Journal of Applied Mathematics and Physics, 2024, 12, 268-276
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Dynamic of Scalar Bosons in Aharonov-Bohm Magnetic Field

Frédéric D. Y. Zongo

Ecole Normale Supérieure, Institut des Sciences et de Technologie, Ouaga, Burkina Faso
Email: zdouny@gmail.com

How to cite this paper: Zongo, F.D.Y. (2024) Dynamic of Scalar Bosons in Aharonov-Bohm Magnetic Field. *Journal of Applied Mathematics and Physics*, **12**, 268-276.
https://doi.org/10.4236/jamp.2024.121021

Received: December 9, 2023
Accepted: January 28, 2024
Published: January 31, 2024

Abstract

We study the dynamic of scalar bosons in the presence of Aharonov-Bohm magnetic field. First, we give the differential equation that governs this dynamic. Secondly, we use variational techniques to show that the following Schrödinger-Newton equation: $\begin{cases} -\Delta_{\mathcal{A}}\phi - \left(\frac{1}{|x|} * |\phi|^2\right)\phi = \lambda\phi, \\ \|\phi\|_2 = N \end{cases}$, where $\mathcal{A}$ is an Aharonov-Bohm magnetic potential, has a unique ground-state solution.

Keywords

Scalar Boson, Aharonov-Bohm Magnetic Field, Schrödinger-Newton Equation, Ground-State Solution

1. Introduction

The stability of matter and the dynamic of many-body systems in quantum mechanics have attracted many studies in the last fifty years (see [1]-[6]) and still stimulate many works today. In the realm of mathematical physics, where equations intricately describe the behavior of subatomic entities, bosons take center stage, offering a lens through which we can peer into the quantum intricacies of our universe. Amidst the array of phenomena that captivate the minds of physicists, the interplay between bosons and magnetic fields, particularly within the framework of the Aharonov-Bohm effect, beckons as a captivating arena for rigorous mathematical exploration. The Aharonov-Bohm effect, a theoretical cornerstone conceived by Aharonov and Bohm in 1959 [7], introduces a distinctive quantum perspective on the interaction of charged particles with magnetic fields. Unlike classical physics, where magnetic fields are confined to regions with non-zero field strength, the Aharonov-Bohm effect asserts that the vector potential of a magnetic field can exert a measurable impact even in regions where the magnetic field itself is absent. It is worth mentioning that a similar phenomenon has been previously described by Ehrenberg and Siday (see [8]). AB effect plays a crucial role in the development of quantum mechanics and has been experimented by Tonomura *et al.* (see [9] [10]). In the existing literature, the main research direction on scalar bosons in magnetic field is the effect of magnetic field on the decay process of scalar bosons into fermions. Indeed, many results have shown that the presence of magnetic field can influence the decay process of particles. The effect of this presence is not unanimous, some authors have shown that magnetic fields enhance the decay process of scalar bosons (see [11] [12] [13] [14]). Unlike, the previous one some authors found that magnetic field inhibits this decay process (see [15] [16]). Regarding the dynamic of scalar bosons in Aharonov-Bohm magnetic, we found few references. [17] studied the relativistic quantum motion of charged scalar particles in the presence of Aharonov-Bohm and Coulomb potentials. The relativistic frame leads them to consider the Duffin-Kemmer-Petiau (DKP) formalism. In a study by Bagrov *et al.* [18], Klein-Gordon and Dirac equations were explored in the context of an AB magnetic field. Another investigation by Castro *et al.* scrutinized the Aharonov-Bohm (AB) problem for vector bosons using the DKP formalism, as outlined in [19]. For more about relativistic scalar bosons, see references therein.

In this study, we first give and justify the model of the motion of scalar bosons in AB magnetic fields and secondly, we prove the existence of a ground-state solution, which means that the system of scalar bosons reaches its lowest possible energy level. Moreover, we show that this state admits a unique wave function. Analyzing AB problems introduces a significant mathematical challenge due to the singularities in the magnetic potential. To address this, integration techniques are employed. Another obstacle lies in the magnetic kinetic energy, which, as will be defined later, appears not to satisfy the decreasing rearrangement inequality. Consequently, the techniques developed by Lieb in [20] to prove the existence of solution cannot be applied. Furthermore, the compactness result fails for a minimizing sequence of (4.2) due to the minus sign. Variational methods are utilized to overcome these difficulties and establish the existence and uniqueness of the ground-state solution. Additionally, a convex inequality for the magnetic Schrödinger operator is demonstrated, contributing to the proof of a unique result for the standard Schrödinger equation.

2. The Model

When describing particles dynamics, two points of view can be considered. The first one is to consider the particles as point-particle (classical mechanics) and the second one, as wave function (quantum mechanics). Since, the point-like particle does not extend in space, the wave-function representation of particles is well suited to study the motion of many-body particles. Therefore, a system of many-particles will be represented by a wave function Ψ, where $\int_\Omega |\Psi|^2 \,\mathrm{d}x$ is the probability of finding this system in the region Ω. Moreover, such a wave function must satisfies the following general time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta_{\mathcal{A}}\Psi(x,t)+V_{ext}(x)\Psi(x,t)+mV_G(x,t)\Psi(x,t), \tag{2.1}$$

where V_{ext} is the external (Coulomb) potential acting on the particles and V_G is the potential energy due to the interaction inside the system, m is the mass of the system, where $\mathcal{A}$ is a magnetic potential and $\Delta_{\mathcal{A}} := (\nabla + i\mathcal{A})^2$ the magnetic laplacian.

We are interested in the dynamics of non-relativistic scalar bosons (Higgs bosons) moving in the presence of an infinitely long solenoid. Higgs bosons have been discovered at CERN in 2013 and have been presented as the missing piece of the puzzle of the understanding of our universe. Because of that some physicists have named it “God’s particle”.

In this paper, we are interested in a system of scalar bosons moving in presence of infinitely long solenoid. This solenoid gives rise to a singular magnetic potential: the Aharonov-Bohm magnetic potential. Since, scalar bosons are free charge particles, then we deduce that the Coulomb potential is identically zero ($V_{ext}\equiv 0$). The self-gravitational potential V_G coming from interaction inside the bosonic system whose mass density is given by $\int_\Omega \rho(x)\mathrm{d}x = m\int_\Omega |\Psi(x,t)|^2\mathrm{d}x$ then, satisfies the following Poisson’s equation:

$$\Delta V_G(x,t)=4\pi Gm|\Psi(x,t)|^2. \tag{2.2}$$

After integration, we get:

$$V_G(x,t)=-Gm\int\frac{|\Psi(y,t)|^2}{|x-y|}\mathrm{d}y. \tag{2.3}$$

Therefore, the dynamics of scalar bosons moving in the presence of Aharonov-Bohm magnetic field is given by:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\Delta_{\mathcal{A}}\Psi(x,t)-Gm\left(\int\frac{|\Psi(y,t)|^2}{|x-y|}\mathrm{d}y\right)\Psi(x,t), \tag{2.4}$$

Such an Equation (2.4) is called magnetic Choquard equation name after P. Choquard who presented it at a Symposium in 1976 in ETH-Lauzanne to describe one component plasma (see [20] [21]). In the literature, equation like (2.4) is also known as the Schrödinger-Newton equation.

In Quantum Mechanics, addressing the singularities arising from the AB magnetic potential is most effectively achieved by imposing a vanishing condition on the eigenfunction at these singularities. Notably, researchers [22] [23] [24] [25], in dealing with the initial Aharonov-Bohm Hamiltonian, employed the natural shielding method and opted for the Dirichlet boundary condition, wherein wave functions vanish at the solenoid. In a recent development, [26] proposed a modification of the AB Hamiltonian that is essentially self-adjoint, signifying a model with a unique self-adjoint extension. The physical interpretation of self-adjointness implies the absence of particle contact with the solenoid. For further insights into the self-adjoint extension of the AB magnetic operator, refer to [27] [28] [29] [30].

3. Preliminaries

In this paper, the symbol C denotes various positive constants whose specific values are irrelevant.

Let $\Omega = O \times \mathbb{R}$, where O is an open subset of $\mathbb{R}^2 \setminus \{0\}$. For the sake of simplicity, we will take $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 > 1\}$. The Aharonov-Bohm magnetic potential $\mathcal{A}$ is defined by: $\mathcal{A} : \Omega :\to \mathbb{R}^3$, $\mathcal{A}(x,y,z) := \frac{1}{\rho^2}(y,-x,0)$ where $\rho = \sqrt{x^2+y^2}$. The magnetic field and the magnetic potential are related by $B = \nabla \times \mathcal{A}$. Then, the magnetic field is perpendicular to the plan (Oxy) and is directed by z-axis. We will denote by $\nabla_{\mathcal{A}} := \nabla + i\mathcal{A}$.

Function spaces. For $1 < p < +\infty$, let $L^p(\mathbb{R}^3)$ be the space of real-valued functions, which are (Lebesgue) measurable and satisfy $\int_{\mathbb{R}^3} |u(x)|^p \,\mathrm{d}x < +\infty$ if $1 \le p < +\infty$ and if $p = +\infty$, $\|u\|_\infty = \inf\{c \ge 0 \mid |u(x)| \le C \text{ a.e.}\}$. We denote dx the Lebesgue measure.

For any p, the $L^p(\mathbb{R}^3)$ space is a Banach space with norm

$\|u\|_p = \left(\int_{\mathbb{R}^3} |u(x)|^p \,\mathrm{d}x\right)^{1/p}$.

In the case $p = 2$, $L^2(\mathbb{R}^3)$ is a separable Hilbert space with scalar product $\langle u,v \rangle = \int_{\mathbb{R}^3} uv\,\mathrm{d}x$ and corresponding norm $\|.\|_2$.

We define the magnetic Sobolev space

$H^1_{\mathcal{A}}(\Omega) := \{u \in L^2(\Omega,\mathbb{C}) : \nabla_{\mathcal{A}} u \in L^2(\Omega,\mathbb{C})\}$. This imply that we must consider functions having compact support in Ω. We thus define $H^1_{\mathcal{A},0}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to the norm:

$$\|u\|_{\mathcal{A}} = \left(\int_\Omega |\nabla_{\mathcal{A}} u|^2 + |u|^2 \,\mathrm{d}x\right)^{1/2}.$$

We also have that $H^1_{\mathcal{A}}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous Sobolev embedding for $p \in [2,6]$. Furthermore, $H^1_{\mathcal{A}}(\Omega) \hookrightarrow L^p(K)$ is compact for any $p \in [2,6]$ and any compact set $K \subset \Omega$.

Lemma 3.1. *(Convexity inequality for magnetic gradient) Let f and g be real-valued functions in* $H^1_{\mathcal{A}}(\mathbb{R}^3)$. *Then*:

$$\int_{\mathbb{R}^3} \left|\nabla_{\mathcal{A}} \sqrt{f^2+g^2}\right|^2 (x)\,\mathrm{d}x \le \int_{\mathbb{R}^3} \left(|\nabla_{\mathcal{A}} f|^2(x) + |\nabla_{\mathcal{A}} g|^2(x)\right)\mathrm{d}x. \tag{3.1}$$

If moreover, $g(x) > 0$ a.e. then, equality holds if and only if there exists a constant c such that $f(x) = cg(x)$ almost everywhere.

Proof. Let f and g be real-valued functions in $H^1_{\mathcal{A}}(\mathbb{R}^3 \setminus \{0\})$. We first show that $\sqrt{f^2+g^2} \in H^1_{\mathcal{A}}(\mathbb{R}^3 \setminus \{0\})$. By the diamagnetic inequality, we have

$\nabla|f|, \nabla|g| \in L^2(\mathbb{R}^3 \setminus \{0\})$. And since, f and g are real-valued functions the convexity inequality for gradient (see [31]) implies:

$$\int_{\mathbb{R}^3} \left|\nabla\sqrt{f^2+g^2}\right|^2 \mathrm{d}x \le \int_{\mathbb{R}^3} \left(|\nabla f|^2 + |\nabla g|^2\right)\mathrm{d}x,$$

thus $\nabla\sqrt{f^2+g^2} \in L^2(\mathbb{R}^3)$. It remains to show that $\mathcal{A}\sqrt{f^2+g^2} \in L^2(\mathbb{R}^3 \setminus \{0\})$ this is obvious since by definition $\mathcal{A}|f|, \mathcal{A}|g| \in L^2(\mathbb{R}^3 \setminus \{0\})$. Recalling that:

$$\nabla_{\mathcal{A}}\sqrt{f^2+g^2} = \nabla\sqrt{f^2+g^2} + i\mathcal{A}\sqrt{f^2+g^2},$$

then, by Theorem 6.17 [31], we have:

$$\nabla\sqrt{f^2+g^2} = \begin{cases} \dfrac{f(x)\nabla f(x) + g(x)\nabla g(x)}{f^2(x)+g^2(x)}, & \text{if } f^2(x)+g^2(x) \ne 0 \\ 0, & \text{otherwise.} \end{cases}$$

And then,

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla_{\mathcal{A}}\sqrt{f^2+g^2}\,\mathrm{d}x + \int_{\{f^2+g^2>0\}} \frac{|g\nabla f - f\nabla g|^2}{f^2+g^2}\mathrm{d}x \\ &= \int_{\mathbb{R}^3} |\nabla f|^2 + |\nabla g|^2\,\mathrm{d}x + \int_{\mathbb{R}^3} |\mathcal{A}|^2\left(|f|^2+|g|^2\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^3} |\nabla_{\mathcal{A}} f|^2 + |\nabla_{\mathcal{A}} g|^2\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

Therefore, (3.1) holds. Now, let assume that $g > 0$ and that equality holds in (3.1). Then, from (3.2) we deduce that:

$$g(x)\nabla f(x) = f(x)\nabla g(x) \tag{3.3}$$

a.e. in $\mathbb{R}^3$. Therefore, following the arguments of the proof of Theorem 7.8 [31], we deduce that $f(x) = cg(x)$ almost everywhere.☐

Remark 3.2. If we consider the standard magnetic Schrödinger equation namely $\Delta_{\mathcal{A}}u + Vu = 0$, where $V \in L^p(\mathbb{R}^3)$ is the electric potential. Then, by the convex inequality on may show that this equation has a unique solution (see [31]).

4. Existence of Ground-State Solution and Uniqueness of the Minimizer

When considering existence of solutions of the time-dependent Schrödinger Equation (2.4), we can seek for solutions of the form $\Psi(x,t) = \phi(x)\mathrm{e}^{-i\lambda t}$. Then, replacing Ψ by $\phi(x)\mathrm{e}^{-i\lambda t}$ in Equation (2.4) with all constant normalized we get the following stationary equation:

$$-\Delta_{\mathcal{A}}\phi - \left(\int \frac{|\phi(y)|^2}{|x-y|}\mathrm{d}y\right)\phi = \lambda\phi, \text{ in } \Omega \tag{4.1}$$

We define the energy functional $\mathcal{E}$ in $H^1_{\mathcal{A}}(\Omega)$ by:

$$\mathcal{E}_{\mathcal{A}}(\phi) = \frac{1}{2}\int_{\Omega} |\nabla_{\mathcal{A}}\phi|^2\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x. \tag{4.2}$$

Let $W(\phi) := \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |\phi|^2 \right) |\phi|^2 \mathrm{d}x$ and $T_{\mathcal{A}}(\phi) := \int_{\Omega} |\nabla_{\mathcal{A}} \phi|^2 \mathrm{d}x$.

Since $\mathcal{E}_{\mathcal{A}}$ is $C^1(H^1_{\mathcal{A}}(\mathbb{R}^3))$ finding that solutions of (4.1) are equivalent to find the solutions of the following variational problem:

$$E(N) = \min\{\mathcal{E}_{\mathcal{A}}(\phi) \mid \phi \in \mathcal{M}_N\}, \text{ where } \mathcal{M}_N = \left\{\phi \in H^1_{\mathcal{A}}(\Omega) : \|\phi\|_2^2 = N\right\}. \quad (4.3)$$

Lemma 4.1. *i)* $\mathcal{E}_{\mathcal{A}}$ *is* $C^1(H^1_{\mathcal{A}}(\Omega))$.

ii) $\mathcal{E}_{\mathcal{A}}$ is bounded from below on $\mathcal{M}_N$.

iii) $E(N) < 0$.

Proof see [32].

We are now ready to state the main result of this article.

Theorem 4.2 *Suppose that* $\Omega = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 > 1\}$. *Then,*

1) (4.3) *has a minimizer which a ground-state solution for Equation* (4.1).

2) *The minimizer* ψ_0 *satisfies* (4.3) *with* $\mathcal{E}_{\mathcal{A}}(\psi_0) = E(N)$. *Moreover,* ψ_0 *is the unique minimizer up to a constant phase and can be chosen to be strictly positive function.*

Remark 4.3. If we replace 1 in the definition of Ω by a positive constant let say $\varepsilon > 0$ we will get the same result. We only took 1 for sake of simplicity.

Proof. 1) Since $\mathcal{E}_{\mathcal{A}}$ is bounded from below, there exists a minimizing sequence $(\phi_j)_j$ of $\mathcal{E}_{\mathcal{A}}$. That is:

$$\lim_{j\to+\infty} \mathcal{E}_{\mathcal{A}}(\phi_j) = E(N) \text{ with } \| \phi_j = N \text{ for any } j. \quad (4.4)$$

Then, from Lemma 4.1, the sequence $(\phi_j)_{j\in\mathbb{N}}$ is bounded in $H^1_{\mathcal{A}}(\mathbb{R}^3)$. By the Banach-Alaoglu theorem there exist a subsequence of $(\phi_j)_{j\in\mathbb{N}}$ still denoted by $(\phi_j)_{j\in\mathbb{N}}$ which converges weakly to ϕ in $H^1_{\mathcal{A}}(\mathbb{R}^3)$.

We know from [33] that:

$$\begin{cases} \left(\frac{1}{|x|} * |\phi_j|^2\right)\phi_j \rightharpoonup \left(\frac{1}{|x|} * |\phi|^2\right)\phi \\ \mathcal{J}_{\frac{1}{|x|}}\left(|\phi_j|^2, |\phi_j|^2\right) \to \mathcal{J}_{\frac{1}{|x|}}\left(|\phi|^2, |\phi|^2\right). \end{cases} \quad (4.5)$$

Which means that the functional $W(.)$ is weakly continuous. Moreover, since $T_{\mathcal{A}}(.)$ is weakly lower semicontinuous, we deduce that the energy functional is weakly lower semicontinuous. Thus,

$$E(N) = \liminf_{j\to+\infty} \mathcal{E}_{\mathcal{A}}(\phi_j) \geq \mathcal{E}_{\mathcal{A}}(\phi) \geq E(N)$$

and this means $\mathcal{E}_{\mathcal{A}}(\phi) = E(N)$.

It remains to show that ϕ satisfies the condition $\|\phi\|_2 = N$. Since, the L^2-norm is weakly lower semi-continuous we have $N = \liminf \|\phi_j\|_2 \geq \|\phi\|_2$. Now suppose that $\|\phi\|_2 = \nu < N$ and let $\varphi = a\phi$ where $a = \frac{N}{\nu} > 1$. Then, $\|\varphi\|_2 = N$ and:

$$\begin{aligned}\mathcal{E}_{\mathcal{A}}(\varphi) &= \frac{a^2}{2}\int_{\Omega}\left|\nabla_{\mathcal{A}}\phi\right|^2 \mathrm{d}x - \frac{a^4}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|\phi|^2\right)|\phi|^2 \mathrm{d}x \\ &= a^2\left[\frac{1}{2}\int_{\Omega}\left|\nabla_{\mathcal{A}}\phi\right|^2 \mathrm{d}x - \frac{a^2}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|\phi|^2\right)|\phi|^2 \mathrm{d}x\right] \\ &\le a^2\mathcal{E}_{\mathcal{A}}(\phi).\end{aligned} \tag{4.6}$$

Therefore, $\mathcal{E}_{\mathcal{A}}(\varphi) \le a^2 E(N) < E(N)$. Absurd! Thus $\|\phi\|_2 \ge N$.

2) Let $u \in \mathcal{M}_N$. Since the AB-potential $\mathcal{A}$ is bounded, we have:

$$\begin{aligned}\mathcal{E}_{\mathcal{A}}(u) &= \frac{1}{2}\int_{\Omega}|\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{\Omega}|\mathcal{A}|^2|u|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|u|^2\right)|u|^2 \mathrm{d}x \\ &\le \frac{1}{2}\int_{\Omega}|\nabla u|^2 \mathrm{d}x + \|\mathcal{A}\|_{\infty}^2\int_{\Omega}|u|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|u|^2\right)|u|^2 \mathrm{d}x \\ &\le \frac{1}{2}\int_{\Omega}|\nabla u|^2 \mathrm{d}x + N\|\mathcal{A}\|_{\infty}^2 - \frac{1}{4}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*|u|^2\right)|u|^2 \mathrm{d}x \\ &:= \mathcal{E}(u) + N\|\mathcal{A}\|_{\infty}^2\end{aligned} \tag{4.7}$$

Suppose u and v are two minimizers of problem (4.3). Then, $|u|$ and $|v|$ belong to $H^1(\mathbb{R}^3)$ and by inequality (4.7) are minimizers of the functional $\mathcal{E}(.) + N\|\mathcal{A}\|_{\infty}^2$. And so, $|u|$ and $|v|$ minimize $\mathcal{E}(.)$. Therefore, by Theorem 10 in [20], we deduce that $u = v$.□

5. Conclusion

We have modeled the dynamic of scalar bosons in the presence of AB-magnetic field. We proved the existence and uniqueness of a ground-state solution, which means the system of scalar bosons has a unique state where it reaches its lowest possible energy level. Physically, these imply stability of the systems of scalar bosons in the presence of AB-magnetic field.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Lieb, E.H. and Seiringer, R. (2010) Stability of Matter in Quantum Mechanics. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511819681

[2] Lieb, E.H. and Yau, H.T. (1987) The Chandrasekhar Theory of Stellar Collapse as the Limit of Quantum Mechanics. *Communications in Mathematical Physics*, **112**, 147-174. https://doi.org/10.1007/BF01217684

[3] Lions, P.-L. (1987) Solutions of Hartree-Fock Equations for Coulomb Systems. *Communications in Mathematical Physics*, **109**, 33-97. https://doi.org/10.1007/BF01205672

[4] Madsen, M.S. and Liddle, A.R. (1990) The Cosmological Formation of Boson Stars. *Physics Letters B*, **251**, 507-510. https://doi.org/10.1016/0370-2693(90)90788-8

[5] Melgaard, M. and Zongo, F. (2022) Solitary Waves and Excited States for Boson Stars. *Analysis and Applications*, **20**, 285-302. https://doi.org/10.1142/S0219530521500147

[6] Sharma, R., Karmakar, S. and Mukherjee, S. (2008) Boson Star and Dark Matter. ArXiv: 0812.3470.

[7] Aharonov, Y. and Bohm, D. (1959) Significance of Electromagnetic Potentials in the Quantum Theory. *Physical Review*, **115**, 485-491. https://doi.org/10.1103/PhysRev.115.485

[8] Ehrenberg, W. and Siday, R.E. (1949) The Refractive Index in Electron Optics and the Principles of Dynamics. *Proceedings of the Physical Society*, *London*, *Section B*, **62**, 8-21. https://doi.org/10.1088/0370-1301/62/1/303

[9] Tonomura, A., *et al*. (1982) Observation of Aharonov-Bohm Effect by Electron Holography. *Physical Review Letters*, **48**, 1443-1446. https://doi.org/10.1103/PhysRevLett.48.1443

[10] Tonomura, A., *et al*. (1986) Evidence for Aharonov-Bohm Effect with Magnetic Field Completely Shielded from Electron Wave. *Physical Review Letters*, **56**, 792-795. https://doi.org/10.1103/PhysRevLett.56.792

[11] Bandyopadhyay, A. and Mallik, S. (2017) Rho Meson Decay in the Presence of a Magnetic Field. *The European Physical Journal C*, **77**, Article No. 771. https://doi.org/10.1140/epjc/s10052-017-5357-9

[12] Piccinelli, G., Jaber-Urquiza, J. and Sànchez, A. (2019) Magnetic Effect on the Decay Process of Neutral Scalar Boson to Charged Fermions. *Astronomical Notes*, **340**, 230-233. https://doi.org/10.1002/asna.201913595

[13] Piccinelli, G. and Sànchez, A. (2017) Scalar Boson Decay in Presence of Magnetic Field. *Astronomical Notes*, **338**, 1136-1141. https://doi.org/10.1002/asna.201713450

[14] Tsai, W. and Erber, T. (1974) Photon Pair Creation in Intense Magnetic Fields. *Physical Review D*, **10**, 492-499. https://doi.org/10.1103/PhysRevD.10.492

[15] Sogut, K., Yanar, H. and Havare, A. (2017) Production of Dirac Particles in External Electromagnetic Fields. *Acta Physica Polonica B*, **48**, 1493-1505. https://doi.org/10.5506/APhysPolB.48.1493

[16] Kawaguchi, M. and Matsuzaki, S. (2017) Lifetime of Rho Meson in Correlation with Magnetic-Dimensional Reduction. *The European Physical Journal A*, **53**, Article No. 68. https://doi.org/10.1140/epja/i2017-12254-1

[17] Boumali, A. and Aounallah, H. (2018) Exact Solutions of Scalar Bosons in the Presence of the Aharonov-Bohm and Coulomb Potentials in the Gravitational Field of Topological Defects. *Advances in High Energy Physics*, **2018**, Article ID: 1031763. https://doi.org/10.1155/2018/1031763

[18] Bagrov, V.G., Gitman, D.M. and Tlyachev, V.B. (2001) Solutions of Relativistic Wave Equations in Superpositions of Aharonov, Bohm Magnetic and Electric Fields. *Journal of Mathematical Physics*, **42**, 1933-1959. https://doi.org/10.1063/1.1353182

[19] Castro, L.B. and Silva, E.O. (2015) Relativistic Quantum Dynamics of Vector Bosons in Aharonov-Bohm Potential. ArXiv: 1507.07790.

[20] Lieb, E.H. (1977) Existence and Uniqueness of the Minimizing Solution of Choquard's Nonlinear Equation. *Studies in Applied Mathematics*, **57**, 93-105. https://doi.org/10.1002/sapm197757293

[21] Lions, P.-L. (1980) The Choquard Equation and Related Equations. *Nonlinear Analysis*, **4**, 1063-1073. https://doi.org/10.1016/0362-546X(80)90016-4

[22] Kretzschmar, M. (1965) Aharonov-Bohm Scattering of a Wave Packet of Finite Extension. *Zeitschrift für Physik*, **185**, 84-96. https://doi.org/10.1007/BF01381305

[23] Magni, C. and Valz-Gris, F. (1995) Can Elementary Quantum Mechanics Explain the Aharonov-Bohm Effect? *Journal of Mathematical Physics*, **36**, 177-186. https://doi.org/10.1063/1.531298

[24] Oliveira, C.R. and Pereira, M. (2008) Mathematical Justification of the Aharonov-Bohm Hamiltonian. *Journal of Statistical Physics*, **133**, 1175-1184. https://doi.org/10.1007/s10955-008-9631-y

[25] De Oliveira, C.R. and Pereira, M. (2011) Impenetrability of Aharonov-Bohm Solenoids. Proof of Norm Resolvent Convergence. *Letters in Mathematical Physics*, **95**, 41-51. https://doi.org/10.1007/s11005-010-0444-y

[26] De Oliveira, C.R. and Romano, R.G. (2017) Aharonov-Bohm Effect without Contact with the Solenoid. *Journal of Mathematical Physics*, **58**, Article ID: 102102. https://doi.org/10.1063/1.4992123

[27] Hansson, A.M. (2005) On the Spectrum and Eigenfunctions of the Schrödinger Operator with Aharonov-Bohm Magnetic Field. *International Journal of Mathematics and Mathematical Sciences*, **23**, 3751-3766. https://doi.org/10.1155/IJMMS.2005.3751

[28] Adami, R. and Teta, A. (1998) On the Aharonov-Bohm Effect. *Letters in Mathematical Physics*, **43**, 43-54. https://doi.org/10.1023/A:1007330512611

[29] Melgaard, M., Ouhabaz, E.-M. and Rozenblum, G. (2004) Negative Discrete Spectrum of Perturbed Multivortex Aharonov-Bohm Hamiltonians. *Annales Henri Poincaré*, **5**, 979-1012. https://doi.org/10.1007/s00023-004-0187-3

[30] Pankrashkin, K. and Richard, S. (2011) Spectral and Scattering Theory for the Aharonov-Bohm Operators. *Reviews in Mathematical Physics*, **23**, 53-81. https://doi.org/10.1142/S0129055X11004205

[31] Lieb, E.H. and Loss, M. (2001) Analysis. Graduate Studies in Mathematics, Vol. 14. 2nd Edition, American Mathematical Society, Providence. https://doi.org/10.1090/gsm/014

[32] Zongo, D.Y.F. (2023) Existence of Ground-State Solution for Schrödinger-Newton Equation with Aharonov-Bohm Magnetic Field. *AIP Conference Proceedings*, **2872**, Article ID: 060029. https://doi.org/10.1063/5.0163541

[33] Melgaard, M. and Zongo, F. (2012) Multiple Solutions of the Quasirelativistic Choquard Equation. *Journal of Mathematical Physics*, **53**, Article ID: 033709. https://doi.org/10.1063/1.3695991