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Motion and Special Relativity in Complex Spaces

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Abstract

A natural extension of the Lorentz transformation to its complex version was constructed together with a parallel extension of the Minkowski M^4 model for special relativity (SR) to complex C^4 space-time. As the [signed] absolute values of complex coordinates of the underlying motion's characterization in C^4 one obtains a *Newtonian-like type* of motion whereas as the real parts of the complex motion's description and of the complex Lorentz transformation, all the SR theory as modeled by M^4 real space-time can be recovered. This means all the SR theory is preserved in the real subspace M^4 of the space-time C^4 while becoming simpler and clearer in the new complex model's framework. Since velocities in the complex model can be determined geometrically, with no primary use of time, time turns out to be definable within the equivalent theory of the reduced complex C^4 model to the C^3 "para-space" model. That procedure allows us to separate time from the (para)space and consider all the SR theory as a theory of C^3 alone. On the other hand, the complex time defined within the C^3 theory is interpreted and modeled by the single separate C^1 complex plane. The possibility for application of the C^3 model to quantum mechanics is suggested. As such, the model C^3 seems to have **unifying** abilities for application to different physical theories.

Keywords

Special Relativity, Complex Space and Time Models and Dramatic SR Simplification, Complex Time and Space Separation, Complex Time Interpretation

MOTTO: "*The complex numbers* (*measurement operators*?) *are more fundamental than the real numbers—they precede the real numbers, are more God—given*".

Roger Penrose.

1. Introduction

1) Two main aspects of this work related to some contemporary physics literature's features are worth emphasizing at the beginning.

The first aspect is related to the way of using complex numbers and models in physics.

Of course, they are applied widely in many areas of theoretical physics, but almost always are only treated as convenient mathematical tools having no physical meaning. Almost always the results of calculations involving complex numbers and functions are real, either as the real and imaginary parts of complex outcomes or, eventually, as their [signed] absolute values.

Very seldom one tries to interpret physically complex results when obtained. Some exceptions to this rule can be found in [1].

Generally, however, there is a chronic lack of trust in any physical interpretation of complex (and especially purely imaginary) numbers' related results or models.

In association with that, considerably little attention among physicists is paid to the complex spacetimes, which, nevertheless, are present in the literature (see, for example, [2]-[9]) although rarely. It's very common to view them as mathematical notions only with no real physical meaning, even if sometimes there is a strong suggestion that the real physical phenomena have causes situated somewhere inside of the complex interior of the complex space. It looks as if some, not directly observable, "complex matter" was included outside of the real space causing real physical processes in a predictable manner. An interesting example of that kind of phenomenon can be found in [1].

This may suggest that our real physical space is emersed in a wider (nonreal complex) environment, which is not directly observable and eventually may be [or not?] of some, say "paraphysical" (not just mathematical) nature.

The second aspect of this paper is that, as it turns out, a proper use of complex models for the physical space and for the time very dramatically simplifies some underlying physical theories, first of all, special relativity.

In this theory, considered as theory of the constructed complex model, all the relatively complicated tensor calculus can simply be omitted as well as many of calculus' descriptions can be replaced by simpler algebraic and trigonometric. Moreover, any unidimensional motion with various constant velocities (that ARE typically considered in SR) can be pictured on a complex plane emersed in C^3 (see **Figure 1** in coming text), essentially more transparently than it is done using the typical plane, emersed in the Minkowski's M^4, whose vertical coordinate is the time coordinate.

This "para-space's" motion representation provides a very clear picture of what happens with the motion when velocities change. From that point of view, it is important for clarity of the description that motion with each particular speed has a different trajectory while the trajectories only differ by simple circular rotations. In turn, the angles of such rotations are strictly determined by underlying changes in speeds. Unlike with the real Minkowski's model description, all this can be "seen" in the same coordinate system with the vertical axis being the imaginary distance (instead of time axis). In turn, any imaginary "distance" may (also) be understood as an amount of kinetic energy that makes a given physical body move.

Dramatic simplification of mathematical apparatus can be achieved not only for special relativity, when modeling, but possibly also one could model quantum mechanics by the C^3 para-space. For that, hypothetical, possibility, see [10], Appendix 3, page 75.

Realize that, it is commonly known that mathematical tools as contemporarily applied in theoretical physics are overblown in their complexity. This often causes the physical content to be almost lost so that, as it is customary to say: "no physicist (really) understands quantum mechanics" with its current mathematical descriptions.

For many physicists, it might be attractive to use functional analysis and advanced abstract algebra for their beauty, but that is very far from simplicity which, in turn, is of another, perhaps even higher, esthetic value.

Besides, it seems to be obvious that the researchers should be aware of what they really want to investigate: physical phenomena or beauty of mathematical structures.

As it is typical, in cases like that, it may turn out in the end that the abnormal growth of complexity of physical theories in their mathematical descriptions is the result of a *hidden error* that was committed sometime in the past at the very beginning of modern physics.

Perhaps, among others, that error results from too strong "believe in real numbers".

It was not necessarily true, as it appeared to some physicists or mathematicians from the beginning of twenty century (see, Von Neuman for example), that "new physics", especially quantum physics, must act parallelly to the "new mathematics" [*i.e.* set-theoretical approach in functional analysis, abstract algebra, topology and others], often unnecessarily merging the two while overlooking other, simpler, and more efficient possibilities. All this, probably, caused that the special role of complex numbers and models for the new physics was not fully recognized.

Nota bene, it is common for any research that, in general, simplicity indicates correctness while constantly growing complexity is the signal that, probably, "something went wrong".

Thus, maybe somewhen in the past, more relevant for the "new physics" would be transition from real to complex mathematical models (like, roughly speaking, $R^n \to C^n$, for $n = 1, 2, \ldots, \infty$) rather than to new mathematics with, mostly real, functional analysis as the starting point. The underlying complex models are much less "complex" than too high levels of abstraction, and, first of all, complexity of modern mathematics whose role could be reasonably limited by the "complex option". In my opinion, the only (but very serious) obstacle for choosing complex models is the *ontological fear* of many to fully understand and adopt the idea of true "complex modeling".

2) Of course, it is fundamental that proper application of complex models requires proper construction of them. Not every construction of complex space-time may yield fruitful physical consequences. It is not enough to just add to a real quantity an imaginary part nor to extend real models to complex ones via analytical continuations.

Typically, the construction should be "right" to be fruitful.

I hope that the extension from real to complex models, for example, the way proposed in this paper, taking as the starting point and extension of the Lorentz transformation to its complex version, is right.

After the first step was done (completing the coefficient $\cos\theta$ (which was defined as equal to the reciprocal of the Lorentz factor) by its so natural complement "$i\sin\theta$"), the physical consequences of that movement were striking. First of all, the "mysterious" phenomenon of the Lorentz contraction found immediate simple explanation as the result of a rotation in the so-created complex plane.

Other striking consequences also justify the choice of the so-obtained simple model for both SR with no tensors (!), and classical Newtonian mechanics. Namely, the real parts of the underlying complex quantities satisfy classical (real) SR theory while their signed absolute values satisfy the Newtonian (with one exception, namely velocities composition was slightly different than the arithmetic sum).

Moreover, there are strong indications that the same model may also serve (through complex values) as a model for quantum mechanics (QM) and possibly for other areas of quantum physics.

Also, as I would anticipate, it was a good model for classical electrodynamics.

Anticipating further, I strongly hope that the complex model described in this paper may be a good candidate as an *elementary* model for the unified physics.

In the proposed model for SR theory, all the theorems of Einstein's SR are preserved while many intuitively difficult-to-grasp facts and "paradoxes" (such as the universality of the speed of light or the twin paradox) look much clearer in this complex framework.

The construction of the model has two stages. At first, the complex C^4 space-time is constructed through an extension of the Lorentz transformation to its complex version. Then it turns out that time can be defined within the theory of the "para-space" C^3. Consequently, *time* turned out not to be a primitive notion and the time transformation can be obtained from the spatial part of the complex Lorentz. The time definition within the SR theory of C^3 was possible because all velocities in that theory can easily be defined geometrically with no primary use of time.

Anticipating, at this stage, the following text let me give a short and simple explanation of the important fact of the possibility of determining speeds and velocities (in macroscale framework) geometrically with no use of time (which therefore can be defined as simple ratio of a distance and speed (for example, the speed of light c). Thus, since we may always substitute inverse to the Lorentz coefficient by $\cos\theta$, where, after naturally completing it to: $\cos\theta + i\sin\theta$, θ turns out to be an angle of rotation (say, of the real x-axis) in the so-constructed complex plane, we follow the fact that in this framework $\sin\theta = u/c$, where u is speed corresponding to the angle θ while c is the speed of light. Now any speed u may be defined as $c\sin\theta$ with no time involved in this definition. On the other hand, speed of light c can also be defined geometrically according to the geometrical notion of orthogonality. Namely, $c = 1 = \sin\pi/2$. As it will be discussed in the paper, the natural ["Galilean"] direction of light is the vertical direction, *i.e.* parallel to the imaginary axis.

Thus, any speed is defined as equivalent to the pair of angles $\pi/2$ and θ whose geometric nature is obvious and no time is needed. One can then conclude that the primary phenomenon in the Nature is motion [or, somehow more generally, "*energy*" to which, mathematically, probably corresponds "imaginarity"] while time is the derivative of the motion.

Since in the obtained version of complex Lorentz transformation $C^4 \to C^4$ (see Formula (23)) time part became separated from the space part I obtained, as the conclusion, the complex version of the Lorentz transformation as a $C^3 \to C^3$, spatial only transformation (see (26*) and (26**)) from which the common real Lorentz $M^4 \to M^4$ transformation can be recovered.

The quite exciting hypothetical fact is a possibility to model with C^3 a single elementary particle, actually, with no use of operators nor the underlying Hilbert spaces. Again, see [10], Appendix 3, page 75.

In short, the particle's analytical description can be given as the complex, time-dependent, value $r\exp[i\omega t]$ where r, ω are positive reals and t is real time. The value may be considered, for example, as a complex eigenvalue of a normal (instead of Hermitian) operator being, say, an extended "momentum operator" on the Hilbert $L^2(C^3)$ space and not on the Hilbert $L^2(R^3)$.

Physically, in the complex paraphysical space C^3, the considered value could be pictured as a small "stick" of length r [or one of its endpoints] vibrating with the angular velocity ω within a single complex plane.

As the first approach, such a stick can be considered a "particle" but more likely it would represent one of the three (sticks) interconnected quarks, each rotating in a separate complex plane.

The actual structure of those three "sticks" formatting the particle may not be straightforward but anyway that structure would represent a complex "**particle**".

On the other hand, the projections of shapes of the rotating sticks on the underlying real axes x, y, z will be given in their lengths as their real parts $\mathrm{Re}\{r_i\exp[i\omega_i t]\}$ = "$r_i\cos\omega_i t$" ($i = 1, 2, 3$), which apparently are observed as **waves**.

Additionally, the projections of the considered three complex quantities on corresponding imaginary axis are "$r_i\sin\omega_i t$" ($i = 1, 2, 3$) which may represent free particle velocity's coordinates, easily obtainable from the given lengths (a reasonable assumption seems to be that $r_i = r$, $\omega_i = \omega$, for $i = 1, 2, 3$).

Such a simple formalism may, eventually, clarify "dark" basic notions and difficulties in QM such as wave-like nature of the particles and the Heisenberg's uncertainty principle. It seems then that the Heisenberg's rule only works in real models. For that and for more details, see again [10] in Appendix 3.

3) In Section 2, we extend the real Lorentz transformation $M^4 \to M^4$ to its complex $C^4 \to C^4$ version, together with the extension of the real Minkowski M^4 spacetime to the complex Minkowski C^4 spacetime.

As mentioned, in Section 2.3, we introduce geometric definition of speeds (as coordinates of underlying velocities) with no use of time.

We obtain there both the relativistic observed speeds and the unbounded, unobserved, complex speeds whose [signed] absolute values gain the **Galilean** (Newtonian) speeds interpretation. The latter is known in SR as "**proper speeds**" but with no reference to Galilean kinematics.

We stress at that point the natural relationship between the Newtonian classical theory and SR, both having the same complex model. This topic is further developed in Section 3.

In Section 4, time transformation is discussed. As the result of deriving Galilean (proper) speeds on one side and the invariance of the absolute value of complex lengths (distances) on the other, we arrive at the notion of **proper time** along the radial line in the interior of the complex plane [say, within the moving "rocket"] of time.

The proper time' phenomenon inclined us to adjust the time transformation within the complex Lorentz transformation.

In Section 5, we found it necessary [for preservation of the metrics] to inessentially reduce the C^4 spacetime model to the C^3 "para-space" model with the time complex plane "eliminated" or rather separated.

What is important, however, in the SR theory of C^3 is that all facts taking place in the C^4 reality are recoverable as the **complex time** is definable and the fourth row of the complex Lorentz turns out to be obtainable from the spatial part of the transformation.

Under all that, the final form of the $C^3 \to C^3$ Lorentz transformation is given in Section 5 by Formulae (26*) and (26**).

At the end, in Section 6, we discuss (Euclidean) isometric properties of the new complex Lorentz.

(Compare these considerations with [11].)

Some discussion on the nature of the complex time was provided in Section 4. See especially Formulae (15) and (16).

2. Preliminary Descriptions

2.1. Lorentz Transformation and Its Complex Extension

To build up a new SR framework, first recall the formula for the common (real) Lorentz transformation $M^4 \to M^4$, where M^4 is the Minkowski real space-time with the invariant hyperbolic Lorentz norm.

Unless stated otherwise, we restrict the attention to uniform motion along the real *x*-axis, with a constant speed *u*. Consider the following Lorentz transformation:

$$\begin{aligned} x - ut &= x'\left(1 - u^2/c^2\right)^{1/2} \\ y &= y' \\ z &= z' \\ t - ux/c^2 &= t'\left(1 - u^2/c^2\right)^{1/2} \end{aligned} \tag{1}$$

where t and t' denote times at rest and within the moving object, respectively.

Under the trigonometric substitutions:

$$\left(1 - u^2/c^2\right)^{1/2} = \cos\theta \text{ and } u/c = \sin\theta \text{, (A)}$$

Formula (1) becomes:

$$\begin{aligned} x - ut &= x'\cos\theta \\ y &= y' \\ z &= z' \\ t - ux/c^2 &= t'\cos\theta \end{aligned} \tag{2}$$

where, initially, a geometric and an associated physical meaning of the "angle" θ are not yet known.

An initial assumption only is that θ is a circular real angle in the Euclidean sense.

That assumption follows the simple fact that:

$$0 \le \left(1 - u^2/c^2\right)^{1/2} \le 1,$$

while, in general, $\cosh\theta > 1$ and $\sinh\theta$ can be arbitrarily high.

For that reason, θ is not an imaginary nor hyperbolic angle.

An **educated guess** may incline one to test the following hypothesis:

Does any significant physical meaning results when the "coefficient" $\cos\theta$ in (2) will be completed by its very natural, from a pure mathematical viewpoint, term to the expression $\cos\theta + i\sin\theta$, where $i^2 = -1$.

But as $\cos\theta + i\sin\theta = \exp[i\theta]$, geometrically, the right-hand sides of the first and fourth rows in Formula (2) become exactly the circular rotations by the angle θ in the so-constructed complex plane, which extends the real *x*-axis and the real *t*-axis of the former M^4.

Of course, in the same manner (for the full form of the Lorentz transformation) one may extend the *y*-axis and the *z*-axis to their corresponding complex planes. This results with C^4 space-time with the hyperbolic Lorentz-like norm.

Anticipating further constructions, as C^4 will later be reduced to C^3 [for a "time free model"], realize that in that case the hyperbolic metric essentially reduces to the Euclidean on the corresponding C^3.

According to my first [false] impression (see [2]), time (like position *x*) will also be rotated by the nonnegative angle $|\theta|$ in the "complex time" plane.

According to the above assumption, the Lorentz transformation (2) becomes extended to its complex form:

$$\begin{aligned} x - ut &= x' \exp[i\theta] \\ y &= y' \\ z &= z' \\ t - ux/c^2 &= t' \exp[i|\theta|] \end{aligned} \tag{3}$$

where $\pi/2 < \theta < \pi/2$ and $|\theta|$ denote the absolute value of the angle θ (presence of that absolute value in fourth row is dictated by the fact that time "goes" in one only (say, positive) direction).

As, however, deeper analysis revealed (see [12]) the last row of (3) was not yet correct. This problem was the main reason for me to write [12] in order to give justification (and to correct the error committed in [10]) for the new complex version of (2).

This version of the complex extension of the real Lorentz transformation (1) or (2) initially is given by what follows:

$$\begin{aligned} x - ut &= x' \exp[i\theta] \\ y &= y' \\ z &= z' \\ t - ux/c^2 &= (t' \cos\theta) \exp[i|\theta|] \end{aligned} \tag{3*}$$

Further Anticipation:

Unfortunately, (3*) still turned out **not** to be **totally correct** since [relativistic] speeds u and c in (3*) must be replaced by *bigger* ["Galilean"] real speeds U_θ and C_θ (known under the name "celerity") which will be defined by Formulae (8) and (9) in the text below, and explained in Section 2.3. Here notice only, the "direction" of the speeds U_θ and C_θ is along the line OB′ in **Figure 1** in below, *i.e.* within the interior of the proper complex plain and not along the real axis.

Thus, one will obtain **the correct $C^4 \to C^4$ formula** for the "complex Lorentz transformation" by replacing (3*) by the following (here just anticipated) Formula (3**):

$$\begin{aligned} x_{c,\theta} - U_\theta t_{c,\theta} &= x' \exp[i\theta] \\ y &= y' \\ z &= z' \\ t_{c,\theta} - U_\theta x_{c,\theta}/C_\theta^2 &= (t' \cos\theta) \exp[i|\theta|] \end{aligned} \tag{3**}$$

where the meaning of the complex position $x_{c,\theta}$ is explained in the text following **Figure 1** and the meaning of the complex time $t_{c,\theta}$ in **Figure 3**.

Formula (3**) will be explained throughout this paper. In my opinion, however, it is purposeful to introduce it here to signal the difference between the correct version (3**) and the not yet quite correct Formula (3*) as present in [12]. So, **(3**) need not yet necessarily be entirely understood at this stage of**

the paper.

Anticipating a bit, for the complex quantities (all having the argument θ): the position $x_{c,\theta}$, time $t_{c,\theta}$, the considered complex speed of a physical body $U_{c,\theta}$, and the corresponding to it complex speed of light $C_{c,\theta}$, we denote:

$$x_\theta = \left| x_{c,\theta} \right|,$$

$$t_\theta = \left| t_{c,\theta} \right|,$$

$$U_\theta = \left| U_{c,\theta} \right|,$$

$$C_\theta = \left| C_{c,\theta} \right|,$$

where the operator $|.|$ is the absolute value of any considered complex quantity.

Thus, for the real ["Galilean"] speeds U_θ and for the real ["semi-Galilean"] speeds of light C_θ in the direction determined on the underlying complex plane by the angle θ, one, in general, has:

$$U_\theta > u \text{ and } C_\theta > c.$$

Moreover, in the real case, *i.e.* when $\theta = 0$, we have:

$$U_0 = u \text{ and } C_0 = c,$$

where u is the considered real relativistic speed in the direction of x'-axis while c is the common [relativistic] speed of light in the real subspace of the complex space.

The symbols $x_{c,\theta}$ and $t_{c,\theta}$ are applied in (3**) instead of the symbols x, t in (3*) and, in both cases, denote complex position and complex time, respectively, while x' and t' are the real position and real time as present in **Figure 1** and **Figure 3**.

Notice also that:

$$x_{c,\theta} = x_\theta \exp[i\theta],$$

$$t_{c,\theta} = t_\theta \exp[i|\theta|],$$

$$U_{c,\theta} = U_\theta \exp[i\theta],$$

$$C_{c,\theta} = C_\theta \exp[i\theta],$$

are the complex, say *(para)-physical*, quantities.

For more detailed justification of (3*), see [12].

In this paper, the reasoning for the (3**)'s validity will be given in Section 4, and for further development of (3**) towards its equivalent $C^3 \to C^3$ versions (26*) and (26**), see Section 5. □

Remark 0. Now, notice that, actually, (3*) and (3**) are no more (complex) "Lorentz transformations" since the space-time interval, as defined in SR, is no longer preserved under (3**). Nevertheless, realize that the common real Lorentz transformation can be recovered from (3**) by taking real parts from both sides of spatial and the absolute value of the fourth row of (3**).

The unpleasant fact of the space-time interval noninvariant behavior can be amended since, as we will see, the fourth row turns out to be dependent (proportional to the first) and thus redundant. This allows to reduce the C^4 model to an equivalent C^3, and so the hyperbolic norm to the Euclidean. Under this circumstance the Euclidean geometry, with Euclidean metric preserved, is recovered in the (para)physical C^3 model.

The final form of the complex "**space-velocity** Lorentz transformation", as equivalent to space-time transformation, is given by (26) in coming text. □□

Consider the spatial part of transformation (3**) which in this particular case reduces to the first row.

Now, the question is what is the physical meaning of this row?

To answer this in a simple way, consider only the special case when $t_{c,\theta} = 0$.

Then, the spatial part of (3**) reduces to:

$$x_{c,\theta} = x' \exp[i\theta],$$

where, after the rotation, x' represents the new real axis [now lying horizontally in place of the former x-axis] and $x_{c,\theta}$ represents the complex ["slant", "radial"] "axis", after transforming it from its previous position that was along the former x-axis.

Since the above represents a <u>circular</u> rotation about the origin of the complex plane, its radius x', which here may represent the physical length of a considered moving body, is invariant!

The analytic counterparts of the geometric fact, of the rotation, in the case $U_\theta t_{c,\theta} = 0$, are the following equalities:

$$|x_{c,\theta}| = x_\theta = |x' \exp[i\theta]| = |x'|. \tag{4}$$

<u>**Notice:**</u> When $t_\theta > 0$, *i.e.* $t_{c,\theta} \neq 0$, the rotation will be performed about the complex point $U_\theta t_{c,\theta}$ (corresponding to the known in the literature real value ut), which fact would unnecessarily spoil clarity and simplicity of the above considerations.

Realize too, that according to the first row of (3**) U_θ is real and $t_{c,\theta}$ is complex, and thus their product is complex.

Nevertheless, in the general case, when $U_\theta t_{c,\theta} \neq 0$ the length is preserved too. □

The conclusion from both geometric and analytic facts is that in <u>this</u> complex <u>version</u> of SR the, originally real, length x [after rotation becoming the length x_θ of a rocket] is preserved as the absolute value of its complex representation and is a real constant independent on θ, *i.e.* or, equivalently, on the speed u.

Thus, in a sense, the Lorentz contraction is eliminated when length is considered within the complex "reality". The contraction only takes place in the real part of the complex plane.

The length, as "observed" in the real space by physical instruments [if such a measurement ever would be performed], is only equal to Re $x_{c,\theta}$, *i.e.* to the real part of the complex chord $x_{c,\theta}$.

Thus, in this complex version of SR, the Galilean (Newtonian) length invariance is recovered!

In my opinion, this recognition, and other described later, motivate the need of the extension of the real Lorentz transformation (1) to its complex form (3**). The form of time transformation in (3**) will be derived and explained later or see [12].

The fact of the "length invariance" with respect to the regular Euclidean norm in C^3, without any time involvement (impossible in Minkowski's M^4), suggests the possibility and need to return to the regular Euclidean (norm) geometry, instead of the somewhat less natural hyperbolic Lorentz geometry in M^4 (see [11]) with the necessity of time involvement.

This invariance on each of the three basic spatial coordinate complex planes of C^3 model [which form the algebraic basis of C^3] is the same.

As this is argued later, the C^4 model (where one of the four base complex planes is the "complex time plane") reduces to its spatial C^3 part as time turns out not to be a primitive notion, but instead is definable within the [SR] theory of the C^3 model (never within R^3 model alone).

Thus, the so obtained C^3 model of the "para-space" is just the regular complex Euclidean space with the usual complex Euclidean geometry.

Remark 1. Anticipating a bit actual consideration that leads to the C^4 reduction to C^3, here only notice that all velocities, within any complex plane, can be defined geometrically (see Formula (5)) with no reference to time and therefore time can always be defined as usually by means of distances and speeds (especially "proper speeds") which are given in advance.□

2.2. Some Graphic Illustration of Unidirectional Uniform Motion across One Complex Plane

As now we restrict the motion to one real direction only along the real x-axis (or just within R^3) with constant speed u, we analyze the complex counterpart of this motion on one complex plane of the positions, whose real axis is the mentioned x-axis. The extension of this motion to the whole C^3 is immediate as the rules of the motion (with velocity's coordinates, say u_x, u_y, u_z in place of only one $u_x = u$ considered here) on each of the three complex coordinate planes are the same.

The ("first") complex coordinate plane that the considered motion takes place in is illustrated in the following **Figure 1**. See also discussion for the same **Figure 1** in [10].

According to the analytical counterpart of **Figure 1**, *i.e.* the first row of (**3****) with $t_{c,\theta} = 0$, the former real x-axis (now being the $x_{c,\theta}$ line, where the subscript "c, θ" will always mean: "… complex with the argument θ", while subscript "θ" alone means: "absolute value of a considered complex quantity whose corresponding geometric segment's (here OB') angle of inclination is the argument θ". Realize that the segment OB was subjected to the rotation "$\exp[i\theta]$" from its original horizontal position by the angle θ, whereas in place of it, at horizontal position, we now have the new, formerly complex, horizontal, x'-axis. The metric of the x'-line is (according to moving observer, *i.e.* "situated" along OB′ line)

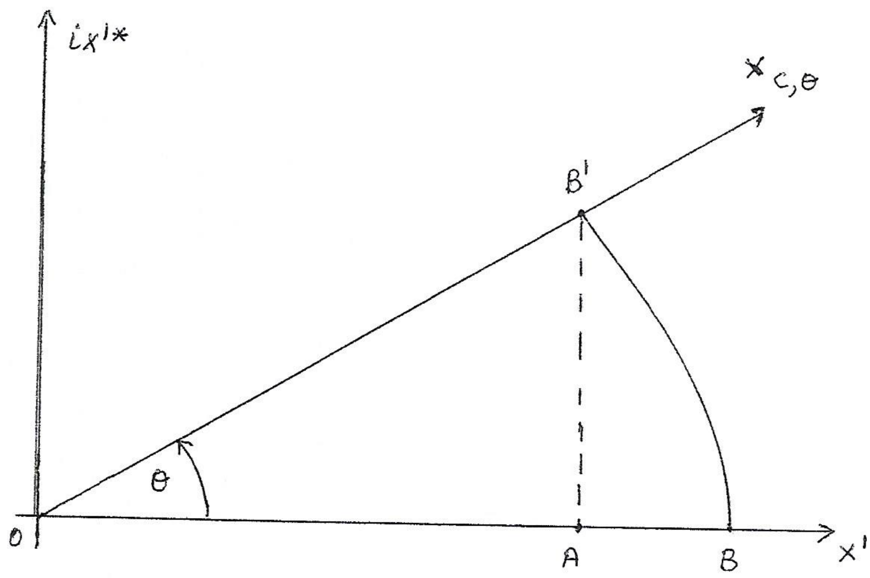

Figure 1. Real (A) and radial (B′) motions across the complex plane.

contracted by the constant (at all points of x') coefficient $\cos\theta$.

Geometrically, this contraction is due to orthogonal projection of the "new" complex $x_{c,\theta}$-axis [*i.e.* "radial axis" along OB′line], with invariant [according to the moving observer] metric, to the real x'-axis. This projection geometrically explains the Lorentz contraction phenomenon.

For each single motion with speed u, consider two distinct trajectories in the complex model illustrated by **Figure 1**.

One is the observed path along the real 0A line and the other along 0B′line within the interior of the complex plane. The direction of the first we will call the "observable [real] direction" and of the second the "natural direction" which, nevertheless, is beyond the reach of human senses and of physical instruments while still present within the easily accessible mathematical model.

If we consider motion of a rocket, our assumption is that, at the initial time epoch $t_{c,\theta} = 0$, its actual, unobserved, length spreads out between the points 0 and B′, whereas the observed "image" of this rocket's length spreads between 0 and A.

According to the Lorentz contraction, as illustrated by **Figure 1**, the image is shorter than the actual rocket by the proportion |0A|/|0B′| = $\cos\theta$ where, according to primary assumptions (A),

$$\cos\theta = \left(1 - u^2/c^2\right)^{1/2}.$$

As one can say, the measured [if that ever "happened"], by physical instruments, length |0A| of the rocket is just the length of the observed "rocket's shadow" whereas its "true" invariant length is |OB′| = |OB|.

2.3. On Galilean Speeds along the Natural (Complex) Directions

Let us now reinterpret a bit (**Figure 1**). Instead of the "rocket" spread out between some points consider a classical particle at the moment $t = 0$ situated at point B′ while its real "image" has position A.

The image of this particle moves along the real x'-axis with the relativistic speed u.

In accordance with the first assumption of (A), which follows Formula (1), we have:

$$u = c\sin\theta . \tag{5}$$

The question now is, what is the speed of the "actual particle" as it moves along the radial line 0B'?

To answer this, notice two facts, one of geometric and the other of analytic nature.

First, the "two objects" (actually it is the same particle considered at point A and at point B′ separately) are "instantaneous" in at least two meanings: geometric as both always lying on the same vertical line connecting them and analytic, since for the real parts we always have:

$$\mathrm{Re}\,\mathrm{B}' = \mathrm{Re}\,\mathrm{A} = \mathrm{A} .$$

Second, the ratio of the distances is:

$$|\mathrm{OB}'|/|\mathrm{OA}| = \sec\theta . \tag{6}$$

The conclusion from these two facts is that the [real] speed U_θ along the [complex] radial $x_{c,\theta}$-axis 0B′ (known in literature as the proper velocity or celerity, but with no complex space framework nor geometric interpretation) is bigger than u and from (6) it follows that:

$$U_\theta = u\sec\theta , \tag{7}$$

where according to the admitted convention if $\theta = 0$, then

$$u = U_0 . \tag{7*}$$

Realize that since, in accordance with (A), $\sec\theta$ equals the Lorentz factor, the proper velocity U_θ here was obtained geometrically with no use of [proper] time.

Remark 2. The premises for conclusion (7) are both the geometric and analytic "instantaneity" described above.

The above considerations do not yet require any primary use of the concept of time since the instantaneity is defined independently of this concept.

Nevertheless, as it will be shown later, the times which elapse for either of the "two" objects to shift from 0 to A and from 0 to B′, will turn out to be the same. Thus, the instantaneity in the sense of "same time" also takes place but will be a conclusion rather than a premise. For more on that, see [12]. □

Combining (7) with (5), we obtain:

$$U_\theta = c\tan\theta . \tag{8}$$

As mentioned, the real speed $U_\theta = |U_{c,\theta}|$, is known as the "proper speed", where $U_{c,\theta} = U_\theta \exp[i\theta]$ is the corresponding "complex speed".

As for the complex speed $C_{c,\theta}$ of a light beam that is sent ahead from the rocket, which itself moves with speed U_θ in 0B′ radial direction, the same as before geometric argument yields to the conclusion that since $|0B|/|0A| = \sec\theta$, we have:

$$\left|C_{c,\theta}\right| = C_\theta = c\sec\theta, \tag{9}$$

where c is the ordinary relativistic ["small"] speed of light in vacuum of the real subspace.

Consequently, we have, for the complex speed of light:

$$C_{c,\theta} = C_\theta \exp[i\theta]. \tag{9*}$$

In the complex quantities' framework, an overwhelming suggestion yields to the conclusion that the proper speeds U_θ as defined by (7) or (8) and C_θ defined by (9) should be considered "Galilean" or "Newtonian" since they are speeds of (*para*)-physical bodies which move along complex paths such as the 0B′ path. The metrics of the paths are invariant [no Lorentz contractions], so each of the radial trajectory [as seen by observers situated on them] is a Euclidean line.

Notice, that the source of boundness of the relativistic speeds u [including speed c of light in R^3] is the Lorentz contraction of distances, which, geometrically, are projections of Euclidean distances in any OB′-like radial line onto the real x'-axis. Parallelly, any speed U_θ or C_θ along that radial line projected (or, analytically, multiplied by $\cos\theta$) into the real x' direction results in its bounded relativistic counterparts u or c. Realize that c as the projection of C_θ is always the same, regardless the angle [speed] θ. This and the Euclidean character of all the radial lines inclines one to treat the speeds U_θ and C_θ as Galilean (or Newtonian).

The relativistic speeds and contracted distances may roughly be considered as deformations of the Galilean and the Euclidean, respectively.

Notice that each Galilean (proper) speed U_θ is finite but unbounded. As $\theta \to \pi/2$, we have, according to (5) and (8), $u \to c$ and $U_\theta \to \infty$.

In the papers [10] [12], the last infinite limit, I considered to be the [actual] "Galilean speed of light" $C_{\pi/2} = C = \infty$. This just is in spirit of the Newtonian theory and supports the "Newtonian version" of Einstein's universality of speed of light (any finite speed U_θ is "infinitely smaller" than the infinite speed of light $C = C_{\pi/2}$.). The "*direction*" of this (full) speed of light is vertical since, in this case, $\theta = \pi/2$.

As mentioned, the Galilean speeds U_θ, are known in SR under the name "proper" but the novum of this presentation relies on different [basically geometric] derivation of them and, first of all, on their Newtonian interpretation.

They do not exceed the corresponding semi-Galilean speeds of light C_θ, for any θ such that:

$$|\theta| < \pi/2.$$

Recall that, unlike the (infinite) "Galilean speed of light" $C_{\pi/2} = \infty$, we call all finite speeds of light C_θ, for $|\theta| < \pi/2$ "semi-Galilean" treating them as the slant projections of the infinite Galilean into the OB′-like slant radial lines. Physically, they are the same as speeds of the light sent ahead or backwards from the rocket that moves along that slant line with the speed U_θ.

Their magnitudes are defined by (9).

This semi-Galilean speed of light is known in SR as the time-like coordinate of four-velocity.

In this case, we always have $C_\theta > U_\theta$, and the "difference" between the two is (independently of θ) the same in the sense that we always have:

$$C_\theta^2 - U_\theta^2 = c^2 . \tag{10}$$

For more on that, see [10] or [12].

Notice too that, in SR terminology, the left hand side of (10) represents the [invariant] squared hyperbolic norm of four-velocity. In the framework presented here, (10) relates to the universality of semi-Galilean speed of light as the squared "difference" between the two speeds is always the constant c^2. Thus, (10) represents the "generalized Einstein's universality of the velocity c".

3. More on Velocities

In SR, the proper velocities [or speeds, here U_θ, and C_θ] are obtained in a quite different way than we did (geometrically), by purely analytic considerations, as the convenient convention with no explicit reference to Newtonian theory.

Recall, shortly, some basic facts to compare the ideas presented here, associated with the complex Euclidean C^3 model for SR, with the common "hyperbolic" approach to SR as related to its M^4 model.

For simplicity, we will consider one real direction motions, and thus we analyze the C^1 model in place of C^3 and M^2 (with one axis being the time axis) in place of M^4. So only one coordinate U_θ of the velocities (U_θ, 0, 0) will now be analyzed.

As the starting point for the comparison of both approaches to [the same] SR, consider any pair (c, u), where c is the usual relativistic speed of light and (u, 0, 0) any relativistic (with u bounded by the speed of light c) velocity, which in this unidimensional case reduces to its speed u.

In SR [considered as the M^4's theory], there is adopted the following hyperbolic trigonometric speed's representation $u = c\tanh\lambda$, where the "rapidity" λ is the hyperbolic angle corresponding to speed u.

By contrast, in association with our Euclidean model (here the complex plane C^1), we applied the trigonometric substitution $u = c\sin\theta$, where θ is the circular angle, the argument of a corresponding complex physical quantity (here the complex Galilean speed $U_{c,\theta}$ so that $U_\theta = |U_{c,\theta}|$ and $\theta = \arg(U_{c,\theta})$).

As it is the common procedure with the full four-dimensional M^4 development of SR, the four-velocity (C_θ, $U_{\theta 1}$, $U_{\theta 2}$, $U_{\theta 3}$) is obtained from the quadruple (c, u_1, u_2, u_3) (where the triple (u_1, u_2, u_3) denotes an ordinary relativistic velocity) by multiplying (c, u_1, u_2, u_3) by the Lorentz factor:

$$\gamma(u) = 1\Big/\sqrt{\left(1-(u/c)^2\right)} .$$

With the simplified two dimensional versions of the SR models, we have instead (for "two-speeds"):

$$\left(C_\theta, U_\theta\right) = \gamma(u)(c, u),$$

and depending on the model (M^2 or C^1) applied, we may substitute:

$$\gamma(u) = \cosh\lambda \quad \text{or} \quad \gamma(u) = \sec\theta .$$

The two representations of the same Lorentz factor must be equal, although the corresponding hyperbolic and circular angles λ and θ are not and are different kinds of mathematical objects.

The question may occur which of the two representations brings more information on the nature of the obtained velocities U_θ and C_θ.

It should be clear that the second representation as $U_\theta = c\tan\theta$ and $C_\theta = c\sec\theta$ (see (8) and (9)) reveals the Galilean (Newtonian) nature of U_θ and C_θ, while the first does not suggest anything like that.

In the first case, however, the invariance of the "two-speeds" with respect to its magnitude immediately follows as according to the (hyperbolic) squared norm's definition in M^2, we have:

$$\left\|\left(C_\theta, U_\theta\right)\right\|_h^2 = C_\theta^2 - U_\theta^2 = c^2 = \text{constant, indepently of } \theta \tag{B}$$

due to the mathematical identity $\cosh^2\lambda - \sinh^2\lambda = 1$.

In C^1, the norm has a different definition, but Equality (B) holds too, this time due to the trigonometric identity $\sec^2\theta - \tan^2\theta = 1$.

However, in the latter case, Equality (B), instead of the squared norm, expresses the universality of speed of light in its generalized form in any of the complex plane directions θ. For more on this, see [10].

In both cases the right-hand side of Equality (B) does not depend on velocity U_θ nor C_θ, but in the second ("circular") case, (B) seems to bring some additional information (some generalization of the basic axiom of SR, between others) exhibiting, in possibly new way, the association between the relativity principle [the independence from θ, *i.e.* from the speed] and the speed of light universality.

The association is then expressed both by the hyperbolic version of SR and by the speed of light universality in the second (circular) version here being under the consideration.

The latter seems to indicate existence of "logical" dependence [implication] of the universality of the speed of light on the relativity principle.

4. On Time Transformation

Return now to **Figure 1**. Recall that according to the second interpretation of this figure, a classical point-like particle moves across this complex plane. It goes along the radial line 0B′ while its observed "image" [or "shadow"] goes along the real line 0A to the right. We may assume that whenever the "complex particle" is positioned, say, at B′ its real image is at A. The <u>actual</u> [proper] distance traveled by the particle from 0 to B′ is <u>invariant</u> and equals its real counterpart OB = ||0B||. On the other hand, its speed U_θ is increased if compared with the observed relativistic speed u in the real direction so that for the speeds ratio we have $U_\theta/u = \sec\theta$, where the angle θ is equivalent to both speeds according to (5) and (8).

Denote the invariant [proper] distance $||0B|| = ||0B'||$ by d.

This distance is traveled by the real particle with the speed u whereas the same distance is traveled by the complex particle with speed U_θ. Denote the time that elapsed when the real particle changed its position from point 0 to point B on the real axis by t, and by τ the time the complex particle traveled between 0 and B′.

Then, we have:

$$d = ut = U_\theta \tau . \tag{11}$$

From (11), it follows that:

$$\tau = t\left(u/U_\theta\right) = t\cos\theta . \tag{12}$$

$$\cos\theta = \left(1 - u^2/c^2\right)^{1/2} .$$

Time t is observed by the "earth" (real) observer at rest, while the shorter [due to the Einstein's time dilation phenomenon] time τ, as measured by a "proper clock" on the rocket, is the familiar "proper time".

So, while the distances in both real and radial directions are the same, the times are different. This phenomenon relates to the proportion:

$$\tau/t = u/U_\theta . \tag{13}$$

Relation (11) indicates that within the complex Lorentz transformation, time transformation must be different than the spatial part of the transformation which is simply the rotation.

According to the complex Lorentz transformation first version (3), also see [10] time, like distance, was only subjected to a rotation in the complex time plane. Such [pure] rotation was illustrated in [10] by the following **Figure 2**.

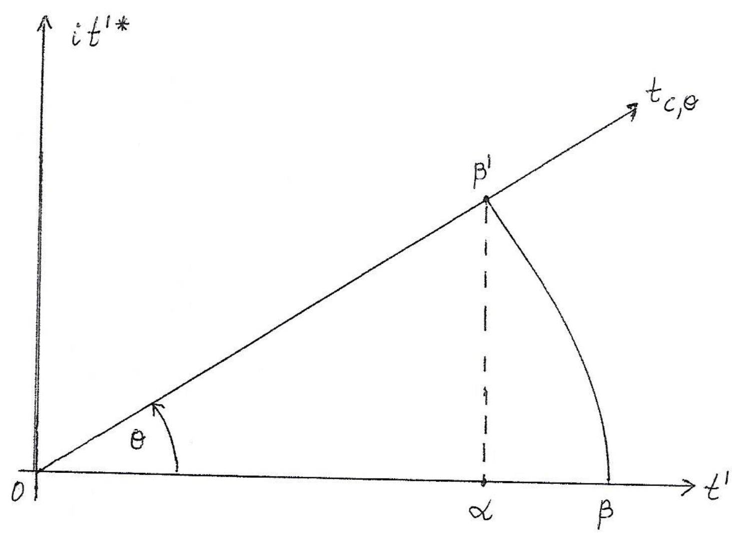

Figure 2. Rotation of time in the complex plane.

This, unfortunately, was wrong since in such a case time (actually, the absolute value of the complex time) would be invariant (as $||O\beta|| = ||O\beta||$), and then the speeds u and U_θ would be equal and this is impossible by the geometric arguments (the "geometric instantaneity") and by the inequality $||0A|| < ||0B||$). What really happens in the time plane is the composition of the rotation by the angle θ in that complex time plane, and, at the same time, dilation of the rotation's radius by the coefficient $\cos\theta$.

Graphically, this [correct] time transformation is illustrated in the following **Figure 3**, which depicts the underlying plane of the complex time.

In accordance with the above quantitative considerations, any real time epoch β (here treated as a positive real variable) is transported by the complex time transformation [the fourth row of (3**)] to the complex time epoch α' on the radial line of, say, the absolute values of time $t_\theta = |t_{c,\theta}|$.

Here, of course, the complex time satisfies $t_{c,\theta} = t_\theta \exp[i|\theta|]$.

Figure 3 is a graphical illustration of the "Lorentz complex time-transformation" which relies on the "generalized rotation" by the angle $|\theta|$, which is the composition of the ordinary rotation and radius contraction by the coefficient $\cos\theta$. This composition analytically is expressed by the product "$\cos\theta\exp[i|\theta|]$" and, finally, turns out to be just the orthogonal **projection** of the real line t' onto the radial (complex) line $t_{c,\theta}$ as determined by the speed-equivalent argument θ.

All these recognitions yield the fourth row of Formula (3**) for the second improved version of the complex Lorentz transformation. The situation expressed by **Figure 3** is described by the fourth row of (3**), under the assumption that $x_{c,\theta} = 0$. Otherwise, the vertex of the rotation (the origin, when $x_{c,\theta} = 0$) will be at the complex point $U_\theta x_{c,\theta} / C_\theta^2$ (corresponding to the, known in the literature, real point ux/c^2) on the same plane, which is a complex number as $x_{c,\theta}$ is nonzero complex and U_θ real.

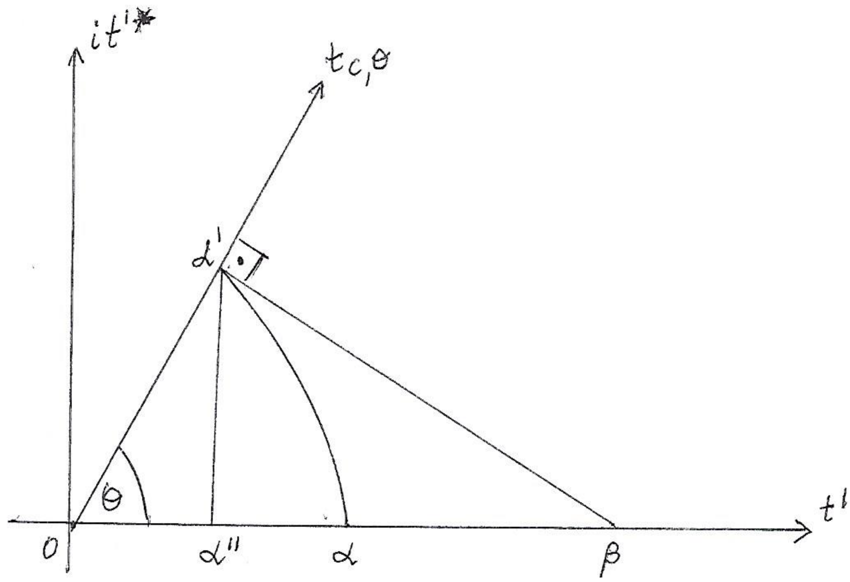

Figure 3. Time transformation in the complex plane.

So, finally, we have arrived at Formula (3**).

For more detailed analysis of these facts, see [12].

Remark 3. Derivation of the Galilean speed U_θ, as determined by its relativistic counterpart u and by (7), was based on the introduced above equivalent notions of geometric and analytic instantaneity of the particle positions A and B′ (Figure 1).

Recall, the analytical instantaneity of the "events" A and B′ relies on equality Re B′= Re A, while its equivalent geometric description relies on both points lying on the same vertical line.

No time concept for these two notions of instantaneity was applied. However, these two equivalent notions of the instantaneity stand as a premise for the (para) physical concept of the Galilean speed U_θ.

Finally, the concept of proper time τ was based on two above considered concepts: that of u and U_θ magnitudes, and the distance d invariance, with the latter statement being a geometric fact.

Everything was then based on the (Euclidean) geometry of the C^1 plane.

It is a nice fact that, after all the geometric constructions, it turns out that the introduced geometric instantaneity implies the ordinary [physical] "time instantaneity".

Namely, the time elapsed when the particle shifts it from 0 to B′ equals the proper time $\tau = t\cos\theta$, whereas, since the time of the real shifting from 0 to B is t, the time for the observed real shifting from 0 to A [with speed u] equals "$t \times ||0A||/||0B||$" = $t\cos\theta$, which is the same for both shifts (O to B′ and O to A), proper time τ.

The time instantaneity follows the equality $\tau = t\cos\theta$, and, as was expected, the geometric and the temporal concepts of instantaneity of the "events" B′ and A turned out to be equivalent. □

Remark 4. Figure 3 represents the complex plane of time which can be constructed using the model C^3 for the full SR theory. This (algebraically) three-dimensional complex model does not explicitly contain the time plane.

In this version of SR theory, time is not a primitive notion and is **not** used for defining velocities or speeds. The latter definitions were based on the **geometry** of the complex (para)-space C^3.

The geometric concepts, applied for speeds definitions (5) and (8), are the angles θ, *i.e.* the arguments of complex physical quantities such as, in this case, lengths. The quantity c (the real speed of light in vacuum) accompanying these formulas one can identify as the right angle or as the (geometric) concept of orthogonality (with the assumption that $c = 1$).

Thus, time definition [within the SR theory of C^3] can be based on the geometrically derived concept of speed. This definition actually is trivial as relying on arithmetic division of distances by speeds (especially by real c or by semi-Galilean speeds C_θ of light).

Here, recall the common fact that time can be measured in distance units.

On the other hand, the relativistic speed of light (when the convention $c = 1$ is adopted) can be characterized as $c = 1 = \sin(\pi/2)$ and all the speeds u can be represented by sines of some acute angles. Also, the corresponding Galilean speeds can be represented as tangents of the same angles whereas the semi-Galilean speeds of light as secants of these angles.

In such a way, we gain a purely geometric interpretation of speeds and consequently times.

Finally, as for the construction of the "complex time" plane, *i.e.* for the inessential extension of the C^3 SR model to the C^3 x C^1 = C^4 space-time model for the same SR theory, we proceed as follows:

1) For the real time t', we divide the real distance variable, say x' (**Figure 1**) by the real part u of the complex speed, say $U_{c,\theta} = u + iu^*$, where u^* is the imaginary part of the complex speed, and

$$\left|U_{c,\theta}\right| = \sqrt{u^2 + u^{*2}} = U_\theta$$

so that:

$$t' = x'/u. \tag{14}$$

Here, the variable x' is assumed to be any distance traveled with the speed u.

2) Suppose the point B′ on the "distances plane" (**Figure 1**) satisfies: $B' = x' + ix^*$.

Then, the imaginary time is defined as:

$$it^* = ix^*/u^*. \tag{15}$$

3) The complex time is then defined as $t' + it^*$, where t' is given by (14) and it^* by (15).

In short:

$$t' + it^* = \left(x' + ix^*\right) \backslash \left(u + iu^*\right), \tag{16}$$

where the latter "division" "\" is defined as "coordinate-wise" (real part divide by real part and imaginary by imaginary, the way it is defined in (14) and (15)) which is different from the usual arithmetic division "/" in the field of complex numbers.

In words, the imaginary time definition (15) is an answer to the question of how much [imaginary] time will elapse to cover the imaginary distance ix^* with the real speed u^*, where u^* is the imaginary part of the complex speed $U_{c,\theta} = u + iu^*$.

Notice, the above determination of complex time did not require any [earlier] primary concept of time.□

Remark 5. In the time definitions (14) and (15), speeds u and u^* were arbitrary, including as particular case, speeds of light. The distances denoted by x and x^* were the distances "traveled" by some particle with those speeds.

One can (without necessity, however) instead of any moving particle consider a photon as a particle of light, and x', ix^* as distances traveled by that photon. This convention will unify definitions (14) and (15) reducing them to the following:

$$t' = x'/c \tag{14*}$$

$$it^* = ix^*/c^* , \tag{15*}$$

where the complex speed of light has the form:

$$C_{c,\theta} = c + ic^* , \tag{17}$$

where $c^* = c\tan\theta$ and c is the ordinary real speed of light.

Notice also that Formula (17) can be rewritten into the form:

$$C_{c,\theta} = C_\theta \exp[i\theta], \tag{17*}$$

where C_θ is the [real] semi-Galilean speed of light in the radial direction OB′ (**Figure 1**).□

Resuming above, the geometric definition of the complex speeds combined with the geometric concept of distance do not require any primary time concept, and are sufficient to define the real and complex time.

This assertion possibly sheds some more light on genesis of the time concept.

The conclusion is that "physical time" relies on the phenomenon of some combination of a complex Euclidean geometry and motion as changes of a position. So that space and motion seems to be prior to time. On the other hand, it seems that the "imaginarity" of (part of) the C^3 space is a source of motion (or, more generally, of "energy").

Thus, the need for using complex space models in physics seems to be essential.

As for practical (in some wider sense) applications of introducing complex and imaginary time, besides theoretical and especially philosophical benefits, first realize that for more strictly practice we mostly use real measures of the time such as real part of complex time but even more importantly its absolute value which turns out to be the so obtained [real] proper time (interval (O a') in **Figure 3**).

Recall, this "proper time" was obtained, in classical SR [understood as the theory of Minkowski's M^4 model], in a quite different, I would say less intuitive, method. Besides, some relativistic phenomena that involve time can be seen as more transparent on a plane. As an example [Here, I must refer to my "paper" that I did not finish yet but the same results can easily be obtained eventually by those who would follow the model here presented] the famous "twin paradox" finds very clear geometric illustration on the complex planes [of time and of space]. Especially, the "what will happen" when one of the twins [the traveling one] starts to return to the Earth is pretty clearly seen as geometrical facts seen on the plane.

These, I hope, make quite useful, also from practical viewpoint, introducing the complex model and, in particular, the complex time.

5. Further Options for Reduced Complex Lorentz Transformation on C³: Time "Elimination"

The obtained by us form (3**) of the complex "Lorentz" $C^4 \to C^4$ transformation is already "correct", but, because of this new [comparing to (3)] form of the time transformation, the "spacetime interval", *i.e.* the "square" of the hyperbolic norm of quadruples (x, y, z, t) $\in C^4$ will not be preserved under (3**).

Therefore, the complex quadruples (x, y, z, t) are not four-vectors anymore.

This situation, occurring whenever the complex model C^4 is introduced, forces us to resign from the four-vector formalism and from the hyperbolic geometry [11]. Our goal then is to recover Euclidean geometry and the Euclidean metric to describe SR theory by means of some other (the C^3) complex model in which the (Euclidean) metric is preserved under a reduced complex

$C^3 \to C^3$ Lorentz transformation (as given further by (26*) and (26**).

As one will see, that procedure will not reduce the SR theory as primarily governed by (3**).

To achieve the goal, *i.e.* proper reduction of the C^4 model to C^3, and the accompanying proper reduction of the complex Lorentz transformation to a new one, we proceed with the following steps:

STEP 1. In this step, we finally will explain why we applied in (3**) the bigger Galilean and semi-Galilean speeds U_θ and C_θ, respectively, instead of the relativistic speeds u and c, where c is, say "small" speed of light.

The reason is, as it is well seen from (3**), that those "large speeds" (U_θ and C_θ) are the speeds of a considered body in the complex directions of $x_{c,\theta}$ and $t_{c,\theta}$, whereas for u and c the real directions x' and t' must be "reserved".

That is why Formula (3*), unfortunately, also was wrong, see [12].

One can consider it as an intermediate step on the way to obtain the correct Formula (3**).

An additional argument for applying the speeds U_θ and C_θ in (3**) instead of u and c is contained in the following Step 2.

STEP 2. Consider the last row of (3**).

First notice that, according to the convention $x = ct$, we have in our case:

$$\left|x_{c,\theta}\right|/c = t'. \tag{K}$$

Multiplying both sides of the convention (K) by $\cos\theta$, one obtains the equality:

$$\left|x_{c,\theta}\right|/\left(c\sec\theta\right) = t'\cos\theta, \tag{18}$$

where the product $t'\cos\theta$, is the proper time τ elapsing as measured within the rocket moving in the θ-direction on the positions' complex plane.

Recall, that $c\sec\theta = C_\theta$.

Under the assumption on time (generalized) "rotation" about the plane's origin 0 (when $U_\theta x_{c,\theta}/C_\theta^2 = 0$ for $x_{c,\theta} = 0$ in **Figure 3**), the fourth row in (3**) is equal to:

$$t_{c,\theta} = \left(t'\cos\theta\right)\exp\left[i\theta\right].$$

Getting back to "full reality", we subtract from the left-hand side of latter equality present in (3**), expression $U_\theta x_{c,\theta}/C_\theta^2$.

Realize that:

$$U_\theta x_{c,\theta}/C_\theta^2 = (U_\theta/C_\theta)(x_{c,\theta}/C_\theta) = (\sin\theta)t_{c,\theta}. \tag{19}$$

Finally, one obtains the last row of (3**) in the form:

$$t_{c,\theta}(1-\sin\theta) = (t'\cos\theta)\exp[i|\theta|], \tag{20}$$

which explicitly is "spatial part free" formula.

This means (20) is a "pure" time transformation.

Now, realize that:

$$U_\theta/C_\theta = u/c = \sin\theta,$$

so here the speeds U_θ and C_θ "work" in the same way as u and c. However, the presence of C_θ (instead of c) in the fraction $x_{c,\theta}/C_\theta$ is essential for obtaining the above time $t_{c,\theta} = x_{c,\theta}/C_\theta$ whose absolute value is contracted as the proper time. Otherwise, if to apply (3*), the fraction $x_{c,\theta}/c$ would express the noncontracted time $t' = |x_{c,\theta}/c|$ which was wrong. This is that additional argument (signalized yet at step 1) for using in (3**) speed C_θ instead of c.

This, in turn, agrees with the fact of the necessity of the presence of the coefficient $\cos\theta$ at the right-hand side of the fourth row of (3**). Recall, this multiplication by $\cos\theta$ was necessary for the existence of the Galilean and semi-Galilean speeds as well as directly for the existence of the proper time.

STEP 3. Now, look at first row of (3**). First realize that we have the relation:

$$x_{c,\theta} = t_{c,\theta}C_\theta, \tag{21}$$

but not

$$x_{c,\theta} = t_{c,\theta}c.$$

This follows from the fact that $|x_{c,\theta}|$ is not contracted while $|t_{c,\theta}|$ is, by the coefficient $\cos\theta$.

This time contraction makes the quotient $|x_{c,\theta}|/|t_{c,\theta}|$ increased by $\sec\theta$ comparing to c with the latter only be present if no time contraction took place.

Thus, the true value of the proper quotient is $c\sec\theta = C_\theta$.

Next, realize that from (21), we also have:

$$U_\theta t_{c,\theta} = (U_\theta/C_\theta)(C_\theta t_{c,\theta}) = (\sin\theta)x_{c,\theta}. \tag{22}$$

Combining (22) with first row of (3**), we obtain this row in the form:

$$x_{c,\theta}(1-\sin\theta) = x'\exp[i\theta],$$

and thus, we finally arrive at the complex Lorentz $C^4 \to C^4$ transformation (3**) (with the reduced second and third rows) in the form:

$$\begin{aligned} &x_{c,\theta}(1-\sin\theta) = x'\exp[i\theta] \\ &y = y' \\ &z = z' \\ &t_{c,\theta}(1-\sin\theta) = (t'\cos\theta)\exp[i|\theta|] \end{aligned} \tag{23}$$

□

Resuming, in the above three steps procedure, we have obtained the $C^4 \to C^4$ transformation (23) as equivalent to (3**).

Realize that transformation (23) is more uniform than (3**). So, first of all, time is not anymore explicitly present in the first row, and the space-like part is not (explicitly) present in the time transformation, so here time and space became separated!

This feature of (23) gives sufficient reason to "split" the Lorentz $C^4 \to C^4$ transformation (23) into the $C^3 \to C^3$ (para)space transformation (the first three rows of (23)) and, separately, $C^1 \to C^1$ time transformation (the fourth row), the latter being derivable from the C^3's SR theory.

Nevertheless, time is implicitly (and not directly) present in the first row. The space-like part of the transformation is implicitly present in fourth row of (23). The "common cause" of this implicit dependence is the argument θ present in both rows. Recall, the angle θ is uniquely equivalent to speeds, both relativistic and Galilean as well as to the semi-Galilean speed of light. The speed information θ allows to recover both time from the first and the space-like part of the transformation from the fourth row.

However, this new linear form (23) of the same (3**) affine transformation does not preserve the space-time interval, so both (3**) and (23) are not isometries in the hyperbolic sense. This is, roughly speaking, because space and time (first and fourth rows) transform slightly differently. Moreover, (the separate) time $C^1 \to C^1$ transformation is a similarity but not a Euclidean isometry.

To recover the idea of isometry with the complex model we are forced to reduce the C^4 model to C^3 with the Euclidean metric which, as we will see in the next section, will be preserved under the new form of the complex Lorentz $C^3 \to C^3$ transformation. This inessential reduction of the model's dimension is possible also because, for a fixed value of the speed u, the fourth row in (23) is proportional to the first with the real proportionality coefficient cosθ only depending on an arbitrary θ, which is assumed to be fixed.

To check this, divide both sides of first row of (23) by the corresponding sides of the fourth. Then we obtain:

$$x_{c,\theta}/t_{c,\theta} = x'/t' \cos\theta \,. \tag{24}$$

Since both the complex numbers $x_{c,\theta}$, $t_{c,\theta}$ have the same argument, we have for their exponential representations:

$$x_{c,\theta}/t_{c,\theta} = |x_{c,\theta}|/|t_{c,\theta}| = C_\theta$$

on the left of (24).

Here, realize that $|t_{c,\theta}|$ is the proper time, *i.e.* the time along the radial (complex) direction and the distance $|x_{c,\theta}|$ is invariant.

On the other hand, since $x'/t' = c$, we have:

$$x'/t' \cos\theta = c \sec\theta$$

on the right.

So, we arrived at the obvious identity:

$$C_\theta = c \sec\theta \,, \tag{25}$$

which only depends on [by assumption arbitrary fixed] θ.

Thus, after the division, the so obtained Formula (25) is the equality of two constants [of the proportionality] for every fixed value of θ.

Recall at this point that it is customary in SR to consider an arbitrary fixed value of the velocity.

Now, combining (23) with (25), we obtain the following Lorentz $C^3 \times R^1 \to C^3 \times R^1$ transformation:

$$\begin{aligned} x_{c,\theta}(1-\sin\theta) &= x'\exp[i\theta] \\ y &= y' \\ z &= z' \\ C_\theta &= c\sec\theta \end{aligned} \tag{26}$$

Transformation (26) is equivalent to both (23) and (3**)!

The "unnecessary" fourth row of (23) and of (3**) can be obtained back from (26) by dividing the first row of (26) by the identity (of constant real numbers) $C_\theta = c\sec\theta$, which always holds.

The latter fact inclines one to drop this identity as "obvious" [*i.e.* known always to hold] and now one obtains the FINAL VERSION of the complex Lorentz $C^3 \to C^3$ transformation:

$$\begin{aligned} x_{c,\theta}(1-\sin\theta) &= x'\exp[i\theta] \\ y &= y' \\ z &= z' \end{aligned} \tag{26*}$$

This formula, however, only describes the simplified [but very common in literature] version of the transformation under the assumption that the real part of the motion takes place exactly along the x'-axis, while the full motion is within the $x' + ix^*$ complex plane. Now, we provide an inessential (from the theory viewpoint) extension of (26*) by taking under consideration motion along a straight line in an arbitrary direction in R^3 with a velocity, say (u_x, u_y, u_z) such that $u_x^2 + u_y^2 + u_z^2 = u^2$, where u is the corresponding this velocity speed.

Now, we have:

$$\begin{aligned} u_x &= c\sin\theta_x \\ u_y &= c\sin\theta_y \\ u_z &= c\sin\theta_z \end{aligned}$$

and (26*) is replaced by its more general $C^3 \to C^3$ form:

$$\begin{aligned} x_{c,\theta_x}(1-\sin\theta_x) &= x'\exp[i\theta_x] \\ y_{c,\theta_y}(1-\sin\theta_y) &= y'\exp[i\theta_y] \\ z_{c,\theta_z}(1-\sin\theta_z) &= z'\exp[i\theta_z] \end{aligned} \tag{26**}$$

where the triple $\left(x_{c,\theta_x}, y_{c,\theta_y}, z_{c,\theta_z}\right)$ is an arbitrary point in C^3.

Notice that in this general framework the (complex) semi-Galilean light velocity is given as:

$$\left(C_{c,\theta_x}, C_{c,\theta_y}, C_{c,\theta_z}\right) = \left(C_{\theta_x}\exp[i\theta_x], C_{\theta_y}\exp[i\theta_y], C_{\theta_z}\exp[i\theta_z]\right),$$

where the real coordinates vector $\left(C_{\theta_x}, C_{\theta_y}, C_{\theta_z}\right)$ corresponds to the light speed we previously considered as C_θ (with $C_\theta^2 = C_{\theta_x}^2 + C_{\theta_y}^2 + C_{\theta_z}^2$).

Realize too that time one can always recover from any of the three rows of (26**) using one of the real speeds C_{θ_x}, C_{θ_y}, C_{θ_z} (see the fourth row of (26)).

Namely, we have:

$$x_{c,\theta_x} / C_{\theta_x} = y_{c,\theta_y} / C_{\theta_y} = z_{c,\theta_z} / C_{\theta_z} = t_{c,\theta}, \tag{R1}$$

where $t_{c,\theta}$ is the common for all rows "complex proper time", while

$$\theta = \arcsin\left(\sin^2\theta_x + \sin^2\theta_y + \sin^2\theta_z\right)^{1/2} \tag{R2}$$

under the assumption $c = 1$.

Also notice, that the eventual "fourth row", which can always be added to (26**), can be recovered, and will have the form:

$$t_{c,\theta}(1 - \sin\theta) = (t'\cos\theta)\exp\left[i|\theta|\right], \tag{R3}$$

where the complex time $t_{c,\theta}$ is given by (R1) and the common speed's argument θ is given by (R2).

As now is evident, time complex plane and time transformation (R3) on it, can always be derived from the $C^3 \to C^3$ spatial transformation (26**), but may be considered separately.

This means (26**) contains all the information about (R3) so in this sense (R3), as the fourth row, became "redundant".

On the other hand, it is obvious that transformation (26**) can always be reduced to (26*) by proper change of the coordinate system, setting the real part of the motion's direction along the x'-axis.

Hint: As above mentioned, an additional argument for the validity of the reduction of the C^4 model to C^3 is the fact that, in the equivalent version of the complex Lorentz transformation (23), space-like variables in the first three rows of (23) and time-like variables in the fourth row are explicitly separated, and, therefore, there is no necessity to consider them jointly as a $C^4 \to C^4$ transformation. Such necessity obviously exists when the space and time variables are essentially "mixed", as takes place in the case of the real transformations $M^4 \to M^4$, see (1) or (2).

So, in our framework, time can be treated as any other physical quantity such as velocity, mass, energy, temperature, and so on, *i.e.* separately from the more "basic" (para)space behavior.

6. Complex Lorentz Transformation as Euclidean Isometry

The time "elimination" (or rather "separation") that we performed above, reduced our primary (hyperbolic) model C^4 to a more proper model C^3 with the usual Euclidean metrics. Now, the isometric (in Euclidean sense) character of transformation (26**) and all the more reason of (26*) can readily be seen arguing as follows.

Consider transformation (26**) as transport from any point (x', y', z') belonging to the real subspace $R^3 \subset C^3$ (of rest states) to some point $\left(x_{c,\theta_x}, y_{c,\theta_y}, z_{c,\theta_z}\right) \in C^3$ at which the considered physical object continues moving on with the complex velocity vector $\left(U_{c,\theta_x}, U_{c,\theta_y}, U_{c,\theta_z}\right)$.

Now realize that the considered transformation is the composition of (first) the circular rotations (27), corresponding to ["quickly"] setting a particle from the point (x', y', z') into the motion:

$$\begin{aligned} x_{c,\theta_x} &= x' \exp\left[i\theta_x\right] \\ y_{c,\theta_y} &= y' \exp\left[i\theta_y\right] \\ z_{c,\theta_z} &= z' \exp\left[i\theta_z\right] \end{aligned} \tag{27}$$

and (second), after some real time t_θ elapses, the translation along the complex vector:

$$\left(x_{c,\theta_x} \sin\theta_x, y_{c,\theta_y} \sin\theta_y, z_{c,\theta_z} \sin\theta_z\right).$$

Obviously, the circular rotations, translations, and their compositions as the isometries preserve the Euclidean distance.

As it is customary, one may extend the notion of Lorentz transformation from, say, (26**) to any isometry in C^3.

Notice. At this point realize that, in the C^3 framework, the so called in SR boosts, geometrically, do not differ from other isometries (as compositions of [circular] rotations and translations only, with no reflections nor time reverse) in C^3. The kinematic character of the boosts is, in a sense, "absorbed" by the geometric notion of [circular] rotations.

Of course, finite compositions of rotations in coordinate complex planes (and not in the R^3 subspace) correspond to boosts, but, mathematically, need not necessarily be considered in terms of speeds.

One then can say that all the Lorentz (also all the Poincare) transformations (but in C^3 instead of C^4) belong to complex geometry, rather than [only] to mechanics. By this purely geometric interpretation more uniformity of the theory is gained.

In this framework, only some of the rotations [in complex planes, but not in R^3], correspond to speed changes, but, geometrically, do not differ from other [no boost] rotations. □

As for the translation, both the complex point $\left(x_{c,\theta_x}, y_{c,\theta_y}, z_{c,\theta_z}\right) \in C^3$, as determined by (27), and the origin (0, 0, 0) are shifted by the same value so their mutual distances after that shift remains the same.

It can be shown closer that transformation (26**) as well as (26*) preserve all Euclidean distances.

Thus, after the three rotations (27) (one rotation on each coordinate plane) about zero of the point:

$$\left(x', y', z'\right) \in R^3 \subset C^3 \quad \text{toward the point} \quad \left(x_{c,\theta_x}, y_{c,\theta_y}, z_{c,\theta_z}\right) \in C^3$$

as it is given by (27), we have that:

$$d\left[(x',y',z');(0,0,0)\right]=d\left[\left(x'\exp[i\theta_x],y'\exp\left[i\theta_y\right],z'\exp[i\theta_z]\right);(0,0,0)\right],$$

i.e. the distance from zero is preserved.

For this, realize that:

$$\left|x'\exp[i\theta_x]\right|=|x'|,$$

$$\left|y'\exp\left[i\theta_y\right]\right|=|y'|,$$

$$\left|z'\exp[i\theta_z]\right|=|z'|.$$

Next, realize that for the superposition of that rotation with a translation we have the equality:

$$\begin{aligned}&d\left[\left(x'\exp[i\theta_x],y'\exp\left[i\theta_y\right],z'\exp[i\theta_z]\right);(0,0,0)\right]\\&=d\left[\left(x'\exp[i\theta_x],y'\exp\left[i\theta_y\right],z'\exp[i\theta_z]\right)+\left(x_{c,\theta_x}\sin\theta_x,y_{c,\theta_y}\sin\theta_y,z_{c,\theta_z}\sin\theta_z\right);\right.\\&\left.(0,0,0)+\left(x_{c,\theta_x}\sin\theta_x,y_{c,\theta_y}\sin\theta_y,z_{c,\theta_z}\sin\theta_z\right)\right]\end{aligned}\quad(28)$$

Here, the semicolon (;) was used to separate the two variables of the metric function *d*(;).

Realize too, that according to (26**), the point (0, 0, 0) is transported by the shift to the point $\left(x_{c,\theta_x}\sin\theta_x,y_{c,\theta_y}\sin\theta_y,z_{c,\theta_z}\sin\theta_z\right)$, whereas it is fixed under the pure rotation.

Recall, that our term "rotation" means the composition of three rotations, each on a separate (complex) plane. As such, composition (27) is an isometry. Also, according to (28), (26**) is the Euclidean **isometry** as the composition of two Euclidean isometries.

This isometry was temporarily viewed as a transformation $R^3 \to C^3 - R^3$ since the points of R^3 are considered as "rest points". However, this transformation can easily be extended to a whole $C^3 \to C^3$ complex Lorentz transformation. One achieves this by substituting in (26**) the vector $\left(x_{c,\theta_x},y_{c,\theta_y},z_{c,\theta_z}\right)$ in place of (*x*', *y*', *z*'), and a new vector, say $\left(x_{c,\theta_x^*1},y_{c,\theta_y^*1},z_{c,\theta_z^*1}\right)$ in place of the previous vector $\left(x_{c,\theta_x},y_{c,\theta_y},z_{c,\theta_z}\right)$ now with some "new" angles (speeds), say θ_x^*, θ_y^*, θ_z^*.

In the C^3 case, the associated (full) Lorentz group is to be defined as the group of all <u>linear</u> isometries of C^3 which contains all the boosts, say (27). The bigger Poincare group of the affine (hence, in general, nonlinear) isometries, one obtains when adding all the translations in C^3 to the elements of the Lorentz group.

Recall again, that here all the boosts as given by (27) are just rotations. The latter recognition provides more uniformity into the (SR) theory of C^3.

Remark 6. Elimination of time in (26**) [comparing to (3**)] may give an impression that the underlying translation is somewhat not clear towards the end or artificial as dependent on the vector $\left(x_{c,\theta_x},y_{c,\theta_y},z_{c,\theta_z}\right)$ that is translated. To provide more clarification, consider (26*) as simpler and all that now can be said about (26*) can be directly extended to the more general case (26**).

It, however, becomes quite clear if we realize that the first row of transformation (26*) is equivalent to the first row of (3**) where time is explicitly present. So let me show again that for the underlying shift in (26*), we have:

$$x_{c,\theta} \sin\theta = U_\theta t_{c,\theta} \,. \tag{29}$$

As we stressed, time is always recoverable from either transformation (26*) or (26**).

Realize then that, as $\sin\theta = U_\theta / C_\theta$, we obtain from (29):

$$x_{c,\theta} \sin\theta = x_{c,\theta}\left(U_\theta / C_\theta\right) = \left(x_{c,\theta} / C_\theta\right) U_\theta \,, \tag{30}$$

which determines the other equivalent expression for the same shift.

Perhaps it would be more transparent if one rewrites (26**) into another equivalent form:

$$\begin{aligned} x_{c,\theta_x}\left(1 - U_{\theta_x} / C_{\theta_x}\right) &= x' \exp\left[i\theta_x\right] \\ y_{c,\theta_y}\left(1 - U_{\theta_y} / C_{\theta_y}\right) &= y' \exp\left[i\theta_y\right] \\ z_{c,\theta_z}\left(1 - U_{\theta_z} / C_{\theta_z}\right) &= z' \exp\left[i\theta_z\right] \end{aligned} \tag{31}$$

which is also time-free.☐

Remark 7. A second benefit of introducing the C^3 model for SR is the possibility that this model has also a chance to serve as a proper model for quantum mechanics or even for some more general quantum physics. That may set a bridge between these two theories.

For more on that, see [10] in Appendix 3.

7. Conclusions

1) As already mentioned in Section 1, it seems to be worth revising the role and use of some complex (number) mathematical models. Namely, it may turn out that, often, applying them instead of some existing real models such as, between others, R^1, R^3 or M^4 may simplify physical theories such as SR, QM and possibly others.

The only price for the possibility of extremely dramatic simplification and clarification of physical theories and problems is a need for a more literal understanding of complex physical quantities.

(See, for example, the concept of the macroscopic "imaginary mass" [measured in "i kilogram"] as described in Section 8 of [2].)

This may create ontological questions on the status of the introduced entities and epistemological problems on some new relations between mathematics and physics (or "para-physics").

On the other hand, in contemporary physics, the situation of "exponentially" growing levels of abstraction and first of all growing complexity of mathematical tools, actually, darkens the investigated physical content. This should incline one to make a "one step" ontological effort to overcome a kind of *superstition* when considering some use of complex numbers or models in place of real.

The use of mostly real number-based models [such as the real domain R^3 for the elements of the Hilbert spaces, so $L^2(R^3)$ instead of, say, $L^2(C^3)$ and Hermitian operators, instead of normal on them] may turn out not to be an actual simplification nor necessity. As often happens, "cheaper" may turn out to be more expensive.

New complex models that admit new approaches to complex physical quantities (like space, time, or mass) may bring a dramatic simplification to physics and not many actual difficulties.

2) As an example, such simplification and clarification of many key SR problems and phenomena are possible if the real Minkowski's M^4 model is replaced by the alternative C^3 complex "para-space" model.

Notice that the new theory of the C^3 para-space contains all the SR theory and possibly some extension of it as well.

It allows for an easy reach to some facts difficult to understand within SR as the theory of M^4. An example of that is an explanation of the universality of the speed of light which, within the real SR is an axiom that seems to require more clarification as being not very intuitive (see [10], Section 6).

3) Also, this paper (together with Reference [10]) seems to promise some applications to other physical areas, first of all, toward quantum mechanics.

On the other hand, it seems that, at least partially, the C^3 model may also serve as a model for classical mechanics [12] with possibly only one exception when velocity addition is of concern. Such *mechanics*, we propose to name "semi-Newtonian". It seems that the semi-Newtonian mechanics only differs from the Newtonian by the nonarithmetic addition of velocities which for "small" velocities [approximately] reduces to the arithmetic.

As we will argue in our next paper such "semi-Newtonian theory" may turn out to be the "true Newtonian" as different properties of high velocities non-arithmetic addition could simply be overlooked by eighteen and nineteenth centuries scientists (but with no actual harm to their results). The reason probably was that, for relatively small Galilean velocities, being that time the only at hand, arithmetic addition seemed to be the natural and proper operation with no experimental evidence available that it was different.

As it turns out, however, the "*true addition*" of the Galilean velocities is generated by the Lorentz-Einstein addition of relativistic velocities.

4) As a byproduct, this work may open some, possibly new, ontological questions and perspectives related to the nature and ontological status of the space, (complex) time and all the *reality* that as modeled by the nonreal interior of the C^3 seems to be transcendent to the real physical space, which, in classical theories, is usually modeled by R^3.

There emerge several possible interpretations of, say, new "transcendental" *reality* that is beyond the human senses as well as beyond the reach of physical instruments, but, nevertheless, easily accessible, directly, by human mind through some simple mathematical structures.

Among two major interpretations, the one that first comes to mind, is "dynamicity", here understood as the source of motion or of energy. In the light of the above theory: anything to move must have some imaginary part.

In turn, this interpretation may [but not necessarily] imply a spiritual [3] understanding of that reality or even personalistic, see [3]. To be careful, however, I would choose as the name for that reality "paraphysical".

Nevertheless, the ontological as well as epistemological problems seems to be open, but whatever would be an interpretation the *physics* of that reality will, basically, remain the same.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Kaiser, G. (2003) Physical Wavelets and Their Sources: Real Physics in Complex Spacetime. *Journal of Physics A*: *Mathematical and General*, **36**, R291. https://www.wavelets.com/ https://doi.org/10.1088/0305-4470/36/30/201

[2] Filus, J.K. (2015) A Complex "Time Free" Model for both Classical and Relativistic Motion. A Shorter (Slightly Modified) Version of the Paper Earlier Published in Research Gate. https://www.researchgate.net/publication/285589522_A_COMPLEX_TIME_FREE_MODEL_FOR_BOTH_CLASSICAL_AND_RELATIVISTIC_MOTION?_tp=eyJjb250ZXh0Ijp7ImZpcnN0UGFnZSI6InByb2ZpbGUiLCJwYWdlIjoicHJvZmlsZSJ9fQ

[3] Lee, S. (2019) Complex Space-Time and Complex Quantum Mind—An Unified Platform to Explain the Large, Medium, and Small Scaled Mysteries of Universe and Consciousness. *Philosophy Study*, **9**, 246-252. https://doi.org/10.17265/2159-5313/2019.05.003

[4] Amoroso, R. and Rauscher, E.A. (2018) Structure, Properties and Implications of Complex Minkowski Spaces. World Scientific, Singapore City.

[5] Amoroso, R. and Rauscher, E.A. (2009) Relativistic Physics in Complex Minkowski Space, Non-Locality, Ether Model and Quantum Physics. In: Duffy, M.C. and Levy, J., Eds., *Ether Space-Time and Cosmology* 3, *Physical Vacuum*, *Relativity and Quantum Physics*, Aperion, Montreal, 23-47.

[6] Rauscher, E. and Targ, R. (2006) Investigation of a Complex Space-Time Metric to Describe Precognition of the Future. *AIP Conference Proceedings*, **283**, 121-146. https://doi.org/10.1063/1.2388752

[7] Kaiser, G. (1990) Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis. North Holland Publishing Company, Amsterdam.

[8] Newman, E.T. (1973) Maxwell's Equations in Complex Minkowski Space. *Journal of Mathematical Physics*, **14**, 202-203. https://doi.org/10.1063/1.1666160

[9] Rauscher, E. and Targ, R. (2001) Complex Space-Time Metrics. *Journal of Scientific Exploration*, **15**, 331-354.

[10] Filus, J. (2016) Complex "Para-Space" Model for Special Relativity Theory. Scholar's Press, Saarbrucken.

[11] Naber, G. (1992) The Geometry of Minkowski Spacetime. Springer-Verlag, New York.

[12] Filus, J. (2019) Newtonian Mechanics Extension (to Arbitrary Speeds) and Its Relation to Special Relativity in Complex Model (An Improved Version). https://www.researchgate.net/publication/332849845_NEWTONIAN_MECHANICS_EXTENSION_TO_ARBITRARY_SPEEDS_AND_ITS_RELATION_TO_SPECIAL_RELATIVITY_IN_COMPLEX_MODEL_An_Improved_Version?_tp=eyJjb250ZXh0Ijp7ImZpcnN0UGFnZSI6InByb2ZpbGUiLCJwYWdlIjoicHJvZmlsZSJ9fQ