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Boundary Conditions for Momentum and Vorticity at an Interface between Two Fluids

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Abstract

Boundary conditions for momentum and vorticity have been precisely derived, paying attention to the physical meaning of each mathematical expression of terms rigorously obtained from the basic equations: Navier-Stokes equation and the equation of vorticity transport. It has been shown first that a contribution of fluid molecules crossing over a conceptual surface moving with fluid velocity due to their fluctuating motion is essentially important to understanding transport phenomena of momentum and vorticity. A notion of surface layers, which are thin layers at both sides of an interface, has been introduced next to elucidate the transporting mechanism of momentum and vorticity from one phase to the other at an interface through which no fluid molecules are crossing over. A fact that a size of δV, in which reliable values of density, momentum, and velocity of fluid are respectively defined as a volume-averaged mass of fluid molecules, a volume-averaged momentum of fluid molecules and a mass-averaged velocity of fluid molecules, is not infinitesimal but finite has been one of the key factors leading to the boundary conditions for vorticity at an interface between two fluids. The most distinguished characteristics of the boundary conditions derived here are the zero-value conditions for a normal component of momentum flux and tangential components of vorticity flux, at an interface.

Keywords

Boundary Condition for Vorticity, Surface Layer, Interface, Momentum Flux, Vorticity Flux

1. Introduction

It is believed in a world of present fluid dynamics that the values of vorticity fluxes at both sides of an interface between two fluids are generally different from each other due to the baroclinic generation of vorticity at an interface where the value of density suddenly changes in an infinitesimal thickness, and that the value of baroclinic generation is a matter of *a posteriori* to be obtained after the flow field is determined otherwise [1]. Hence, the boundary condition for vorticity has been considered to provide no information in analyzing a flow field. This seems to be true for the conventional vorticity defined as a rotation of a velocity vector. In fact, many researchers have treated subjects relating to boundary conditions for vorticity since Thom's paper [2] appeared in 1933, aiming at utilization in numerical analysis based on the equation of vorticity transport. However, proposed boundary conditions for vorticity were devised to give the value of vorticity at an interface fitting to the existing velocity field, and were not capable of affecting the flow field by nature.

The main purpose of this work is to precisely derive the boundary condition for the vorticity newly defined as a rotation of a momentum vector, paying attention to physical meanings of mathematical expressions of terms rigorously obtained from basic equations: Navier-Stokes equation and the equation of vorticity transport. The boundary conditions for vorticity obtained in this work essentially affect the flow field and work as new boundary conditions other than those for velocity and force at an interface between two fluids. The basic equations are described in terms of density, ρ, and a vector of velocity, $\boldsymbol{u}$. We start our investigation by pointing out that the value of ρ is a volume-averaged mass of fluid molecules in a small volume δV, and the value of $\rho\boldsymbol{u}$ is a volume-averaged momentum of fluid molecules in a small volume δV. A size of δV for reliable values of ρ, $\rho\boldsymbol{u}$ and $\boldsymbol{u}$ is very small, and these values are usually treated as point values. However, we should keep it in mind that a size of δV is not infinitesimal but finite, especially in considering boundary conditions at an interface, as precisely described later.

Section 2 is set up to establish an approach to obtain boundary conditions at an interface based on physical images behind the mathematical expressions of terms rigorously obtained from the basic equation and focused on obtaining boundary conditions for momentum. In Subsection 2.1, we start from Navier-Stokes equation to figure out a transporting aspect of momentum through a conceptual surface S moving with fluid velocity $\boldsymbol{u}$, and it is shown that a contribution of fluid molecules randomly crossing over S due to their fluctuating movement is essentially important. It is pointed out next in Subsection 2.2 that a certain understanding of physical phenomena at an interface is inevitable to relate the momentum fluxes at both surfaces of an interface through which no fluid molecules are crossing over. A notion of a surface layer, which is a thin layer parallel to an interface with a thickness "δV", is introduced following a physical understanding of fluid surface [3]. A notation "δV" means a size of δV for reliable values of ρ, $\rho\boldsymbol{u}$ and $\boldsymbol{u}$, and will be used for simple expression, hereafter. In Subsection 2.3, boundary conditions for momentum are obtained based on a physical understanding of mathematical expressions of terms treated.

In Section 3, boundary conditions for vorticity are obtained based on physical images behind the mathematical expressions of terms rigorously obtained from the basic equation, the equation of vorticity transport. In Subsection 3.1, vorticity is re-defined as a rotation of a momentum vector, and it is shown that the newly defined vorticity is a point value of angular momentum of fluid. In Subsection 3.2, mathematical expressions relating vorticity flux on a conceptual surface S moving with fluid velocity $\boldsymbol{u}$ are obtained. Boundary conditions for vorticity are obtained in Subsection 3.3, based on a physical understanding of mathematical expressions of terms treated. The availability of the boundary conditions for vorticity is summarized in Subsection 3.4.

Concluding remarks are given in Section 4, where the most distinguished characteristics of the boundary conditions derived in this work are summarized, and a prospective view beyond the boundary conditions for vorticity is described.

2. Boundary Conditions for Momentum at an Interface between Two Fluids

2.1. Basic Concepts Relating to Boundary Conditions for Momentum

Navier-Stokes equation is given as (1).

$$\frac{\partial}{\partial t}(\rho \boldsymbol{u}) + \boldsymbol{u} \cdot \nabla (\rho \boldsymbol{u}) = -\nabla \cdot (\boldsymbol{\tau} + P\boldsymbol{I}) + \rho \boldsymbol{g} \tag{1}$$

Here, ρ, $\boldsymbol{u}$, $\boldsymbol{\tau}$, P, $\boldsymbol{g}$ and $\boldsymbol{I}$ are density of fluid, vector of fluid velocity, stress tensor, static pressure, vector of gravitational acceleration and unit tensor, respectively. The value of ρ is a volume-averaged mass of fluid molecules in a small volume δV, and the value of $\rho \boldsymbol{u}$ is a volume-averaged momentum of fluid molecules in a small volume δV.

$$\rho \equiv \frac{\sum m_i}{\delta V} \tag{2}$$

$$\rho \boldsymbol{u} \equiv \frac{\sum m_i \boldsymbol{u}_i}{\delta V} \tag{3}$$

Here, m_i and $\boldsymbol{u}_i$ are a molecular mass and a velocity vector of i-th molecule in δV. It is known from (2) and (3) that the value of $\boldsymbol{u}$ is a mass-averaged velocity of fluid molecules in δV.

$$\boldsymbol{u} \equiv \frac{\sum m_i \boldsymbol{u}_i}{\sum m_i} \tag{4}$$

The stress tensor, $\boldsymbol{\tau}$, is given by (5).

$$\boldsymbol{\tau} = -\mu \left\{ (\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^{\mathrm{T}} \right\} \tag{5}$$

Here, μ is a viscosity of fluid.

Navier-Stokes equation is recognized as a basic equation describing a spatial distribution of fluid velocity, $\boldsymbol{u}$. The stress tensor, $\boldsymbol{\tau}$, represents viscous forces caused by momentum transportation due to fluctuating movement of fluid molecules. Static pressure, P, in a moving fluid is a scalar quantity defined by the equation of state, $P = P(\rho, T)$ [3]. A size of δV for reliable values of ρ, $\rho\boldsymbol{u}$ and $\boldsymbol{u}$ is very small, and these values are usually treated as point values. However, we should note a fact in considering boundary conditions at an interface between two fluids that a size of δV is not infinitesimal but finite.

Since the left-hand side of (1) is a substantial derivative of momentum vector, (6) is obtained by integrating (1) in a volume V surrounded by a conceptual closed surface S moving with fluid velocity $\boldsymbol{u}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (\rho\boldsymbol{u})\mathrm{d}V = -\oint_S \left\{\boldsymbol{n}\cdot(\boldsymbol{\tau} + P\boldsymbol{I})\right\}\mathrm{d}S + \int_V \rho\boldsymbol{g}\mathrm{d}V \tag{6}$$

Here, $\boldsymbol{n}$ is a unit normal vector outwardly directed on S. Equation (6) is mathematically obtained by applying Gauss' law relating volume integral and surface integral. The left-hand side of (6) is an increasing rate of fluid momentum in V. The first term on the right-hand side of (6) is a sum of forces acting on S. The second term on the right-hand side is a gravitational force acting on fluid in V. Equation (6) shows Newton's second law of motion.

A mathematical expression, $(\boldsymbol{n}\cdot\boldsymbol{\tau})$, in the first term on the right-hand side of (6) is recognized as a viscous force working on unit area on S, which is also interpreted as a diffusive momentum flux caused by fluid molecules cutting across S in unit time. Let us think about physical phenomena behind the mathematical expression $(\boldsymbol{n}\cdot\boldsymbol{\tau})$. Since S is moving with fluid velocity $\boldsymbol{u}$, there are no fluid flow through S. However, fluid molecules are freely moving through S from outside to inside, or from inside to outside of S due to their fluctuating motion. Fluid molecules moving through S directly transport momentum and cause a diffusive momentum flux, which is a total amount of fluid momentum flowing into or out of unit area on S in unit time. There also are many couples of fluid molecules which collide and bounce at points on S causing normal and tangential forces at both sides of S. However, a normal force resulted from "collide and bounce" on S has already been counted in pressure as illustrated in the kinetic theory of molecules. Then, the value of normal component of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ on S should be a result of fluid molecules passing across S in unit time. Hence, the value of normal component of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ on a conceptual surface moving with $\boldsymbol{u}$ is zero, if no fluid molecules are cutting across the conceptual surface, such as an interface between two fluids. A tangential force resulted from "collide and bounce" can be regarded as a part of diffusive momentum flux caused by fluctuating motion of fluid molecules cutting across S.

A mathematical expression, $-\oint_S (\boldsymbol{n}\cdot P\boldsymbol{I})\mathrm{d}S$, in the first term on the right-hand side of (6) is a sum of pressure acting on S, which does not concern momentum transport.

2.2. Interface between Two Fluids

An interface between two fluids is moving with fluid velocity because there is no fluid flow through an interface. However, the arguments on the value of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ as a diffusive momentum flux cannot be directly applied for an interface because no fluid molecules are cutting across an interface. Therefore, certain understanding on physical phenomena at an interface is inevitable to relate the momentum fluxes at both surfaces of the interface.

Physics of liquid surface was precisely described by Davies and Rideal [4]. The net attraction between neighboring molecules is fulfilled most completely in the interior of the fluid phase, while those molecules at the surface are attracted less completely than they would have been in the bulk. They called this thin layer at the surface as a surface phase, which always tends to contract spontaneously and causes surface tension. Since the surface phase tends to contract, it works like a flak vest to suppress a fluctuating movement of a liquid surface and keeps its stationary shape under given circumstance of hydrodynamical forces and a surface tension. Though Davies and Rideal did not give detailed discussion on a thickness of the surface phase [4], it should be larger than a size of fluid molecule but is seemed to be considerably smaller than "δV", that is a size of δV. Let us introduce a notion of a surface layer with thickness "δV" beneath the surface to consider a transporting aspect of momentum at an interface. **Figure 1** is a schematical illustration of a surface layer at liquid surface following that of Davies and Rideal, though they did not show an interior boundary between liquid surface and bulk of fluid. Circles represent fluid molecules, and arrows mean attractive forces. Fluctuating fluid molecules are cutting across the interior boundary, and momentum is transferred through the interior boundary as a diffusive momentum flux.

Similarly, surface phases seem to exist at both sides of an interface between two fluids, say fluid A and fluid B, as shown in **Figure 2**. The boundary condition for momentum is to be given as a relation between momentum fluxes at interior boundaries A and B. In considering the relation, we may suppose a flat interface because the thickness of the surface phase is very thin. However, we should keep it in mind that there exists Laplace pressure for a curved interface due to an interfacial tension, which works as a normal force per unit area.

2.3. Boundary Condition for Momentum

In advance to evaluate the value of diffusive momentum flux, $(\boldsymbol{n}\cdot\boldsymbol{\tau})$, it should be noted that the mathematical expression, $(\boldsymbol{n}\cdot\boldsymbol{\tau})$, is supposed on S moving

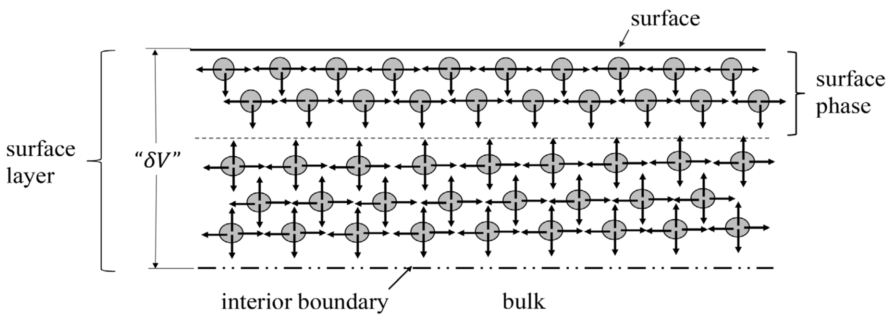

Figure 1. Surface layer at liquid surface.

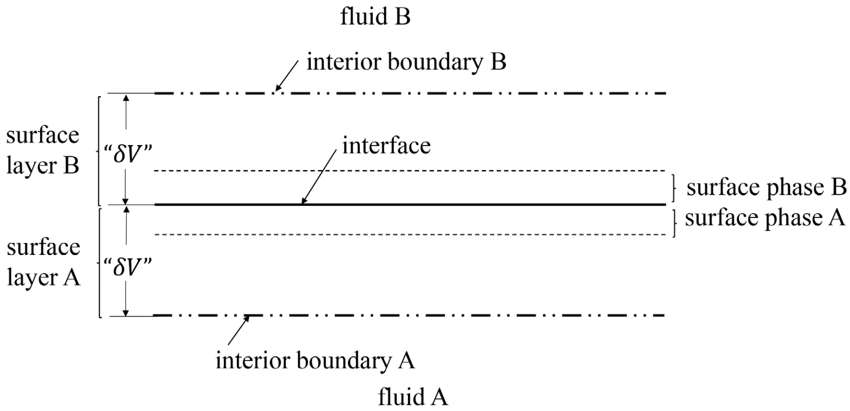

Figure 2. Interface between fluid A and fluid B.

with fluid velocity through which no fluid flow exists. Hence, the value of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ should be evaluated by using a velocity vector relative to a point where the term $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ in concern is assigned. However, tangential components of velocity play an important role in evaluating the value of vorticity flux on a surface moving with fluid velocity as shown later at the end of Subsection 2.2. Then, let us use a velocity vector $\boldsymbol{U}=\boldsymbol{u}-u_n\boldsymbol{n}$ in calculating the value of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$. Here, u_n is a normal velocity component at a point in concern on *S*.

A term $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ means an amount of momentum being transported through unit area on *S* in unit time, and the value of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ is to be assigned at a point in concern on *S*, just like the values of volume-averaged terms in Navier-Stokes equation are assigned at a point in concern. A point value of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ on a curved interface coincides with a point value of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ on a flat plane which touches at the point in concern. Then, we can evaluate the term $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ on an interface by using a flat interface with general applicability for curved interface. Considering that the surface layer is very thin, let us suppose a flat interface between fluids A and B with interior boundaries A and B parallel to the interface, and use a rectangular coordinate system setting *X*- and *Y*-axes on the interface. Our target is to obtain relations between momentum fluxes to *Z*-direction at both interior boundaries A and B by using a velocity vector $\boldsymbol{U}=\boldsymbol{u}-u_n\boldsymbol{n}$. **Figure 3** is a cross-sectional view of an interface between fluids A and B.

The value of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ to *Z*-direction at an interior boundary A is given by (7).

$$\boldsymbol{M}_Z^A=\boldsymbol{i}_Z\cdot\left[-\mu^A\left\{\left(\nabla\boldsymbol{U}^A\right)+\left(\nabla\boldsymbol{U}^A\right)^{\mathrm{T}}\right\}\right] \tag{7}$$

Here, a super subscript *A* means a physical quantity of fluid A, and $\boldsymbol{U}^A=\boldsymbol{u}^A-w_0\boldsymbol{i}_Z$, where w_0 is a normal velocity component at a point in concern on an interface. Equation (8) is obtained by introducing $\boldsymbol{U}^A=\boldsymbol{u}^A-w_0\boldsymbol{i}_Z$ and $\boldsymbol{u}^A=u^A\boldsymbol{i}_X+v^A\boldsymbol{i}_Y+w^A\boldsymbol{i}_Z$ into (7).

$$\begin{aligned}\boldsymbol{M}_Z^A&=\boldsymbol{i}_Z\cdot\left[-\mu^A\left\{\left(\nabla\boldsymbol{U}^A\right)+\left(\nabla\boldsymbol{U}^A\right)^{\mathrm{T}}\right\}\right]\\&=-\mu^A\frac{\partial u^A}{\partial Z}\boldsymbol{i}_X-\mu^A\frac{\partial v^A}{\partial Z}\boldsymbol{i}_Y-2\mu^A\frac{\partial w^A}{\partial Z}\boldsymbol{i}_Z\end{aligned} \tag{8}$$

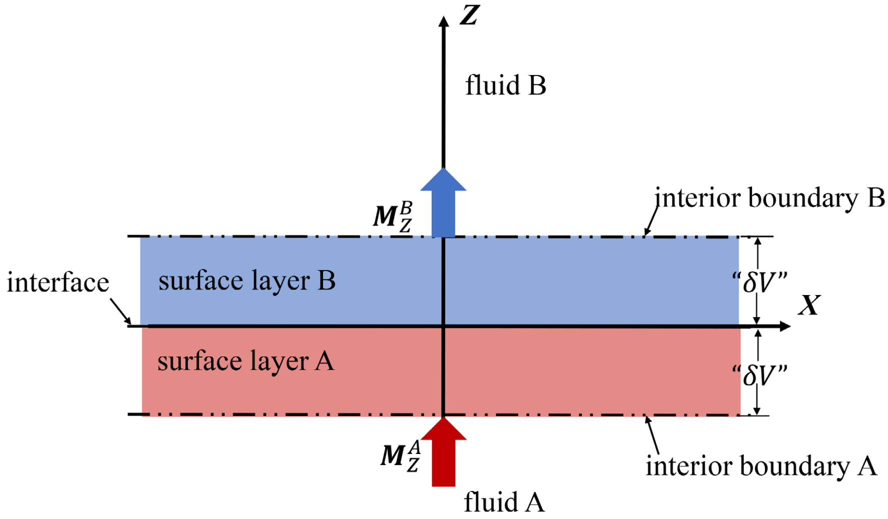

Figure 3. Momentum transport through an interface between fluid A and fluid B.

Here, relations $\frac{\partial W^A}{\partial X} = \frac{\partial\left(w^A - w_0\right)}{\partial X} = 0$, $\frac{\partial W^A}{\partial Y} = \frac{\partial\left(w^A - w_0\right)}{\partial Y} = 0$ and $\frac{\partial w_0}{\partial Z} = 0$ have been used. Similarly, (9) is obtained for a momentum flux to Z-direction at an interior boundary B, $\boldsymbol{M}_Z^B$.

$$\boldsymbol{M}_Z^B = -\mu^B \frac{\partial u^B}{\partial Z} \boldsymbol{i}_X - \mu^B \frac{\partial v^B}{\partial Z} \boldsymbol{i}_Y - 2\mu^B \frac{\partial w^B}{\partial Z} \boldsymbol{i}_Z \tag{9}$$

Here, a super subscript B means a physical quantity of fluid B. Though the right-hand sides of (8) and (9) are diffusive fluxes of momentum on the interior boundaries caused by fluid molecules cutting across the interior boundaries, terms on the right-hand sides of (8) and (9) are conventionally interpreted as forces acting on a unit area of interior boundaries A and B, respectively.

The first and second terms on the right-hand side of (8) are shear forces working on a surface layer A at unit area of an interior boundary A, and those on the right-hand side of (9) are shear forces acting on bulk of fluid B at unit area of an interior boundary B. Remembering an argument in Subsection 2.1 on a physical image of a mathematical expression $(\boldsymbol{n} \cdot \boldsymbol{\tau})$ at a conceptual surface moving with fluid velocity, it is known that the shear force at an interface, through which no fluid molecules are cutting across, is caused by collision and bounce of fluid molecules at an interface. Since a surface layer A is very thin, the value of shear force working at an interior boundary A should be in a balance with the shear force at an interface to maintain surface layer A. Similarly, it can be deduced that a shear force working at an interior boundary B should be in a balance with a shear force at an interface to maintain surface layer B. Hence, the shear forces working on a unit area of interior boundaries A and B should be balanced to assure stationary surface layers. Then, tangential components of $\boldsymbol{M}_Z^A$ and $\boldsymbol{M}_Z^B$ are equal regardless of the slip or no-slip condition of velocity at an interface.

$$\mu^A \frac{\partial u^A}{\partial Z} = \mu^B \frac{\partial u^B}{\partial Z} \tag{10}$$

$$\mu^A \frac{\partial v^A}{\partial Z} = \mu^B \frac{\partial v^B}{\partial Z} \tag{11}$$

Though the values on the left-hand sides of (10) and (11) are given on an interior boundary A and those on the right-hand sides of (10) and (11) are given on an interior boundary B, (10) and (11) can be regarded as boundary conditions at an interface because surface layers A and B are very thin. Equations (10) and (11) are identical with balances of shear force at an interface which have been used in hydrodynamical analysis.

The third terms on the right-hand sides of (8) and (9) are normal forces working on a unit area of a surface layer A at an interior boundary A and that working on bulk of fluid B at a unit area of an interior boundary B, respectively. Since a thickness of surface layer is very thin, the value of total normal force acting on a unit area of interior boundary A and that of interior boundary B should be in a balance. Then, (12) should hold as a balance of total normal forces acting on a unit area of interior boundaries A and B.

$$-2\mu^A \frac{\partial w^A}{\partial Z} + P^A = -2\mu^B \frac{\partial w^B}{\partial Z} + P^B + P^L \tag{12}$$

Here, P^A and P^B are static pressures acting on interior boundaries A and B, respectively, and P^L is a Laplace pressure given by (13), which is a normal force per unit area on an interface caused by an interfacial tension.

$$P^L = \frac{2}{R}\sigma \tag{13}$$

Notations R and σ are a curvature radius of an interface and an interfacial tension, respectively. A term P^L has been conveniently added to the right-hand side of (12) assuming fluid B is on a convex side of an interface. Equation (12) resembles to the normal force balance which has been used in previous works, though the Laplace pressure term is often neglected because the value of P^L is usually very small comparing with the value of static pressure.

Now, let us examine (12) from a viewpoint of momentum transportation. The first term on the left-hand side of (12) is a normal component of diffusive momentum flux flowing into surface layer A through an interior boundary A, which is a sum of momentum carried into surface layer A through unit area of an interior boundary A in unit time due to fluctuating motion of fluid molecules. There may occur collisions of fluid molecules at an interior boundary A, however resulted normal component has already been counted as a static pressure working on a surface of interior boundary A. Then, the first term on the left-hand side of (13), $-2\mu^A \frac{\partial w^A}{\partial Z}$, is a pure diffusive momentum flux caused by fluid molecules passing across the interior boundary A due to their fluctuating motion. Considering that there exists no diffusive momentum flux at an interface where no fluid molecules are cutting across, the value of a normal component of diffusive momentum flux is zero at an interface. If the value of $-2\mu^A \frac{\partial w^A}{\partial Z}$ is not zero at an interior boundary A, we should figure out a mechanism to increase or decrease the value of a normal component of diffusive momentum flux from $-2\mu^A \frac{\partial w^A}{\partial Z}$ to zero in a surface layer A against a fact that momentum is a conservative quantity. Then, the value of $-2\mu^A \frac{\partial w^A}{\partial Z}$ should be zero at an interior boundary A. Similarly, the value of $-2\mu^B \frac{\partial w^B}{\partial Z}$ is zero at an interior boundary B.

$$\frac{\partial w^A}{\partial Z} = \frac{\partial w^B}{\partial Z} = 0 \tag{14}$$

Though (14) is not familiar to us, it is a logical consequence matching with physical image behind the mathematical expressions of terms in a basic equation; Navier-Stokes equation. Equation (15) is obtained by introducing (14) into (12).

$$P^A = P^B + P^L \tag{15}$$

Equations (10), (11), (14) and (15) are boundary conditions, which are derived as relations between the mathematical expressions on an interior boundary A and those on an interior boundary B. These relations can be regarded as relations at an interface because surface layers A and B are very thin.

3. Boundary Conditions for Vorticity at an Interface between Two Fluids

3.1. Angular Momentum and Vorticity

A relation between the angular momentum of fluid in a conceptual sphere with radius R and the local value of the vorticity in the conceptual sphere will be clearly derived here, by using a spherical coordinate system with its origin at a center of the conceptual sphere, as shown in **Figure 4**, where the angular coordinates θ and ϕ are defined with respect to the Cartesian coordinate system X, Y and Z. Let us calculate the Z-component of angular momentum of fluid in the conceptual sphere. The X- and Y-components can be calculated in the same way.

The Z-component of the angular momentum of fluid in a torus, $T_{r\sin\theta}$, is given by (16).

$$\boldsymbol{A}^T_{r\sin\theta,Z} = r\mathrm{d}\theta\mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi\left(r^2\sin^2\theta\rho u_\phi\right) \tag{16}$$

Here, $T_{r\sin\theta}$ is a torus perpendicular to Z-axis with radius $r\sin\theta$ and an infinitesimal cross-sectional area $r\mathrm{d}\theta\mathrm{d}r$, and u_ϕ is the ϕ-component of the velocity of fluid.

Next, let us denote a spherical shell with radius r and an infinitesimal thickness dr as S_r^{Shell}. The Z-component of the angular momentum of fluid in S_r^{Shell}

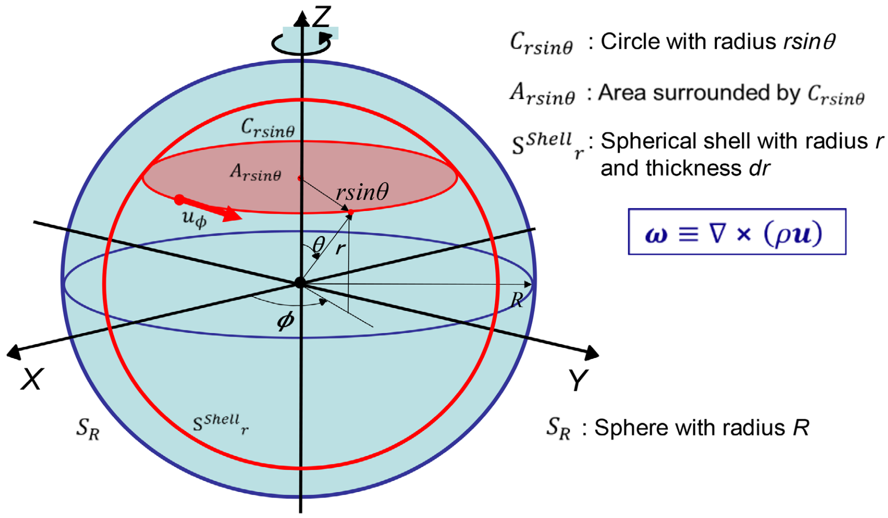

Figure 4. Coordinate system.

is given by (17).

$$\boldsymbol{A}_{r,Z}^{Shell} = r\mathrm{d}r\int_0^{\pi}\mathrm{d}\theta\left\{\int_0^{2\pi}\mathrm{d}\phi\left(r^2\sin^2\theta\rho u_\phi\right)\right\} \tag{17}$$

Then, the Z-component of the angular momentum of fluid in a conceptual sphere with radius R is given by (18).

$$\boldsymbol{A}_{R,Z}^{S} = \int_0^R r\mathrm{d}r\left[\int_0^{\pi}\mathrm{d}\theta\left\{\int_0^{2\pi}\mathrm{d}\phi\left(r^2\sin^2\theta\rho u_\phi\right)\right\}\right] \tag{18}$$

Equation (19) generally holds (Stokes' theorem).

$$\int_0^{2\pi} r\sin\theta\mathrm{d}\phi\left(\rho u_\phi\right) = \int_{A_{r\sin\theta}}\left\{\nabla\times\left(\rho\boldsymbol{u}\right)\right\}_Z\mathrm{d}S \tag{19}$$

A left-hand side of (19) is a curvilinear integral of a momentum along a circle with radius $r\sin\theta$, and a right-hand side is a surface integral of the Z-component of a rotation of momentum vector, $\left\{\nabla\times\left(\rho\boldsymbol{u}\right)\right\}_Z$, on a circular area, $A_{r\sin\theta}$, surrounded by the circle with radius $r\sin\theta$.

Equation (20) is obtained by introducing (19) into (18).

$$\boldsymbol{A}_{R,Z}^{S} = \int_0^R \mathrm{d}r\left[\int_0^{\pi} r^2\sin\theta\mathrm{d}\theta\left\{\int_{A_{r\sin\theta}}\left\{\nabla\times\left(\rho\boldsymbol{u}\right)\right\}_Z\mathrm{d}S\right\}\right] \tag{20}$$

Now, let us re-define the vorticity as a rotation of a vector of momentum, as (21).

$$\boldsymbol{\omega} \equiv \nabla\times\left(\rho\boldsymbol{u}\right) \tag{21}$$

Equation (20) shows that an angular momentum of fluid in a conceptual sphere with radius R can be calculated by using a distribution of the newly defined vorticity in the conceptual sphere. Hence, the angular momentum of fluid is represented by a local value of the newly defined vorticity, $\boldsymbol{\omega} \equiv \nabla\times\left(\rho\boldsymbol{u}\right)$, and transported in fluid as described by the equation of vorticity transport.

3.2. Basic Concepts Relating to Boundary Conditions for Vorticity

Equation (22) is obtained from (19).

$$\{\nabla\times(\rho\boldsymbol{u})\}_Z=\lim_{R\to 0}\frac{\int_0^{2\pi} r\sin\theta\mathrm{d}\phi\left(\rho u_\phi\right)}{\pi R^2} \tag{22}$$

Equation (22) is known as an implicit definition of a rotation and shows that a local value of the vorticity is a circulation of momentum along an infinitesimal circle divided by an area of a circular surface surrounded by the infinitesimal circle. Remembering that the value of fluid momentum is a volume-averaged momentum of fluid molecules in a small volume δV, a diameter of the circle along which fluid momentum is integrated is not infinitesimal but larger than **2** × "δV" to avoid double counting of a contribution of fluid molecules in δV. Then, the newly defined vorticity is an angular momentum of fluid volume-averaged in a very small volume with a diameter **2** × "δV" or larger, and assigned at a center of the volume, which is regarded as a point value of angular momentum of fluid. The vorticity thus defined does not represent a circulating motion, but a man-caused circulation defined by a curvilinear integration of momentum component along a very small, circle with a diameter **2** × "δV" or larger. So, the vorticity does not exist in an area where a circle for the curvilinear integration cannot be set.

Equation of vorticity transport is obtained by taking a rotation of each term in (1).

$$\begin{aligned}\frac{\partial\boldsymbol{\omega}}{\partial t}&=\nabla\cdot\{(\nabla\times\boldsymbol{u})(\rho\boldsymbol{u})\}-\nabla\cdot\{(\rho\boldsymbol{u})(\nabla\times\boldsymbol{u})\}\\&\quad+\nabla\cdot\left[\nabla\{\nabla\times(\mu\boldsymbol{u})\}\right]-\frac{1}{2}(\nabla\rho)\times\{\nabla(\boldsymbol{u}\cdot\boldsymbol{u})\}\\&\quad-\nabla\times\left[\boldsymbol{u}\{\nabla\cdot(\rho\boldsymbol{u})\}\right]-\nabla\times\left[\nabla\cdot\{(\nabla\mu)\boldsymbol{u}+\boldsymbol{u}(\nabla\mu)\}\right]\end{aligned} \tag{23}$$

Here, $\boldsymbol{\omega}$ is a vorticity vector defined by $\boldsymbol{\omega}\equiv\nabla\times(\rho\boldsymbol{u})$. Rather lengthy calculation leading to (23) is precisely shown by Ueyama [5]. Equation (23) can be considerably simplified to (24) for fluid with constant ρ and μ.

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\boldsymbol{u}\cdot(\nabla\boldsymbol{\omega})=\nabla\cdot(\boldsymbol{\omega}\boldsymbol{u})+\frac{\mu}{\rho}\nabla\cdot(\nabla\boldsymbol{\omega}) \tag{24}$$

The left-hand side of (24) is a substantial derivative of $\boldsymbol{\omega}$. Then, (25) is obtained by integrating (24) in a volume V surrounded by a conceptual closed surface S moving with fluid velocity $\boldsymbol{u}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V(\boldsymbol{\omega})\mathrm{d}V=\oint_S\{\boldsymbol{n}\cdot(\boldsymbol{\omega}\boldsymbol{u})\}\mathrm{d}S+\frac{\mu}{\rho}\oint_S\{\boldsymbol{n}\cdot(\nabla\boldsymbol{\omega})\}\mathrm{d}S \tag{25}$$

Here, $\boldsymbol{n}$ is a unit normal vector outwardly directed on S. Equation (25) is mathematically obtained by applying Gauss' law relating volume integral and surface integral. The left-hand side of (25) is an increasing rate of vorticity in V. The first term on the right-hand side of (25) is a sum of fluxes of vorticity flowing through S from outside to inside of S due to a synergistic effect of vorticity and velocity. Detailed mechanism of this term is not known at present, but it surely exists as appearing in (25). The second term is a sum of diffusive fluxes of vorticity flowing through S from inside to outside of S. Equation (25) clearly shows that a vorticity flux on a conceptual surface moving with $\boldsymbol{u}$ is composed of two terms, $\boldsymbol{n}\cdot(\boldsymbol{\omega u})$ and $\frac{\mu}{\rho}\boldsymbol{n}\cdot(\nabla\boldsymbol{\omega})$, the values of which are given for a velocity vector $\boldsymbol{U}=\boldsymbol{u}-u_n\boldsymbol{n}$ just like the calculation of a term $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ for momentum flux. The value of a term $\boldsymbol{n}\cdot(\boldsymbol{\omega u})$ is a product of a normal component of $\boldsymbol{\omega}$ and tangential components of velocity, which directly shows a contribution of tangential components of velocity. This is the reason why only a normal component of velocity vector is subtracted from a velocity vector and a velocity vector $\boldsymbol{U}=\boldsymbol{u}-u_n\boldsymbol{n}$ has been used in evaluating the value of $(\boldsymbol{n}\cdot\boldsymbol{\tau})$ for momentum flux.

3.3. Boundary Condition for Vorticity at an Interface between Two Fluids

Let us investigate the values of $-\boldsymbol{n}\cdot(\boldsymbol{\omega u})$ and $\frac{\mu}{\rho}\boldsymbol{n}\cdot(\nabla\boldsymbol{\omega})$ at a point in concern on an interface between fluid A and fluid B by using a rectangular coordinate system setting X- and Y-axes on a flat interface with interior boundaries A and B parallel to an interface. **Figure 5** is a cross-sectional view of an interface between fluids A and B, assuming a flat interface on X-Y plane. By substituting a fluid velocity vector $\boldsymbol{u}$ with a velocity vector $\boldsymbol{U}=\boldsymbol{u}-u_n\boldsymbol{n}$, a vorticity at an interior boundary A, $\boldsymbol{\omega}^A$, is given by (26).

$$\begin{aligned}\boldsymbol{\omega}^A &= \nabla\times\left\{\rho^A\left(\boldsymbol{u}^A-w_0\boldsymbol{i}_Z\right)\right\}\\ &=\rho^A\left(\boldsymbol{i}_X\frac{\partial}{\partial X}+\boldsymbol{i}_Y\frac{\partial}{\partial Y}+\boldsymbol{i}_Z\frac{\partial}{\partial Z}\right)\times\left\{\boldsymbol{i}_X u^A+\boldsymbol{i}_Y v^A+\boldsymbol{i}_Z\left(w^A-w_0\right)\right\}\\ &=\rho^A\left\{\boldsymbol{i}_Z\left(\frac{\partial v^A}{\partial X}-\frac{\partial u^A}{\partial Y}\right)+\boldsymbol{i}_Y\frac{\partial u^A}{\partial Z}-\boldsymbol{i}_X\frac{\partial v^A}{\partial Z}\right\}\end{aligned}\tag{26}$$

Here, relations $\frac{\partial\left(w^A-w_0\right)}{\partial X}=0$ and $\frac{\partial\left(w^A-w_0\right)}{\partial Y}=0$ have been used. Then,

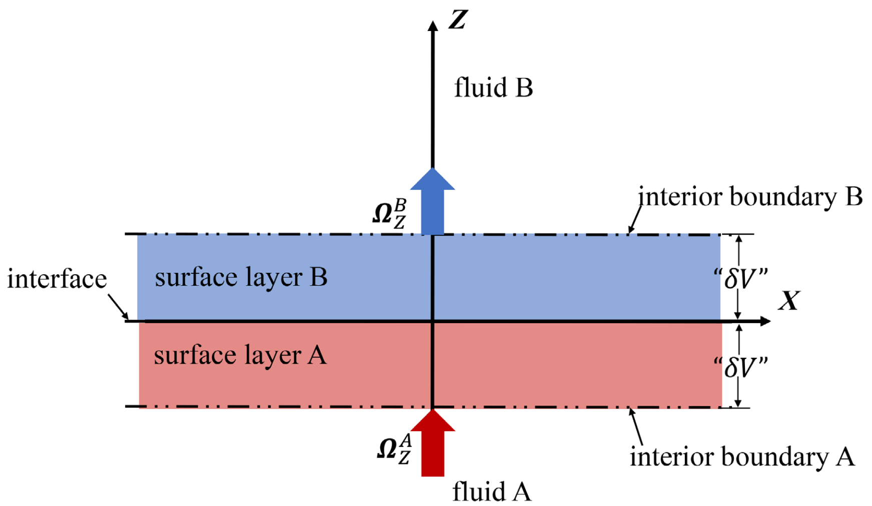

Figure 5. Vorticity transport through an interface between fluid A and fluid B.

a vorticity flux to Z-direction at an interior boundary A is given by (27).

$$
\begin{aligned}
\mathbf{\Omega}_Z^A &= \boldsymbol{i}_Z \cdot \left\{ \boldsymbol{\omega}^A \left(\boldsymbol{u}^A - w_0 \boldsymbol{i}_Z \right) \right\} + \frac{\mu^A}{\rho^A} \boldsymbol{i}_Z \cdot \left(\nabla \boldsymbol{\omega}^A \right) \\
&= \left\{ -\rho^A u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) + \mu^A \frac{\partial^2 v^A}{\partial Z^2} \right\} \boldsymbol{i}_X \\
&\quad + \left\{ -\rho^A v^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) + \mu^A \frac{\partial^2 u^A}{\partial Z^2} \right\} \boldsymbol{i}_Y + \mu^A \left\{ \frac{\partial}{\partial X} \left(\frac{\partial v^A}{\partial Z} \right) - \frac{\partial}{\partial Y} \left(\frac{\partial u^A}{\partial Z} \right) \right\} \boldsymbol{i}_Z
\end{aligned} \tag{27}
$$

Here, a relation $w^A - w_0 = 0$ has been used.

Similarly, a vorticity flux to Z-direction at an interior boundary B, $\mathbf{\Omega}_Z^B$, is given by (28).

$$
\begin{aligned}
\mathbf{\Omega}_Z^B &= \left\{ -\rho^B u^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right) + \mu^B \frac{\partial^2 v^B}{\partial Z^2} \right\} \boldsymbol{i}_X \\
&\quad + \left\{ -\rho^B v^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right) + \mu^B \frac{\partial^2 u^B}{\partial Z^2} \right\} \boldsymbol{i}_Y + \mu^B \left\{ \frac{\partial}{\partial X} \left(\frac{\partial v^B}{\partial Z} \right) - \frac{\partial}{\partial Y} \left(\frac{\partial u^B}{\partial Z} \right) \right\} \boldsymbol{i}_Z
\end{aligned} \tag{28}
$$

Since the newly defined vorticity is a point value of angular momentum of fluid and the angular momentum is conservative, the value of vorticity flux flowing into an interior boundary A and that flowing out of interior boundary B should be the same.

$$
\begin{aligned}
&\left\{ -\rho^A u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) + \mu^A \frac{\partial^2 v^A}{\partial Z^2} \right\} \boldsymbol{i}_X \\
&+ \left\{ -\rho^A v^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) + \mu^A \frac{\partial^2 u^A}{\partial Z^2} \right\} \boldsymbol{i}_Y + \mu^A \left\{ \frac{\partial}{\partial X} \left(\frac{\partial v^A}{\partial Z} \right) - \frac{\partial}{\partial Y} \left(\frac{\partial u^A}{\partial Z} \right) \right\} \boldsymbol{i}_Z \\
&= \left\{ -\rho^B u^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right) + \mu^B \frac{\partial^2 v^B}{\partial Z^2} \right\} \boldsymbol{i}_X \\
&+ \left\{ -\rho^B v^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right) + \mu^B \frac{\partial^2 u^B}{\partial Z^2} \right\} \boldsymbol{i}_Y + \mu^B \left\{ \frac{\partial}{\partial X} \left(\frac{\partial v^B}{\partial Z} \right) - \frac{\partial}{\partial Y} \left(\frac{\partial u^B}{\partial Z} \right) \right\} \boldsymbol{i}_Z
\end{aligned} \tag{29}
$$

Three relations are obtained from (29).

$$
\rho^A u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) - \mu^A \frac{\partial^2 v^A}{\partial Z^2} = \rho^B u^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right) - \mu^B \frac{\partial^2 v^B}{\partial Z^2} \tag{30}
$$

$$
\rho^A v^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) - \mu^A \frac{\partial^2 u^A}{\partial Z^2} = \rho^B v^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right) - \mu^B \frac{\partial^2 u^B}{\partial Z^2} \tag{31}
$$

$$
\mu^A \left\{ \frac{\partial}{\partial X} \left(\frac{\partial v^A}{\partial Z} \right) - \frac{\partial}{\partial Y} \left(\frac{\partial u^A}{\partial Z} \right) \right\} = \mu^B \left\{ \frac{\partial}{\partial X} \left(\frac{\partial v^B}{\partial Z} \right) - \frac{\partial}{\partial Y} \left(\frac{\partial u^B}{\partial Z} \right) \right\} \tag{32}
$$

The values of the left-hand sides of (30), (31) and (32) are given at an interior boundary A and those of the right-hand sides are given at an interior boundary B. Let us examine whether these relations are physically sound or not.

It is easily seen that (10) and (11), which are the boundary condition for momentum, assure (32) to hold. Then, a normal component of vorticity is exactly transferred from fluid A to fluid B due to shear forces working at an interface.

The left-hand side of (30) is a vorticity flux of X-component flowing into surface layer A through an interior boundary A. If the value of the left-hand side of (30) is not zero, it is necessary to understand the mechanism to transfer the X-component of vorticity from an interior boundary A to an interface. Let us evaluate terms in (30) one by one. The first term on the left-hand side of (30), $\rho^A u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right)$, is given by using tangential components of fluid velocity, which are assigned at a center of a surface layer A with a thickness, " δV ", in which reliable values of fluid density and fluid velocity are defined. Then the value of a term $\rho^A u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right)$ can be given in a surface layer A. The second term on the left-hand side of (30), $-\mu^A \frac{\partial^2 v^A}{\partial Z^2}$, is a normal gradient of a shear force working on a plane parallel to an interface. Considering that a shear force working at an interior boundary is in a balance with a shear force working at an interface, the absolute value of its normal gradient should be negligibly small comparing with the absolute value of $\rho^A u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right)$, in a surface layer A. Then, (33) is obtained.

$$\left| \rho^A u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) \right| \gg \left| -\mu^A \frac{\partial^2 v^A}{\partial Z^2} \right| \quad \text{in a surface layer A} \tag{33}$$

Similarly, (34) is obtained.

$$\left| \rho^B u^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right) \right| \gg \left| -\mu^B \frac{\partial^2 v^B}{\partial Z^2} \right| \quad \text{in a surface layer B} \tag{34}$$

Equation (35) is obtained from (30) by using relations (33) and (34).

$$\rho^A u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) \doteqdot \rho^B u^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right) \tag{35}$$

The value of the left-hand side of (35) is given at a center of the surface layer A, and that of the right-hand side of (35) is given at a center of the surface layer B. However, (35) does not generally hold, because

$u^A \left(\frac{\partial v^A}{\partial X} - \frac{\partial u^A}{\partial Y} \right) = u^B \left(\frac{\partial v^B}{\partial X} - \frac{\partial u^B}{\partial Y} \right)$ for no-slip condition of velocity and

$\rho^A \neq \rho^B$. Then, (30) cannot be accepted as a new boundary condition.

The non-zero-value assumption for the left-hand side of (30) has been thus denied. Then, the value of the left-hand side of (30) should be zero. Similarly, it can be concluded that the values of the right-hand side of (30) and both sides of (31) are zero. These conclusions can be precisely given by (36).

$$\boldsymbol{n} \times \left\{ \boldsymbol{n} \cdot (\boldsymbol{\omega u}) + \frac{\mu}{\rho} \boldsymbol{n} \cdot (\nabla \boldsymbol{\omega}) \right\} = 0 \quad \text{at an interior boundary} \tag{36}$$

Equation (36) means that tangential components of vorticity are not flowing into a surface layer, and we need not worry about the transporting mechanism of tangential components of vorticity in a surface layer. Though (36) is obtained as a condition at an interior boundary, it can be regarded as a boundary condition at an interface between two fluids, because the surface layer is very thin.

It seems to be meaningful to point out here that the tangential components of vorticity cannot be defined in a surface layer. Remembering an implicit definition of a rotation, (22), a tangential component of vorticity is defined as a circulation of momentum along a very small circle perpendicular to an interface divided by an area of the circle. Though (22) is a mathematical expression supposing an infinitesimal circle, we should remind a fact that the values of ρ, $\rho\boldsymbol{u}$ and $\boldsymbol{u}$ respectively are volume-averaged mass, volume-averaged momentum and mass-averaged velocity, of fluid molecules in a very small volume, δV. Then, a diameter of the circle for the vorticity should be larger than double of "δV" to avoid overlapping of contributions of fluid molecules. Therefore, the tangential components of vorticity cannot be defined in a surface layer because a circle for the tangential component of momentum, which is perpendicular to an interface, cannot be set in a surface layer with thickness "δV". This fact fits well to (36), which shows that the tangential components of vorticity do not flow into a surface layer.

Equation (36) is a boundary condition for vorticity to be generally adopted together with those for velocity and momentum in analyzing flow fields. Ueyama proposed the same boundary conditions for vorticity in 2020 focusing on physical soundness of transporting phenomena of the newly defined vorticity [5]. However, the derivation of the boundary condition was not fully matured because the zero-value condition for the tangential components of vorticity flux were concluded based on rather intuitive arguments that a circle along which a circulation of momentum is calculated cannot be set stepping across an interface. In this work, (36) has been obtained through detailed discussion on transporting aspect of vorticity in thin surface layers at both sides of an interface. Key factors in the discussion are a contribution of fluctuating motion of fluid molecules and a fact that a size of δV for reliable values of ρ, $\rho\boldsymbol{u}$ and $\boldsymbol{u}$ is not infinitesimal but finite.

Since a conventional vorticity defined as a rotation of velocity vector, $\nabla\times\boldsymbol{u}$, is not a conservative quantity, arguments for boundary conditions could not be made similarly to those for the newly defined vorticity, $\nabla\times(\rho\boldsymbol{u})$. There was no way for the conventional vorticity other than concluding that the disagreement between the values of vorticity flux at both surfaces of an interface is caused by a baroclinic generation of vorticity at the interface, and the value of baroclinic generation is a matter of *a posteriori* to be obtained after the flow field is determined otherwise [1]. With or without a density is an either-or choice in defining the vorticity. The definition of vorticity as a rotation of momentum vector leads to (36), which can be taken as a boundary condition for the vorticity in addition to the boundary conditions for velocity and force at an interface.

3.4. Availability of the Boundary Condition for Vorticity at an Interface between Two Fluids

Equation (36) has already realized remarkable progress in analytical investigation and numerical calculation.

One is an analytical investigation of a flow field surrounding a spherical fluid particle set in a simple shear flow [5]. Ueyama obtained a general solution of Navier-Stokes equation with a convection term represented by the creeping flow solution, that is, Saffman's equation for the first-order inner expansion [6], and the boundary condition for the newly defined vorticity made it possible to determine the values of 49 integral constants in the general solution. By using the solution of Saffman's equation, the value of the lift coefficient for a spherical bubble was obtained to be 0.4, which agrees well with experimental data for small bubbles in a simple shear flow [7].

The other is a drastic abbreviation of numerical calculation to obtain a velocity distribution in a laminar boundary layer on a flat plate. A basic equation for the velocity distribution in a boundary layer on a flat plate was proposed in 1908 by Blasius with its numerical solution [8]. However, it was very hard to obtain reliable numerical results by using two boundary conditions for velocity at a plate surface and at a point infinitely far from a plate surface, because a matching process between an analysis from surface to infinity and that from infinity to surface was quite troublesome and difficult. It took 30 years before Howarth's solution appeared in 1938 [9] as the most reliable solution of Blasius's equation beyond successive improvements made by several researchers [10] [11] [12] [13], as briefly summarized by Schlichting [14]. By using the boundary condition for the newly defined vorticity together with those for velocity, Ueyama obtained a numerical solution of Blasius's equation through one-way calculation from surface to infinity without the matching process [15], the numerical values of which agree in three digits or more with Howarth's solution.

4. Concluding Remarks

Transporting aspects of momentum and vorticity have been illustrated focusing on physical phenomena behind mathematical expressions of terms in basic equations: Navier-stokes equation and the equation of vorticity transport, and boundary conditions at an interface between two fluids have been derived.

The most distinguished characteristics of the boundary conditions derived here are the zero-value conditions at an interface for a normal component of momentum flux and tangential components of vorticity flux. These fluxes concern a velocity component normal to an interface. The fact that no fluid molecules are cutting across an interface leads to the zero-value condition for the normal component of momentum flux, and the zero-value condition for tangential components of vorticity flux fits very well with the fact that there exist very thin layers at both sides of an interface where the tangential components of vorticity cannot be defined. On the other hand, the tangential components of momentum flux and a normal component of vorticity flux, which are solely concerning velocity components parallel to an interface, have been shown to be perfectly transferred from one phase to the other at an interface as diffusive fluxes due to shear forces working at an interface.

These boundary conditions have been obtained by considering transporting aspects of momentum and vorticity in thin surface layers at both sides of interface, paying attention to physical meanings of mathematical expressions of terms rigorously obtained from basic equations: Navier-Stokes equation and the equation of vorticity transport. Hence, the boundary conditions obtained in this work are to be generally applied in analyzing flow fields.

The boundary condition for the newly defined vorticity provides us a prospective view to develop a new approach in analyzing flow fields by obtaining a velocity distribution first based on the equation of vorticity transport and a pressure distribution next by introducing the obtained velocity distribution into Navier-Stokes equation.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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