Journal of Applied Mathematics and Physics, 2024, 12, 181-193
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Global Existence and Decay of Solutions for a Class of a Pseudo-Parabolic Equation with Singular Potential and Logarithmic Nonlocal Source

Xiaoxin Yang

School of Mathematics and Statistics, Changchun University of Science and Technology, Changchun, China
Email: yang_xiaoxin4523@163.com

How to cite this paper: Yang, X.X. (2024) Global Existence and Decay of Solutions for a Class of a Pseudo-Parabolic Equation with Singular Potential and Logarithmic Nonlocal Source. *Journal of Applied Mathematics and Physics*, **12**, 181-193.
https://doi.org/10.4236/jamp.2024.121014

Received: December 25, 2023
Accepted: January 26, 2024
Published: January 29, 2024

Abstract

This article investigates the well posedness and asymptotic behavior of Neumann initial boundary value problems for a class of pseudo-parabolic equations with singular potential and logarithmic nonlinearity. By utilizing cut-off techniques and combining with the Faedo Galerkin approximation method, local solvability was established. Based on the potential well method and Hardy Sobolev inequality, derive the global existence of the solution. In addition, we also obtained the results of decay.

Keywords

Nonlocal Parabolic Equation, Singular Potential, Logarithmic Nonlocal Source, Global Existence, Decay

1. Introduction

In this paper, we focus on the Neumann initial boundary problem:

$$\begin{cases} |x|^{-2} u_t - \Delta u - \Delta u_t = u \ln|u| - \dfrac{|x|^{-2}}{\int_\Omega |x|^{-2} \mathrm{d}x} \int_\Omega u \ln|u| \mathrm{d}x, & x \in \Omega, t > 0; \\ \dfrac{\partial u(x,t)}{\partial n} = 0, & x \in \partial\Omega, t > 0; \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^N \ (N \geq 3)$ is a bounded domain with smooth boundaries $\partial\Omega$, n is the outer normal vector of $\partial\Omega$ while $0 \neq u_0(x) \in H_* = \left\{u \in H_0^1(\Omega) : \int_\Omega |x|^{-2} u(x,t) \mathrm{d}x = 0\right\}$. $x = (x_1, x_2, \cdots, x_N) \in R^N$

with $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$. As is well known, according to the law of conservation, many diffusion processes with reactions can be described by the following equation (see [1]):

$$u_t - \nabla \cdot (D\nabla u) = f(x,t,u,\nabla u). \tag{2}$$

Among them, $u(x,t)$ represents the mass concentration in the chemical reaction process or the temperature in thermal conduction. At position x and time t in the diffusion medium, the function D is called the diffusion coefficient or thermal diffusion rate, the term $\nabla cdot(D\nabla u)$ represents the rate of change caused by diffusion, and $f(x,t,u\nabla u)$ is the rate of change caused by the reaction.

In the past few years, many researchers have paid attention to Equation (2). For source $f(x,t,u,\nabla u) = u^q$ and $D = 1$, a lot of work has been obtained. Many scholars have studied the global existence [2] [3], blow-up conditions, blow-up time estimates, and asymptotic behavior of solutions to such problems. Interested individuals can read reference materials [4] [5] [6].

Yan *et al.* [7] considered the following parabolic equation:

$$\begin{cases} u_t - \Delta u = u \ln|u| - \dfrac{1}{|\Omega|}\int_\Omega u \ln|u| \mathrm{d}x, & x \in \Omega, t > 0; \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0; \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3}$$

According to the logarithmic Sobolev inequality and energy estimation method, the results of blow-up and non-extinction of solutions under appropriate conditions are given, which generalizes some recent results.

Taking inspiration from these studies, we will consider the problem with logarithmic nonlocal sources in this paper. As far as we know, this is the first work to consider the singular parabolic Laplace equation with strong damping and logarithmic nonlocal sources. This work has great significance and can fill the research gap in this area.

The organizational structure of this article is as follows. In Section 2, we will introduce some symbols, definitions and basic lemmas that will be used in this paper. In Section 3, we present the main results of the paper, which are the local existence of weak solutions and the global existence and decay estimation of weak solutions under certain conditions.

2. Preliminaries

In this section, we will introduce some symbols and lemmas that will run through this paper. In the following text, we denote by $\|\cdot\|_r (r \ge 1)$ the norm in $L^r(\Omega)$ and by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product. First, for Problem (1), we introduce the potential energy functional:

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\|u\|_2^2 - \frac{1}{2}\int_\Omega |u|^2 \ln|u| \mathrm{d}x, \tag{4}$$

and the Nehari functional:

$$I(u)=\|\nabla u\|_2^2-\int_\Omega |u|^2\ln|u|\,\mathrm{d}x, \tag{5}$$

by a direct computation:

$$J(u)=\frac{1}{2}I(u)+\frac{1}{4}\|u\|_2^2. \tag{6}$$

By $I(u)$ and $J(u)$, we define the potential well:

$$W=\{u\in H_*:J(u)<d,I(u)>0\}\cup\{0\},$$

$$V=\{u\in H_*:J(u)<d,I(u)<0\},$$

and the Nehari manifold:

$$N=\{u\in H_0^1(\Omega)\setminus\{0\}:I(u)=0\}.$$

The depth of potential well is defined as:

$$d=\inf_{u\in N}J(u).$$

Lemma 1. [8] [9] Let μ be a positive number. Then we have the following inequalities:

$$s^p\ln s\le(e\mu)^{-1}s^{p+\mu},\quad \text{for all } s\ge 1,$$

and

$$\left|s^p\ln s\right|\le(ep)^{-1},\quad \text{for all } 0<s<1.$$

Lemma 2. [8] Let Ω is a bounded smooth region in R^N, then for any $u\in H_0^1(\Omega)$ and $a>0$, we have:

$$2\int_\Omega |u(x)|^2\ln\left(\frac{|u(x)|}{\|u\|_{L^2(\Omega)}}\right)\mathrm{d}x+n(1+\ln a)\|u\|_{L^2(\Omega)}^2\le\frac{a^2}{\pi}\int_\Omega|\nabla u|^2\,\mathrm{d}x.$$

Lemma 3. [7] [9] [10] Let $u\in H_0^1(\Omega)$. Then, $\frac{u}{|x|}\in L^2(\Omega)$ and there exists a constant $H_N=H(N,\Omega)>0$ such that:

$$\int_\Omega\frac{|u|^2}{|x|^2}\mathrm{d}x\le H_N\int_\Omega|\nabla u|^2\,\mathrm{d}x, \tag{7}$$

Lemma 4. [11] For any $u\in H_0^1(\Omega)$, we have the following inequality:

$$\|u\|_{2+\mu}^{2+\mu}\le C_G\|\nabla u\|_2^{(2+\mu)\theta}\|u\|_2^{(1-\theta)(2+\mu)}. \tag{8}$$

where $\theta=\frac{N(\mu)}{2(2+\mu)}$, $0<\mu<\frac{4}{N-2}$.

Lemma 5. [8] [11] Let $f:R^+\to R^+$ be a nonincreasing function and σ be a positive constant such that:

$$\int_t^{+\infty}f^{1+\sigma}(s)\,\mathrm{d}s\le\frac{1}{\omega}f^\sigma(0)f(t),\ \forall t\ge 0.$$

Then, we have:

1) $f(t)\le f(0)\mathrm{e}^{1-\omega t}$, for all $t\ge 0$, whenever $\sigma=0$.

2) $f(t) \le f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, for all $t \ge 0$, whenever $\sigma > 0$.

Lemma 6. Assume that $u \in H_0^1(\Omega)\setminus\{0\}$, then:

1) $\lim_{\lambda\to 0^+} J(\lambda u) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.

2) There exists a unique $\lambda^* = \lambda^*(u) > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u)\Big|_{\lambda=\lambda^*} = 0$.

3) $J(\lambda u)$ is increasing on $0 < \lambda < \lambda^*$, decreasing on $\lambda^* < \lambda < +\infty$, and attains the maximum at $\lambda = \lambda^*$.

4) $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$, and $I(\lambda^* u) = 0$.

The following is the definition of weak solution for Problem (1). To avoid confusion, we also write $u(x,t)$ as $u(t)$.

Definition 7. [7] [12] (Weak solution) A function $u = u(x,t) \in L^\infty(0,T;H_*)$ with:

$$u_t \in L^2\left(0,T;H_0^1(\Omega)\right), \int_0^T \left\|\frac{u_t}{|x|}\right\|_2^2 \mathrm{d}t < \infty,$$

is called a weak solution of Problem (1) on $\Omega\times[0,T)$ if $u(x,0) = u_0(x)$ in H_* and:

$$\left\langle \frac{u_t}{|x|^2}, w \right\rangle + \langle \nabla u, \nabla w\rangle + \langle \nabla u_t, \nabla w\rangle$$
$$= \langle u\ln|u|, w\rangle - \left\langle \frac{|x|^{-s}}{\int_\Omega |x|^{-s}\,\mathrm{d}x}\int_\Omega u\ln|u|\,\mathrm{d}x, w \right\rangle, \quad \text{a.e. } t \in 0,T.$$

for any $v \in H_*$.

3. Main Results

In this section, we present two theorems. Firstly, we present the local existence and uniqueness theorems for weak solutions to Problem (1). Next, we present the existence theorem for the global weak solution of Problem (1), and also provide an estimate of the exponential decay of the solution in the theorem.

Theorem 8. Let $u_0 \in H_*\setminus\{0\}$. Then, there exist a $T > 0$ and a unique weak solution $u(x,t) \in L^\infty(0,T;H_*)$ of (1) with:

$$u_t \in L^2\left(0,T;H_0^1(\Omega)\right), \int_0^T \left\|\frac{u_t}{|x|}\right\|_2^2 \mathrm{d}t < \infty,$$

satisfying $u(0) = u_0$. Moreover, $u(x,t)$ satisfies the energy equality:

$$\int_0^t \left(\left\|\frac{u_t}{|x|}\right\|_2^2 + \|\nabla u_t\|_2^2\right)\mathrm{d}t + J(u) = J(u_0), \quad 0 \le t \le T.$$

Proof. We divide the proof of Theorem 8 into 3 steps.

Step 1. Local existence

To deal with the influence of singular potentials, we introduce a cut-off function:

$$\rho_n(x) = \min\left\{|x|^{-2}, n\right\}, \forall n \in N^+.$$

We denote the solutions corresponding to ρ_n of Problem (1) as u_n. We can know that:

$$0 \neq u_{n0}(x) \in \tilde{H}_* = \left\{u_n \in H_0^1(\Omega): \int_\Omega \rho_n(x) u_n(x,t)\mathrm{d}x = 0\right\},$$

where $H_* = \tilde{H}_*$ as $n \to \infty$. Let $\{\omega_j\}_{j=1}^{\infty}$ be a linear independent basis in $\tilde{H}_*$ and construct the approximate solution:

$$u_n^k(x,t) = \sum_{j=1}^{k} a_{nj}^k(t)\omega_j(x) \quad \text{for } k = 1,2,\cdots;\ j = 1,2,\cdots,k,$$

solving the problem

$$\begin{aligned} &\left\langle \rho_n(x) u_{nt}^k, \omega_j \right\rangle + \left\langle \nabla u_n^k, \nabla \omega_j \right\rangle + \left\langle \nabla u_{nt}^k, \nabla \omega_j \right\rangle \\ &= \left\langle \left|u_n^k\right|^{q-2} u_n^k \ln\left|u_n^k\right|, \omega_j \right\rangle - \left\langle \frac{\rho_n(x)}{\int_\Omega \rho_n(x)\mathrm{d}x} \int_\Omega u_n^k \ln\left|u_n^k\right| \mathrm{d}x, \omega_j \right\rangle, \end{aligned} \tag{9}$$

and

$$u_n^k(x,0) = \sum_{j=1}^{k} b_{nj}^k \omega_j(x) = u_{n0}^k \to u_0(x) \text{ in } H_* \tag{10}$$

as $k \to +\infty, n \to +\infty$. Noticing that $\omega_j(x) \in \tilde{H}_*$, it is not hard to verify for any fixed j:

$$\begin{aligned} \int_\Omega \rho_n(x) u_n^k(t)\mathrm{d}x &= \int_\Omega \rho_n(x) \sum_{j=1}^{k} a_{nj}^k(t)\omega_j(x)\mathrm{d}x \\ &= \sum_{j=1}^{k} a_{nj}^k(t) \int_\Omega \rho_n(x)\omega_j(x)\mathrm{d}x = 0. \end{aligned}$$

From above equality, we know that $\left\{a_{nj}^k\right\}_{j=1}^{k}$ is determined by the following Cauchy problem:

$$\begin{cases} \sum_{j=1}^{k} \left(\int_\Omega \rho_n(x)\omega_j(x)\omega_j \mathrm{d}x\right)\left[a_{nj}^k(t)\right]_t + \lambda_j\left[a_{nj}^k(t)\right]_t = G_{nj}^k(t), \\ a_{nj}^k(0) = b_{nj}^k, \end{cases}$$

where

$$\begin{aligned} G_{nj}^k(t) = &\int_\Omega \sum_{j=1}^{k} a_{nj}^k(t)\omega_j(x) \ln\left|\sum_{j=1}^{k} a_{nj}^k(t)\omega_j(x)\right| \omega_j \mathrm{d}x \\ &- \int_\Omega \sum_{j=1}^{k} a_{nj}^k(t)\nabla\omega_j(x)\nabla\omega_j \mathrm{d}x. \end{aligned}$$

The standard theory of ODE states that there exists a $T > 0$ such that $a_{nj}^k(t) \in C^1([0,T])$.

Multiply (9) by $a_{nj}^k(t)$, sum for $j = 1,\cdots,k$ and recall $u_n^k(x,t)$ to find:

$$\left\langle \rho_n(x) u_{nt}^k, u_n \right\rangle + \left\langle \nabla u_n^k, \nabla u_n^k \right\rangle + \left\langle \nabla u_{nt}^k, \nabla u_n^k \right\rangle$$
$$= \left\langle u_n^k \ln\left|u_n^k\right|, u_n^k \right\rangle - \left\langle \frac{\rho_n}{\int_\Omega \rho_n \mathrm{d}x} \int_\Omega u_n^k \ln\left|u_n^k\right| \mathrm{d}x, u_n^k \right\rangle, \quad \text{a.e. } t \in (0,T). \tag{11}$$

Integrating both sides of (11) in $\Omega \times [0,t]$, we get:

$$S_n^k(t) \le S_n^k(0) + \int_0^t \int_\Omega \left|u_n^k(t)\right|^2 \ln\left|u_n^k(t)\right| \mathrm{d}x\mathrm{d}s, \tag{12}$$

where

$$S_n^k(t) = \frac{1}{2}\left\|(\rho_n(x))^{\frac{1}{2}} u_n^k(t)\right\|_2^2 + \frac{1}{2}\left\|\nabla u_n^k(t)\right\|_2^2 + \int_0^t \left\|\nabla u_n^k(s)\right\|_2^2 \mathrm{d}s. \tag{13}$$

From Lemma 1, we get:

$$\begin{aligned}
&\int_\Omega \left|u_n^k(t)\right|^2 \ln\left|u_n^k(t)\right| \mathrm{d}x \\
&= \int_{\Omega_1 = \left\{x\in\Omega; \left|u_n^k(x)\right| \ge 1\right\}} \left|u_n^k(t)\right|^2 \ln\left|u_n^k(t)\right| \mathrm{d}x \\
&\quad + \int_{\Omega_2 = \left\{x\in\Omega; \left|u_n^k(x)\right| < 1\right\}} \left|u_n^k(t)\right|^2 \ln\left|u_n^k(t)\right| \mathrm{d}x \\
&\le (e\mu)^{-1} \int_{\Omega_1 = \left\{x\in\Omega; \left|u_n^k(x)\right| \ge 1\right\}} \left|u_n(t)\right|^{2+\mu} \mathrm{d}x \\
&\le (e\mu)^{-1} \left\|u_n^k(t)\right\|_{2+\mu}^{2+\mu}.
\end{aligned} \tag{14}$$

Let $0 < \mu < \frac{4}{N}$, then from (14), Lemma 2 and Young's inequality, we obtain:

$$\begin{aligned}
&\int_\Omega \left|u_n^k(t)\right|^2 \ln\left|u_n^k(t)\right| \mathrm{d}x \\
&\le (e\mu)^{-1} \left\|u_n^k(t)\right\|_{2+\mu}^{2+\mu} \\
&\le (e\mu)^{-1} C_G \left\|\nabla u_n^k(t)\right\|_2^{\theta(2+\mu)} \left\|u_n^k(t)\right\|_2^{(1-\theta)(2+\mu)} \\
&\le (e\mu)^{-1} C_G \varepsilon \left\|\nabla u_n^k(t)\right\|_2^2 + (e\mu)^{-1} C_G C(\varepsilon) \left\|u_n^k(t)\right\|_2^{\frac{2(1-\theta)(2+\mu)}{2-\theta(2+\mu)}} \\
&\le (e\mu)^{-1} C_G \varepsilon \left\|\nabla u_n^k(t)\right\|_2^2 + (e\mu)^{-1} C_G C(\varepsilon) B_1 \left\|\nabla u_n^k(t)\right\|_2^{\frac{2(1-\theta)(2+\mu)}{2-\theta(2+\mu)}}.
\end{aligned} \tag{15}$$

where $\varepsilon \in (0,1)$, and $\theta = \left(\frac{1}{2} - \frac{1}{2+\mu}\right) N = \frac{\mu N}{(2+\mu)2}$, B_1 is the best embedding constant. We note that since $0 < \mu < \frac{4}{N}$, $\theta(2+\mu) < 2$ holds. Let

$$\alpha = \frac{2(1-\theta)(2+\mu)}{2\left[2-\theta(2+\mu)\right]} = \frac{2(N+2+\mu) - N(2+\mu)}{2(N+2) - N(2+\mu)},$$

then $\alpha > 1$ since $0 < \mu < \frac{4}{N}$.

From (12), (13) and (15), we get:

$$S_n^k(t) \le S_n^k(0) + \int_0^t (e\mu)^{-1} C_G \varepsilon \left\| \nabla u_n^k(t) \right\|_2^2 \mathrm{ds}$$
$$+ \int_0^t (e\mu)^{-1} C_G C(\varepsilon) B_1 \left\| \nabla u_n^k(t) \right\|_2^{2\alpha} \mathrm{ds}$$
$$\le S_n^k(0) + (e\mu)^{-1} C_G \varepsilon S_n^k(t) + (e\mu)^{-1} C_G C(\varepsilon) B_1 \int_0^t \left(S_n^k(t)\right)^\alpha \mathrm{ds},$$

that is

$$S_n^k(t) \le C_1 + C_2 \int_0^t \left(S_n^k(t)\right)^\alpha \mathrm{ds}, \tag{16}$$

where $1-(e\mu)^{-1} C_G \varepsilon > 0$, $C_1 = \frac{S_n^k(0)}{1-(e\mu)^{-1} C_G \varepsilon}$, $C_2 = \frac{(e\mu)^{-1} C_G C(\varepsilon) B_1}{1-(e\mu)^{-1} C_G \varepsilon}$.

From Gronwall inequality, we obtain:

$$S_n^k(t) \le C_3, \tag{17}$$

where C_3 is a constant which dependent on T.

Multiplying (11) by $\left[a_{nj}^k(t)\right]_t$, summing on $j = 1, 2, \cdots, k$ and then integrating on $(0,t)$, we know that, for all $0 \le t \le T$, we have:

$$\int_0^t \left(\left\| (\rho_n(x))^{\frac{1}{2}} u_{nt}^k(t) \right\|_2^2 + \left\| \nabla u_{nt}^k(t) \right\|_2^2 \right) \mathrm{ds} + J\left(u_n^k(t)\right) = J\left(u_{n0}^k\right). \tag{18}$$

By the continuity of the functional J and (10), there exists a constant $C > 0$ satisfying:

$$J\left(u_{n0}^k\right) \le C \text{, for any positive integer } n \text{ and } k. \tag{19}$$

Applying (4), (13), (15), (17), (18) and (19), we obtain:

$$\left(\frac{1}{2} - \frac{C_G \varepsilon}{e\mu 2}\right) \left\| \nabla u_n^k(t) \right\|_2^2 + \frac{1}{4} \left\| u_n^k(t) \right\|_2^2 - C_4 \le J\left(u_n^k(t)\right) \le C, \tag{20}$$

where $C_4 = \frac{2 C_G C(\varepsilon) B_1}{e\mu} (C_3)^\alpha$. From (18) and (20), for any $n, k \in N^+$, it follows that:

$$\left\| u_n^k(t) \right\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \le C, \tag{21}$$

$$\left\| u_n^k(t) \right\|_{L^\infty\left(0,T;L^2(\Omega)\right)} \le C, \tag{22}$$

$$\left\| u_{nt}^k(t) \right\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \le C, \tag{23}$$

$$\left\| (\rho_n(x))^{\frac{1}{2}} u_{nt}^k(t) \right\|_{L^2\left(0,T;L^2(\Omega)\right)} \le C, \tag{24}$$

By (22), (23) and Aubin-Lions-Simon Lemma, we get:

$$u_n^k \to u \text{ in } C\left(0,T;L^2(\Omega)\right). \tag{25}$$

Combining (10) with that $u_n^k(x,0) \to u(x,0)$ in $L^2(\Omega)$, we observe that $u(x,0) = u_0$ in H_*.

By (25), we have $u_n^k \to u$, a.e. $(x,t) \in \Omega \times (0,T)$. That means:

$$u_n^k \ln\left|u_n^k\right| \to u\ln|u| \text{ a.e.} (x,t) \in \Omega\times(0,T).$$

That is to say, there is:

$$\begin{aligned}&\int_\Omega \left|u_n^k \ln|u_n|\right|^2 \mathrm{d}x \\ &= \int_{\Omega_1=\left\{x\in\Omega;\left|u_n^k(x)\right|\ge 1\right\}} \left|u_n^k \ln\left|u_n^k\right|\right|^2 \mathrm{d}x + \int_{\Omega_1=\left\{x\in\Omega;\left|u_n^k(x)\right|<1\right\}} \left|u_n^k \ln\left|u_n^k\right|\right|^2 \mathrm{d}x\end{aligned}$$

From Lemma 1 and Lemma 2, we get:

$$\begin{aligned}&\int_\Omega \left|u_n^k(t)\ln\left|u_n^k(t)\right|\right|^2 \mathrm{d}x \\ &= \int_{\Omega_1} \left|u_n^k(t)\ln\left|u_n^k(t)\right|\right|^2 \mathrm{d}x + \int_{\Omega_2} \left|u_n^k(t)\ln\left|u_n^k(t)\right|\right|^2 \mathrm{d}x \\ &\le \int_{\Omega_1} \left|\left|u_n^k(t)\right|^{-\mu} \ln\left|u_n^k(t)\right| \cdot \left|u_n^k(t)\right|^{1+\mu}\right|^2 \mathrm{d}x + \int_{\Omega_2} \left|\left|u_n^k(t)\right|\ln\left|u_n^k(t)\right|\right|^2 \mathrm{d}x \\ &\le (e\mu)^{-2}\left\|u_n^k(t)\right\|_{2(1+\mu)}^{2(1+\mu)} + e^{-2}|\Omega| \\ &\le (e\mu)^{-2} B_2 \left\|\nabla u_n(t)\right\|_2^{2(1+\mu)} + e^{-2}|\Omega| < C,\end{aligned}$$

where B_2 is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2(1+\mu)}(\Omega)$. Here, we choose $0<\mu\le\frac{2}{N-2}$, we know that:

$$\left\|u_n^k(t)\ln\left|u_n^k(t)\right|\right\|_{L^\infty\left(0,T;L^2(\Omega)\right)} \le C \text{, for any positive integer } n \text{ and } k. \tag{26}$$

According to the Holder inequality, we obtain:

$$\begin{aligned}&\int_\Omega \left(\frac{\rho_n(x)}{\int_\Omega \rho_n(x)\mathrm{d}x}\int_\Omega u_n^k \ln\left|u_n^k\right|\mathrm{d}x\right)^2 \mathrm{d}x \\ &\le \frac{n^2}{\left(\min\left\{R^s, R^{-s}, n\right\}\right)^2|\Omega|}\int_\Omega |\Omega|^{\frac{1}{2}}\left\|u_n^k \ln\left|u_n^k\right|\right\|_2^2 \mathrm{d}x \le C,\end{aligned} \tag{27}$$

where $|x|<R$.

By (21)-(24), (26), and (27), there exist functions u and a subsequence of $\left\{u_n^k\right\}_{k,n=1}^\infty$, which we still denote it by $\left\{u_n^k\right\}_{k,n=1}^\infty$ such that:

$$u_n^k \to u \quad \text{weakly star in } L^\infty\left([0,T];H_0^1(\Omega)\right) \tag{28}$$

$$u_{nt}^k \to u_t \quad \text{weakly in } L^2\left([0,T];H_0^1(\Omega)\right) \tag{29}$$

$$\left(\rho_n(x)\right)^{\frac{1}{2}} u_{nt}^k \to \frac{u_t}{|x|} \quad \text{weakly in } L^2\left([0,T];L^2(\Omega)\right) \tag{30}$$

$$u_n^k \ln\left|u_n^k\right| \to u\ln|u| \quad \text{weakly star in } L^\infty\left([0,T];L^2(\Omega)\right) \tag{31}$$

By (28)-(31), passing to the limit in (9), (10) as $k,n\to+\infty$, it follows that u satisfies the initial condition $u(0)=u_0$:

$$\left\langle \frac{u_t}{|x|^2},\varphi\right\rangle + \langle\nabla u,\nabla\varphi\rangle + \langle\nabla u_t,\nabla\varphi\rangle = \langle u\ln|u|,\varphi\rangle - \left\langle \frac{|x|^{-s}}{\int_\Omega |x|^{-s}\mathrm{d}x}\int_\Omega u\ln|u|\mathrm{d}x,\varphi\right\rangle, \tag{32}$$

for all $\varphi \in H_*$.

Step 2. Energy equality

Multiplying u_t at both ends of Problem (1), integrating from 0 to t and combining (4), we have:

$$\int_0^t \left(\left\| \frac{u_t(t)}{|x|} \right\|_2^2 + \|\nabla u_t(t)\|_2^2 \right) \mathrm{d}s + J(u(t)) = J(u_0), \quad 0 \le t \le T.$$

Step 3. Uniqueness

Assuming u_1 and u_2 are two solutions to Problem (1), we have:

$$\begin{aligned} &\left\langle \frac{u_{1t}}{|x|^2}, v \right\rangle + \langle \nabla u_1, \nabla v \rangle + \langle \nabla u_{1t}, \nabla v \rangle \\ &= \langle u_1 \ln|u_1|, v \rangle - \left\langle \frac{|x|^{-2}}{\int_\Omega |x|^{-2} \mathrm{d}x} \int_\Omega u_1 \ln|u_1| \mathrm{d}x, v \right\rangle, \end{aligned} \tag{33}$$

and

$$\begin{aligned} &\left\langle \frac{u_{2t}}{|x|^2}, v \right\rangle + \langle \nabla u_2, \nabla v \rangle + \langle \nabla u_{2t}, \nabla v \rangle \\ &= \langle u_2 \ln|u_2|, v \rangle - \left\langle \frac{|x|^{-2}}{\int_\Omega |x|^{-2} \mathrm{d}x} \int_\Omega u_2 \ln|u_2| \mathrm{d}x, v \right\rangle. \end{aligned} \tag{34}$$

Let $w = u_1 - u_2$ and $w(0) = 0$, then by subtracting (33) and (34), we can derive:

$$\begin{aligned} &\int_\Omega |x|^{-2} w_t v \mathrm{d}x + \int_\Omega \nabla w \nabla v \mathrm{d}x + \int_\Omega \nabla w_t \nabla v \mathrm{d}x \\ &= \int_\Omega (u_1 \ln|u_1| - u_2 \ln|u_2|) v \mathrm{d}x - \int_\Omega \frac{|x|^{-2}}{\int_\Omega |x|^{-2} \mathrm{d}x} \left(\int_\Omega (u_1 \ln|u_1| - u_2 \ln|u_2|) \mathrm{d}x \right) v \mathrm{d}x. \end{aligned}$$

Let $v = w$, we obtain:

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \frac{w}{|x|} \right\|_2^2 + \|\nabla w\|_2^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla w\|_2^2 \\ &= \int_\Omega \frac{u_1 \ln|u_1| - u_2 \ln|u_2|}{w} w^2 \mathrm{d}x \le \int_\Omega \frac{f(u_1) - f(u_2)}{w} w^2 \mathrm{d}x \end{aligned}$$

Integrating it on $[0, t]$, we obtain:

$$\|\nabla w\|_2^2 \le 2 \int_0^t \int_\Omega \frac{f(u_1) - f(u_2)}{w} w^2 \mathrm{d}x \mathrm{d}t. \tag{35}$$

where $f(s) = s \ln|s|$ and $f: R^n \to R$ satisfy locally Lipschitz continuity. That means:

$$\|\nabla w\|_2^2 \le 2 M_T \int_0^t \|\nabla w\|_2^2 \mathrm{d}t.$$

By Gronwall's inequality, we have $\|\nabla w\|_2^2 = 0$.

The proof of Theorem 8 is complete.□

Theorem 9. Assume that $J(u_0) \le d$ and $I(u_0) > 0$, then Problem (1) admits a global solution $u \in L^\infty(0,\infty;H_*)$, $u_t \in L^2(0,\infty;H_0^1(\Omega))$ with $\frac{u_t}{|x|} \in L^2(0,\infty;L^2(\Omega))$, and $u(t) \in W$ for $0 \le t \le \infty$. Moreover, if $u_0 \in W$, then

$$\|\nabla u(t)\|_2^2 \le \|\nabla u_0\|_2^2 e^{1-\frac{c_1}{c_2}t},\ t \ge 0,$$

where $c_1 = 1 - \frac{a^2}{2\pi}$, $c_2 = \frac{1}{2}H_N + \frac{1}{2}$.

Proof. Now, we prove Theorem 9. In order to prove the existence of global weak solutions, we consider two following cases.

1) Global existence

Case 1. The initial data $J(u_0) < d$ and $I(u_0) < 0$.

Taking a weak solution $u \in L^\infty(0,T_{\max};H_*)$, which satisfies:

$$\int_0^t \left(\left\| \frac{u_t(s)}{|x|} \right\|_2^2 + \|\nabla u_t(s)\|_2^2 \right) ds + J(u(t)) = J(u_0), \quad 0 \le t \le T_{\max}. \tag{36}$$

Among them, $T_{\max}$ is the maximum existence time of the solution $u(t)$. We need to prove that $T_{\max} = +\infty$. Thanks to $J(u_0) < d$ and (36), we obtain:

$$\int_0^t \left(\left\| \frac{u_t(s)}{|x|} \right\|_2^2 + \|\nabla u_t(s)\|_2^2 \right) ds + J(u(t)) < d, \quad 0 \le t \le T_{\max}. \tag{37}$$

We will assert that:

$$u(t) \in W \text{ for all } 0 \le t \le T_{\max}. \tag{38}$$

In fact, using the method of proof to the contrary, assuming that (38) does not hold, let t_* is the minimum time for $u(t_*) \notin W$. So, considering the continuity of $u(t)$, it can be inferred that there is $u(t_*) \in \partial W$. The following conclusion can be drawn:

$$J(u(t_*)) = d, \tag{39}$$

and

$$I(u(t_*)) = 0 \text{ with } u(t_*) \ne 0. \tag{40}$$

It is evident that (39) could not occur by (37) while if (40) holds then, by the definition of *d*, we have:

$$J(u(t_*)) \ge \inf_{u \in N} J(u) = d,$$

which also contradicts with (37). As a consequence, it follows from this fact and the definition of functional *J* that:

$$\int_0^t \left(\left\| \frac{u_t(s)}{|x|} \right\|_2^2 + \|\nabla u_t(s)\|_2^2 \right) ds + \frac{1}{2} I(u(t)) + \frac{1}{4}\|u(t)\|_2^2 < d, \tag{41}$$

namely,

$$\frac{1}{4}\|u(t)\|_2^2 < d. \tag{42}$$

From Lemma 2, we have:

$$\int_\Omega |u(x)|^2 \ln|u(x)|\,\mathrm{d}x \le \frac{a^2}{2\pi}\|\nabla u\|_2^2 - \frac{n}{2}(1+\ln a)\|u\|_2^2 + \|u\|_2^2 \ln\|u\|_2^2, \tag{43}$$

Combining above inequality, (36) and (42), we obtain:

$$\begin{aligned}&\int_0^t \left(\left\|\frac{u_t(s)}{|x|}\right\|_2^2 + \|\nabla u_t(s)\|_2^2\right)\mathrm{d}t + \left(\frac{1}{2} - \frac{a^2}{2\pi}\right)\|\nabla u\|_2^2 + \left(\frac{1}{4} + \frac{n}{2}(1+\ln a)\right)\|u\|_2^2 \\ &< d + 4d\ln 4d = C_d\end{aligned} \tag{44}$$

This estimate allows us to take $T_{\max} = +\infty$. Hence, we can conclude that there exists a unique global weak solution $u(t) \in W$ of Problem (1), which satisfies that:

$$\int_0^t \left(\left\|\frac{u_t(s)}{|x|}\right\|_2^2 + \|\nabla u_t(s)\|_2^2\right)\mathrm{d}s + J(u(t)) = J(u_0), \quad 0 \le t \le +\infty.$$

Case 2. The initial data $J(u_0) = d$ and $I(u_0) > 0$.

Firstly, we choose a sequence $\{\theta_m\}_{m=1}^{\infty} \subset (0,1)$ such that $\lim_{m\to\infty} \theta_m = 1$. Then, we consider the following problem:

$$\begin{cases} \dfrac{u_t}{|x|^2} - \Delta u - \Delta u_t = u\ln|u| - \dfrac{|x|^{-2}}{\int_\Omega |x|^{-2}\,\mathrm{d}x}\displaystyle\int_\Omega u\ln|u|\,\mathrm{d}x, & (x,t) \in \Omega \times R^+, \\ u = 0, & (x,t) \in \partial\Omega \times R^+, \\ u(x,0) = u_{0m}(x), & x \in \Omega, \end{cases} \tag{45}$$

where $u_{0m} = \theta_m u_0$.

Due to $I(u_0) > 0$, it can be inferred from Lemma 3 that $\lambda^* > 1$. Therefore, we obtain $I(u_{0m}) = I(\theta_m u_0) > 0$ and $J(u_{0m}) = J(\theta_m u_0) < J(u_0) = d$. Use arguments similar to Case 1. We found that Problem (45) allows for global weak solutions *u*.

The remainder of the proof can be processed similarly as Case 1.

2) Decay estimates

We are now in a position to prove the algebraic decay results. Thanks to $u_0 \in W$, and Lemma 3, we get $u(t) \in W$. Fro (5), (38) and (40), we have:

$$\begin{aligned} I(u) &= \|\nabla u\|_2^2 - \int_\Omega |u|^2 \ln|u|\,\mathrm{d}x \\ &\ge \|\nabla u\|_2^2 - \frac{a^2}{2\pi}\|\nabla u\|_2^2 + \frac{n}{2}(1+\ln a)\|u\|_2^2 - \|u\|_2^2\ln\|u\|_2^2 \\ &\ge \left(1 - \frac{a^2}{2\pi}\right)\|\nabla u\|_2^2 + \left(\frac{n}{2}(1+\ln a) - \ln\|u\|_2^2\right)\|u\|_2^2 \\ &\ge \left(1 - \frac{a^2}{2\pi}\right)\|\nabla u\|_2^2 + \left(\frac{n}{2}(1+\ln a) - \ln 4d\right)\|u\|_2^2 \\ &\ge \left(1 - \frac{a^2}{2\pi}\right)\|\nabla u\|_2^2 = c_1\|\nabla u(t)\|_2^2. \end{aligned} \tag{46}$$

Combining with the first equality of Problems (1), (5) and Lemma 3, we obtain:

$$\begin{aligned}\int_t^T I(u)\,\mathrm{d}s &= \int_t^T \left(\|\nabla u\|_2^2 - \int_\Omega |u|^2 \ln|u|\,\mathrm{d}x\right)\mathrm{d}s \\ &= -\frac{1}{2}\int_t^T \left(\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{u}{|x|}\right\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)\mathrm{d}s \\ &= \frac{1}{2}\left(\left\|\frac{u(t)}{|x|}\right\|_2^2 + \|\nabla u(t)\|_2^2\right) - \frac{1}{2}\left(\left\|\frac{u(T)}{|x|}\right\|_2^2 + \|\nabla u(T)\|_2^2\right) \\ &\le \frac{1}{2}\left(\left\|\frac{u(t)}{|x|}\right\|_2^2 + \|\nabla u(t)\|_2^2\right) \\ &\le \left(\frac{1}{2}H_N + \frac{1}{2}\right)\|\nabla u(t)\|_2^2 = c_2\|\nabla u(t)\|_2^2,\end{aligned} \tag{47}$$

where $c_2 = \frac{1}{2}H_N + \frac{1}{2}$.

By (46) and (47), we get:

$$\int_t^T \|\nabla u(t)\|_2^2\,\mathrm{d}s \le \frac{c_2}{c_1}\|\nabla u(t)\|_2^2,\ \forall t \in [0,T].$$

Let $T \to +\infty$ in above inequality, by Lemma 5, it follows that:

$$\|\nabla u(t)\|_2^2 \le \|\nabla u_0\|_2^2\,\mathrm{e}^{1-\frac{c_1}{c_2}t},\ t \ge 0,$$

The proof of Theorem 9 is complete.☐

Acknowledgements

Sincere thanks to the professional performance of JAMP members, and special thanks to the editors for their dedication to this article.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Pao, C.V. (2012) Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York.

[2] Hu, B. (2011) Blow-Up Theories for Semilinear Parabolic Equations. Springer, Berlin. https://doi.org/10.1007/978-3-642-18460-4

[3] Levine, H.A. (1990) The Role of Critical Exponents in Blowup Theorems. *SIAM Review*, **32**, 262-288. https://doi.org/10.1137/1032046

[4] Galaktionov, V.A. and Vazquez, J.-L. (2002) The Problem of Blow-Up in Nonlinear Parabolic Equations. *Discrete and Continuous Dynamical Systems*, **8**, 399-434. https://doi.org/10.3934/dcds.2002.8.399

[5] El-Gamel, M., El-Baghdady, G.I. and El-Hady, M.A. (2022) Highly Efficient Method for Solving Parabolic PDE with Nonlocal Boundary Conditions. *Applied Mathemat-*

ics, **13**, 101-119. https://doi.org/10.4236/am.2022.132009

[6] Sheng, L.X, Hu, W.M., Su, Y.H. and Yun, Y.Z. (2023) Existence of Solutions for a Non-Autonomous Evolution Equations with Nonlocal Conditions. *Journal of Applied Mathematics and Physics*, **11**, 4079-4091. https://doi.org/10.4236/jamp.2023.1112260

[7] Yan, L.J. and Yang, Z.D. (2018) Blow-Up and Non-Extinction for a Nonlocal Parabolic Equation with Logarithmic Nonlinearity. *Boundary Value Problems*, **2018**, Article No. 121. https://doi.org/10.1186/s13661-018-1042-7

[8] Le, C.N. and Le, X.T. (2017) Global Solution and Blow-Up for a Class of Pseudo-Laplacian Evolution Equations with Logarithmic Nonlinearity. *Computers and Mathematics with Applications*, **151**, 149-169. https://doi.org/10.1007/s10440-017-0106-5

[9] Hardy, G.H. and Littlewood, J.E. and Polya, G. (1934) Inequalities (Cambridge Mathematical Library). Subsequent Edition, Cambridge University Press, Cambridge.

[10] Do, T.D., Trong, N.N. and Thanh, B.L.T. (2023) On a Higher-Order Reaction-Diffusion Equation with a Special Medium Void via Potential Well Method. *Taiwanese Journal of Mathematics*, **27**, 53-79. https://doi.org/10.11650/tjm/220703

[11] Han, Y.Z. (2018) A New Blow-Up Criterion for Non-Newton Filtration Equations with Special Medium Void. *Rocky Mountain Journal of Mathematics*, **48**, 2489-2501. https://doi.org/10.1216/RMJ-2018-48-8-2489

[12] Tan, Z. (2002) The Reaction-Diffusion Equation with Lewis Function and Critical Sobolev Exponent. *Journal of Mathematical Analysis and Applications*, **272**, 480-495. https://doi.org/10.1016/S0022-247X(02)00166-X