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The Alternating Group Explicit Iterative Method for the Regularized Long-Wave Equation

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Abstract

An Alternating Group Explicit (AGE) iterative method with intrinsic parallelism is constructed based on an implicit scheme for the Regularized Long-Wave (RLW) equation. The method can be used for the iteration solution of a general tridiagonal system of equations with diagonal dominance. It is not only easy to implement, but also can directly carry out parallel computation. Convergence results are obtained by analysing the linear system. Numerical experiments show that the theory is accurate and the scheme is valid and reliable.

Keywords

RLW Equation, AGE Iterative Method, Parallelism, Convergence

1. Introduction

The Regularized Long-Wave (RLW) equation:

$$u_t + u_x + uu_x - \mu u_{xxt} = f(x,t), \tag{1}$$

where $\mu > 0$.

It is a different explanation of nonlinear dispersive waves compared to the famous Korteweg-de Vries (KdV) equation, which has the form:

$$u_t + u_x + uu_x + u_{xxx} = f(x,t). \tag{2}$$

In the study of physical phenomena such as water wave and plasma wave propagation, the RLW equation is regarded as the modified model of the KdV equation. Compared with the KdV equation, the RLW equation has better mathematical properties and has been widely studied.

The RLW equation was first proposed by Peregrine [1] [2] in 1966, which can be as a representative form of nonlinear long wave to describe the behavior of wave-like surging tide. It has been shown to have solitary wave solutions. RLW equation can also describe wave motion to the same order of approximation as the KdV equation, so it plays an important role in the study of nonlinear dispersive waves [3]. In recent years, some numerical methods for (1) have been studied in [4]-[12], because it is a very important equation in many applications. However, parallel algorithms for RLW equation are few.

With the rapid development of the high-performance computing in large-scale scientific and engineering computations, the parallel difference methods for partial differential equations have been studied rapidly. Evans [12] proposed an Alternating Group Explicit (AGE) scheme for solving diffusion equations by Saul'yev [13] asymmetric scheme. Furthermore, the Alternating Segment Explicit-Implicit (ASE-I) scheme [14] and Alternating Segment Crank-Nicolson (ASC-N) scheme [15] were designed. In the past two decades, the alternating algorithms have become one of the effective methods to solve parabolic equations, such as the ASC-N scheme for solving convection-diffusion equations [16], alternating difference scheme for dispersion equations [17] [18], and the fourth-order parabolic equation [19] [20]. Afterward, the alternating technology became a very effective method for some parabolic equations, for instance, the ASC-N and ASE-I scheme of Burgers' equation [21] [22] and so on.

Zhang and Liang [23] proposed a local one-dimensional ASE-I scheme and ASC-N scheme for two-dimensional parabolic equations. It reduces high-dimensional problems into local one-dimensional calculations to improve the computational efficiency. However, it can be seen that more alternating methods with parallelism are proposed for linear partial differential equations. Because the RLW equations contain nonlinear terms, alternating methods cannot be directly applied to RLW equations. The AGE iterative method based on the implicit scheme needs to be constructed, which has some theoretical significance and practical values.

The remainder of this paper is arranged as follows. In Section 2, we construct an AGE iterative method for RLW equation. The idea is that the computing process is designed as a number of small-size independent linear systems, which can be computed independently. In Section 3, we give the analysis of convergence, which is the main content of this paper. Finally, some numerical experiments show the effectiveness of our theoretical results.

2. The AGE Iterative Method

Consider the following initial-boundary value problem for the RLW equation:

$$u_t + u_x + uu_x - u_{xxt} = 0,\ (x,t) \in [x_L, x_R] \times [0,T], \tag{3}$$

$$u(x_L,t) = u(x_R,t) = 0,\ t \in [0,T], \tag{4}$$

$$u(x,0) = u_0(x),\ x \in [x_L, x_R]. \tag{5}$$

where $u_0(x)$ is the known initial function.

Firstly, the computational region $x_j = jh(0 \le j \le J)$ is meshed as follows. Let h and t be the space step and time step. Denote $x_j = jh(0 \le j \le J)$, $t_n = n\tau(0 \le n \le N)$, $u_j^n \approx u(x_j, t_n)$, where $u(x,t)$ represents the exact solution, u_j^n represents the numerical solution.

The implicit scheme of the RLW Equation (3) is given as follows:

$$\begin{aligned}&\frac{1}{\tau}\left(u_j^{n+1}-u_j^n\right)+\frac{1}{4h}\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}+u_{j+1}^n-u_{j-1}^n\right)+\frac{a_j^n}{4h}\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}+u_{j+1}^n-u_{j-1}^n\right)\\&=\frac{1}{\tau h^2}\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}-u_{j+1}^n+2u_j^n-u_{j-1}^n\right),\end{aligned} \tag{6}$$

i.e.

$$\begin{aligned}&(r_1-r_3)u_{j-1}^{n+1}+2r_2u_j^{n+1}+(r_1+r_3)u_{j+1}^{n+1}\\&=(r_1+r_3)u_{j-1}^n+2r_2u_j^n+(r_1-r_3)u_{j+1}^n,\end{aligned} \tag{7}$$

where $r_1 = -2$, $r_2 = 2 + h^2$, $r_3 = \frac{\tau h}{2}\left(1 + a_j^n\right)$, $a_j^n = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)$.

The linear system of the scheme (7) is as following:

$$AU = F, \tag{8}$$

where

$$A=\begin{bmatrix}2r_2 & r_1+r_3 & & & \\ r_1-r_3 & 2r_2 & r_1+r_3 & & \\ & \ddots & \ddots & \ddots & \\ & & r_1-r_3 & 2r_2 & r_1+r_3\\ & & & r_1-r_3 & 2r_2\end{bmatrix}_{(J-1)\times(J-1)},$$

and $U=\left[u_1^{n+1},u_2^{n+1},\cdots,u_{J-1}^{n+1}\right]^{\mathrm{T}}$, $F=\left[f_1^n, f_2^n,\cdots, f_{J-1}^n\right]^{\mathrm{T}}$,

$$f_1^n=(r_1+r_3)u_0^n+2r_2u_1^n+(r_1-r_3)\left(u_2^n-u_0^{n+1}\right),$$

$$f_j^n=(r_1+r_3)u_{j-1}^n+2r_2u_j^n+(r_1-r_3)u_{j+1}^n,\ j=2,3,\cdots,J-2,$$

$$f_{J-1}^n=(r_1+r_3)\left(u_{J-2}^n-u_J^{n+1}\right)+2r_2u_{J-1}^n+(r_1-r_3)u_J^n,\ u_0^{n+1}=u_J^{n+1}=0,\quad n=0,1,2,\cdots.$$

Split the matrix A, we have:

$$A = G_1 + G_2, \tag{9}$$

where

$$G_1=\begin{bmatrix}r_2 & & & & \\ & P & & & \\ & & P & & \\ & & & \ddots & \\ & & & & P\end{bmatrix}_{(J-1)\times(J-1)},\quad G_2=\begin{bmatrix}P & & & & \\ & P & & & \\ & & \ddots & & \\ & & & P & \\ & & & & r_2\end{bmatrix}_{(J-1)\times(J-1)},$$

and the block submatrix is:

$$P=\begin{bmatrix}r_2 & r_1+r_3\\ r_1-r_3 & r_2\end{bmatrix}.$$

Then, the AGE iterative method is constructed as follows:

$$\begin{cases} \left(I+G_1\right)U^{\left(k+\frac{1}{2}\right)} = \left(I-G_2\right)U^{(k)} + F, \\ \left(I+G_2\right)U^{(k+1)} = \left(I-G_1\right)U^{\left(k+\frac{1}{2}\right)} + F, \end{cases} \quad k = 0,1,2,\cdots. \tag{10}$$

Remark 1. The AGE iteration method (10) is a linear system, and the coefficient matrix is quasi-diagonal matrix. This matrix can be divided into several sub-block linear equations systems and calculated independently. Therefore, scheme (10) can do parallel processing calculations.

Remark 2. Obviously, we obtained that G_1 and G_2 are strictly dominance matrices, *i.e.*

$$r_2 - r_1 + r_3 = 4 + h^2 + \frac{\tau h}{2}\left(1 + a_j^n\right) > 0, \tag{11}$$

$$r_2 - r_1 - r_3 = 4 + h^2 - \frac{\tau h}{2}\left(1 + a_j^n\right) > 0. \tag{12}$$

By Gershgorin circle theorem, it follows that G_1 and G_2 are positive definite matrices.

3. The Analysis of the Convergence

In this section, we will discuss the convergence of the AGE iterative method (10). The proof of convergence relies on the following Kellogg lemmas [24].

Lemma 1. If $\gamma > 0$ and $C + C^{\mathrm{T}}$ is nonnegative definite, then $\left(\gamma I + C\right)^{-1}$ exists and

$$\left\|\left(\gamma I + C\right)^{-1}\right\|_2 \le \gamma^{-1}.$$

Lemma 2. Under the conditions of Lemma 1, there is:

$$\left\|\left(\gamma I - C\right)\left(\gamma I + C\right)^{-1}\right\|_2 \le 1.$$

Theorem 1. The AGE iterative method (10) is convergent.

Proof. By eliminating $U^{\left(l+\frac{1}{2}\right)}\left(l = 1,2,\cdots,k\right)$ from (10), we can obtain:

$$U^{(k+1)} = TU^{(k)} + D_k = \cdots = T^{k+1}U^{(0)} + D_0, \tag{13}$$

where $T = \left(I+G_2\right)^{-1}\left(I-G_1\right)\left(I+G_1\right)^{-1}\left(I-G_2\right)$.

Let

$$\hat{T} = \left(I+G_2\right)T\left(I+G_2\right)^{-1} = \left(I-G_1\right)\left(I+G_1\right)^{-1}\left(I-G_2\right)\left(I+G_2\right)^{-1}. \tag{14}$$

From Remark 2, it is easily obtain that the matrices G_1 and G_2 are nonnegative real matrices.

Therefore, we can obtain the following inequalities from the above Kellogg lemmas:

$$\begin{aligned} \rho(T) \le \|T\|_2 = \left\|\hat{T}\right\|_2 &\le \left\|\left(I-G_1\right)\left(I+G_1\right)^{-1}\right\|_2 \left\|\left(I-G_2\right)\left(I+G_2\right)^{-1}\right\|_2 \\ &= \max_{1\le i\le m}\left|\frac{1-\alpha_i}{1+\alpha_i}\right| \cdot \max_{1\le i\le m}\left|\frac{1-\beta_i}{1+\beta_i}\right| < 1, \end{aligned} \tag{15}$$

then

$$\rho\left(T^k\right) \le \left\|T^k\right\|_2 \le \left(\left\|T\right\|_2\right)^k < 1, \tag{16}$$

where $\alpha_i, \beta_i > 0$ are the eigenvalues of the positive definite matrices G_1 and G_2, respectively. We complete the proof.

4. Numerical Experiments

In order to demonstrate the effectiveness and applicability of the proposed AGE iterative method (10) in this paper, we consider the following example:

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{17}$$

$$u\left(x_L, t\right) = u\left(x_R, t\right) = 0, \quad t \in [0, T], \tag{18}$$

$$u(x, 0) = u_0(x), \quad x \in \left[x_L, x_R\right], \tag{19}$$

where

$$-x_L = x_R = 50, \quad u_0(x) = \operatorname{sech}^2\left(\frac{x}{4}\right). \tag{20}$$

The single solitary wave solution of (17)-(19) is:

$$u(x,t) = A\operatorname{sech}^2(kx - wt), \tag{21}$$

where

$$A = \frac{3a^2}{1-a^2}, \quad k = \frac{a}{2}, \quad w = \frac{a}{2\left(1-a^2\right)}, \tag{22}$$

where a is arbitrary constant. This example takes $a = \frac{1}{2}$.

Taking $\tau = 1/10000$ and $h = 1/80$, we compare the errors among scheme (10), C-N scheme and other two algorithms in [6] and [7] under L2 norm. It can be shown in **Table 1**. More nodes are selected by scheme (10), compared to other ones.

In **Table 2**, the space step size h is selected from 1/400 to 1/1600, while the time step size is taken as $\tau = 1/100$. It displays that the CPU time decreases by parallel calculation, when the linear system is divided into K subsystem at $t = 10$. Since the general tridiagonal systems of algorithms in [6] and [7] are not available in parallel computers, the CPU time costs in the calculation are much longer than our scheme (10) in **Table 2**.

Table 1. The comparison of different methods with respect to L2 error, where h = 1/80.

	Scheme (10)	Shao [7]	C-N scheme	Cai [6]
t = 0.2	1.575e−5	0.00056	0.00070	0.00053
t = 0.4	2.625e−5	0.00085	0.03331	0.00113
t = 0.6	6.728e−5	0.00112	0.06337	0.00175
t = 0.8	1.925e−4	0.00141	0.08433	0.00237

Table 2. The comparison of three schemes calculation time.

	K	Scheme (10)	Cai [6]	Shao [7]
$h = 1/100$	2	9.5615 s	16.2625 s	15.3531 s
$h = 1/400$	8	5.0932 s	33.4592 s	32.0037 s
$h = 1/800$	16	5.0946 s	59.3013 s	56.1752 s
$h = 1/1600$	32	5.5882 s	145.3313 s	132.8453 s

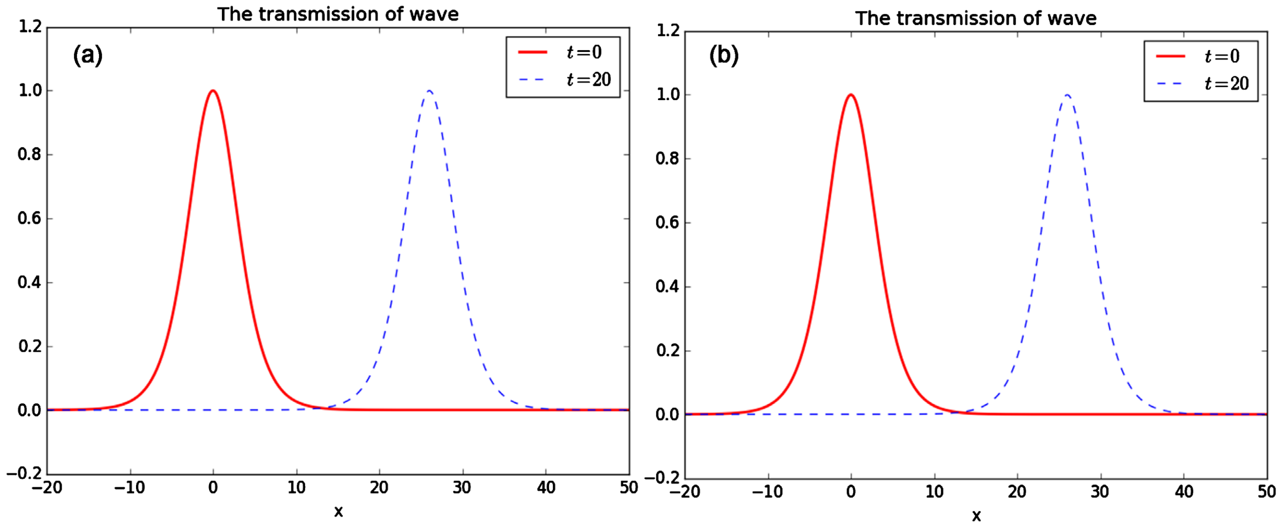

Figure 1. Simulation of single solitary wave transmission: (a) $h = 0.1$; (b) $h = 0.2$.

Since the scheme (10) is designed based on the implicit scheme (7), it obviously obtains the second-order spatial accuracy, *i.e.* $O(\tau + h^2)$. From Table 1 and Table 2, it illustrates that scheme (10) can not only perform parallel computation, but also get higher accuracy than the other three methods.

It is shown that the transmission process of single solitary wave from $t = 0$ to $t = 20$ with a spatial step size of $h = 0.1$ and $h = 0.2$ in Figure 1. We obtain that scheme (10) does not cause wave attenuation with an increase in step size.

The numerical example results above present that the AGE iterative method (10) proposed in this paper has good computational accuracy and parallel efficiency with the same mesh division. Additionally, it can effectively simulate the transmission of single solitary wave, which demonstrates the applicability of our algorithm.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Peregrine, D.H. (1966) Calculations of the Development of an Unduiar Bore. *Journal of Fluid Mechanics*, **25**, 321-330. https://doi.org/10.1017/S0022112066001678

[2] Peregrine, D.H. (1967) Long Waves on Beach. *Journal of Fluid Mechanics*, **27**, 815-825. https://doi.org/10.1017/S0022112067002605

[3] Bona, J.L., Pritchard, W.G. and Scott, L.R. (1985) Numerical Schemes for a Model of Nonlinear Dispersive Waves. *Journal of Computational Physics*, **60**, 167-196. https://doi.org/10.1016/0021-9991(85)90001-4

[4] Xu, J., Xie, S. and Fu, H. (2022) A Two-Grid Block-Centered Finite Difference Method for the Nonlinear Regularized Long Wave Equation. *Applied Numerical Mathematics*, **171**, 128-148. https://doi.org/10.1016/j.apnum.2021.08.008

[5] Wang, B., Sun, T. and Liang, D. (2019) The Conservative and Fourth-Order Compact Finite Difference Schemes for Regularized Long Wave Equation. *Journal of Computational and Applied Mathematics*, **356**, 98-117. https://doi.org/10.1016/j.cam.2019.01.036

[6] Cai, J. (2009) Multisymplectic Numerical Method for the Regularized Long-Wave Equation. *Computer Physics Communications*, **180**, 1821-1831. https://doi.org/10.1016/j.cpc.2009.05.009

[7] Shao, X., Xue, G. and Li, C. (2013) A Conservative Weighted Finite Difference Scheme for Regularized Long Wave Equation. *Applied Mathematics and Computation*, **219**, 9202-9209. https://doi.org/10.1016/j.amc.2013.03.068

[8] Wang, H., Li, S. and Wang, J. (2017) A Conservative Weighted Finite Difference Scheme for the Generalized Rosenau-RLW Equation. *Computational and Applied Mathematics*, **36**, 63-78. https://doi.org/10.1007/s40314-015-0214-7

[9] Gao, J., He, S., Bai, Q. and Liu, J. (2023) A Time Two-Mesh Finite Difference Numerical Scheme for the Symmetric Regularized Long Wave Equation. *Fractal and Fractional*, **7**, Article 487. https://doi.org/10.3390/fractalfract7060487

[10] Yimnet, S., Wongsaijai, B., Rojsiraphisal, T. and Poochinapan, K. (2016) Numerical Implementation for Solving the Symmetric Regularized Long Wave Equation. *Applied Mathematics and Computation*, **273**, 809-825. https://doi.org/10.1016/j.amc.2015.09.069

[11] Hossain, S., Roshid, M., Uddin, M., Ripa, A.A. and Roshid, H.O. (2023) Abundant Time-Wavering Solutions of a Modified Regularized Long Wave Model Using the EMSE Technique. *Partial Differential Equations in Applied Mathematics*, **8**, Article ID: 100551. https://doi.org/10.1016/j.padiff.2023.100551

[12] Evans, D.J. (1985) Alternating Group Explicit Method for the Diffusion Equation. *Applied Mathematical Modelling*, **9**, 201-206. https://doi.org/10.1016/0307-904X(85)90008-3

[13] Saul'yev, V.K. (1964) Integration of Equations of Parabolic Type by Method of Nets. The Macmillan Co., New York, 281-304.

[14] Zhang, B. (1991) Alternating Segment Explicit-Implicit Method for the Diffusion Equation. *International Journal of Numerical Methods and Applications*, **12**, 245-251.

[15] Zhang, B. and Li, W. (1994) On Alternating Segment Crank-Nicolson Scheme. *Parallel Computing*, **20**, 897-902. https://doi.org/10.1016/0167-8191(94)90070-1

[16] Wang, W. (2004) A Class of Alternating Segment Crank-Nicolson Methods for Solving Convection-Diffusion Equations. *Computing*, **73**, 41-55. https://doi.org/10.1007/s00607-004-0077-z

[17] Zhang, Q. and Wang, W. (2009) A Four-Order Alternating Segment Crank-Nicolson

Scheme for the Dispersive Equation. *Computers & Mathematics with Applications*, **57**, 283-289. https://doi.org/10.1016/j.camwa.2008.10.080

[18] Guo, G., Lü, S. and Liu, B. (2015) Unconditional Stability of Alternating Difference Schemes with Variable Time Steplengthes for Dispersive Equation. *Applied Mathematics and Computation*, **262**, 249-259. https://doi.org/10.1016/j.amc.2015.04.037

[19] Guo, G. and Liu, B. (2013) Unconditional Stability of Alternating Difference Schemes with Intrinsic Parallelism for the Fourth-Order Parabolic Equation. *Applied Mathematics and Computation*, **219**, 7319-7328. https://doi.org/10.1016/j.amc.2012.12.010

[20] Guo, G. and Lü, S. (2016) Unconditional Stability of Alternating Difference Schemes with Intrinsic Parallelism for Two-Dimensional Fourth-Order Diffusion Equation. *Computers & Mathematics with Applications*, **71**, 1944-1959. https://doi.org/10.1016/j.camwa.2016.03.004

[21] Wang, W. and Lu, T. (2005) The Alternating Segment Difference Scheme for Burgers' Equation. *International Journal for Numerical Methods in Fluids*, **49**, 1347-1358. https://doi.org/10.1002/fld.1028

[22] Xue, G. and Feng, H. (2020) An Alternating Segment Explicit-Implicit Scheme with Intrinsic Parallelism for Burgers' Equation. *Journal of Computational and Theoretical Transport*, **49**, 15-30. https://doi.org/10.1080/23324309.2019.1709081

[23] Zhang, S.H. and Liang, D. (2015) Locally One-Dimensional Alternating Segment Explicit-Implicit and Locally One-Dimensional Alternating Segment Crank-Nicolson Methods for Two-Dimension Parabolic Equations. *International Journal of Computer Mathematics*, **92**, 513-531. https://doi.org/10.1080/00207160.2014.895336

[24] Kellogg, R.B. (1964) An Alternating Direction Method for Operator Equations. *Journal of the Society for Industrial and Applied Mathematics*, **12**, 848-854. https://doi.org/10.1137/0112072