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Existence of Solutions for a Non-Autonomous Evolution Equations with Nonlocal Conditions

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Abstract

The existence of mild solutions for non-autonomous evolution equations with nonlocal conditions in Banach space is studied in this article. We obtained the existence of at least one mild solution to the evolution equations by using Krasnoselskii's fixed point theorem as well as the theory of the evolution family. The interest of this paper is that any assumptions are not imposed on the nonlocal terms and Green's functions and a new alternative method is applied to prove the existence of mild solutions. The results obtained in this paper may improve some related conclusions on this topic. An example is given as an application of the results.

Keywords

Non-Autonomous Evolution Equation, Nonlocal Conditions, Mild Solution, Evolution Family

1. Introduction

Recently, the evolution equation has been used to describe the state or process that changes with time in physics, mechanics, or other natural sciences. Byszewski [1] first investigated the nonlocal problems, they obtained the existence and uniqueness of mild solutions for nonlocal differential equations without impulsive conditions. Deng [2] pointed out that the nonlocal initial condition can be applied in physics with better effect than the classical initial condition $u(0)=u_0$, and used the nonlocal conditions $u(0)=\sum_{k=1}^{m} c_k u(t_k)$ to describe the diffusion phenomenon on a small amount of gas in a transparent tube. The aforementioned findings encourage more authors to focus on differential equations with nonlocal conditions. The integro-differential equations are usually applied to model processes which are subjected to abrupt changes at a certain time. They have wide applications in control, mechanics, electrical engineering fields, and so on. In 2010, Fan [3] discussed the existence and simulation of positive solutions for m-point fractional differential equations with derivative terms. In 2011, Tai [4] studied the exact controllability of fractional impulsive neutral functional integro-differential systems with nonlocal conditions by using the fractional power of operators and the Banach contraction mapping theorem. Consequently, to describe some physical phenomena, the nonlocal condition can be more useful than the standard initial condition $u(0)=u_0$. The importance of nonlocal conditions has also been discussed in [5] [6] [7] [8] [9].

All of researchers focus on the case that the differential operators in the main parts are independent of time *t*, which means that the problems are autonomous in previous researches. However, when treating some parabolic evolution equations, due to the frequent occurrence of such operators related to time *t* in applications, it is usually assumed that the partial differential operators depend on time *t* on account of this class of operators appears frequently in the applications, for the details please see Fu [10]; Zhu [11] and Wang [12]. So it is meaningful to study the non-autonomous evolution equation, that is, the differential operators of the main part are related to time *t*.

In recent years, the existence and approximate controllability of the mild solution of the evolution equation are widely studied. Kalman [13] in 1963 introduced the concept of controllability firstly, and the concept has become an active area of investigation due to its great applications in the field of physics. There are various works on approximate controllability of systems represented by differential equations, integro-differential equations, differential inclusions, neutral functional differential equations, and impulsive differential equations of integer order in Banach spaces. Mahmudov [14] in 2008 studied the approximate controllability for the abstract evolution equations with nonlocal conditions in Hilbert spaces and obtained sufficient conditions for the approximate controllability of the semi-linear evolution equation. In 2017, Kumar [15] studied mild solution and fractional optimal control of semilinear system with fixed delay in Banach space. In 2018, Chen [16] discussed that approximate controllability of non-autonomous evolution system with nonlocal conditions and introduced a new Green's function and constructed a control function involving Gramian controllability operator. He applied the Schauder's fixed point theorem to proved the mild solutions of the evolution equations.

In 1995, Byszewski [17] obtained the existence and uniqueness of solutions to a class of abstract functional differential equations with nonlocal conditions of the form

$$\begin{cases} u'(t)=f\left(t,u(t),u\left(a(t)\right)\right), & t\in J:=[0,a], \\ u(0)+\sum_{k=1}^{m}c_k u(t_k)=u_0, \end{cases}$$

where $a>0$ is a constant, $0<t_1<t_2<\cdots<t_m<a$, $f:J\times X\times X\to X$ and

$a: J \to J$ are given functions, X is a Banach space, $x_0 \in X$, $c_k \in \mathbb{R}$, $c_k \neq 0(k=1,2,\cdots,m)$, $m \in \mathbb{N}$. It pointed out that if $c_k \neq 0(k=1,2,\cdots,m)$ then we know that this can be applied to the motion of gas in physics.

In 2013, Fu and Zhang [10] obtained the exact null controllability for the following non-autonomous functional evolution system with nonlocal condition

$$\begin{cases} u'(t) = A(t)u(t) + Bv(t) + F\left(t, u\left(h(t)\right)\right), & t \in [0,a] := J, \\ u(0) + g(x) = u_0, \end{cases}$$

where the state variable $u(\cdot)$ takes values in a Hilbert space X and the control function $v(\cdot)$ is given in Banach space $L^2(J;U)$ of admissible control functions, U is also a Hilbert space. The family $\{A(t): t \in J\}$ of linear operators generates a linear evolution system and B is a bounded linear operator from U into X. $h(t) \in C(J,J)$, $F: J \times X \to X$, $g: C(J,X) \to X$ and $g(x) = \sum_{i=0}^{p} c_i u(t_i)$, where $c_i, i = 0,1,\cdots,p$ are given constants and $0 < t_0 < t_1 < \cdots < t_p < a$. A strong assumption is made for the nonlocal function g by [5] but no assumption is made for the nonlocal function in this paper.

In the above literature [18]-[25], the authors make some assumptions about the nonlocal terms. Therefore, it seems natural to remove the strong constraints on the nonlocal function g. Motivated by all of the above-mentioned aspects, in this paper we consider the existence of mild solutions for the following non-autonomous evolution equation with nonlocal conditions

$$\begin{cases} u'(t) = A(t)u(t) + Bv(t) + f\left(t, u(t), u\left(h(t)\right)\right), & t \in [0,a] := J, \\ u(0) = \sum_{k=1}^{m} c_k u(t_k), \end{cases} \tag{1}$$

in Banach space X, where $a > 0$ is a constant. $A(t)$ is a family of (possibly unbounded) linear operators depending on time and having the domains $D(A(t))$ for every $t \in J$, the control function $v(t)$ is given in Banach space $L^2(J;U)$ of admissible control functions, U is also a Banach space, $h(t) \in C(J,J)$, $f: J \times X \times X \to X$ is a continuous nonlinear mapping, and B is a bounded linear operator from U to X, $0 < t_1 < t_2 < \cdots < t_m < a$, $m \in \mathbb{N}$, c_k are real numbers, $c_k \neq 0, k = 1,2,\cdots,m$. The existence of mild solutions to the problem (1) is considered by using the Krasnoselskii's fixed point theorem and the theory of evolution family. In additions, an example is given as an application of the results. We assumed that the existence of the mild solution to the evolution equation is related to its approximative controllability, which can lead to new ideas for the study of the existence and approximate controllability of the mild solution to the evolution equation. The interesting of this paper is that any assumptions are not imposed to the nonlocal terms and Green's functions, and a new alternative method is applied to prove the existence of mild solutions.

The rest of this paper is organized as follows. Some basic definitions, lemmas and properties are introduced in Section 2. In Section 3, the existence of mild solutions to the non-autonomous evolution equations (1) is proved. An example is given to illustrate the main results in Section 4. The main contents of this article are summarized in Section 5.

2. Preliminaries

In this section, we introduce some notations, definitions, and preliminary facts which are used throughout this paper.

Let X and U be two real Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_U$. We denote by $C(J,X)$ the Banach space of all continuous functions from interval J into X equipped with the supremum norm

$$\|u\|_C = \sup_{t\in J}\|u(t)\|,\ \ u\in C(J,X).$$

And by $L(X)$ the Banach space of all linear and bounded operators in E endowed with the topology defined by the operator norm. Let $L^2(J,U)$ be the Banach space of all U value Bochner square integrable functions defined on J with the norm

$$\|u\|_2 = \left(\int_0^a \|u(t)\|_U^2\,\mathrm{d}t\right)^{\frac{1}{2}},\ \ u\in L^2(J,U),$$

Suppose that a family of linear operators $\{A(t):0\le t\le a\}$ satisfies the following assumptions:

(A_1) The family $\{A(t):0\le t\le a\}$ is a closed linear operator;

(A_2) For each $t\in[0,a]$, the resolvent $R(\lambda,A(t))=(\lambda-A(t))^{-1}$ of linear operator $A(t)$ exists for all λ such that $\mathrm{Re}\,\lambda\le 0$, and there also exists $K>0$ such that $\|R(\lambda,A(t))\|\le K/(|\lambda|+1)$;

(A_3) There exist $0<\delta\le 1$ and $K>0$ such that $\|(A(t)-A(s))A^{-1}(\tau)\|\le K|t-s|^{\delta}$ for all t,s and $\tau\in[0,a]$;

(A_4) For each $t\in[0,a]$ and some $\lambda\in\rho(A(t))$, the resolvent set $R(\lambda,A(t))$ of linear operator $A(t)$ is compact.

Because of these conditions, the family $\{A(t):0\le t\le T\}$ generates a unique linear evolution system, or called linear evolution family $\{H(t,s):0\le s\le t\le T\}$, and there exists a family of bounded linear operators $\{\Psi(t,\tau)\,|\,0\le\tau\le t\le T\}$ with norm $\|\Psi(t,\tau)\|\le C|t-\tau|^{\delta-1}$ such that $H(t,s)$ can be represented as

$$H(t,s)=\mathrm{e}^{-(t-s)A(t)}+\int_s^t \mathrm{e}^{-(t-\tau)A(\tau)}\Psi(\tau,s)\,\mathrm{d}\tau, \tag{2}$$

where $\mathrm{e}^{-\tau A(t)}$ denotes the analytic semigroup with infinitesimal generator $(-A(t))$.

Lemma 2.1 [16] *The family of linear operators* $\{H(t,s):0\le s\le t\le T\}$ *satisfies the following conditions:*

1) the mapping $(t,s)\to H(t,s)$ is continuous, for each $x\in X$, $H(t,s)\in L(X)$ and $0\le s\le t\le T$;

2) $H(t,s)H(s,\tau)=H(t,\tau)$ for $0\le\tau\le s\le t\le T$, and $H(t,t)=I$;

3) $H(t,s)$ is a compact operator whenever $t-s>0$;

4) There holds, if $0<h<1$, $0<\gamma<1$, and $t-\tau>h$,

$$\|H(t+h,\tau)-H(t,\tau)\|\le\frac{Kh^{\gamma}}{|t-\tau|^{\gamma}}.$$

Condition (A_4) ensures the generated evolution operator satisfies 3) (see [12], Proposition 2.1). Hence, there exists a constant $M \geq 1$, such that

$$\|H(t,s)\| \leq M \quad \text{for all} \quad 0 \leq s \leq t \leq T. \tag{3}$$

Definition 2.1 [16] *The evolution family* $\{H(t,s): 0 \leq s \leq t \leq T\}$ *is continuous and maps bounded subsets of X into pre-compact subsets of X.*

Lemma 2.2 [18] *Let* $\{H(t,s), 0 \leq s \leq t \leq a\}$ *be a compact evolution system in X. Then for each* $s \in [0,a]$, $t \mapsto H(t,s)$ *is continuous by operator norm for* $t \in (s,a]$.

Let Y be another separable reflexive Banach space, whose norm is also denoted by $\|\cdot\|$, in which the control function $x(t)$ takes its values, E a bounded subset of Y. Denoted by $P_c(Y)$ is a class of nonempty closed and convex subsets of Y. We suppose that the multi-valued map $\psi: J \to P_c(Y)$ is graph measurable, $\psi(\cdot) \subset E$. The admissible control set H_{ad} is defined by

$$H_{ad} = \{x \in L^p(J,E): x(t) \in \psi(t), \text{a.e. } t \in J\}, \quad p > 1.$$

Obviously, $H_{ad} \neq \varnothing$ (see [26]) and $H_{ad} \subset L^p(J,Y)(p>1)$ is bounded, closed and convex.

In order to discuss the existence of solutions to (1), we consider the following linear non-autonomous evolution equations:

$$\begin{cases} u'(t) = A(t)u(t) + f(t), & t \in [0,a], \\ u(0) = u_0, \end{cases} \tag{4}$$

exists a unique solution $u \in C^1((0,a],X) \cap C((0,a],D(A(t))) \cap C([0,a],X)$ expressed by

$$u(t) = H(t,0)u_0 + \int_0^t H(t,s) f(s) \mathrm{d}s, \quad t \in [0,a]. \tag{5}$$

If $f \in L^1([0,a],X)$, the function u given by (5) belongs to $C([0,a],X)$, which is known as a mild solution of the (4).

Assume that the condition

(H_0) $\sum_{k=1}^m |c_k| < \frac{1}{M}$

holds. From (3) and the assumption (H_0) one gets that

$$\left\|\sum_{k=1}^m c_k H(t_k,0)\right\| \leq \sum_{k=1}^m |c_k| \cdot \|H(t_k,0)\| < 1. \tag{6}$$

From the (6), we obtain

$$\mathcal{P} := \left(I - \sum_{k=1}^m c_k H(t_k,0)\right)^{-1}.$$

By operator spectrum theorem, the operator $\mathcal{P} := \left(I - \sum_{k=1}^m c_k H(t_k,0)\right)^{-1}$ exists and is bounded. Moreover, by Neumann expression, we have

$$\|\mathcal{P}\| \leq \sum_{n=0}^{\infty} \left\|\sum_{k=1}^m c_k H(t_k,0)\right\|^n = \frac{1}{1 - \left\|\sum_{k=1}^m c_k H(t_k,0)\right\|} \leq \frac{1}{1 - M\sum_{k=1}^m |c_k|}.$$

To prove our main results, for any $f \in C([0,a],X,X)$, we consider the following nonlocal problem of linear non-autonomous evolution equation

$$\begin{cases} u'(t) = A(t)u(t) + f(t), & t \in [0,a], \\ u(0) = \sum_{k=1}^{m} c_k u(t_k). \end{cases} \tag{7}$$

Lemma 2.3 *If the condition* (H_0) *is satisfied, then a function* $u \in C(J, X)$ *is said to be a mild solution of the system* (7), which is

$$u(t) = \sum_{k=1}^{m} c_k H(t,0) \mathcal{P} \int_0^{t_k} H(t_k, s) f(s) \mathrm{d}s + \int_0^t H(t,s) f(s) \mathrm{d}s, \quad t \in [0,a]. \tag{8}$$

Proof. It is well-known that (7) has a unique mild solution $u \in C([0,a], X)$ expressed by

$$u(t) = H(t,0)u(0) + \int_0^t H(t,s) f(s) \mathrm{d}s, \quad t \in J. \tag{9}$$

Particularly,

$$u(t_k) = H(t_k,0)u(0) + \int_0^{t_k} H(t_k,s) f(s) \mathrm{d}s, \quad k = 1,2,\cdots,m. \tag{10}$$

From (7) and (10), we have

$$u(0) = \sum_{k=1}^{m} c_k H(t_k,0)u(0) + \sum_{k=1}^{m} c_k \int_0^{t_k} H(t_k,s) f(s) \mathrm{d}s. \tag{11}$$

From (9) and (11), we know that $u \in C([0,a], X)$ satisfies (3).

Inversely, we can verify directly that the function $u \in C([0,a], X)$ given by (8) is a mild solution of (7).

Definition 2.2 *A function* $u \in C(J, X)$ *is said to be a mild solution of non-local problem* (1), if for any $v \in L^2(J, U)$, $u(t)$ satisfies the integral equation

$$u(t) = \sum_{k=1}^{m} c_k H(t,0) \mathcal{P} \int_0^{t_k} H(t_k,s) \left[Bv(s) + f(s, u(s), u(h(s))) \right] \mathrm{d}s$$
$$+ \int_0^t H(t,s) \left[Bv(s) + f(s, u(s), u(h(s))) \right] \mathrm{d}s, \quad t \in J.$$

Lemma 2.4 [18] (*Krasnoselskii's fixed point theorem*). *Let W be a closed, convex and nonempty subset of Banach space X. Let operators* $Q_1, Q_2 : W \to X$ *satisfy*

1) if $x, y \in W$, then $Q_1 x + Q_2 y \in W$;

2) Q_1 is a contraction;

3) Q_2 is compact and continuous.

Then the operator $Q := Q_1 + Q_2$ has at least one fixed point in W.

Lemma 2.5 [18] *If* Ω *is a compact subset of a Banach space X, its convex closure is compact.*

3. Existence of Mild Solutions

In this section, the existence of mild solutions of the non-autonomous evolution Equation (1) is we considered. The proof is based on the Krasnoselskii's fixed point theorem and the theory of evolution system. We further assume the following conditions:

(H_1) The function $f : J \times X \times X \to X$ satisfies that for every $t \in J$, the function $f(t, \cdot, \cdot) : X \times X \to X$ is continuous and for each $(x, y) \in X \times X$, the function $f(\cdot, u, v) : J \to X$ is strongly measurable, there exists a function

$\mu \in L(J, \mathbb{R}^+)$ such that $\|f(t,u,u)\| \le \mu(t)$ for all $u \in X$ and $t \in J$,

(H$_2$) There exists a function $\omega \in L(J, \mathbb{R}^+)$ such that $\|Bv(t)\| \le \omega(t)$ for all $v \in L^2(J,U)$ and $t \in J$,

(H$_3$) $\int_0^a [\mu(s)+\omega(s)]\mathrm{d}s = L$, and $\lim_{r\to+\infty} \frac{L}{r} = \sigma < \infty$,

(H$_4$) The function $f: J \times X \times X \to X$ and there exists a constant $M_f > 0$, such that $\left\|\int_0^{t_k} f(s,u(s),u(h(s))) - f(s,y(s),y(h(s)))\mathrm{d}s\right\| \le M_f \|u-y\|$, $u,y \in X$.

Theorem 3.1 *Let the evolution family* $\{H(t,s): 0 \le s \le t \le T\}$ *generated by* $\{A(t): 0 \le t \le a\}$ *is compact. Suppose also that the assumption* (H$_0$)-(H$_4$) *are satisfied. Then the nonlocal problem* (1) *has at least one mild solution on J provided that*

$$\frac{M^2 \sum_{k=1}^m |c_k| \sigma}{1 - M\sum_{k=1}^m |c_k|} + M\sigma < 1.$$

Proof. Defined the operator $Q = Q_1 + Q_2$, where

$$(Q_1 u)(t) = \sum_{k=1}^m c_k H(t,0)\mathcal{P}\int_0^{t_k} H(t_k,s)\left[Bv(s) + f(s,u(s),u(h(s)))\right]\mathrm{d}s, \quad t \in J, \tag{12}$$

$$(Q_2 u)(t) = \int_0^t H(t,s)\left[Bv(s) + f(s,u(s),u(h(s)))\right]\mathrm{d}s, \quad t \in J. \tag{13}$$

By Definition 2.2, we can know that the mild solution of nonlocal problem (1) is equivalent to the fixed point of operator Q. In the following, we will prove that the operator Q admits a fixed point by applying the Krasnoselskii's fixed point theorem. The proof is divided into five steps.

Step 1. $Q(\Omega_r) \subseteq \Omega_r$ for some $r > 0$. For any $r > 0$, let $\Omega_r := \{u \in C(J,X): \|u(t)\| \le r, t \in J\}$.

If this is not true, for each $r > 0$, there exists $u_r \in \Omega_r$, $\|(Qu_r)t\| > r$ for all $t \in J$. From the definition of Q and hypotheses (H$_0$)-(H$_4$), we have

$$\begin{aligned}
r &< \|(Qu_r)t\| \\
&\le \left\|\sum_{k=1}^m c_k H(t,0)\mathcal{P}\int_0^{t_k} H(t_k,s)\left[Bv(s) + f(s,u(s),u(h(s)))\right]\mathrm{d}s\right\| \\
&\quad + \left\|\int_0^t H(t,s)\left[Bv(s) + f(s,u(s),u(h(s)))\right]\mathrm{d}s\right\| \\
&\le \frac{M^2\sum_{k=1}^m |c_k|}{1 - M\sum_{k=1}^m |c_k|}\left\|\int_0^{t_k} H(t_k,s)\left[Bv(s) + f(s,u(s),u(h(s)))\right]\mathrm{d}s\right\| \\
&\quad + M\left\|\int_0^t H(t,s)\left[Bv(s) + f(s,u(s),u(h(s)))\right]\mathrm{d}s\right\| \\
&\le \frac{M^2\sum_{k=1}^m |c_k|}{1 - M\sum_{k=1}^m |c_k|}\int_0^{t_k}\left[\mu(s)+\omega(s)\right]\mathrm{d}s + M\int_0^t \left[\mu(s)+\omega(s)\right]\mathrm{d}s \\
&\le \frac{M^2\sum_{k=1}^m |c_k|}{1 - M\sum_{k=1}^m |c_k|}\int_0^{a}\left[\mu(s)+\omega(s)\right]\mathrm{d}s + M\int_0^a \left[\mu(s)+\omega(s)\right]\mathrm{d}s \\
&= \frac{M^2\sum_{k=1}^m |c_k|}{1 - M\sum_{k=1}^m |c_k|} L + ML.
\end{aligned}$$

Dividing on both sides by r and taking the lower limit as $r \to +\infty$, we obtain $1 \le \frac{M^2 \sum_{k=1}^m |c_k|}{1 - M\sum_{k=1}^m |c_k|}\sigma + M\sigma$, which is a contradiction to the condition in Theorem 3.1. Thus, $Q(\Omega_r) \subseteq \Omega_r$ for some $r > 0$.

Step 2. $Q_1 : \Omega_r \to \Omega_r$ is a contraction operator.

For any $t \in J$, $u, y \in \Omega_r$, (12), and (H_0)-(H_4) imply

$$
\begin{aligned}
&\left\|(Q_1 u)(t) - (Q_1 y)(t)\right\| \\
&= \Bigg\|\sum_{k=1}^m c_k H(t,0)\mathcal{P}\int_0^{t_k} H(t_k,s)\big[Bv(s) + f(s,u(s),u(h(s)))\big]\mathrm{d}s \\
&\quad - \sum_{k=1}^m c_k H(t,0)\mathcal{P}\int_0^{t_k} H(t_k,s)\big[Bv(s) + f(s,y(s),y(h(s)))\big]\mathrm{d}s\Bigg\| \\
&\le \frac{M^2\sum_{k=1}^m |c_k|}{1 - M\sum_{k=1}^m |c_k|}\left\|\int_0^{t_k} f(s,u(s),u(h(s))) - f(s,y(s),y(h(s)))\mathrm{d}s\right\| \\
&\le \frac{M^2\sum_{k=1}^m |c_k| M_f}{1 - M\sum_{k=1}^m |c_k|}\|u - y\|,
\end{aligned}
$$

which yields that $\|Q_1 u - Q_1 y\| \le \frac{M^2\sum_{k=1}^m |c_k| M_f}{1 - M\sum_{k=1}^m |c_k|}\|u - y\|$. Hence Q_1 is a contraction operator in Ω_r.

Step 3. $Q_2 : \Omega_r \to \Omega_r$ is continuous. Let $\{u_n\}_{n=1}^{\infty} \subset C(J,X)$ with $\lim_{n\to+\infty} u_n = u$ in $C(J,X)$. Then by the continuity of *f*, we have

$$\lim_{n\to+\infty} f(s,u_n(s),u_n(h(s))) = f(s,u(s),u(h(s))), \quad \forall s \in J. \tag{14}$$

In addition, since

$$\left\| f(s,u_n(s),u_n(h(s))) - f(s,u(s),u(h(s)))\right\| \le 2\mu(s), \tag{15}$$

and the Lebesgue's dominated convergence theorem follows that

$$
\begin{aligned}
&\left\|(Q_2 u_n)(t) - (Q_2 u)(t)\right\| \\
&\le \int_0^t \|H(t,s)\|\left\| f(s,u_n(s),u_n(h(s))) - f(s,u(s),u(h(s)))\right\|\mathrm{d}s \\
&\le M\int_0^t \left\| f(s,u_n(s),u_n(h(s))) - f(s,u(s),u(h(s)))\right\|\mathrm{d}s \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

which implies that $Q_2 : \Omega_r \to \Omega_r$ is continuous.

Step 4. Q_2 is equi-continuous in Ω_r. For any $u \in \Omega_r$ and $0 \le t_1 \le t_2 \le a$, by (13) and (H_0)-(H_4), we have

$$
\begin{aligned}
&\left\|(Q_2 u)(t_2) - (Q_2 u)(t_1)\right\| \\
&= \Bigg\|\int_0^{t_2} H(t_2,s)\big[f(s,u(s),u(h(s))) + Bv(s)u(s)\big]\mathrm{d}s \\
&\quad - \int_0^{t_1} H(t_1,s)\big[f(s,u(s),u(h(s))) + Bv(s)u(s)\big]\mathrm{d}s\Bigg\| \\
&\le \int_0^{t_1} \|H(t_2,s) - H(t_1,s)\|\left\|Bv(s) + f(s,u(s),u(h(s)))\right\|\mathrm{d}s
\end{aligned}
$$

$$+\int_{t_1}^{t_2}\left\|H(t_2,s)\left[Bv(s)+f\left(s,u(s),u(h(s))\right)\right]\right\|\mathrm{d}s$$
$$:= I_1 + I_2,$$

where

$$I_1=\int_0^{t_1}\left\|H(t_2,s)-H(t_1,s)\right\|\left\|Bv(s)+f\left(s,u(s),u(h(s))\right)\right\|\mathrm{d}s;$$
$$I_2=\int_{t_1}^{t_2}\left\|H(t_2,s)\left[Bv(s)+f\left(s,u(s),u(h(s))\right)\right]\right\|\mathrm{d}s.$$

If $t_1 \equiv 0$, $0<t_2\le a$, the conclusion is obvious. If $0<t_1<a$, we choose $\varepsilon\in(0,t_1)$ small enough, by the conditions (H_0), (H_1) and (H_2), we have

$$\begin{aligned}I_1 &\le \int_0^{t_1-\varepsilon}\left\|H(t_2,s)-H(t_1,s)\right\|\left\|f\left(s,u(s),u(h(s))\right)\right\|\mathrm{d}s\\ &\quad+\int_{t_1-\varepsilon}^{t_1}\left\|H(t_2,s)-H(t_1,s)\right\|\left\|f\left(s,u(s),u(h(s))\right)\right\|\mathrm{d}s\\ &\le \sup_{s\in[0,t_1-\varepsilon]}\left\|H(t_2,s)-H(t_1,s)\right\|_{L(X)}\int_0^{t_1-\varepsilon}\left[\mu(s)+\omega(s)\right]\mathrm{d}s\\ &\quad+M\int_{t_1-\varepsilon}^{t_1}\mu(s)\mathrm{d}s+M\int_{t_1-\varepsilon}^{t_1}\omega(s)\mathrm{d}s\to 0 \text{ as } t_2-t_1\to 0 \text{ and } \varepsilon\to 0,\end{aligned}$$

$I_2\le M\int_{t_1}^{t_2}\mu(s)+\omega(s)\mathrm{d}s\to 0$, as $t_2-t_1\to 0$.

Therefore, $\left\|(Q_2u)(t_2)-(Q_2u)(t_1)\right\|\to 0$, as $t_2-t_1\to 0$, which means that the operator $Q_2:\Omega_r\to\Omega_r$ is equi-continuous.

Step 5. The set $Z(t):=\left\{(Q_2u)(t):u\in\Omega_r\right\}$ s relatively compact in *X* for each $t\in J$. Obviously, the set $Z(0)=\left\{(Q_2x)(0):u\in\Omega_r\right\}$ is relatively compact in *X*. Let $t\in(0,a]$, for any $x\in H_{ad},u\in\Omega_r$ and $\varepsilon\in(0,t-s)$, we define an operator Q_2^{ε} by

$$\left(Q_2^{\varepsilon}u\right)(t):=\int_0^{t-\varepsilon}H(t,s)\left[Bv(s)+f\left(s,u(s),u(h(s))\right)\right]\mathrm{d}s.$$

It follows from the boundedness of H_{ad} and (H_1) that the set $\mathcal{O}_{\varepsilon}=\left\{H(t,s)\left[Bv(s)+f\left(s,u(s),u(h(s))\right)\right]:0\le s<t-\varepsilon\right\}$ is relatively compact and depend on the compactness of $H(t,s)(t-s>0)$. Then, $\overline{co}(\mathcal{O}_{\varepsilon})$ is a compact set depend on Lemma 2.5. By the mean value theorem of Bochner integrals, we can get $\left(Q_2^{\varepsilon}u\right)(t)\in(t-\varepsilon)\overline{co}(\mathcal{O}_{\varepsilon})$ for all $t\in J$. Thus, the set $Z_{\varepsilon}(t)=\left\{\left(Q_2^{\varepsilon}u\right)(t):u\in\Omega_r\right\}$ is relatively compact in *X* for every $t\in J$. Moreover, by (13) and (H_0) and (H_1), we have

$$\begin{aligned}&\left\|(Q_2u)(t)-\left(Q_2^{\varepsilon}u\right)(t)\right\|\\ &=\left\|\int_0^{t}H(t,s)\left[Bv(s)+f\left(s,u(s),u(h(s))\right)\right]\mathrm{d}s\right.\\ &\quad\left.-\int_0^{t-\varepsilon}H(t,s)\left[Bv(s)+f\left(s,u(s),u(h(s))\right)\right]\mathrm{d}s\right\|\\ &\le\int_{t-\varepsilon}^{t}\left\|H(t,s)\left[Bv(s)+f\left(s,u(s),u(h(s))\right)\right]\right\|\mathrm{d}s\\ &\le M\int_{t-\varepsilon}^{t}\mu(s)+\omega(s)\mathrm{d}s,\end{aligned}$$

which means that $\lim_{\varepsilon\to 0}\left\|(Q_2u)(t)-\left(Q_2^{\varepsilon}u\right)(t)\right\|=0$. So we have proved that there is a family of relatively compact sets $Z_{\varepsilon}(t)$ arbitrarily close to the set $Z(t)$. Thus, the set $Z(t)$ is relatively compact in *X* for every $t\in[0,a]$.

By Steps 3-5, thanks to the Ascoli-Arzela theorem, we deduce that the operator $Q_2 : \Omega_r \to \Omega_r$ is compact and continuous in Ω_r. By the Krasnoselskii's fixed point theorem we can get the operator Q has at least one fixed point in Ω_r, which is the mild solution of the evolution Equation (1) on *J*. This completes the proof of Theorem 3.1.

4. Example

In this section, we provide a correct example to illustrate our abstract results.

Example 4.1 Consider the following non-autonomous partial differential equation with nonlocal problem:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial^2 x}u(x,t) + a(t)u(x,t) + \frac{t^2\sin(2\pi t)}{1+|u(x,t)|}\cdot\frac{1}{1+|u(x,\sin t)|} + 2v(x,t), \quad x\in[0,\pi], t\in[0,a], \\ u(0,t) = u(\pi,t) = u(\pi,\sin t) = 0, \quad t\in[0,a], \\ u(x,0) = \sum_{k=1}^{m}\arctan\frac{1}{2k^2}x(x,t_k), \quad x\in[0,\pi], \end{cases} \tag{16}$$

where $a:[0,1]\to\mathbb{R}$ is a continuously differentiable function and satisfies

$$a_{\min} := \min_{t\in[0,1]} a(t) < 1, \tag{17}$$

and $a>0$ is a constant, $v\in L^2(J, L^2(0,\pi;\mathbb{R}))$, $c_k\in\mathbb{R}, k=1,2,\cdots,m$. Let $X = L^2(0,\pi;\mathbb{R})$ with the norm $\|\cdot\|_2$ and inner product $\langle\cdot,\cdot\rangle$. Consider the operator A on X defined by

$$Au := \frac{\partial^2}{\partial x^2}u, \quad u\in D(A),$$

where

$$D(A) := \{u\in L^2(0,\pi;\mathbb{R}), u''\in L^2(0,\pi;\mathbb{R}), u(0)=u(\pi)=0\}.$$

The A generates a compact and analytic C_0-semigroup in C, A has a discrete spectrum, and its eigenvalues are $-n^2, n\in\mathbb{N}^+$ with the corresponding normalized eigenvectors $v_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Define the operator $A(t)$ on A by

$$A(t)u = Au - a(t)u,$$

with domain

$$D(A(t)) = D(A), \quad t\in[0,1].$$

The family $\{A(t): 0\le t\le a\}$ generates an strongly continuous evolution family $\{H(t,s): 0\le s\le t\le a\}$ defined by

$$H(t,s)u = \sum_{n=1}^{\infty} e^{-\left(\int_s^t a(\tau)d\tau + n^2(t-s)\right)}\langle u, v_n\rangle v_n, \quad 0\le s\le t\le 1, u\in X. \tag{18}$$

A direct calculation gives

$$\|H(t,s)\|_{L(X)} \le e^{-(1+a_{\min})(t-s)}, \quad 0\le s\le t\le 1.$$

(17) and (18) means that

$$M := \sup_{0\le s\le t\le a} \left\| H(t,s) \right\|_{\mathcal{L}(X)} = 1.$$

(see [9], [22])

For any $t \in [0,a]$, we define

$$u(t) = u(\cdot,t);$$

$$f\left(t,u(t),u(h(t))\right) = \frac{t^2 \sin(2\pi t)}{1+|u(\cdot,t)|} \cdot \frac{1}{1+|u(x,\sin t)|};$$

$$A : U := X \to X;$$

$$Bv(t) = 2v(\cdot,t);$$

$$c_k = \arctan\frac{1}{2k^2}, \quad t_k = 1,2,\cdots,m.$$

In conclusion, from $\sum_{k=1}^{m} |c_k| \le \sum_{k=1}^{\infty} \arctan\frac{1}{2k^2} = \frac{\pi}{4} < 1$, we know that the assumption (H_0) hold. From the definition of nonlinear term *f* and bounded linear operator *A* combined with the above discussion, we can easily verify that the assumptions (H_1)-(H_4) are satisfied with $\left\| f\left(t,u(t),u(h(t))\right)\right\| \le \sqrt{\pi} t^2 + 1 = \mu(t)$, $\omega(t) = 2v(t)$.

Therefore, the non-autonomous partial differential Equation (16) is equivalent to the problem (1). According to theorem 3.1, we know that (16) has at least one mild solution $u \in \left[C(0,\pi)\times(0,a)\right]$.

5. Conclusion

In this paper, the existence of solutions to non-autonomous evolution equations with nonlocal conditions is studied. By using the development family theory, solution operators and the application of Krasnoselskiis fixed point theorem, the existence of solutions of non-autonomous evolution equations is obtained.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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