathcal{P}(\mathbb{Z}^d)$ and $f\in L^{q()}_{loc}(\Omega)$.

If $\{f^{[n]}\}_{n\ge1}\subset\left(\left(L^{q()},l^{p()}\right)(\Omega),{}_1\|\cdot\|_{q(),p(),\Omega}\right)$ is a sequence of non negative functions such that $f^{[n]}$ increases to a function *f* pointwise always everywhere (a.e.).

Then:

$$\text{either}\quad f\in\left(L^{q()},l^{p()}\right)(\Omega)\ \text{ and}$$
$$_1\left\|f^{[n]}\right\|_{q(),p(),\Omega}\to{}_1\|f\|_{q(),p(),\Omega}$$

or

$$f\notin\left(L^{q()},l^{p()}\right)(\Omega)\ \text{ and}$$
$$_1\left\|f^{[n]}\right\|_{q(),p(),\Omega}\to\infty={}_1\|f\|_{q(),p(),\Omega}.$$

Proof.

First:

$$\rho_{L^{q()}(\Omega)}(f)=\int_{\Omega\setminus\Omega_\infty^{q()}}|f(x)|^{q(x)}\,\mathrm{d}x+\|f\|_{L^\infty\left(\Omega_\infty^{q()}\right)}$$
$$=\int_{\Omega\setminus\Omega_\infty}|f(x)|^{q(x)}\,\mathrm{d}x+\|f\|_{L^\infty(\Omega_\infty)}.$$

Let's decompose *f* as:

$$f=f_1+f_2 \tag{29}$$

where $f_1=f\chi_{\{x\in\Omega:|f(x)|<1\}}$, $f_2=f\chi_{\{x\in\Omega:|f(x)|\ge1\}}$.

Therefore,

$$\rho_{L^{q()}(\Omega)}(f)\le\int_{\Omega\setminus\Omega_\infty}\left(|f_1(x)|+|f_2(x)|\right)^{q(x)}\mathrm{d}x$$
$$+\left\||f_1|+|f_2|\right\|_{L^\infty(\Omega_\infty)}.$$

We have that:

$$\forall q()\in\mathcal{P}(\Omega):\quad 1\le q_-\le q(x)\le q_+\le\infty,$$

in other hand, for any non negative real numbers a,b:

$$(a+b)^{q(x)} \le 2^{q(x)-1}\left(a^{q(x)}+b^{q(x)}\right) \le 2^{q_+-1}\left(a^{q(x)}+b^{q(x)}\right).$$

Then,

$$\begin{aligned}
\rho_{L^{q()}(\Omega)}(f) &\le 2^{q_+-1}\int_{\Omega\setminus\Omega_\infty}\left(\left|f_1(x)\right|^{q(x)}+\left|f_2(x)\right|^{q(x)}\right)\mathrm{d}x+\left\|\,|f_1|+|f_2|\,\right\|_{L^\infty(\Omega_\infty)} \\
&\le 2^{q_+-1}\int_{\Omega\setminus\Omega_\infty}\left(\left|f_1(x)\right|^{q_-}+\left|f_2(x)\right|^{q_+}\right)\mathrm{d}x+\left\|\,|f_1|+|f_2|\,\right\|_{L^\infty(\Omega_\infty)} \\
&\le 2^{q_+-1}\int_{\Omega\setminus\Omega_\infty}\left(\left|f_1(x)\right|^{q_-}\right)\mathrm{d}x+\left\|f_1\right\|_{L^\infty(\Omega_\infty)}+2^{q_+-1}\int_{\Omega\setminus\Omega_\infty}\left(\left|f_2(x)\right|^{q_+}\right)\mathrm{d}x+\left\|f_2\right\|_{L^\infty(\Omega_\infty)} \\
&\le \left[2^{q_+-1}\left\|\left(f_1\chi_{\Omega\setminus\Omega_\infty}\right)^{q_-}\right\|_1+\left\|f_1\right\|_{L^\infty(\Omega_\infty)}\right]+\left[2^{q_+-1}\left\|\left(f_2\chi_{\Omega\setminus\Omega_\infty}\right)^{q_+}\right\|_1+\left\|f_2\right\|_{L^\infty(\Omega_\infty)}\right] \\
&\le \left[2^{q_+-1}\left\|\left(f_1\chi_{\Omega\setminus\Omega_\infty}\right)\right\|_{q_-}^{q_-}+\left\|f_1\right\|_{L^\infty(\Omega_\infty)}\right]+\left[2^{q_+-1}\left\|\left(f_2\chi_{\Omega\setminus\Omega_\infty}\right)\right\|_{q_+}^{q_+}+\left\|f_2\right\|_{L^\infty(\Omega_\infty)}\right].
\end{aligned}$$

Therefore, for any $f\in L_{loc}^{q()}(\Omega)$, we can decompose it as $f=f_1+f_2$ such that:

$$\begin{aligned}
\rho_{q()}(f) \le &\left[C(q(\cdot))\left\|\left(f_1\chi_{\Omega\setminus\Omega_\infty}\right)\right\|_{q_-}^{q_-}+\left\|f_1\right\|_{L^\infty(\Omega_\infty)}\right] \\
&+\left[C(q(\cdot))\left\|\left(f_2\chi_{\Omega\setminus\Omega_\infty}\right)\right\|_{q_+}^{q_+}+\left\|f_2\right\|_{L^\infty(\Omega_\infty)}\right]
\end{aligned}$$

that is:

$$\begin{aligned}
\rho_{q()}(f) \le &\left[C(q())\left\|f_1\right\|_{L^{q_-}(\Omega\setminus\Omega_\infty)}^{q_-}+\left\|f_1\right\|_{L^\infty(\Omega_\infty)}\right] \\
&+\left[C(q())\left\|f_2\right\|_{L^{q_+}(\Omega\setminus\Omega_\infty)}^{q_+}+\left\|f_2\right\|_{L^\infty(\Omega_\infty)}\right]
\end{aligned} \tag{30}$$

where $C(q())=2^{q_+-1}$.

Now, we begin the proof:

$f^{[n]}$ increases to the function f a.e. (by hypothesis), we can estimate ${}_1\left\|f-f^{[n]}\right\|_{q(),p(),\Omega}$, for any $k\in\mathbb{Z}^d$ and $0<r<\infty$:

$$\begin{aligned}
&{}_1\left\|f-f^{[n]}\right\|_{q(),p(),\Omega} \\
&=\left\|\left\{\left\|\left(f-f^{[n]}\right)\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}\left(\mathbb{Z}^d\right)}
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\left\|\left(f-f^{[n]}\right)\chi_{I_k^r}\right\|_{L^{q()}(\Omega)} \\
&=\inf\left\{\lambda>0:\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)\le 1\right\}.
\end{aligned} \tag{32}$$

To estimate $\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$, we use (29) and (31), to get:

$f=f_1+f_2$ and $f^{[n]}=f_1^{[n]}+f_2^{[n]}$

$$\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$$

$$\leq\left[C\left(q\left(\right)\right)\times\left\|\frac{\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}}{\lambda}\right\|_{L^{q_-}(\Omega\setminus\Omega_\infty)}^{q_-}+\left\|\frac{\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}}{\lambda}\right\|_{L^\infty(\Omega_\infty)}\right]$$

$$+\left[C\left(q\left(\right)\right)\times\left\|\frac{\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}}{\lambda}\right\|_{L^{q_+}(\Omega\setminus\Omega_\infty)}^{q_+}+\left\|\frac{\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}}{\lambda}\right\|_{L^\infty(\Omega_\infty)}\right]$$

$$\leq\left[C\left(q\left(\right)\right)\times\lambda^{-q_-}\left\|\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}\right\|_{L^{q_-}(\Omega\setminus\Omega_\infty)}^{q_-}+\lambda^{-1}\left\|\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}\right\|_{L^\infty(\Omega_\infty)}\right]$$

$$+\left[C\left(q\left(\right)\right)\times\lambda^{-q_+}\left\|\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}\right\|_{L^{q_+}(\Omega\setminus\Omega_\infty)}^{q_+}+\lambda^{-1}\left\|\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}\right\|_{L^\infty(\Omega_\infty)}\right]$$

that is

$$\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$$

$$\leq\left[C\left(q\left(\right)\right)\times\lambda^{-q_-}\left\|\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}\right\|_{L^{q_-}(\Omega\setminus\Omega_\infty)}^{q_-}+\lambda^{-1}\left\|\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}\right\|_{L^\infty(\Omega_\infty)}\right]$$

$$+\left[C\left(q\left(\right)\right)\times\lambda^{-q_+}\left\|\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}\right\|_{L^{q_+}(\Omega\setminus\Omega_\infty)}^{q_+}+\lambda^{-1}\left\|\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}\right\|_{L^\infty(\Omega_\infty)}\right].$$

Now, if we use Lemma 9, since

$1\leq q_-\leq q_+\leq\infty$ and $I_k^1\subset\Omega$, we get:

$$\begin{aligned}\left\|\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}\right\|_{L^{q_-}(\Omega\setminus\Omega_\infty)}&\leq\left\|\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}\right\|_{L^{q_-}(\Omega)}\\&=\left\|f_1-f_1^{[n]}\right\|_{L^{q_-}\left(I_k^1\right)}\\&\leq\left[\left|I_k^1\right|\right]^{\frac{1}{q_-}}\times\left\|f_1-f_1^{[n]}\right\|_{L^{+\infty}\left(I_k^1\right)}.\end{aligned}$$

But $\left|I_k^1\right|=1$, therefore

$$\left\|\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}\right\|_{L^{q_-}(\Omega\setminus\Omega_\infty)}\leq\left\|f_1-f_1^{[n]}\right\|_{L^\infty\left(I_k^1\right)}\tag{33}$$

by the same way, we also have that:

$$\left\|\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}\right\|_{L^{q_+}(\Omega\setminus\Omega_\infty)}\leq\left\|f_2-f_2^{[n]}\right\|_{L^{+\infty}\left(I_k^1\right)}\tag{34}$$

Substituting (33) and (34) in $\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$, we get:

$$\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$$
$$\leq\left[C\left(q(\,)\right)\times\lambda^{-q_-}\left\|f_1-f_1^{[n]}\right\|_{L^\infty\left(I_k^1\right)}^{q_-}+\lambda^{-1}\left\|\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}\right\|_{L^\infty(\Omega_\infty)}\right]$$
$$+\left[C\left(q(\,)\right)\times\lambda^{-q_+}\left\|f_2-f_2^{[n]}\right\|_{L^\infty\left(I_k^1\right)}^{q_+}+\lambda^{-1}\left\|\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}\right\|_{L^\infty(\Omega_\infty)}\right].$$

Thus,

$$\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$$
$$\leq\left[C\left(q(\,)\right)\times\lambda^{-q_-}\left\|f_1-f_1^{[n]}\right\|_{L^\infty\left(I_k^1\right)}^{q_-}+\lambda^{-1}\left\|f_1-f_1^{[n]}\right\|_{L^\infty\left(I_k^1\cap\Omega_\infty\right)}\right]$$
$$+\left[C\left(q(\,)\right)\times\lambda^{-q_+}\left\|f_2-f_2^{[n]}\right\|_{L^\infty\left(I_k^1\right)}^{q_+}+\lambda^{-1}\left\|f_2-f_2^{[n]}\right\|_{L^\infty\left(I_k^1\cap\Omega_\infty\right)}\right]$$

this implies that:

$$\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$$
$$\leq\left[C\left(q(\,)\right)\times\lambda^{-q_-}\left\|f_1-f_1^{[n]}\right\|_{L^\infty\left(I_k^1\right)}^{q_-}+\lambda^{-1}\left\|f_1-f_1^{[n]}\right\|_{L^\infty\left(I_k^1\right)}\right]$$
$$+\left[C\left(q(\,)\right)\times\lambda^{-q_+}\left\|f_2-f_2^{[n]}\right\|_{L^\infty\left(I_k^1\right)}^{q_+}+\lambda^{-1}\left\|f_2-f_2^{[n]}\right\|_{L^\infty\left(I_k^1\right)}\right]$$

$f^{[n]}$ increases to a function f pointwise, a.e., then $f^{[n]}\leq f$ for any n, since $0\leq\left(f-f^{[n]}\right)=\left(f_1-f_1^{[n]}\right)+\left(f_2-f_2^{[n]}\right)$, we have:

$$0\leq\left(f-f^{[n]}\right)\chi_{I_k^1}(y)=\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}(y)+\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}(y)$$
$$\leq\left|f\chi_{I_k^1}-f^{[n]}\chi_{I_k^1}\right|(y)$$

that is:

$$0\leq\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}(y)+\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}(y)\leq\left|\left(f-f^{[n]}\right)\chi_{I_k^1}\right|(y)\qquad(35)$$

$f^{[n]}$ converges pointwise to f always everywhere *i.e.*

$\forall y\in\Omega:\lim_{n\to\infty}\left(f(y)-f^{[n]}(y)\right)=0$, therefore

for any $y\in\Omega$, if n is sufficiently large, (35) gives:

$$0\leq\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}(y)+\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}(y)\leq\left|\left(f-f^{[n]}\right)\chi_{I_k^1}\right|(y)\to0$$

which implies for n sufficiently large:

$$\begin{cases}\left(f_1-f_1^{[n]}\right)\chi_{I_k^1}(y)\to0\\\left(f_2-f_2^{[n]}\right)\chi_{I_k^1}(y)\to0\end{cases},\text{ then }\begin{cases}\left\|f_1-f_1^{[n]}\right\|_{L^\infty\left(I_k^1\right)}\to\|0\|_{L^\infty\left(I_k^1\right)}=0\\\left\|f_2-f_2^{[n]}\right\|_{L^\infty\left(I_k^1\right)}\to\|0\|_{L^\infty\left(I_k^1\right)}=0\end{cases}$$

therefore:

$$\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)\to 0 \quad \text{as} \quad n\to 0$$

If we replace $\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$ by its value in (32), we get:

$$\left\|\left(f-f^{[n]}\right)\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\to 0 \quad \text{as} \quad n\to\infty .$$

Replacing $\left\|\left(f-f^{[n]}\right)\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}$ by its value in (31), we will find:

$$_1\left\|f-f_n\right\|_{q(),p(),\Omega}\to 0 \quad \text{as} \quad n\to\infty ,$$

therefore

$$0\le\left|{}_1\left\|f^{[n]}\right\|_{q(),p(),\Omega}-{}_1\left\|f\right\|_{q(),p(),\Omega}\right|\le{}_1\left\|f-f^{[n]}\right\|_{q(),p(),\Omega}\to 0 \quad \text{as} \quad n\to 0$$

□

Remark 11.

In the calculation of $\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)$ (in the above proof), we allow the possibility $\rho_{L^{q()}(\Omega)}\left(\frac{\left(f-f^{[n]}\right)\chi_{I_k^1}}{\lambda}\right)=\infty$, it is the case when $f\notin\left(L^{q()},l^{p()}\right)(\Omega)$ or $q_+=\infty$ ($C\left(q(\,)\right)=2^{q_+-1}=\infty$) then $_1\left\|f_n\right\|_{q(),p()}\to\infty={}_1\left\|f\right\|_{q(),p(),\Omega}$.

Remark 12.

If $f\notin\left(L^{q()},l^{p()}\right)(\Omega)$, we have defined $_1\left\|f\right\|_{q(),p(),\Omega}=\infty$, so in every case, we may write $_1\left\|f_n\right\|_{q(),p(),\Omega}\to{}_1\left\|f\right\|_{q(),p(),\Omega}$.

Lemma 13. (*Lemma of Fatou*)

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, given $q(\,)\in\mathcal{P}(\Omega)$, $p(\,)\in\mathcal{P}\left(\mathbb{Z}^d\right)$ and $f\in L_{loc}^{q()}(\Omega)$.

If $\left(f_n\right)_n\subset\left(\left(L^{q()},l^{p()}\right)(\Omega),{}_1\left\|\cdot\right\|_{q(),p(),\Omega}\right)$ is a sequence of non negative functions such that $f_n\to f$ pointwise a.e.

If $\liminf_{n\to\infty}{}_1\left\|f_n\right\|_{q(),p(),\Omega}<\infty$.

Then, $f\in\left(L^{q()},l^{p()}\right)(\Omega)$ and $_1\left\|f\right\|_{q(),p(),\Omega}\le\liminf_{n\to+\infty}{}_1\left\|f_n\right\|_{q(),p(),\Omega}$.

Proof.

Define a sequence $g_n(x)=\inf_{m\ge n}\left|f_m(x)\right|$, $x\in\Omega$.

Then, for all $m\ge n$, $g_n(x)\le\left|f_m(x)\right|$ and so $g_n\in\left(L^{q()},l^{p()}\right)(\Omega)$. By definition, $\left(g_n\right)_n$ is an increasing sequence and

$\lim_{n\to\infty}g_n(x)=\lim_{n\to\infty}\inf_{m\ge n}\left|f_m(x)\right|=\liminf_{m\to\infty}\left|f_m(x)\right|=\left|f(x)\right|$ a.e $x\in\Omega$.

Therefore, by monotone convergence lemma:

$_1\|f\|_{q(),p(),\Omega} = \lim_{n\to+\infty} {_1\|g_n\|_{q(),p(),\Omega}} \le \lim_{n\to+\infty}\left(\inf_{m\ge n} {_1\|f_m\|_{q(),p(),\Omega}}\right) = \liminf_{n\to\infty} {_1\|f_n\|_{q(),p(),\Omega}} < \infty$, therefore $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$.

□

Lemma 14. (*Lemma of Riesz-Fischer*)

Let Ω be a set such that $\mathbb{Z}^d \subset \Omega$, given $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ and $f \in L^{q()}_{loc}(\Omega)$.

If $(f_n)_n \subset \left(\left(L^{q()}, l^{p()}\right)(\Omega), {_1\|\cdot\|_{q(),p(),\Omega}}\right)$ is a sequence such that:

$$\sum_{n=1}^{\infty} {_1\|f_n\|_{q(),p(),\Omega}} < \infty.$$

Then, there exists $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$ such that: $F_i = \sum_{n=1}^{i} f_n \to f$ in norm as $i \to \infty$ and $_1\|f\|_{q(),p(),\Omega} \le \sum_{n=1}^{\infty} {_1\|f_n\|_{q(),p(),\Omega}}$.

Proof.

Define the function F on Ω by:

$F(x) = \sum_{n=1}^{\infty} |f_n(x)|$ and define the sequence $(F_i)_i$ by: $F_i(x) = \sum_{n=1}^{i} |f_n(x)|$.

The sequence $(F_i)_i$ is non-negative and increases pointwise almost everywhere to *F*. Further, for each *i*, $F_i \in \left(L^{q()}, l^{p()}\right)(\Omega)$ and its norm is uniformly bounded, since $_1\|F_i\|_{q(),p(),\Omega} \le \sum_{n=1}^{i} {_1\|f_n\|_{q(),p(),\Omega}} \le \sum_{n=1}^{\infty} {_1\|f_n\|_{q(),p(),\Omega}} < \infty$ by hypothesis

By the monotone convergence theorem $F \in \left(L^{q()}, l^{p()}\right)(\Omega)$. In particular from Proposition 8-1) *F* is finite a.e.

Hence, if we define the sequence $(G_i)_i$ by $G_i(x) = \sum_{n=1}^{i} f_n(x)$.

Then, this sequence also converges pointwise almost everywhere since absolute convergence implies convergence. Denote its sum by *f* ($G_i = \sum_{k=1}^{i} f_k \to f$ as $i \to \infty$).

Let $G_0 = 0$, then for any $j \ge 0$, $G_i - G_j \to f - G_j$ pointwise almost everywhere.

Furthermore,

$\liminf_{i\to\infty} {_1\|G_i - G_j\|_{q(),p(),\Omega}} \le \liminf_{i\to\infty} \sum_{n=j+1}^{i} {_1\|f_n\|_{q(),p(),\Omega}} = \sum_{n=j+1}^{\infty} {_1\|f_n\|_{q(),p(),\Omega}} < \infty$. By Fatou's lemma, if we take $j = 0$ then:

$$_1\|f\|_{q(),p(),\Omega} \le \liminf_{i\to+\infty} {_1\|G_i\|_{q(),p(),\Omega}} \le \sum_{n=1}^{\infty} {_1\|f_n\|_{q(),p(),\Omega}} < \infty$$

More generally, for each *j*, the same argument shows that:

$$_1\|f - G_j\|_{q(),p(),\Omega} \le \liminf_{i\to\infty} {_1\|G_i - G_j\|_{q(),p(),\Omega}} \le \sum_{n=j+1}^{\infty} {_1\|f_n\|_{q(),p(),\Omega}}.$$

Since the sum in the right-hand side tends to zero, we see that $G_j \to f$ in norm, which completes the norm.☐

Proposition 15.

Let Ω be a set such that $\mathbb{Z}^d \subset \Omega$, given $q(\,) \in \mathcal{P}(\Omega)$, $p(\,) \in \mathcal{P}(\mathbb{Z}^d)$.

$$\left(\left(L^{q()}, l^{p()}\right)(\Omega), {}_1\|\cdot\|_{q(),p()}\right) \text{ is a Banach space}$$

Proof.

It is sufficient to show that every Cauchy sequence in $\left(\left(L^{q()}, l^{p()}\right)(\Omega), {}_1\|\cdot\|_{q(),p(),\Omega}\right)$ converges in norm.

Let $(f_n) \subset \left(\left(L^{q()}, l^{p()}\right)(\Omega), {}_1\|\cdot\|_{q(),p(),\Omega}\right)$ be a Cauchy sequence.

Choose n_1 such that: ${}_1\|f_i - f_j\|_{q(),p(),\Omega} < 2^{-1}$ for $i, j \ge n_1$

Choose $n_2 > n_1$ such that: ${}_1\|f_i - f_j\|_{q(),p(),\Omega} < 2^{-2}$ for $i, j \ge n_2$

and so on...

This construction yields a subsequence $\left(f_{n_j}\right)_j, n_{j+1} \ge n_j$ such that:

$${}_1\left\|f_{n_{j+1}} - f_{n_j}\right\|_{q(),p(),\Omega} < 2^{-j}$$

Define a new sequence $(g_j)_j$ by:

$$\begin{cases} g_1 = f_{n_1} \\ g_j = f_{n_j} - f_{n_{j-1}} \quad \text{if } j > 1 \end{cases}$$

Then, for all *j*, we get the sum:

$$\sum_{i=1}^{j} g_i = f_{n_j}.$$

Further, we have that:

$$\sum_{j=1}^{\infty} {}_1\|g_j\|_{q(),p(),\Omega} \le {}_1\|f_{n_1}\|_{q(),p(),\Omega} + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Therefore, by the Riesz-Fischer lemma, there exists $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$ such that:

$$f_{n_j} \to f \quad \text{in norm.}$$

Finally, by the triangle inequality, we have that:

$${}_1\|f - f_n\|_{q(),p(),\Omega} \le {}_1\left\|f - f_{n_j}\right\|_{q(),p(),\Omega} + {}_1\left\|f_{n_j} - f_n\right\|_{q(),p(),\Omega}$$

Since $(f_n)_n$ is a Cauchy sequence, for *n* sufficiently large we can choose n_j to make the right-hand side as small as desired.

Hence, $f_n \to f$ in norm.

☐

We will need the following lemma.

Lemma 16. [27]

Given $q(\,), q_1(\,), q_2(\,) \in \mathcal{P}(\Omega)$.

1) $\|f\|_{q_1()} \le K\|f\|_{q_2()} \Leftrightarrow \begin{cases} q_1(x) \le q_2(x) \quad a.e.\ x \in \Omega \\ \int_{\{x\in\Omega:\, q_1(x)<q_2(x)\}} \lambda^{-\frac{q_2(x)\times q_1(x)}{q_2(x)-q_1(x)}} \mathrm{d}x < \infty \ \text{ for some } \lambda > 1 \end{cases}$

In particular, if Ω is bounded set:

$$\left.\begin{matrix} q_1(x) \le q_2(x) \quad a.e.\ x \in \Omega \\ \left|\Omega \setminus \Omega^{q_1()}\right| < \infty \end{matrix}\right\} \Rightarrow \|f\|_{q_1()} \le \left(1 + \left|\Omega \setminus \Omega^{q_1()}\right|\right)\|f\|_{q_2()}$$

2) $\|f\|_{q()} \le \|f\|_\infty \Leftrightarrow L^\infty(\Omega) \subset L^{q()} \Leftrightarrow 1 \in L^{q()} \Leftrightarrow \int_{\Omega\setminus\Omega_\infty^{q()}} \lambda^{-q(x)}\mathrm{d}x < \infty$ for some $\lambda > 0$.

In particular, the embedding holds if $|\Omega| < \infty$.

Proposition 17.

Let Ω be a set such that $\mathbb{Z}^d \subset \Omega$, given $q(),q_1(),q_2() \in \mathcal{P}(\Omega)$, $p(), p_1(), p_2() \in \mathcal{P}(\mathbb{Z}^d)$ and $f \in L_{loc}^{q()}(\Omega)$.

1)

If $\max\left\{\frac{1}{q_-},\frac{1}{p_-}\right\} \le s < \infty$, $\left|\Omega_\infty^{q()}\right| = 0$ and $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$, then ${}_1\left\||f|^s\right\|_{q(),p(),\Omega} = {}_1\|f\|_{sq(),sp(),\Omega}^s$

2)

•) Let $f \in \left(L^{q_2()}, l^{p()}\right)(\Omega)$.

If $q_1() \le q_2()$ on Ω, then $f \in \left(L^{q()}, l^{p_1()}\right)(\Omega)$ and $\|f\|_{q_1(),p(),\Omega} \le K_{q_1(),q_2()} \times \|f\|_{q_2(),p(),\Omega}$ or

$$\left(L^{q_2()}, l^{p()}\right)(\Omega) \subset \left(L^{q_1()}, l^{p()}\right)(\Omega).$$

••) In particular, when $\left|\Omega \setminus \Omega_\infty^{q_1()}\right| < \infty$, we have:

$$\|f\|_{q_1(),p(),\Omega} \le \left(1 + \left|\Omega \setminus \Omega_\infty^{q_1()}\right|\right)\|f\|_{q_2(),p(),\Omega} \le \left(1+|\Omega|\right)\|f\|_{q_2(),p(),\Omega}$$

3)

•) Let $f \in \left(L^{q()}, l^{p_2()}\right)(\Omega)$ and $p_1() \le p_2()$ on $\mathbb{Z}^d$.

Then, $f \in \left(L^{q()}, l^{p_1()}\right)(\Omega)$ and ${}_1\|f\|_{q(),p_2(),\Omega} \le C \times {}_1\|f\|_{q(),p_1(),\Omega}$ or

$$\left(L^{q()}, l^{p_1()}\right)(\Omega) \subset \left(L^{q()}, l^{p_2()}\right)(\Omega).$$

••) In particular when $|\Omega| < \infty$, we have:

$$\|f\|_{q(),p(),\Omega} \le \left(1+|\Omega|\right)\|f\|_{q_+,p_-,\Omega}$$

4)

If $\begin{cases} q() = q = \text{constant real} \\ p() = p = \text{constant real} \end{cases}$

both in $[1,\infty]$ then $\left(L^{q()}, l^{p()}\right)(\Omega) = \left(L^q, l^p\right)(\Omega)$ with constant exponents.

$\left(L^q, l^p\right)(\Omega)$ with constant exponents have been widely studied by many researchers (see [1] [15] [16] [17] [18]).

5)

If $|\Omega| < \infty$. Then, there exist positive constant reals c, C such that:

$$c \times {}_1\|f\|_{q_-, p_+, \Omega} \le {}_1\|f\|_{q(), p(), \Omega} \le C \times {}_1\|f\|_{q_+, p_-, \Omega}$$

otherwise,

$$\left(L^{q_+}, l^{p_-}\right)(\Omega) \subset \left(L^{q()}, l^{p()}\right)(\Omega) \subset \left(L^{q_-}, l^{p_+}\right)(\Omega).$$

6)

If

$$\left.\begin{array}{l} \dfrac{1}{q_1()} + \dfrac{1}{q_2()} = \dfrac{1}{q()} \le 1 \\ \dfrac{1}{p_1()} + \dfrac{1}{p_2()} = \dfrac{1}{p()} \le 1 \\ (f, g) \in \left(L^{q_1()}, l^{p_1()}\right)(\Omega) \times \left(L^{q_2()}, l^{p_2()}\right)(\Omega) \end{array}\right\}$$

Then,

$$\begin{cases} fg \in \left(L^{q()}, l^{p()}\right)(\Omega) \\ {}_1\|fg\|_{q(), p(), \Omega} \le C \times {}_1\|f\|_{q_1(), p_1(), \Omega} \times {}_1\|g\|_{q_2(), p_2(), \Omega} \end{cases}$$

7)

Let $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$.

Then,

$${}_1\|f\|_{q_+ + q_-, p(), \Omega} \le C(p())\left[{}_1\|f_1\|_{q_+, p(), \Omega} + {}_1\|f_2\|_{q_-, p(), \Omega}\right] \le 2 \times {}_1\|f\|_{q(), p(), \Omega}$$

with $f_1 \in \left(L^{q_+}, l^{p()}\right)(\Omega)$ and $f_2 \in \left(L^{q_-}, l^{p()}\right)(\Omega)$,
otherwise,

$$\left(L^{q()}, l^{p()}\right)(\Omega) \subset \left(L^{q_+}, l^{p()}\right)(\Omega) + \left(L^{q_-}, l^{p()}\right)(\Omega) \subset \left(L^{q_+ + q_-}, l^{p()}\right)(\Omega).$$

8)

For any $r > 0$, the norms ${}_1\|\cdot\|_{q(), p()}$ and ${}_r\|\cdot\|_{q(), p()}$ are equivalent.

Proof.

1) Under the hypotheses of 1), we know that $\left\||f|^s\right\|_{L^{q()}(\Omega)} = \|f\|^s_{L^{sq()}(\Omega)}$:

$$f \in \left(L^{q()}, l^{p()}\right)(\Omega) \Rightarrow \left\|f \chi_{I_k^1}\right\|_{L^{q()}(\Omega)} < \infty, \quad k \in \mathbb{Z}^d$$

Combining these two results, we will get:

$$\left\|\left|f \chi_{I_k^1}\right|^s\right\|_{L^{q()}(\Omega)} = \left\|f \chi_{I_k^1}\right\|^s_{L^{sq()}(\Omega)}$$

$$\Rightarrow \left\|\left\{\left\|\left|f \chi_{I_k^1}\right|^s\right\|_{L^{q()}(\Omega)}\right\}_{k \in \mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)} = \left\|\left\{\left\|f \chi_{I_k^1}\right\|^s_{L^{sq()}(\Omega)}\right\}_{k \in \mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$, using properties 4-2), we get:

$$\left\|\left\{\left\|\left|f \chi_{I_k^1}\right|^s\right\|_{L^{q()}(\Omega)}\right\}_{k \in \mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)} = \left\|\left\{\left\|f \chi_{I_k^1}\right\|_{L^{sq()}(\Omega)}\right\}\right\|^s_{l^{sp()}(\mathbb{Z}^d)}$$, therefore

$$\left\|\left\{\left\|\left|f\chi_{I_k^1}\right|^s\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}=\left(\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{sq()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{sp()}(\mathbb{Z}^d)}\right)^s$$

that is

$${}_1\left\|\left|f\right|^s\right\|_{q(),p(),\Omega}={}_1\left\|f\right\|^s_{sq(),sp(),\Omega}$$

2)

•) $q_1()\le q_2()$ on Ω $\Rightarrow$ $\frac{1}{q_1()}\ge\frac{1}{q_2()}$ on Ω, therefore, there exists $q_3()>0$ on Ω such that $\frac{1}{q_1()}=\frac{1}{q_2()}+\frac{1}{q_3()}$ on Ω, from Holder's inequality $\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}=\left\|f\chi_{I_k^1}\times\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\le K\times\left\|f\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}\times\left\|\chi_{I_k^1}\right\|_{L^{q_3()}(\Omega)}$, but $\left|I_k^1\right|=1<\infty$, therefore $\left\|\chi_{I_k^1}\right\|_{L^{q_3()}(\Omega)}\le\left|I_k^1\right|+1=2$ (Lemma 2.39 of [27]), then we get: $\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\le 2K\times\left\|f\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}$

$$\Rightarrow\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\le 2K\times\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)},$$

that is

$${}_1\left\|f\right\|_{q_1(),p(),\Omega}\le C\times{}_1\left\|f\right\|_{q_2(),p(),\Omega}\quad\text{or}\quad\left(L^{q_2()},l^{p()}\right)(\Omega)\subset\left(L^{q_1()},l^{p()}\right)(\Omega).$$

this result generalizes (3).

••) In the particular case, when $\left|\Omega\setminus\Omega_\infty^{q_1()}\right|<\infty$, we have:

$$\left\|f\chi_{I_k^1}\right\|_{q_1()}\le\left(1+\left|\Omega\setminus\Omega_\infty^{q_1()}\right|\right)\left\|f\chi_{I_k^1}\right\|_{q_2()}$$

then follows the inequality:

$$\left\|f\right\|_{q_1(),p(),\Omega}\le\left(1+\left|\Omega\setminus\Omega_\infty^{q_1()}\right|\right)\times\left\|f\right\|_{q_2(),p(),\Omega}\overset{\Omega\setminus\Omega_\infty^{q_1()}\subset\Omega}{\le}\left(1+\left|\Omega\right|\right)\left\|f\right\|_{q_2(),p(),\Omega}$$

3)

•) $f\in\left(L^{q()},l^{p_2()}\right)(\Omega)\Rightarrow\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}<\infty$. Under the hypotheses of 3), since $p_1()\le p_2()$ on $\mathbb{Z}^d$, we have from (17):

$$\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_2()}(\mathbb{Z}^d)}\le\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_1()}(\Omega)},$$

that is

$${}_1\left\|f\right\|_{q(),p_2(),\Omega}\le{}_1\left\|f\right\|_{q(),p_1(),\Omega}\quad\text{or}\quad\left(L^{q()},l^{p_1()}\right)(\Omega)\subset\left(L^{q()},l^{p_2()}\right)(\Omega).$$

This result generalizes (2)

••)

$$\begin{cases} q(\ ) \le q_+ \\ p(\ ) \ge p_- \\ \Omega_\infty \subset \Omega \Rightarrow |\Omega \setminus \Omega_\infty| \le |\Omega| < \infty \end{cases},$$

if we apply 2) ••) and3) •), we get:

$$\|f\|_{q(),p(),\Omega} \le (1+|\Omega\setminus\Omega_\infty|)\times\|f\|_{q_+,p_-,\Omega} \le (1+|\Omega|)\times\|f\|_{q_+,p_-,\Omega}$$

4)

We know that:

$${}_r\|f\|_{q(),p(),\Omega} = \left\| \left\{ \left\| f\chi_{I_k^r} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}.$$

If $p(\ ) = p = \text{constant} \in [1,\infty]$, then from (5) this last equality becomes:

$${}_r\|f\|_{q(),p(),\Omega} = \begin{cases} \left[\sum_{k\in\mathbb{Z}^d} \left\| f\chi_{I_k^r} \right\|_{q()}^p \right]^{\frac{1}{p}} & \text{if } p<\infty \\ \sup_{k\in\mathbb{Z}^d} \left\| f\chi_{I_k^r} \right\|_{q()} & \text{if } p=\infty \end{cases}.$$

Suppose that both $q(\ ) = q$ and $p(\ ) = p$ are constants belonging to $[1,\infty]$, the last equality gives:

$${}_r\|f\|_{q(),p(),\Omega} = \begin{cases} \left[\sum_{k\in\mathbb{Z}^d} \left\| f\chi_{I_k^r} \right\|_{q}^p \right]^{\frac{1}{p}} & \text{if } p<\infty \\ \sup_{k\in\mathbb{Z}^d} \left\| f\chi_{I_k^r} \right\|_{q} & \text{if } p=\infty \end{cases} = {}_r\|f\|_{q,p,\Omega}, \text{ see (1).}$$

We conclude that our space $(L^{q()}, l^{p()})(\Omega)$ generalizes both: $(L^{q()}, l)(\Omega)$ studied in [19] [20] [21]; and $(L^q, l^p)(\Omega)$ in [16] [17] [18].

5)

We have $\begin{cases} q_- \le q_{()} \le q_+ \\ p_- \le p_{()} \le p_+ \end{cases}$, we will use 2) and 3) to get:

$${}_1\|f\|_{q_-,p_+,\Omega} \le K \times {}_1\|f\|_{q(),p(),\Omega} \le K\times C\times {}_1\|f\|_{q_+,p_-,\Omega}$$

$$\text{or}\quad c\times {}_1\|f\|_{q_-,p_+,\Omega} \le {}_1\|f\|_{q(),p(),\Omega} \le C\times {}_1\|f\|_{q_+,p_-,\Omega}$$

otherwise,

$$(L^{q_+}, l^{p_-})(\Omega) \subset (L^{q()}, l^{p()})(\Omega) \subset (L^{q_-}, l^{p_+})(\Omega)$$

6)

$(f,g) \in (L^{q_1()}, l^{p_1()})(\Omega) \times (L^{q_2()}, l^{p_2()})(\Omega)$, from Holder's inequality:

$$\left\| (fg)\chi_{I_k^1} \right\|_{L^{q()}(\Omega)} \le C\times \left\| f\chi_{I_k^1} \right\|_{L^{q_1()}(\Omega)} \times \left\| g\chi_{I_k^1} \right\|_{L^{q_2()}(\Omega)},\ k\in\mathbb{Z}^d \tag{36}$$

Case 1: $p_1(\ ) = p_2(\ ) = \infty$

This implies that $p(\ ) = \infty$.

$\|\cdot\|_{l^{p()}(\mathbb{Z}^d)}$ $(1 \le p(\ ) \le \infty)$ is order preserving, therefore the last inequality (36) implies that:

$$\left\|\left\{\left\|(fg)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{\infty}(\mathbb{Z}^d)}$$

$$\leq C\times\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\times\left\{\left\|g\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{\infty}(\mathbb{Z}^d)}$$

now we apply (21) with $p()=q()=r()=\infty$ to get:

$$\left\|\left\{\left\|(fg)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{+\infty}(\mathbb{Z}^d)}$$

$$\leq C\times\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{\infty}(\mathbb{Z}^d)}\times\left\|\left\{\left\|g\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{\infty}(\mathbb{Z}^d)}$$

that is

$${}_1\|fg\|_{q(),\infty}\leq C\times{}_1\|f\|_{q_1(),\infty}\times{}_1\|g\|_{q_2(),\infty}$$

Case 2: $p_1()<\infty$, $p_2()=\infty$

This implies that $p()=p_1()<\infty$.

(36) $\Rightarrow$

$$\left\|\left\{\left\|(fg)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq C\times\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\times\left\{\left\|g\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

now we apply (21) with $p()=p()$, $q()=p_1()$, $r()=p_2()$ to get:

$$\left\|\left\{\left\|(fg)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq C\times\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_1}(\mathbb{Z}^d)}\times\left\|\left\{\left\|g\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_2()}(\mathbb{Z}^d)}$$

Therefore,

$${}_1\|fg\|_{q(),p()}\leq C\times{}_1\|f\|_{q_1(),p_1()}\times{}_1\|g\|_{q_2(),\infty}$$

Case 3: $p_1()=\infty$, $p_2()<\infty$

This implies that $p()=p_2()<\infty$.

An analogous reasoning gives:

$${}_1\|fg\|_{q(),p()}\leq C\times{}_1\|f\|_{q_1(),\infty}\times{}_1\|g\|_{q_2(),p_2()}$$

Case 4: $p_1()<\infty$, $p_2()<\infty$

This implies that $p()<\infty$.

(36) and the order-preservation of $\|\cdot\|_{l^{p()}(\mathbb{Z}^d)}$ $\Rightarrow$

$$\left\|\left\{\left\|(fg)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\leq C\times\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\times\left\|g\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

applying (21) to the right hand side on the last inequality, we get:

$$\left\|\left\{\left\|(fg)\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\leq C\times\left\|\left\{\left\|f\chi_{I_k^1}\right\|_{L^{q_1()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_1()}(\mathbb{Z}^d)}\times\left\|\left\{\left\|g\chi_{I_k^1}\right\|_{L^{q_2()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p_2()}(\mathbb{Z}^d)},$$

that is:

$$_1\|fg\|_{q(),p()}\leq C\times{}_1\|f\|_{q_1(),p_1()}\times{}_1\|g\|_{q_2(),p_2()}$$

7)

$$f\in\left(L^{q()},l^{p()}\right)(\Omega)\Rightarrow\left\|f\chi_{I_k^1}\right\|_{q()}<\infty,\ k\in\mathbb{Z}^d$$

By the homogeneity of $\|\cdot\|_{L^{q()}(\Omega)}$, we may assume without loss of generality that:

$$\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}=1.$$

The consequence is that:

$$\rho_{L^{q()}(\Omega)}\left(f\chi_{I_k^1}\right)=\int_{\Omega\backslash\Omega_\infty^{q()}}\left|f\chi_{I_k^1}(x)\right|^{q(x)}\mathrm{d}x+\left\|f\chi_{I_k^1}\right\|_{L^\infty\left(\Omega_\infty^{q()}\right)}\leq\left\|f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)}=1$$

we decompose $f\chi_{I_k^1}$ as f_1+f_2 such that:

$$f_1=f\chi_{\left\{y\in\Omega:\left|f\chi_{I_k^1}(y)\right|<1\right\}}\quad\text{and}\quad f_2=f\chi_{\left\{y\in\Omega\backslash\Omega_\infty^{q()}:\left|f\chi_{I_k^1}(y)\right|\geq1\right\}}$$

- Case $q_+<\infty$

$q_+<\infty\Rightarrow q()<\infty$ on Ω $\Rightarrow\Omega_\infty^{q()}=\varnothing$, therefore, in this case, the decomposition of $f\chi_{I_k^1}$ becomes:

$$f_1=f\chi_{\left\{y\in\Omega:\left|f\chi_{I_k^1}(y)\right|<1\right\}}\quad\text{and}\quad f_2=f\chi_{\left\{y\in\Omega:\left|f\chi_{I_k^1}(y)\right|\geq1\right\}}\quad\text{and}$$

$$\rho_{q()}\left(f\chi_{I_k^1}\right)=\int_\Omega\left|f\chi_{I_k^1}(y)\right|^{q(y)}\mathrm{d}y$$

In this case:

$$\int_\Omega\left|f_1\chi_{I_k^1}(y)\right|^{q_+}\mathrm{d}y\leq\int_\Omega\left|f_1\chi_{I_k^1}(y)\right|^{q(y)}\mathrm{d}y\leq\int_\Omega\left|f\chi_{I_k^1}(y)\right|^{q(y)}\mathrm{d}y$$
$$=\rho_{q()}\left(f\chi_{I_k^1}\right)\leq\left\|f\chi_{I_k^1}\right\|_{q()}=1,$$

then

$$\int_\Omega\left|f_1\chi_{I_k^1}(y)\right|^{q_+}\mathrm{d}y\leq1\ \text{ otherwise }\ \left\|f_1\chi_{I_k^1}\right\|_{L^{q^+}}\leq1.$$

$$\int_\Omega\left|f_2\chi_{I_k^1}(y)\right|^{q_-}\mathrm{d}y\leq\int_\Omega\left|f_2\chi_{I_k^1}(y)\right|^{q(y)}\mathrm{d}y\leq\int_\Omega\left|f\chi_{I_k^1}(y)\right|^{q(y)}\mathrm{d}y$$
$$=\rho_{q()}\left(f\chi_{I_k^1}\right)\leq\left\|f\chi_{I_k^1}\right\|_{q()}=1,$$

then $\int_\Omega \left| f_2\chi_{I_k^1}(y)\right|^{q_-} dy \le 1$ otherwise $\left\| f_2\chi_{I_k^1}\right\|_{L^{q^-}(\Omega)} \le 1$, then

$$\begin{cases} \left\| f_1\chi_{I_k^1}\right\|_{L^{q^+}(\Omega)} \le 1 \\ \left\| f_2\chi_{I_k^1}\right\|_{L^{q^-}(\Omega)} \le 1 \end{cases} \Rightarrow \begin{cases} \left\| f_1\chi_{I_k^1}\right\|_{L^{q^+}(\Omega)} \le \left\| f\chi_{I_k^1}\right\|_{q(\cdot)} \\ \left\| f_2\chi_{I_k^1}\right\|_{L^{q^-}(\Omega)} \le \left\| f\chi_{I_k^1}\right\|_{q()} \end{cases}$$

$$\Rightarrow \begin{cases} \left\| \left\{ \left\| f_1\chi_{I_k^1}\right\|_{L^{q^+}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \le \left\| \left\{ \left\| f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \\ \left\| \left\{ \left\| f_2\chi_{I_k^1}\right\|_{L^{q^-}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \le \left\| \left\{ \left\| f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)} \right\}\right\|_{l^{p()}(\mathbb{Z}^d)} \end{cases}$$

which gives:

$$\left\| \left\{ \left\| f_1\chi_{I_k^1}\right\|_{L^{q^+}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} + \left\| \left\{ \left\| f_2\chi_{I_k^1}\right\|_{L^{q^-}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\le 2\times \left\| \left\{ \left\| f\chi_{I_k^1}\right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\Rightarrow {}_1\| f\|_{q_+ + q_-, p(),\Omega} \overset{\text{by def.}}{\le} \left[{}_1\| f_1\|_{q_+, p(),\Omega} + {}_1\| f_2\|_{q_-, p(),\Omega} \right] \le 2\times {}_1\| f\|_{q(), p(),\Omega}$$

otherwise,

$$\left(L^{q()}, l^{p()}\right)(\Omega) \subset \left(L^{q_+}, l^{p()}\right)(\Omega) + \left(L^{q_-}, l^{p()}\right)(\Omega) \subset \left(L^{q_+ + q_-}, l^{p()}\right)(\Omega)$$

- Case $q_+ = \infty$

  As the same way (case $q_+ < \infty$) we prove the claim.

  8)

  a) Let's consider two positive real numbers r_1, r_2 such that $r_1 \ne r_2$, without loss of generality we can suppose that $r_1 < r_2$ and we consider the following two subsets of $\mathbb{Z}^d$:

$$L = \left\{ k \in \mathbb{Z}^d : I_k^{r_2} \cap I_l^{r_1} \ne \varnothing \right\} \text{ and}$$

$T = \left\{ l \in \mathbb{Z}^d : I_l^{r_1} \cap I_k^{r_2} \ne \varnothing \right\}$, and $\text{Card}(L), \text{Card}(T)$ will be the number of the elements of the sets L and T, suppose temporary that they are finite and greater than 1, we will prove it later.

Given $f \in L_{loc}^{q()}(\Omega)$.

Remind that ${}_r\| f\|_{q(), p(),\Omega} = \left\| \left\{ \left\| f\chi_{I_k^r}\right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}$ and the notations p'_+ or p'_- should be understand in the sense that: $p'_+ = \left(p'(\,)\right)_+$, $p'_- = \left(p'(\,)\right)_-$.

- $\forall k \in \mathbb{Z}^d$:

$$\left\| f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)} \le \sum_{l\in T} \left\| f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)} \overset{(20)}{\le} K_{p()} \left\| \left\{ \left\| f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)} \right\}_{l\in T} \right\|_{l^{p()}(T)} \times \left\| \{1\}_{k\in T} \right\|_{l^{p'()}(T)}.$$

that is:

$$\left\| f\chi_{I_k^{r_2}} \right\|_{L^{q(\cdot)}(\Omega)} \le K_{p(\cdot)} \left\| \left\{ \left\| f\chi_{I_l^{r_1}} \right\|_{L^{q(\cdot)}(\Omega)} \right\}_{l\in T} \right\|_{l^{p(\cdot)}(T)} \times \left\| \{1\}_{k\in T} \right\|_{l^{p'(\cdot)}(T)}. \tag{37}$$

First case: $p'(k) < \infty$ on $\mathbb{Z}^d$.

$$\left\| \{1\}_{k\in T} \right\|_{l^{p'(\cdot)}(T)} = \inf\left\{ \lambda > 0 : \sum_{k\in T} \left(\frac{1}{\lambda}\right)^{p'(k)} \le 1 \right\} = \inf\left\{ \lambda > 0 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\}.$$

$$\forall k \in T : p'_- \le p'(k) \le p'_+ \quad \text{therefore} \quad -p'_+ \le -p'(k) \le -p'_-.$$

The function $x \mapsto \lambda^x$ is $\begin{cases} \text{decreasing} & \text{if } 0<\lambda<1 \\ \text{increasing} & \text{if } \lambda \ge 1 \end{cases}$, therefore

$$0<\lambda<1 \Rightarrow \lambda^{-p'_-} \le \lambda^{-p'(k)} \le \lambda^{-p'_+};\quad \lambda \ge 1 \Rightarrow \lambda^{-p'_+} \le \lambda^{-p'(k)} \le \lambda^{-p'_-}.$$

$$\left\{ 0<\lambda<1 : \sum_{k\in T} \lambda^{-p'_+} \le 1 \right\} \subset \left\{ 0<\lambda<1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \tag{38}$$

$$\left\{ \lambda \ge 1 : \sum_{k\in T} \lambda^{-p'_-} \le 1 \right\} \subset \left\{ \lambda \ge 1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \tag{39}$$

$$\left\{ \lambda > 0 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} = \left\{ 0<\lambda<1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \cup \left\{ \lambda \ge 1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\},$$

Since $\inf(A \cup B) \le \min\{\inf A, \inf B\}$, we get:

$$\begin{aligned} &\inf\left\{ \lambda > 0 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \\ &= \inf\left\{ \left\{ 0<\lambda<1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \cup \left\{ \lambda \ge 1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \right\} \\ &\le \min\left\{ \inf\left(\left\{ 0<\lambda<1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \right), \inf\left(\left\{ \lambda \ge 1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \right) \right\}, \end{aligned}$$

(38) $\Rightarrow$

$$\begin{aligned} &\inf\left\{ 0<\lambda<1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \le \inf\left\{ 0<\lambda<1 : \sum_{k\in T} \lambda^{-p'_+} \le 1 \right\} \\ &= \inf\left\{ \lambda > 0 : (\operatorname{Card} T)^{\frac{1}{p'_+}} \le \lambda < 1 \right\} = (\operatorname{Card} T)^{\frac{1}{p'_+}}. \end{aligned}$$

that is:

$$\inf\left\{ 0<\lambda<1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \le (\operatorname{Card} T)^{\frac{1}{p'_+}} \tag{40}$$

$$(39) \Rightarrow \begin{aligned} &\inf\left\{ \lambda \ge 1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \le \inf\left\{ \lambda \ge 1 : \sum_{k\in T} \lambda^{-p'_-} \le 1 \right\} \\ &= \inf\left\{ \lambda \ge 1 : \lambda \ge (\operatorname{Card} T)^{\frac{1}{p'_-}} \right\} \end{aligned}$$

$T \ne \varnothing \Rightarrow \operatorname{Card} T \ge 1 \Rightarrow (\operatorname{Card} T)^{\frac{1}{p'_-}} \ge 1$, therefore

$$\inf\left\{ \lambda \ge 1 : \sum_{k\in T} \lambda^{-p'(k)} \le 1 \right\} \le (\operatorname{Card} T)^{\frac{1}{p'_-}} \tag{41}$$

(40) and (41) $\Rightarrow \inf\left\{\lambda>0:\sum_{k\in T}\lambda^{-p'(k)}\le 1\right\}\le \min\left\{(\operatorname{Card}T)^{\frac{1}{p'_+}},(\operatorname{Card}T)^{\frac{1}{p'_-}}\right\}$.

In other hand, we have that for any $k\in T$: $1\le p'_-\le p'(k)\le p'_+$ therefore $\frac{1}{p'_+}\le\frac{1}{p'_-}$, $\operatorname{Card}T\ge 1$ then the function $x\mapsto(\operatorname{Card}T)^x$ is increasing therefore $(\operatorname{Card}T)^{\frac{1}{p'_+}}\le(\operatorname{Card}T)^{\frac{1}{p'_-}}$, finally:

$$\left\|\{1\}_{k\in T}\right\|_{l^{p'()}(T)}\le(\operatorname{Card}T)^{\frac{1}{p'_+}} \tag{42}$$

Thus, $\forall k\in\mathbb{Z}^d$: (37) and (42) $\Rightarrow$

$\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\le K_{p()}\times(\operatorname{Card}T)^{\frac{1}{p'_+}}\times\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in T}\right\|_{l^{p()}(T)}$, this implies that:

$$\begin{aligned}&\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\\&\le K_{p()}(\operatorname{Card}T)^{\frac{1}{p'_+}}\left\|\left\{\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in T}\right\|_{l^{p()}(T)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\\&=K_{p()}(\operatorname{Card}T)^{\frac{1}{p'_+}}\left\|\left\{\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\right\}_{l\in T}\right\|_{l^{p()}(T)}\\&=K_{p()}(\operatorname{Card}T)^{\frac{1}{p'_+}}\left\|\{1\}_{k\in T}\right\|_{l^{p()}(T)}\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\end{aligned}$$

We proceed analogously as we did to prove (42), to demonstrate that:

$$\left\|\{1\}_{k\in T}\right\|_{l^{p()}(T)}\le(\operatorname{Card}T)^{\frac{1}{p_+}}$$

therefore

$$\begin{aligned}&\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\\&\le K_{p()}\times(\operatorname{Card}T)^{\frac{1}{p'_+}}\times(\operatorname{Card}T)^{\frac{1}{p_+}}\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)},\end{aligned}$$

that is

$${}_{r_2}\|f\|_{q(),p(),\Omega}\le K_{p()}\times(\operatorname{Card}T)^{\frac{1}{p'_+}}\times(\operatorname{Card}T)^{\frac{1}{p_+}}\ {}_{r_1}\|f\|_{q(),p(),\Omega} \tag{43}$$

where $(\operatorname{Card}T)^{\frac{1}{p'_+}}\times(\operatorname{Card}T)^{\frac{1}{p_+}}=(\operatorname{Card}T)^{\frac{1}{p'_+}+\frac{1}{p_+}}$

- $\forall l\in\mathbb{Z}^d$:

$$\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\le\sum_{k\in L}\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\overset{(20)}{\le}K_{p()}\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in L}\right\|_{l^{p()}(L)}\left\|\{1\}_{k\in L}\right\|_{l^{p'()}(L)}$$

that is

$$\left\| f\chi_{I_l^{r_1}} \right\|_{L^{q()}(\Omega)} \le K_{p()} \left\| \left\{ \left\| f\chi_{I_k^{r_2}} \right\|_{L^{q()}(\Omega)} \right\}_{k\in L} \right\|_{l^{p()}(L)} \left\| \{1\}_{k\in L} \right\|_{l^{p'()}(L)} \quad (44)$$

from (42), we get:

$$\left\| f\chi_{I_l^{r_1}} \right\|_{L^{q()}(\Omega)} \le K_{p()} (\operatorname{Card} L)^{\frac{1}{p'_+}} \left\| \left\{ \left\| f\chi_{I_k^{r_2}} \right\|_{L^{q()}(\Omega)} \right\}_{k\in L} \right\|_{l^{p()}(L)}$$

this implies that:

$$\begin{aligned} &\left\| \left\{ \left\| f\chi_{I_l^{r_1}} \right\|_{L^{q()}(\Omega)} \right\}_{l\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \\ &\le K_{p()} (\operatorname{Card} L)^{\frac{1}{p'_+}} \left\| \left\{ \left\| \left\{ \left\| f\chi_{I_k^{r_2}} \right\|_{L^{q()}(\Omega)} \right\}_{k\in L} \right\|_{l^{p()}(L)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \\ &= K_{p()} (\operatorname{Card} L)^{\frac{1}{p'_+}} \left\| \left\{ \left\| \left\{ \left\| f\chi_{I_k^{r_2}} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \right\}_{l\in L} \right\|_{l^{p()}(L)} \\ &= K_{p()} (\operatorname{Card} L)^{\frac{1}{p'_+}} \left\| \{1\}_{l\in L} \right\|_{l^{p()}(L)} \left\| \left\{ \left\| f\chi_{I_k^{r_2}} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \end{aligned}$$

as we see above:

$$\left\| \{1\} \right\|_{l^{p()}(L)} \le (\operatorname{Card} L)^{\frac{1}{p_+}}$$

then

$$\begin{aligned} &\left\| \left\{ \left\| f\chi_{I_l^{r_1}} \right\|_{L^{q()}(\Omega)} \right\}_{l\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)} \\ &\le K_{p()} (\operatorname{Card} L)^{\frac{1}{p'_+}} (\operatorname{Card} L)^{\frac{1}{p_+}} \left\| \left\{ \left\| f\chi_{I_k^{r_2}} \right\|_{L^{q()}(\Omega)} \right\}_{k\in\mathbb{Z}^d} \right\|_{l^{p()}(\mathbb{Z}^d)}, \end{aligned}$$

that is:

$$_{r_1}\|f\|_{q(),p(),\Omega} \le K_{p()} (\operatorname{Card} L)^{\frac{1}{p'_+}} (\operatorname{Card} L)^{\frac{1}{p_+}} \, _{r_2}\|f\|_{q(),p(),\Omega} \quad (45)$$

(43) and (45) imply that:

$$_{r_1}\|f\|_{q(),p(),\Omega} \le c \times \, _{r_2}\|f\|_{q(),p(),\Omega} \le C \times \, _{r_1}\|f\|_{q(),p(),\Omega} \quad (46)$$

where $c = K_{p()} (\operatorname{Card} L)^{\frac{1}{p'_+}} (\operatorname{Card} L)^{\frac{1}{p_+}}$ and

$$C = K_{p()}^2 (\operatorname{Card} L)^{\frac{1}{p'_+}} (\operatorname{Card} L)^{\frac{1}{p_+}} K_{p()} \times (\operatorname{Card} T)^{\frac{1}{p'_+}} \times (\operatorname{Card} T)^{\frac{1}{p_+}}.$$

Second case: $p'(k) = \infty$ on $\mathbb{Z}^d$.

- From (37), we have:

$$\forall k\in\mathbb{Z}^d:\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\le K_{p()}\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in T}\right\|_{l^{p()}(T)}\times\left\|\{1\}_{l\in T}\right\|_{l^{p'()}(T)},$$

$$\left\|\{1\}_{l\in T}\right\|_{l^{p'()}(T)}=\left\|\{1\}_{l\in T}\right\|_{l^{\infty}(T)}=\inf\left\{\lambda<0:\sup_{l\in T}\left(\frac{1}{\lambda}\right)\le 1\right\}$$
$$=\inf\left\{\lambda>0:\left(\frac{1}{\lambda}\right)\le 1\right\}=1$$

therefore,

$$\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\le K_{p()}\times\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in T}\right\|_{l^{p()}(T)},$$

this implies that:

$$\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$
$$\le K_{p()}\times\left\|\left\{\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in T}\right\|_{l^{p()}(T)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$
$$=K_{p()}\times\left\|\left\{\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\right\}_{l\in T}\right\|_{l^{p()}(T)}$$
$$=K_{p()}\times\left\|\{1\}_{l\in T}\right\|_{l^{p()}(T)}\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

that is:

$$\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\le K_{p()}\times\left\|\{1\}_{l\in T}\right\|_{l^{p()}(T)}\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)},$$

$$p'(\,)=\infty\Rightarrow p(\,)=1\quad\text{then}$$

$$\left\|\{1\}_{l\in T}\right\|_{l^{p()}(T)}=\left\|\{1\}_{l\in T}\right\|_{l^{1}(T)}=\inf\left\{\lambda>0:\sum_{l\in T}\left(\frac{1}{\lambda}\right)\le 1\right\}$$
$$=\inf\left\{\lambda>0:\operatorname{Card}(T)\left(\frac{1}{\lambda}\right)\le 1\right\}=\operatorname{Card}T$$

Therefore,

$$\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\le K_{p()}\times\operatorname{Card}(T)\times\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}, \text{ that is}$$

$$_{r_2}\|f\|_{q(),p(),\Omega}\le K_{p()}\times\operatorname{Card}(T)\times{}_{r_1}\|f\|_{q(),p(),\Omega}\tag{47}$$

- From (44), we have:

$$\forall l\in\mathbb{Z}^d:\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\le K_{p()}\times\left\|\{1\}_{l\in L}\right\|_{l^{p'()}(L)}\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in L}\right\|_{l^{p()}(L)}$$

$$\left\|\{1\}_{l\in L}\right\|_{l^{p'()}(L)} = \left\|\{1\}_{l\in L}\right\|_{l^{\infty}(L)} = \inf\left\{\lambda>0:\sup_{k\in L}\left(\frac{1}{\lambda}\right)\le 1\right\}$$

$$= \inf\left\{\lambda>0:\left(\frac{1}{\lambda}\right)\le 1\right\} = 1$$

therefore,

$$\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)} \le K_{p()}\times\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in L}\right\|_{l^{p()}(L)},$$

this implies that:

$$\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$\le K_{p()}\times\left\|\left\{\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(L)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$

$$= K_{p()}\times\left\|\left\{\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}\right\}_{k\in L}\right\|_{l^{p()}(L)}$$

$$= K_{p()}\times\left\|\{1\}_{k\in L}\right\|_{l^{p()}(L)}\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}.$$

Therefore,

$$\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)} \le K_{p()}\times\left\|\{1\}_{k\in L}\right\|_{l^{p()}(L)}\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$$ but

$p'(\,)=\infty \Rightarrow p(\,)=1$, then

$$\left\|\{1\}_{k\in L}\right\|_{l^{p()}(L)} = \left\|\{1\}_{k\in L}\right\|_{l^{1}(L)} = \inf\left\{\lambda>0:\sum_{k\in L}\left(\frac{1}{\lambda}\right)\le 1\right\}$$

$$= \inf\left\{\lambda>0:\operatorname{Card}T\left(\frac{1}{\lambda}\right)\le 1\right\} = \operatorname{Card}L$$

thus,

$$\left\|\left\{\left\|f\chi_{I_l^{r_1}}\right\|_{L^{q()}(\Omega)}\right\}_{l\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)} \le K_{p()}\times\operatorname{Card}L\times\left\|\left\{\left\|f\chi_{I_k^{r_2}}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)},$$

that means:

$$_{r_1}\|f\|_{q(),p(),\Omega} \le K_{p()}\times\operatorname{Card}(L)\times{}_{r_2}\|f\|_{q(),p(),\Omega} \tag{48}$$

(47) and (48) imply that:

$$_{r_1}\|f\|_{q(),p(),\Omega} \le c\times{}_{r_2}\|f\|_{q(),p(),\Omega} \le C\times{}_{r_1}\|f\|_{q(),p(),\Omega} \tag{49}$$

where $c = K_{p()}\times\operatorname{Card}(L)$ and $C = K_{p()}^2\operatorname{Card}(L)\times\operatorname{Card}(T)$.

Thus, in all cases, there exist two positive real numbers:

$c = c(r_1, r_2, \text{Card}(L), \text{Card}(T)) > 0$ and

$C = C(r_1, r_2, \text{Card}(L), \text{Card}(T)) > 0$ such that for any $f \in L_{loc}^{q()}(\Omega)$:

$c \times {}_{r_1}\|f\|_{q(),p(),\Omega} \le {}_{r_2}\|f\|_{q(),p(),\Omega} \le C \times {}_{r_1}\|f\|_{q(),p(),\Omega}$.

b)

Let's consider a positive real number *r*.

The result in a) proves that there exists two positive numbers *a* and *A* such that for any $f \in L_{loc}^{q()}(\Omega)$:

$a \times {}_1\|f\|_{q(),p(),\Omega} \le {}_r\|f\|_{q(),p(),\Omega} \le A \times {}_1\|f\|_{q(),p(),\Omega}$, therefore the claim is proved.

□

The proof of 8) has a sense only if we prove that $\text{Card}\, L, \text{Card}\, T$ are finite quantities, we do so in 1) in the following lemma, the 2) and 3) of the next lemma also play a great role in the different proofs in this work:

Lemma 18.

1)

Let $r_1, r_2 > 0$ such that $r_1 \ne r_2$, the cardinal of the sets

$L = L(r_1, r_2) = \{k \in \mathbb{Z}^d : I_l^{r_1} \cap I_k^{r_2} \ne \varnothing\}$ and

$T = T(r_1, r_2) = \{l \in \mathbb{Z}^d : I_l^{r_1} \cap I_k^{r_2} \ne \varnothing\}$ are both finite.

2)

For a fixed positive real number *r*, if $k_1, k_2 \in \mathbb{Z}^d$ and $k_1 \ne k_2$ then $I_{k_1}^r \cap I_{k_2}^r = \varnothing$.

3) If *r* is a fixed positive real number, then $\bigcup_{k \in \mathbb{Z}^d} I_k^r = \Omega$.

Proof.

1) Only the proof for L is given, the proof of T is similar Let $m = (m_j)_{1 \le j \le d}$ and $n = (n_j)_{1 \le j \le d}$ be in $\mathbb{Z}^d$, without loss of generality, we can suppose that $r_1 < r_2$.

$$\begin{cases} m \in I_l^{r_1} \\ n \in I_k^{r_2} \end{cases} \Rightarrow \begin{cases} l_j \times r_1 \le m_j < (l_j + 1) \times r_1 \\ k_j \times r_2 \le n_j < (k_j + 1) \times r_2 \end{cases} \quad \forall j \in \{1, 2, \cdots, d\}$$

$$\Rightarrow \begin{cases} l_j \times r_1 \le m_j < (l_j + 1) \times r_1 \\ -(k_j + 1) \times r_2 < -n_j \le -k_j \times r_2 \end{cases} \quad \forall j \in \{1, 2, \cdots, d\} \Rightarrow$$

$$l_j \times r_1 - (k_j + 1) \times r_2 < m_j - n_j < (l_j + 1) \times r_1 - k_j \times r_2 \tag{50}$$

If $m = n$ *i.e.* $\forall j \in \{1, 2, \cdots, d\}$ $m_j = n_j$ then $m = n \in I_k^{r_2} \cap I_l^{r_1}$, (50) becomes: $l_j \times r_1 - (k_j + 1) \times r_2 < 0 < (l_j + 1) \times r_1 - k_j \times r_2$. In the case of $L = L(r_1, r_2)$ the variable to find is k_j, $j = 1, 2, \cdots, d$, the others r_1, r_2, l_j are supposed known. The precedent double inequality becomes:

$$l_j \times r_1 - r_2 < k_j \times r_2 < l_j \times r_1 + r_1.$$

this implies that:

$$l_j \times \frac{r_1}{r_2} - 1 < k_j < l_j + 1$$

from this double inequality, we can say that $\text{Card}\,L \geq 1$ since $\text{Card}\,L$ is the number of $k = (k_j)_{1\leq j\leq d}$ such that the last double inequality holds, indeed let c_j be the integer such that $l_j \times \frac{r_1}{r_2} - 1 < c_j < l_j + 1$, $C = (c_j)_{1\leq j\leq d} \in \mathbb{Z}^d \cap L$.

If $E(x)$ is the great integer less than or equal to $x \in \mathbb{R}$, the double inequality $l_j \times \frac{r_1}{r_2} - 1 < k_j < l_j + 1$ gives:

$$l_j \times E\left(\frac{r_1}{r_2}\right) - 1 < k_j < l_j + 1$$

$$\Rightarrow k_j \in \left]a = l_j \times E\left(\frac{r_1}{r_2}\right) - 1, b = l_j + 1\right[\cap \mathbb{Z} = \left]a = -1, b = l_j + 1\right[$$

whose elements number is $b - a - 1 = l_j + 1$ due to the fact that $E\left(\frac{r_1}{r_2}\right) = 0$ since $\frac{r_1}{r_2} \in]0,1[$, the number of such k_j doesn't exceed $|l_j| + 1 \leq \max_{1\leq j\leq d} |l_j| + 1$.

Therefore, $1 \leq \text{Card}\,L \leq \left(\max_{1\leq j\leq d} |l_j| + 1\right)^d < \infty$.

2) Suppose that $I_{k_1}^r \cap I_{k_2}^r \neq \varnothing$ and $k_1 \neq k_2$,

let $m = (m_j)_{1\leq j\leq d}$ and $n = (n_j)_{1\leq j\leq d}$ be in $\mathbb{Z}^d$.

$$\begin{cases} m \in I_{k_1}^r \\ n \in I_{k_2}^r \end{cases} \Rightarrow \begin{cases} k_{1j} \times r \leq m_j < (k_{1j} + 1) \times r \\ k_{2j} \times r \leq n_j < (k_{2j} + 1) \times r \end{cases} \Rightarrow \begin{cases} k_{1j} \times r \leq m_j < (k_{1j} + 1) \times r \\ -(k_{2j} + 1) \times r < -n_j \leq -k_{2j} \times r \end{cases}$$

$$\Rightarrow k_{1j} \times r - (k_{2j} + 1) \times r < m_j - n_j < (k_{1j} + 1) \times r - k_{2j} \times r.$$

If $m = n$ that is $\forall j \in \{1, 2, \cdots, d\}$: $m_j = n_j$, then $m = n \in I_{k_1}^r \cap I_{k_2}^r$, therefore

$k_{1j} - (k_{2j} + 1) < 0 < (k_{1j} + 1) - k_{2j} \Rightarrow k_{1j} - 1 < k_{2j} < k_{1j} + 1$

$\Rightarrow k_{2j} \in]k_{1j} - 1, k_{1j} + 1[\Rightarrow k_{2j} = k_{1j}, \forall j \in \{1, 2, \cdots, d\} \Rightarrow k_1 = k_2$

contradiction with $k_1 \neq k_2$, therefore $I_{k_1}^r \cap I_{k_2}^r = \varnothing$.

3)

- $\forall k \in \mathbb{Z}^d$: $I_k^r \subset \Omega$ therefore $\bigcup_{k\in\mathbb{Z}^d} I_k^r \subset \Omega$, it remains to prove that:
- $\Omega \subset \bigcup_{k\in\mathbb{Z}^d} I_k^r$, let $E(x)$ be the great integer less than or equal to $x \in \mathbb{R}$.

We choose $z \in \Omega$, we should prove that $z \in \bigcup_{k\in\mathbb{Z}^d} I_k^r$, that is there exists one $k_0 = (k_{0_j})_{1\leq j\leq d} \in \mathbb{Z}^d$ such that $z \in I_{k_0}^r$.

$$z \in I_{k_0}^r = \prod_{j=1}^d \left[k_{0_j} \times r, (k_{0_j} + 1) \times r\right[\Rightarrow k_{0_j} \times r \leq z_j < (k_{0_j} + 1) \times r$$

$\Rightarrow k_{0_j} \leq \frac{z_j}{r} < k_{0_j} + 1 \Rightarrow k_{0_j} = E\left(\frac{z_j}{r}\right) \Rightarrow k_0 = \left(E\left(\frac{z_j}{r}\right)\right)_{1\leq j\leq d}$. This is the wanted k_0, therefore the claim holds. Therefore, the claim holds.

□

4. The Dual of $\left(L^{q()},l^{p()}\right)(\Omega)$

In the classical wiener amalgam, $\left(L^{q'},l^{p'}\right)(\Omega)$ is isometrically isomorphic to the dual $\left[\left(L^{q},l^{p}\right)(\Omega)\right]^{*}$ of $\left(L^{q()},l^{p()}\right)(\Omega)$ when $1\le q,p<\infty$ (see the introduction paragraph). The behavior of the variable amalgam spaces is analogous if $q_+,p_+<\infty$.

Definition 19.

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, given $q(\,)\in\mathcal{P}(\Omega)$, $p(\,)\in\mathcal{P}\left(\mathbb{Z}^d\right)$ and $r>0$ we define:

$$
{}_r\|f\|_{q(),p(),\Omega}=\sup_{\substack{g\in\left(L^{q'()},l^{p'()}\right)(\Omega)\\ {}_r\|g\|_{q'(),p'(),\Omega}\le1}}\int_\Omega f(x)g(x)\mathrm{d}x. \tag{51}
$$

We denote $M^{q(),p()}(\Omega)$ by:

$$
M^{q(),p()}(\Omega)=\left\{f\in L^0(\Omega):{}_r\|f\|_{q(),p(),\Omega}<\infty\right\}. \tag{52}
$$

Proposition 20.

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, given $q(\,)\in\mathcal{P}(\Omega)$, $p(\,)\in\mathcal{P}\left(\mathbb{Z}^d\right)$ and $r>0$, the set $M^{q(),p()}(\Omega)$ is a normed vector space with respect to the norm ${}_r\|\cdot\|'_{q(),p(),\Omega}$. Furthermore, the norm is order preserving:

given $f,g\in M^{q(),p()}(\Omega)$ such that $|f|\le|g|$ then

${}_r\|f\|'_{q(),p(),\Omega}\le{}_r\|g\|'_{q(),p(),\Omega}$.

Proof.

It is immediate that $M^{q(),p()}(\Omega)$ is a vector space. The fact that ${}_r\|\cdot\|'_{q(),p(),\Omega}$ is an order preserving norm is a consequence of the properties of integrals and supremums and the following equivalent characterization of ${}_r\|\cdot\|'_{q(),p()}$. First, note that it is immediate from this definition that for all measurable functions *f*:

$$
{}_r\|f\|'_{q(),p(),\Omega}\le\sup_{\substack{g\in\left(L^{q'()},l^{p'()}\right)(\Omega)\\ {}_r\|g\|_{q'(),p'(),\Omega}\le1}}\left|\int_\Omega f(x)g(x)\mathrm{d}x\right|\le\sup_{\substack{g\in\left(L^{q'()},l^{p'()}\right)(\Omega)\\ {}_r\|g\|_{q'(),p'(),\Omega}\le1}}\int_\Omega\left|f(x)g(x)\right|\mathrm{d}x \tag{53}
$$

but in fact all of these are equal. To see this, it suffices to note that for any $g\in M^{q'(),p'()}(\Omega)$ with ${}_r\|g\|_{q'(),p'(),\Omega}\le1$, (remark that for any real number *x*: $|x|=\operatorname{sgn}(x)\times x$),

we have:

$$
\left|f(x)g(x)\right|=\left|f(x)\right|\left|g(x)\right|=f(x)\times\operatorname{sgn}\left(f(x)\right)\times\left|g(x)\right|=f(x)\times h(x),
$$

where

$$
h(x)=\operatorname{sgn}\left(f(x)\right)\times\left|g(x)\right|,
$$

then

$$
h(x)\le\left|g(x)\right|\Rightarrow{}_r\|h\|_{q'(),p'(),\Omega}\le{}_r\|g\|_{q'(),p'(),\Omega}\le1,
$$

hence

$$\int_{\Omega}\left|f(x)g(x)\right|\mathrm{d}x=\int_{\Omega}f(x)h(x)\mathrm{d}x\leq {}_{r}\|f\|_{q(),p(),\Omega}.$$

This last inequality with (53) implies that:

$$ {}_{r}\|f\|'_{q(),p(),\Omega}=\sup_{\substack{g\in\left(L^{q'()},l^{p'()}\right)(\Omega)\\ {}_{r}\|g\|_{q'(),p'(),\Omega}\leq 1}}\left|\int_{\Omega}f(x)g(x)\mathrm{d}x\right|=\sup_{\substack{g\in\left(L^{q'()},l^{p'()}\right)(\Omega)\\ {}_{r}\|g\|_{q'(),p'(),\Omega}\leq 1}}\int_{\Omega}\left|f(x)g(x)\right|\mathrm{d}x$$

□

Remark 21.

As a consequence of the proof of Proposition 20, we get another version of Hölder's inequality:

$$\int_{\Omega}\left|f(x)g(x)\right|\mathrm{d}x\leq {}_{r}\|f\|_{q(),p(),\Omega}\times {}_{r}\|g\|'_{q'(),p'(),\Omega}. \tag{54}$$

In the next result, we show that:

$$M^{q(),p()}(\Omega)=\left(L^{q()},l^{p()}\right)(\Omega)$$

and that the norms ${}_{r}\|\cdot\|_{q(),p(),\Omega}$ and ${}_{r}\|\cdot\|'_{q(),p(),\Omega}$ are equivalent. We will refer to the norm ${}_{r}\|\cdot\|'_{q(),p(),\Omega}$ as the associate norm on $\left(L^{q()},l^{p()}\right)(\Omega)$.

Theorem 22. (*Norm conjugacy inequality*)

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, given $q()\in\mathcal{P}(\Omega)$, $p()\in\mathcal{P}(\mathbb{Z}^d)$, $r>0$, and a measurable function *f*.

Then, $f\in\left(L^{q()},l^{p()}\right)(\Omega)$ if and only if $f\in M^{q(),p()}(\Omega)$; furthermore:

$$c\times {}_{r}\|f\|_{q(),p(),\Omega}\leq {}_{r}\|f\|'_{q(),p(),\Omega}\leq C\times {}_{r}\|f\|_{q(),p(),\Omega}. \tag{55}$$

Proof.

$$\int_{\Omega}f(x)g(x)\mathrm{d}x=\sum_{k\in\mathbb{Z}^d}\int_{I_k^r}f(x)g(x)\mathrm{d}x={}_{r}\|fg\|_{1,1,\Omega}.$$

From Proposition 17-6), we get:

$$\int_{\Omega}f(x)g(x)\mathrm{d}x={}_{r}\|fg\|_{1,1,\Omega}\leq C\times {}_{r}\|f\|_{q(),p(),\Omega}\times {}_{r}\|g\|_{q'(),p'(),\Omega}$$

if we pass to the supremum over all g such that $g\in\left(L^{q'()},l^{p'()}\right)(\Omega)$ and ${}_{r}\|g\|_{q'(),p'(),\Omega}\leq 1$, we get:

$${}_{r}\|f\|'_{q(),p(),\Omega}\leq C\times {}_{r}\|f\|_{q(),p(),\Omega}. \tag{56}$$

It remains to prove that:

$${}_{r}\|f\|_{q(),p(),\Omega}\leq K\times {}_{r}\|f\|'_{q(),p(),\Omega}$$

If $f=0$ then ${}_{r}\|f\|_{q(),p(),\Omega}=0={}_{r}\|f\|'_{q(),p(),\Omega}$ and the last inequality becomes an equality.

Suppose that $f\neq 0$, $f\in\left(L^{q()},l^{p()}\right)(\Omega)$.

Consider an element *t* of $\left]0,{}_{r}\|f\|_{q(),p(),\Omega}\right[$, then $t<{}_{r}\|f\|_{q(),p(),\Omega}$.

First case: $p()=\infty$ on $\mathbb{Z}^d$.

Since $0 < t < {}_r\|f\|_{q(),\infty,\Omega} = \sup_{k\in\mathbb{Z}^d}\left\|f\chi_{I_k^r}\right\|_{q(),\Omega}$ there exists $\tilde{k}\in\mathbb{Z}^d$ such that:

$$t < \left\|f\chi_{I_{\tilde{k}}^r}\right\|_{q(),\Omega} = \sup\left\{\left|\int_\Omega\left(f\chi_{I_{\tilde{k}}^r}(x)\right)g(x)\mathrm{d}x\right| : g\in L^{q'()}(\Omega), \|g\|_{q'(),\Omega}\le 1\right\},$$

therefore there exists an element $h\in L^{q'()}(\Omega)$ such that:

$$t < \left|\int_\Omega f(x)h(x)\chi_{I_{\tilde{k}}^r}(x)\mathrm{d}x\right| \text{ and } \|h\|_{q'(),\Omega}\le 1 \tag{57}$$

Let $g_0 = h\chi_{I_{\tilde{k}}^r}$.

we have:

$$\begin{aligned}
{}_r\|g_0\|_{q'(),1,\Omega} &= \left\|\left\{\left\|g_0\chi_{I_k^r}\right\|_{L^{q'()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^1(\mathbb{Z}^d)} = \sum_{k\in\mathbb{Z}^d}\left\|g_0\chi_{I_k^r}\right\|_{L^{q'()}(\Omega)} \\
&= \left\|h\chi_{I_{\tilde{k}}^r}\right\|_{L^{q'()}(\Omega)} + \sum_{\substack{k\in\mathbb{Z}^d\\ k\neq\tilde{k}}}\left\|h\chi_{I_k^r}\right\|_{L^{q'()}(\Omega)} = \left\|h\chi_{I_{\tilde{k}}^r}\right\|_{L^{q'()}(\Omega)} + 0 \text{ from (57)}, \\
&= \left\|h\chi_{I_{\tilde{k}}^r}\right\|_{L^{q'()}(\Omega)} \le \|h\|_{L^{q'()}(\Omega)} \le 1
\end{aligned}$$

that is:

$$\begin{cases}
{}_r\|g_0\|_{q'(),1,\Omega} = {}_r\left\|h\chi_{I_{\tilde{k}}^r}\right\|_{q'(),1,\Omega} \le \|h\|_{L^{q'()}(\Omega)} \le 1 \\
t < \left|\int_\Omega f(x)g_0(x)\mathrm{d}x\right| = \left|\int_\Omega f(x)h\chi_{I_{\tilde{k}}^r}(x)\mathrm{d}x\right|
\end{cases} \text{ from (57).}$$

therefore,

$$t < \sup\left\{\left|\int_\Omega f(x)g(x)\mathrm{d}x\right| : g\in\left(L^{q'()}, l^{p'(),\Omega}\right)(\Omega), {}_r\|g\|_{q'(),p'(),\Omega}\le 1\right\}$$

t being any element of $\left]0, {}_r\|f\|_{q(),p(),\Omega}\right[$, if we tend t to ${}_r\|f\|_{q(),p(),\Omega}$, we get ${}_r\|f\|_{q(),p(),\Omega} \le \sup\left\{\left|\int_\Omega f(x)g(x)\mathrm{d}x\right| : g\in\left(L^{q'()}, l^{p'(),\Omega}\right)(\Omega), {}_r\|g\|_{q'(),p'(),\Omega}\le 1\right\}$ that is:

$${}_r\|f\|_{q(),p(),\Omega} \le {}_r\|f\|'_{q(),p(),\Omega}.$$

Second case: $p(\,) < \infty$ on $\mathbb{Z}^d$.

$$\begin{aligned}
& f\in\left(L^{q()}, l^{p()}\right)(\Omega) \\
&\Rightarrow {}_r\|f\|_{q(),p(),\Omega} = \left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)} < \infty \\
&\Rightarrow \left\{\left\|f\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d} \in l^{p()}(\mathbb{Z}^d).
\end{aligned}$$

Since $t < {}_r\|f\|_{q(),p(),\Omega} = \left\|\left\{\left\|f\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)}$ and

$\left\{\left\| f\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d} \in l^{p()}\left(\mathbb{Z}^d\right)$, there exists $\{c_k\}_{k\in\mathbb{Z}^d} \in l^{p'()}\left(\mathbb{Z}^d\right)$ such that $\left\|\{c_k\}_{k\in\mathbb{Z}^d}\right\|_{l^{p'()}\left(\mathbb{Z}^d\right)} \le 1$ and $t < \sum_{k\in\mathbb{Z}^d} c_k \left\| f\chi_{I_k^r}\right\|_{q(),\Omega}$ (see Proposition 5).

In other hand, from the theory on Variable Exponent Lebesgue Spaces, for any $k\in\mathbb{Z}^d$, there exists $g_k \in L^{q'()}\left(I_k^r, dx\right)$ verifying:

$$\|g_k\|_{q'(),I_k^r} \le 1 \text{ and } \int_{I_k^r} g_k(x) f(x)\mathrm{d}x = \left\| f\chi_{I_k^r}\right\|_{q(),\Omega}.$$

Let's consider the function $g\in L^0(\Omega)$ defined by:

for any $k\in\mathbb{Z}^d$: $g\chi_{I_k^r} = c_k g_k$, we have:

$$\begin{cases} {}_r\|g\|_{q'(),p'(),\Omega} = \left\|\left\{\left\|g\chi_{I_k^r}\right\|_{L^{q'()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p'()}\left(\mathbb{Z}^d\right)} \le \left\|\{c_k\}_{k\in\mathbb{Z}^d}\right\|_{l^{p'()}\left(\mathbb{Z}^d\right)} \le 1 \\ t < \sum_{k\in\mathbb{Z}^d} c_k \left\| f\chi_{I_k^r}\right\|_{q()} = \sum_{k\in\mathbb{Z}^d} c_k \int_{I_k^r} g_k(x) f(x)\mathrm{d}x = \sum_{k\in\mathbb{Z}^d} \int_{I_k^r} g(x) f(x)\mathrm{d}x = \int_\Omega g(x) f(x)\mathrm{d}x \end{cases},$$

then $g\in\left(L^{q'()}, l^{p'()}\right)(\Omega)$, ${}_r\|g\|_{q'(),p'(),\Omega} \le 1$ and

$$t < \sup\left\{\left|\int_\Omega f(x) g(x)\mathrm{d}x\right| : g\in\left(L^{q'()}, l^{p'()}\right)(\Omega),\ {}_r\|g\|_{q'(),p'(),\Omega} \le 1\right\},$$

since $t\in\left]0, {}_r\|f\|_{q(),p(),\Omega}\right[$, if we tend t to ${}_r\|f\|_{q(),p(),\Omega}$, we will get:

$$\begin{aligned} {}_r\|f\|_{q(),p(),\Omega} &\le \sup\left\{\left|\int_\Omega f(x) g(x)\mathrm{d}x\right| : g\in\left(L^{q'()}, l^{p'()}\right)(\Omega),\ {}_r\|g\|_{q'(),p'(),\Omega} \le 1\right\} \\ &= {}_r\|f\|'_{q(),p(),\Omega}, \end{aligned}$$

this completes the proof.

□

Definition 23.

In this section, we consider the dual of $\left(L^{q()}, l^{p()}\right)(\Omega)$: that is the Banach space $\left[\left(L^{q()}, l^{p()}\right)(\Omega)\right]^*$ of continuous linear functionals $T:\left(L^{q()}, l^{p()}\right)(\Omega)\to\mathbb{R}$ with norm:

$$\|T\| = \sup_{{}_r\|f\|_{q(),p(),\Omega}\le 1} |T(f)|$$

We begin by constructing a large family of continuous linear functionals and showing that they are induced by elements of $\left(L^{q'()}, l^{p'()}\right)(\Omega)$. Given a measurable function g, define the linear functional T_g on $\left(L^{q()}, l^{p()}\right)(\Omega)$ by:

$$T_g(f) = \int_\Omega f(x) g(x)\mathrm{d}x \tag{58}$$

for any $f\in L^0(\Omega)$ for which fg is integrable.

Proposition 24.

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, given $q()\in\mathcal{P}(\Omega)$, $p()\in\mathcal{P}\left(\mathbb{Z}^d\right)$, suppose that $g\in\left(L^{q'()}, l^{p'()}\right)(\Omega)$.

Then, T_g belongs to the dual $\left[\left(L^{q()}, l^{p()}\right)(\Omega)\right]^*$ of $\left(L^{q()}, l^{p()}\right)(\Omega)$ and verifies for any real number $r>0$:

$$\|T_g\| := \sup\left\{|T_g(f)| : f \in \left(L^{q()}, l^{p()}\right)(\Omega),\ {}_r\|f\|_{q(),p(),\Omega} \le 1\right\} \le {}_r\|g\|_{q'(),p'(),\Omega}$$

Proof.

Let $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$, we have:

$$\int_\Omega |f(x)g(x)|\mathrm{d}x = \sum_{k\in\mathbb{Z}^d} \int_{I_k^r} |f(x)g(x)|\mathrm{d}x = \sum_{k\in\mathbb{Z}^d} \left\|fg\chi_{I_k^r}\right\|_1 = {}_r\|fg\|_{1,1}.$$

From Proposition 17-6), we get:

$$\int_\Omega |f(x)g(x)|\mathrm{d}x = {}_r\|fg\|_{1,1} \le C \times {}_r\|f\|_{q(),p(),\Omega} \times {}_r\|g\|_{q'(),p'(),\Omega} < \infty$$

by consequence $f \times g$ is integrable and

$$|T_g(f)| = \left|\int_\Omega f(x)g(x)\mathrm{d}x\right| \le \int_\Omega |f(x)g(x)|\mathrm{d}x$$

that is:

$$|T_g(f)| \le C \times {}_r\|f\|_{q(),p(),\Omega} \times {}_r\|g\|_{q'(),p'(),\Omega}. \tag{59}$$

Therefore, T_g is a functional from $\left(L^{q()}, l^{p()}\right)(\Omega)$ to $\mathbb{C}$.

The linearity of T_g comes from the integral's properties.

From (59), T_g is continuous and

$$\|T_g\| \le C \times {}_r\|g\|_{q'(),p'(),\Omega}$$

□

Notation 25.

Remind that:

We have defined the $S = S(\Omega)$ to be the collection of all simple functions, that is, functions whose range is finite: $s \in S(\Omega)$ if:

$$s(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$$

where the numbers a_j are distinct and the sets $E_j \subset \Omega$ are pairwise disjoint. The family $S_0(\Omega)$ is the collection of $s \in S$ with the additional property that:

$$\left|\bigcup_{j=1}^{n} E_j\right| < \infty$$

Theorem 26. (*Dominated convergence theorem*)

Let Ω be a set such that $\mathbb{Z}^d \subset \Omega$ and $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$, with $q_+, p_+ < \infty$. If the sequence $\{f_n\} \subset \left(L^{q()}, l^{p()}\right)(\Omega)$ is such that $f_n \to f$ pointwise almost everywhere and there exists $g \in \left(L^{q()}, l^{p()}\right)(\Omega)$ such that $|f_n(x)| \le g(x)$ almost everywhere, then

$$f \in \left(L^{q()}, l^{p()}\right)(\Omega) \text{ and } {}_r\|f - f_n\|_{q(),p(),\Omega} \to 0 \text{ as } n \to \infty$$

Proof.

$q_+ < \infty$, $g \in \left(L^{q()}, l^{p()}\right)(\Omega) \subset L^{q()}_{loc}(\Omega) \Rightarrow g\chi_{I_k^r} \in L^{q()}(\Omega) \Rightarrow \rho_{L^{q()}(\Omega)}\left(g\chi_{I_k^r}\right) < \infty$,

then we get $\left(g\chi_{I_k^r}(x)\right)^{q(x)} \in L^1(\Omega)$.

Again, from the fact that $q_+ < \infty$:

$$\left|(f-f_n)\chi_{I_k^r}(x)\right|^{q(x)} \le 2^{q(x)-1}\left(\left|f\chi_{I_k^r}(x)\right|^{q(x)}+\left|f_n\chi_{I_k^r}(x)\right|^{q(x)}\right) \le 2^{q_+}\left(g\chi_{I_k^r}(x)\right)^{q(x)} \in L^1(\Omega) \quad ,$$

that is $\left|(f-f_n)\chi_{I_k^r}(x)\right|^{q(x)} \le 2^{q_+}\left(g\chi_{I_k^r}(x)\right)^{q(x)} < \infty$.

Since $f_n \to f$ pointwise almost everywhere, we have:

$$\left.\begin{array}{l}\left|(f-f_n)\chi_{I_k^r}(x)\right| \to 0, \text{ as } n\to\infty \\ q(x)\ge 1\end{array}\right\} \Rightarrow \left|(f-f_n)\chi_{I_k^r}(x)\right|^{q(x)} \to 0 \text{ as } n\to\infty.$$

we recapitulate:

$$\begin{cases}\left|(f-f_n)\chi_{I_k^r}(x)\right|^{q(x)} \le 2^{q_+}\left(g\chi_{I_k^r}(x)\right)^{q(x)} \\ \left|(f-f_n)\chi_{I_k^r}(x)\right|^{q(x)} \to 0 \text{ as } n\to\infty\end{cases},$$

therefore, by the classical dominated convergence theorem:

$$\rho_{L^{q()}(\Omega)}\left((f-f_n)\chi_{I_k^r}\right) = \int_\Omega \left|(f-f_n)\chi_{I_k^r}(x)\right|^{q(x)} \mathrm{d}x \to 0 \text{ as } n\to\infty,$$

by Norm-modular unit ball property stated above, since $q_+ < \infty$.

$\left\|(f-f_n)\chi_{I_k^r}\right\|_{q()} \to 0$ as $n\to\infty$ therefore, we have:

$$\left\{\left\|(f-f_n)\chi_{I_k^r}\right\|_{q()}\right\}_{k\in\mathbb{Z}^d} \to \{0\}_{k\in\mathbb{Z}^d} \text{ as } n\to\infty$$

$$\Rightarrow \rho_{l^{p()}(\mathbb{Z}^d)}\left(\left\{\left\|(f-f_n)\chi_{I_k^r}\right\|_{q()}\right\}_{k\in\mathbb{Z}^d}\right) \to \rho_{l^{p()}(\mathbb{Z}^d)}\left(\{0\}_{k\in\mathbb{Z}^d}\right) = 0 \text{ as } n\to\infty.$$

By Norm-modular unit ball property again, since $p_+ < \infty$, we get:

$$\left\|\left\{\left\|(f-f_n)\chi_{I_k^r}\right\|_{L^{q()}(\Omega)}\right\}_{k\in\mathbb{Z}^d}\right\|_{l^{p()}(\mathbb{Z}^d)} \to 0 \text{ as } n\to\infty$$

this means that:

$$_r\left\|f-f_n\right\|_{q(),p(),\Omega} \to 0 \text{ as } n\to\infty.$$

As an immediate corollary to the dominated convergence theorem we can give stronger version of the monotone convergence theorem.

Corollary 1.

Let Ω be a set such that $\mathbb{Z}^d \subset \Omega$, $r>0$, $q() \in \mathcal{P}(\Omega)$ and $p() \in \mathcal{P}(\mathbb{Z}^d)$ with $q_+, p_+ < \infty$, suppose that the sequence of non-negative functions $\{f_n\} \subset \left(L^{q()}, l^{p()}\right)(\Omega)$ increases pointwise almost everywhere to a function $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$, then:

$$_r\left\|f-f_n\right\|_{q(),p()} \to 0 \text{ as } n\to\infty$$

Theorem 27.

Given an open set Ω such that $\mathbb{Z}^d \subset \Omega$, $q() \in \mathcal{P}(\Omega)$, $p() \in \mathcal{P}(\mathbb{Z}^d)$ and $q_+, p_+ < \infty$.

Then the set of bounded functions of $\left(L^{q()},l^{p()}\right)(\Omega)$ with compact support and $\operatorname{supp}(f)\subset\Omega$ is dense in $\left(L^{q()},l^{p()}\right)(\Omega)$.

Proof.

Let K_n be a nested sequence of compact subsets of Ω such that $\Omega=\bigcup_n K_n$.

(For instance, $K_n=\left\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)\geq\frac{1}{k}\right\}\cap\overline{B}_n(0)$).

Fix $f\in\left(L^{q()},l^{p()}\right)(\Omega)$ and define the sequence $\{f_n\}$ by:

$$f_n(x)=\begin{cases} n & \text{if } f_n(x)>n \\ f(x) & \text{if } -n\leq f(x)\leq n \\ -n & \text{if } f_n(x)<-n \end{cases}$$

and let $g_n(x)=f_n(x)\chi_{K_n}(x)$. $f\in\left(L^{q()},l^{p()}\right)(\Omega)\Rightarrow f$ is finite almost everywhere, $g_n\to f$ pointwise almost everywhere; since $f\in\left(L^{q()},l^{p()}\right)(\Omega)$ and $|g_n(x)|\leq|f(x)|$, $g_n\in\left(L^{q()},l^{p()}\right)(\Omega)$. Therefore, since $q_+<+\infty$, by the dominated convergence theorem (Theorem 26), $g_n\to f$ in norm.

As a corollary to Theorem 27, we get two additional dense subsets.

Corollary 2.

Let Ω be a non void open set, $q(\,)\in\mathcal{P}(\Omega)$ and $p(\,)\in\mathcal{P}(\mathbb{Z}^d)$, $\mathbb{Z}^d\subset\Omega$ and $q_+,p_+<\infty$.

Then $C_c(\Omega)$ and $S_0(\Omega)$ are dense in $\left(L^{q()},l^{p()}\right)(\Omega)$.

Proof.

We will prove this for $C_c(\Omega)$; the proof for $S_0(\Omega)$ is identical.

Fix $f\in\left(L^{q()},l^{p()}\right)(\Omega)$ and $\varepsilon>0$. We will find a function $h\in C_c(\Omega)$ such that ${}_r\|f-h\|_{q(),p(),\Omega}<\varepsilon$. By Theorem 27 there exists a bounded function in $\left(L^{q()},l^{p()}\right)(\Omega)$ of compact support *g*, such that:

$$ {}_r\|f-g\|_{q(),p(),\Omega}<\frac{\varepsilon}{2} \tag{60}$$

Let $\operatorname{supp}(g)\subset B\cap\Omega$ for some open ball *B*. Then, since $q_+<\infty$, $C_c(B\cap\Omega)$ is dense in $\left(L^{q_+},l^{p_-}\right)(B\cap\Omega)$ (see Introduction chapter, "Denseness of some subsets in amalgam spaces $\left(L^q,l^p\right)(\mathbb{R}^d)$ with constant exponents"); thus there exists $h\in C_c(B\cap\Omega)\subset C_c(\Omega)$ such that:

$$ {}_r\|g-h\|_{q_+,p_-,\Omega}={}_r\|g-h\|_{q_+,p_-,B\cap\Omega}<\frac{\varepsilon}{2\left(1+|B\cap\Omega|\right)}$$

Therefore, by Proposition 17-3)-••):

$$ {}_r\|g-h\|_{q(),p(),\Omega}={}_r\|g-h\|_{q(),p(),B\cap\Omega}\leq\left(1+|B\cap\Omega|\right){}_r\|g-h\|_{q_+,p_-,B\cap\Omega}<\frac{\varepsilon}{2} \tag{61}$$

and so using (60) and (61):

$$ {}_r\|f-h\|_{q(),p(),\Omega}\leq{}_r\|f-g\|_{q(),p(),\Omega}+{}_r\|g-h\|_{q(),p(),\Omega}<\varepsilon$$

□

As consequence of Theorem 22 and Corollary 2, we have the following.

Corollary 3.

Let Ω be a non void open set, $q(\ ) \in \mathcal{P}(\Omega)$ and $p(\ ) \in \mathcal{P}(\mathbb{Z}^d)$, $\mathbb{Z}^d \subset \Omega$, with $1 < q(\ ), p(\ )$. For any positive real number *r*,

$$_r\|f\|_{q(),p(),\Omega} = \sup\left\{\left|\int_\Omega f(x)g(x)\mathrm{d}x\right| : \varphi \in C_c(\Omega), {}_r\|\varphi\|_{q'(),p'(),\Omega} \le 1\right\}$$

We will need the following lemma.

Lemma 28.

Let Ω be a non void open set such that $\mathbb{Z}^d \subset \Omega$, suppose that q, p are constant such that $1 \le q, p < \infty$ then $(L^q, l^p)(\Omega)$ is separable.

Proof.

We should find a countable dense subset of $(L^q, l^p)(\Omega)$. Let $f \in (L^q, l^p)(\Omega)$.

We partition Ω as follows:

$$\Omega = \bigcup_{n=1}^{\infty} B(0,n) \cap \Omega$$

$B(0,n) \cap \Omega$ is an open set, therefore there exists a bounded function g belonging to $(L^q, l^p) B(0,n) \cap \Omega$ with compact support such that:

$$_r\|f - g\|_{q,p,B(0,n)\cap\Omega} < \frac{\varepsilon}{2}$$

$\mathrm{supp}(g) \subset B(0,n) \cap \Omega$, since $q, p < \infty$, $C_c(B(0,n) \cap \Omega)$ is dense in $(L^q, l^p)(B(0,n) \cap \Omega)$ (see Introduction chapter, paragraph "Denseness of some subsets in amalgam spaces $(L^q, l^p)(\Omega)$ with constant exponents") therefore:

$$\exists h \in C_c(B(0,n) \cap \Omega) \text{ such that}$$

$$_r\|g - h\|_{q,p,B(0,n)\cap\Omega} < \frac{\varepsilon}{2}$$

From these two last inequalities, we get:

$$_r\|f - h\|_{q,p,B(0,n)\cap\Omega} \le {}_r\|f - g\|_{q,p,B(0,n)\cap\Omega} + {}_r\|f - g\|_{q,p,B(0,n)\cap\Omega}$$

thus $C_c(B(0,n) \cap \Omega)$ is dense in $(L^q, l^p)(B(0,n) \cap \Omega)$.

In other hand, each element of $C_c(B(0,n) \cap \Omega)$ can be numbered by the positive integer *n*, therefore $(L^q, l^p)(B(0,n) \cap \Omega)$ is separable, and the union of all these sets is countable and dense in $(L^q, l^p)(\Omega)$, we conclude that $(L^q, l^p)(\Omega)$ is separable.

□

Theorem 29.

Given an open set Ω such that $\mathbb{Z}^d \subset \Omega$, $q(\ ) \in \mathcal{P}(\Omega)$, $p(\ ) \in \mathcal{P}(\mathbb{Z}^d)$. If $q_+, p_+ < \infty$ then $(L^{q()}, l^{p()})(\Omega)$ is separable.

Proof.

$q_+ < \infty$. Then the proof of separability is almost identical to the proof of Corollary 2, so we describe roughly only the key details.

We partition Ω as follows:

$$\Omega = \bigcup_{k=1}^{\infty} B(0,k) \cap \Omega$$

Since $B(0,k)\cap\Omega$ is open, $\left(L^{q_+},l^{p_-}\right)\left(B(0,k)\cap\Omega\right)$ is separable (see Lemma 28) and so contains a countable dense subset. The union of all of these sets is a countable set contained in $\left(L^{q()},l^{p()}\right)(\Omega)$.

Arguing exactly as we did before, we see that this set is also dense in $\left(L^{q()},l^{p()}\right)(\Omega)$.

□

Proposition 30.

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, $q(\,)\in\mathcal{P}(\Omega)$, $p(\,)\in\mathcal{P}\left(\mathbb{Z}^d\right)$, and a measurable function *g*, then T_g is a continuous linear functional on $\left(L^{q()},l^{p()}\right)(\Omega)$ if and only if $g\in\left(L^{q'()},l^{p'()}\right)(\Omega)$.

Furthermore, $\left\|T_g\right\| = {}_r\left\|g\right\|_{q'(),p'(),\Omega}$ and so

$$c\times{}_r\left\|g\right\|_{q'(),p'(),\Omega}\le\left\|T_g\right\|\le C\times{}_r\left\|g\right\|_{q'(),p'(),\Omega}. \tag{62}$$

Proof.

Given any measurable function *g*, it follows from the definitions that $\left\|T_g\right\| = {}_r\left\|g\right\|_{q'(),p'(),\Omega}$ and so by Theorem 22 (with the roles of *f* and *g* exchanged in the statement and $q(\,),p(\,)$ replaced by $q'(\,),p'(\,)$), T_g is continuous if and only if $g\in\left(L^{q'()},l^{p'()}\right)(\Omega)$ and we get inequality (62).

□

Theorem 31.

Let Ω be a set such that $\mathbb{Z}^d\subset\Omega$, $q(\,)\in\mathcal{P}(\Omega)$, $p(\,)\in\mathcal{P}\left(\mathbb{Z}^d\right)$, if $q_+,p_+<\infty$, then the map $g\mapsto T_g$ is an isomorphism: given any $g\in\left(L^{q'()},l^{p'()}\right)(\Omega)$, the functional T_g is continuous and linear; conversely, given any continuous linear functional $T\in\left[\left(L^{q()},l^{p()}\right)(\Omega)\right]^*$, there exists a unique $g\in\left(L^{q'()},l^{p'()}\right)(\Omega)$ such that $T=T_g$ and ${}_r\left\|g\right\|_{q'(),p'(),\Omega}\approx\left\|T\right\|$.

Proof.

$q_+,p_+<\infty$, fix $T\in\left[\left(L^{q()},l^{p()}\right)(\Omega)\right]^*$, we will find $g\in\left(L^{q'()},l^{p'()}\right)(\Omega)$ such that $T=T_g$. Note that by (62) we immediately get that ${}_r\left\|g\right\|_{q'(),p'(),\Omega}\approx\left\|T_g\right\|$.

We consider two cases.

First case: $|\Omega|<\infty$.

Define the set function μ by $\mu(E)=T\left(\chi_E\right)$ for all measurable $E\subset\Omega$. Since *T* is linear and $\chi_{E\cup F}=\chi_E+\chi_F$ if $E\cap F=\varnothing$, μ is additive. To see that it is countably additive, let

$$E=\bigcup_{j=1}^{\infty}E_j,$$

where the sets $E_j\subset\Omega$ are pairwise disjoint, and let

$$F_n=\bigcup_{j=1}^{n}E_j,$$

Since $p_+<\infty\Rightarrow p_-<\infty$ and $p(\,)<\infty$ on $\mathbb{Z}^d$, Then by Proposition 17-3)-••)

$$
\begin{aligned}
{}_r\left\|\chi_E-\chi_{F_n}\right\|_{q(),p(),\Omega} &\le \left(1+|\Omega|\right)_r\left\|\chi_E-\chi_{F_n}\right\|_{q_+,p_-,\Omega} \\
&= \left(1+|\Omega|\right)\left[\sum_{k\in\mathbb{Z}^d}\left\|\left(\chi_E-\chi_{F_n}\right)\chi_{I_k^r}\right\|_{q_+}^{p_-}\right]^{\frac{1}{p_-}} \\
&= \left(1+|\Omega|\right)\left[\sum_{k\in\mathbb{Z}^d}\left|\left(E\setminus F_n\right)\cap I_k^r\right|^{\frac{p_-}{q_+}}\right]^{\frac{1}{p_-}} \\
&\le \left(1+|\Omega|\right)\left[\sum_{k\in\mathbb{Z}^d}\left|E\setminus F_n\right|^{\frac{p_-}{q_+}}\right]^{\frac{1}{p_-}}.
\end{aligned}
$$

Since $|E|<\infty$, (because $E\subset\Omega$, $|\Omega|<\infty$); $|E\setminus F_n|$ tends to 0 as $n\to\infty$, therefore ${}_r\left\|\chi_E-\chi_{F_n}\right\|_{q(),p(),\Omega}\to+\infty$ as $n\to\infty$ that is $\chi_{F_n}\to\chi_E$ in norm as $n\to\infty$. Therefore, by the continuity of *T*, $T\left(\chi_{F_n}\right)\to T\left(\chi_E\right)$; equivalently,

$$\sum_{j=1}^{\infty}\mu\left(E_j\right)=\mu(E)$$

therefore, μ is countably additive. In other words, μ is a measure on Ω.

Further, μ is absolutely continuous with respect to the Lebesgue measure $|\cdot|$, we prove it: let $E\subset\Omega$ such that $|E|=0$ then $\chi_E=0$ and so:

$$\mu(E)=T\left(\chi_E\right)=T(0)=0$$

By the Radon-Nikodym theorem [28], absolutely continuous measures are gotten from L^1 functions. More precisely, there exists $g\in L^1(\Omega)$:

$$T\left(\chi_E\right)=\mu(E)=\int_\Omega\chi_E(x)g(x)\mathrm{d}x$$

By the linearity of *T*, for every simple function $f=\sum a_j\chi_{E_j}$, $E_j\subset\Omega$,

$$T(f)=\int_\Omega f(x)g(x)\mathrm{d}x$$

By Corollary 2, the simple functions are dense in $\left(L^{q()},l^{p()}\right)(\Omega)$, and so T and T_g agree on a dense subset. Thus, by continuity $T=T_g$, and so by Proposition 30, $g\in\left(L^{q'()},l^{p'()}\right)(\Omega)$.

Finally, to see that *g* is unique, it is enough to note that if $g,\tilde{g}\in\left(L^{q'()},l^{p'()}\right)(\Omega)$ are such that $T_g=T_{\tilde{g}}$, then for all $f\in\left(L^{q()},l^{p()}\right)(\Omega)$:

$$\int_\Omega f(x)\left(g(x)-\tilde{g}(x)\right)\mathrm{d}x=0. \tag{63}$$

Since $|\Omega|<\infty$, by Proposition 17-5):

$g-\tilde{g}\in\left(L^{q'()},l^{p'()}\right)(\Omega)\subset\left(L^{q'()_-},l^{p'()_+}\right)(\Omega)$ simply denoted by $\left(L^{q'_-},l^{p'_+}\right)(\Omega)$,

and since (63) holds for all $f\in\left(L^{q()},l^{p()}\right)(\Omega)\subset\left(L^{q_-},l^{p_+}\right)$,
then we have:

$$\begin{cases} g-\tilde{g}\in\left(L^{q'_-},l^{p'_+}\right)(\Omega)\approx\left[\left(L^{q_-},l^{p_+}\right)\right]^* \\ f\in\left(L^{q_-},l^{p_+}\right)\end{cases}$$

by the duality theorem, (see Introduction chapter and paragraph "Duality of the wiener amalgam spaces $\left(L^q, l^p\right)\left(\mathbb{R}^d\right)$ with constant exponents").

$g - \tilde{g} = 0$ almost everywhere.

Second case: $|\Omega| = \infty$.

In this case, we write:

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n$$

where for each *n*, $|\Omega_n| < \infty$ and $\overline{\Omega_n} \subset \Omega_{n+1}$. Given $T \in \left[\left(L^{q()}, l^{p()}\right)(\Omega)\right]^*$, by restriction T induces a bounded linear functional on $\left(L^{q()}, l^{p()}\right)(\Omega_n)$ for each *n*. Therefore, by the above argument, there exists $g_n \in \left(L^{q'()}, l^{p'()}\right)(\Omega_n)$ such that for all $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$, $\operatorname{supp}(f) \subset \overline{\Omega_n}$,

$$T(f) = \int_{\Omega_n} f(x) g_n(x) \mathrm{d}x$$

Further, ${}_r\|g_n\|_{q'(),p'(),\Omega} \le C\|T\|$. Since the sets Ω_n are nested, we must have that for all *f* with support in Ω_n:

$$\int_{\Omega_n} f(x) g_n(x) \mathrm{d}x = \int_{\Omega_{n+1}} f(x) g_{n+1}(x) \mathrm{d}x$$

Since the functions g_n are unique, we must have that $g_n = g_{n+1}\chi_{\Omega_n}$ for all $x \in \Omega_n$. Since $\operatorname{supp}(g_n) \subset \overline{\Omega_n}$ closure of the set Ω_n, the sequence g_n increases to $|g|$; hence, by the monotone convergence Lemma 10:

$${}_r\|g\|_{q'(),p'(),\Omega} = \lim_{n\to\infty} {}_r\|g_n\|_{q'(),p'(),\Omega} \le C\|T\| < \infty$$

thus $g \in \left(L^{q'()}, l^{p'()}\right)(\Omega)$.

Now, fix $f \in \left(L^{q()}, l^{p()}\right)(\Omega)$ and let $f_n = f\chi_{\Omega_n}$. Then $f_n \to f$ pointwise almost everywhere and $|f - f_n| \le f$ so by the dominated convergence Theorem 26:

$f_n \to f$ in norm. Further, $f_n g \to fg$ pointwise, and by Holder's inequality $|f_n g| \le |fg| \in L^1(\Omega)$. Therefore, by the classical dominated convergence theorem and the continuity of *T*:

$$\begin{aligned}\int_{\Omega} f(x) g(x) \mathrm{d}x &= \lim_{n\to\infty} \int_{\Omega_n} f(x) g_n(x) \mathrm{d}x \\ &= \lim_{n\to\infty} \int_{\Omega_n} f_n(x) g_n(x) \mathrm{d}x \\ &= \lim_{n\to\infty} T(f_n) = T(f).\end{aligned}$$

Finally, since the restriction of *g* to each Ω_n is uniquely determined, *g* itself is the unique element of $\left(L^{q'()}, l^{p'()}\right)(\Omega)$ with this property. This completes the proof of the theorem.

□

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Authors' Contributions

We contributed entirely to writing this paper. We read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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