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Global Stability in a Graph *p*-Laplacian SIR Epidemic Model

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Abstract

A *p*-Laplacian ($p>2$) reaction-diffusion system on weighted graphs is introduced to a networked SIR epidemic model. After overcoming difficulties caused by the nonlinear *p*-Laplacian, we show that the endemic equilibrium is globally asymptotically stable if the basic reproduction number r_0 is greater than 1, while the disease-free equilibrium is globally asymptotically stable if r_0 is lower than 1. We extend the stability results of SIR models with graph Laplacian ($p=2$) to general graph *p*-Laplacian.

Keywords

p-Laplacian, Network, Global Stability, Lyapunov Function

1. Introduction

In recent years, reaction-diffusion systems on complex networks have been used to study epidemic processes [1] [2]. A network is mathematically a graph $G=(V,E)$, which contains a set $V=\{1,2,\cdots,n\}$ of **vertices** and a set E of **edges**. If vertices x and y are connected by an edge (also called **adjacent**), we write $x\sim y$. G is called a finite-dimensional graph if it has a finite number of edges and vertices. A graph G is **weighted** if each adjacent x and y is assigned a weight function $\omega(x,y)$. Here $\omega: V\times V\to[0,+\infty)$ is a symmetric, nonnegative and bounded function, and $\omega(x,y)>0$ if and only if $x\sim y$.

A graph $G=(V,E)$ is called **connected**, if for every pair of vertices $x,y\in V$, there exists a sequence (called a path) of vertices $\{x_0=x,x_1,\cdots,x_n=y\}\subset V$ such that $x_{j-1}\sim x_j$ for $j=1,\cdots,n$. For a finite subset $\Omega\subset V$, let $\partial\Omega$ denote the boundary of Ω and Ω^0 denote the interior of Ω, which are defined by

$$\partial\Omega := \{x\in\Omega : \exists\, y\in V\setminus\Omega \text{ such that } y\sim x\} \text{ and } \Omega^0 := \Omega\setminus\partial\Omega, \tag{1}$$

respectively. Throughout this paper, $G=(V,E)$ is assumed to be a connected weighted finite-dimensional graph without self-loops. We also assume that Ω^0 is a nonempty connected subset.

In this paper, we use discrete p-Laplacian operators defined on a network to describe the movements of mosquitoes in each vertex which depend on the topological structure of the network. In order to describe our problem more conveniently, we first introduce the following discrete p-Laplacian operators defined on a network.

Definition 1.1 *For a function* $u:\Omega^0 \to \mathbb{R}$ *and* $p \in (2,+\infty)$, *the graph* p-Laplacian Δ_ω^p *on* Ω^0 *is defined by*

$$\Delta_\omega^p u(x) := \sum_{y\sim x, y\in\Omega^0} |u(y)-u(x)|^{p-2}(u(y)-u(x))\omega(x,y). \tag{2}$$

For $x\in\Omega$, the degree $D_\omega(x)$ on Ω^0 is defined by $D_\omega(x) := \sum_{y\sim x, y\in\Omega^0} \omega(x,y)$.

When $p=2$, it is called the discrete Laplacian $\Delta_\omega := \Delta_\omega^2$ on Ω, which is defined by

$$\Delta_\omega v(x) := \sum_{y\sim x, y\in\Omega} (v(y)-v(x))\omega(x,y). \tag{3}$$

Recently, classical Laplacian Δ is substituted by the discrete Laplacian Δ_ω in graph Laplacian problems, and various methods and techniques to study the existence and qualitative properties of solutions have been developed [2]-[7]. Here we should emphasize that the discrete p-Laplacian operator $\Delta_\omega^p (p>2)$ is actually **nonlinear**, which is different from the classical Laplacian Δ or the discrete Laplacian Δ_ω or the discrete Laplacian Δ_ω.

We consider the following nonlinear SIR model with p-Laplacian defined on networks

$$\begin{cases} \dfrac{\partial S}{\partial t} - d\Delta_\omega^p S = \lambda - \beta SI - \mu S, & (x,t)\in\Omega^0\times(0,+\infty), \\ \dfrac{\partial I}{\partial t} - d\Delta_\omega^p I = \beta SI - (\mu+\alpha) I, & (x,t)\in\Omega^0\times(0,+\infty), \\ S(x,t) = I(x,t) = 0, & (x,t)\in\partial\Omega\times[0,\infty), \\ S(x,0) = S_0(x) \ge (\not\equiv) 0, I(x,0) = I_0(x) \ge (\not\equiv) 0, & x\in\Omega. \end{cases} \tag{4}$$

Here S, I represent the population sizes of susceptible and infectious compartments, respectively. The recovery R is omitted, due to the fact that $S+I+R$ is assumed to be a constant. Parameter d is the diffusion rate of individuals, λ indicates the recruitment rate of S and parameter β the contact rate between susceptible and infectious populations. Population S and I die at a rate of μ and $\mu+\alpha$, respectively, here α is the additional death rate caused by infectious disease.

In this paper, when $p>2$, we overcome the difficulties caused by the nonlinear operators p-Laplacian Δ_ω^p and study the global stability for the solution of system (4). First, we prove the Green formula of nonlinear operators Δ_ω^p. Then we construct the maximum principle and stronge maximum principle of the graph Laplacian equations. With the help of the priori estimate, we present the global existence result. At last, we investigate the asymptotical behavior of the system by the method of Lyapunov function.

2. Preliminaries

Lemma 2.1 (*Green Formula*). *For any functions* $u,v:\Omega\to\mathbb{R}$, *then* $2\sum_{x\in\Omega^0}v(x)\Delta_\omega^p u(x) = -\sum_{x,y\in\Omega^0}|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x))\omega(x,y)$ *holds. In particular, in case of* $u=v$, *the following holds*

$$2\sum_{x\in\Omega^0}u(x)\Delta_\omega^p u(x) = -\sum_{x,y\in\Omega^0}|u(y)-u(x)|^p\,\omega(x,y). \tag{5}$$

Proof 1 *Using* (2), *we have*

$$\begin{aligned}\sum_{x\in\Omega^0}v(x)\Delta_\omega^p u(x) &= \sum_{x\in\Omega^0}v(x)\sum_{y\sim x,y\in\Omega^0}|u(y)-u(x)|^{p-2}(u(y)-u(x))\omega(x,y)\\ &= \sum_{x,y\in\Omega^0}v(x)|u(y)-u(x)|^{p-2}(u(y)-u(x))\omega(x,y)\\ &= \sum_{x,y\in\Omega^0}v(y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\omega(x,y).\end{aligned}$$

From the above equality, we deduce $2\sum_{x\in\Omega^0}v(x)\Delta_\omega^p u(x) = -\sum_{x,y\in\Omega^0}|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x))\omega(x,y)$, which completes the proof.

It's worth noting that the existence of nonlinear operators $\Delta_\omega^p\,(p>2)$ causes difficulties when we construct the Maximun principle of system (4).

Lemma 2.2 (*Maximun Principle*). *Suppose that* $d>0$ *and* K *are constants. For any* $T>0$, *assume that* $u(x,t)$ *is continuous with respect to t in* $\Omega\times[0,T]$, *is differentiable with respect to t in* $\Omega\times[0,T]$, *and further satisfies*

$$\begin{cases}\dfrac{\partial u}{\partial t}-d\Delta_\omega^p u+Ku\ge 0, & (x,t)\in\Omega^0\times(0,T],\\ u(x,t)\ge 0, & (x,t)\in\partial\Omega^0\times(0,T],\\ u(x,0)\ge 0, & x\in\Omega^0,\end{cases} \tag{6}$$

then $u(x,t)\ge 0$ in $\Omega\times[0,T]$.

Proof 2 *By setting* $v=\mathrm{e}^{-K_0t}u(x,t)$, *where* K_0 *is a positive constant satisfying* $K_0+K>0$, *we deduce* $\Delta_\omega^p u=\Delta_\omega^p(\mathrm{e}^{K_0t}v)=\mathrm{e}^{(p-1)K_0t}\Delta_\omega^p v$. *Thus we have*

$$\frac{\partial v}{\partial t}-d\mathrm{e}^{(p-2)K_0t}\Delta_\omega^p v+(K_0+K)v\ge 0 \text{ for } (x,t)\in\Omega^0\times(0,T]. \tag{7}$$

Notice that $v(x,t)$ are continuous on $[0,T]$ for each $x\in\Omega$ and Ω is finite, we can find $(x_0,t_0)\in\Omega\times[0,T]$ such that $v(x_0,t_0)=\min_{x\in\Omega}\min_{t\in[0,T]}v(x,t)$.

For the case that $x_0\in\partial\Omega$, in view of the boundary condition of *u* in (6), we have $v(x_0,t_0)=\mathrm{e}^{-K_0t_0}u(x_0,t_0)\ge 0$. Thus we have $v(x,t)\ge 0$ in $\Omega\times[0,T]$, which implies $u(x,t)\ge 0$ in $\Omega\times[0,T]$.

For the case that $x_0\in\Omega^0$, the above equation implies $v(x_0,t_0)\le v(y,t_0)$ for any $y\in\Omega$. In view of the definition of Δ_ω^p, we have

$$\Delta_\omega^p v(x_0,t_0)=\sum_{y\sim x_0,y\in\Omega^0}\left|u(y,t_0)-u(x_0,t_0)\right|^{p-2}\left(u(y,t_0)-u(x_0,t_0)\right)\omega(x_0,y)\geq 0. \quad (8)$$

Meanwhile it follows from the differentiability of $v(x,t)$ in $(0,T]$ that

$$\frac{\partial v}{\partial t}(x_0,t_0)\leq 0. \quad (9)$$

By substituting (8) and (9) into (7), we have $(K_0+K)v(x_0,t_0)\geq 0$. Noting that $K_0+K>0$, we deduce $v(x_0,t_0)\geq 0$, which means $\min_{x\in\Omega}\min_{t\in[0,T]}v(x,t)$. Therefore, we have $v(x,t)\geq 0$ in $\Omega\times[0,T]$. That is $u(x,t)\geq 0$ in $\Omega\times[0,T]$.

Lemma 2.3 (*Strong Maximun Principle*). *Suppose that* $d>0$ *and K are constants. For any* $T>0$, *assume that* $u(x,t)$ *is continuous with respect to t in* $\Omega\times[0,T]$, *is differentiable with respect to t in* $\Omega\times[0,T]$, *and satisfies* (6). *If* $u(x^*,0)>0$ *for some* $x^*\in\Omega^0$, *then* $u(x,t)>0$ *in* $\Omega^0\times[0,T]$.

Proof 3 *Using the above maximum principle, we have* $u(x,t)\geq 0$ *in* $\Omega\times[0,T]$. *By setting* $v=\mathrm{e}^{-K_0t}u(x,t)$, *where* K_0 *is defined as in the proof of Lemma* 2.2, *which satisfying* $K_0+K>0$, *we have*

$$\left.\left(\frac{\partial v}{\partial t}-d\mathrm{e}^{(p-2)K_0t}\Delta_\omega^p v+(K_0+K)v\right)\right|_{(x^*,t)}\geq 0. \quad (10)$$

Notice that $v(x,t)$ are continuous on $[0,T]$ for each $x\in\Omega$ and Ω is finite, we deduce that $M:=\max_{x\in\Omega}\max_{t\in[0,T]}v(x,t)<+\infty$. Plugging (2) into (10), we have

$$\begin{aligned}\frac{\partial v(x^*,t)}{\partial t}&\geq -d\mathrm{e}^{(p-2)K_0t}\sum_{y\sim x^*,y\in\Omega^0}\left|v(y,t)-v(x^*,t)\right|^{p-2}v(x^*,t)\omega(x^*,y)\\&\quad-(K_0+K)v(x^*,t)\\&\geq -d(2M)^{p-2}\mathrm{e}^{(p-2)K_0T}\sum_{y\sim x^*,y\in\Omega^0}\omega(x^*,y)v(x^*,t)-(K_0+K)v(x^*,t)\\&\geq -d(2M)^{p-2}\mathrm{e}^{(p-2)K_0T}D_\omega(x^*)v(x^*,t)-(K_0+K)v(x^*,t)\\&\geq -\left[d(2M)^{p-2}\mathrm{e}^{(p-2)K_0T}D_\omega(x^*)+(K_0+K)\right]v(x^*,t)\ \text{for } t\in(0,T].\end{aligned} \quad (11)$$

Since $v(x^*,0)=u(x^*,0)>0$, (11) implies that

$$v(x^*,t)\geq v(x^*,0)\mathrm{e}^{-\left[d(2M)^{p-2}\mathrm{e}^{(p-2)K_0T}D_\omega(x^*)+(K_0+K)\right]t}>0\ \text{for } t\in(0,T]. \quad (12)$$

We prove the result by contradiction. If $u(x,t)>0$ in $\Omega^0\times[0,T]$ cannot hold, there would exists a point $(x_0,t_0)\in\Omega^0\times[0,T]$ such that $u(x_0,t_0)=0$, which implies $v(x_0,t_0)=\min_{x\in\Omega}\min_{t\in[0,T]}v(x,t)=0$. By (7), we have

$$\left.\left(\frac{\partial v}{\partial t}-d\mathrm{e}^{(p-2)K_0t_0}\Delta_\omega^p v+(K_0+K)v\right)\right|_{(x_0,t_0)}\geq 0. \quad (13)$$

Since v is differential with respect to t in $\Omega\times(0,T]$, it follows that $\left.\frac{\partial v}{\partial t}\right|_{(x_0,t_0)}\leq 0$.

Thus (13) implies that

$$de^{(p-2)K_0t_0}\Delta_\omega^p v(x_0,t_0)\le \frac{\partial v}{\partial t}\Big|_{(x_0,t_0)}+(K_0+K)v(x_0,t_0)\le 0 \tag{14}$$

By (2), we also have $\Delta_\omega^p v(x_0,t_0)\ge 0$. In view of $v(x_0,t_0)=0$ and $v(x,t)\ge 0$, we have $\Delta_\omega^p v(x_0,t_0)=\sum_{y\sim x_0,y\in\Omega^0} v^{p-1}(y,t_0)\omega(x_0,y)=0$. The above inequality implies that

$$v(y,t_0)=0 \text{ for any } y\in\Omega^0 \text{ and } y\sim x_0. \tag{15}$$

On the other hand, since Ω^0 is connected, for $x^*\in\Omega^0$, there exists a path $x_0\sim x_1\sim\cdots\sim x_n=x^*$. By (15), we obtain that $v(x_1,t_0)=0$. Employing the above argument repeatedly, we shall induce $u(x^*,t_0)=0$ in order, which contradicts with (12). The proof is completed.

Using the strong maximum principle, we easily have the following Lemma.

Lemma 2.4 *Suppose that for each* $x\in\Omega^0$, $w(x,\cdot)\in C([0,+\infty))$ *is differentiable in* $(0,+\infty)$. *Assume that* $d>0,\alpha\ge 0,\beta>0$ *are constants. If w satisfies*

$$\begin{cases}\dfrac{\partial w}{\partial t}-d\Delta_\omega^p w\ge(\le)\alpha-\beta w, & (x,t)\in\Omega^0\times(0,+\infty),\\ w(x,0)=w_0(x)\ge(\not\equiv)0, & x\in\Omega^0,\end{cases} \tag{16}$$

or

$$\begin{cases}\dfrac{\partial w}{\partial t}-d\Delta_\omega^p w\ge(\le)w(\alpha-\beta w), & (x,t)\in\Omega^0\times(0,+\infty),\\ w(x,0)=w_0(x)\ge(\not\equiv)0, & x\in\Omega^0,\end{cases} \tag{17}$$

then

$$\liminf_{t\to+\infty} w(x,t)\ge\frac{\alpha}{\beta}\left(\limsup_{t\to+\infty} w(x,t)\le\frac{\alpha}{\beta}\right) \text{ uniformly for } x\in\Omega^0. \tag{18}$$

Lemma 2.5 *Let* (S,I) *be a solution to the system* (4) *defined for* $t\in[0,T]$ *for any* $T\in(0,+\infty)$. *There exist a constant* M *such that* $0\le S(x,t)\le M$, $0\le I(x,t)\le M$ for $(x,t)\in\Omega^0\times[0,T]$.

Proof 4 *Applying Lemma* 2.2 *to the system* (4), *we obtain* $S(x,t)\ge 0$, $I(x,t)\ge 0$ *for* $(x,t)\in\Omega^0\times[0,T]$. *Consequently,* $S+I$ *satisfies* $\dfrac{\partial(S+I)}{\partial t}-d\Delta_\omega^p(S+I)\le\lambda-\mu(S+I)$ *for* $(x,t)\in\Omega^0\times[0,T]$ *with* $(S+I)(x,0)=S_0(x)+I_0(x)$ *for* $x\in\Omega^0$. *By choosing* $M:=\max\left\{\dfrac{\lambda}{\mu},\max_{x\in\Omega_0}(S_0(x)+I_0(x))\right\}$, *and applying Lemma* 2.2 *again, we have* $S(x,t)+I(x,t)\le M$ *for* $(x,t)\in\Omega^0\times[0,T]$.

Owing to the priori estimate of Lemma 2.5, we present the following global existence theorem.

Theorem 2.6 *System* (4) *possesses a unique solution for all* $t\in[0,+\infty)$.

3. Global Stability of the Disease-Free and Endemic Equilibria

Theorem 3.1 *Define the basic reproduction number* $r_0:=\dfrac{\lambda\beta}{\mu(\mu+\alpha)}$, *and the* *endemic equilibrium* (S^*, I^*) *with* $S^* = \frac{\mu+\alpha}{\beta}$, $I^* = \frac{\lambda}{\mu+\alpha} - \frac{\mu}{\beta}$. *The disease-free equilibrium* $\left(\frac{\lambda}{\mu}, 0\right)$ *is globally asymptotically stable if* $r_0 < 1$, *but the endemic equilibrium* (S^*, I^*) *is globally asymptotically stable if* $r_0 > 1$.

Proof 5 We first prove the case of the disease-free equilibrium $\left(\frac{\lambda}{\mu}, 0\right)$. By Lemma 2.5, we have $0 \le S(x,t), I(x,t) \le M$ for $(x,t) \in \Omega^0 \times [0, +\infty)$. Then we find that *S* satisfies

$$\begin{cases} \frac{\partial S}{\partial t} - d\Delta_\omega^p S \le \lambda - \mu S, & (x,t) \in \Omega^0 \times (0, +\infty), \\ S(x,0) = S_0(x) \not\equiv 0, & x \in \Omega^0. \end{cases} \tag{19}$$

Applying Lemma 2.4, for any small $\varepsilon_1 > 0$, there exists $t_1 > 0$ such that

$$S(x,t) < \frac{\lambda}{\mu} + \varepsilon_1, \text{ for } (x,t) \in \Omega^0 \times [t_1, +\infty). \tag{20}$$

Moreover, we have

$$\limsup_{t\to+\infty} S(x,t) \le \frac{\lambda}{\mu} \text{ uniformly for } x \in \Omega^0. \tag{21}$$

Plugging (20) into system (4), we see that *I* satisfies

$$\begin{cases} \frac{\partial I}{\partial t} - d\Delta_\omega^p I \le I\left(\frac{\beta\lambda}{\mu} + \beta\varepsilon_1 - \mu - \alpha\right), & (x,t) \in \Omega^0 \times (t_1, +\infty), \\ I(x,t_1) = I(x,t_1), & x \in \Omega^0. \end{cases}$$

Since $r_0 < 1$, we choose $\varepsilon_1 < \frac{\mu(\mu+\alpha) - \beta\lambda}{\mu\beta}$ such that $\frac{\beta\lambda}{\mu} + \beta\varepsilon_1 - \mu - \alpha < 0$. Following Lemma 2.4, we have $\limsup_{t\to+\infty} I(x,t) \le 0$ uniformly in $x \in \Omega^0$. In view of the positivity of *I*, we have $\lim_{t\to+\infty} I(x,t) = 0$ uniformly in $x \in \Omega^0$. Consequently, for any small $\varepsilon_2 > 0$, there exists $t_2 > t_1$ such that $I(x,t) < \varepsilon_2$, for $(x,t) \in \Omega^0 \times [t_2, +\infty)$. Plugging this into system (4), we see that *S* satisfies

$$\begin{cases} \frac{\partial S}{\partial t} - d\Delta_\omega^p S \le \lambda - S(\mu + \beta\varepsilon_2), & (x,t) \in \Omega^0 \times (t_2, +\infty), \\ S(x,t_2) = S(x,t_2), & x \in \Omega^0. \end{cases}$$

It follows from Lemma 2.4 that, for any small $\varepsilon_3 > 0$, there exists $t_3 > t_2$ such that $S(x,t) > \frac{\lambda}{\mu + \beta\varepsilon_2} + \varepsilon_3$ for $(x,t) \in \Omega^0 \times [t_3, +\infty)$. Due to the arbitrariness of ε_2 and ε_3, it follows immediately that

$$\liminf_{t\to+\infty} S(x,t) > \frac{\lambda}{\mu} \text{ uniformly for } x \in \Omega^0. \tag{22}$$

Combining (21) with (22), we obtain $\lim_{t\to+\infty} S(x,t) = \frac{\lambda}{\mu}$ uniformly for $x \in \Omega^0$. Thus we prove that $\left(\frac{\lambda}{\mu}, 0\right)$ is globally asymptotically stable.

We show the endemic equilibrium $\left(S^*,I^*\right)$ by using Lyapunov functions. Define a Lyapunov function

$$E(t)=\sum_{x\in\Omega^0}\left(S(x,t)-S^*-S^*\ln\left(\frac{S(x,t)}{S^*}\right)\right)+\sum_{x\in\Omega^0}\left(I(x,t)-I^*-I^*\ln\left(\frac{I(x,t)}{I^*}\right)\right).$$

Then $E(t)\geq 0$ for all $t\geq 0$, and $E(t)=0$ if and only if $(S,I)=\left(S^*,I^*\right)$. We can compute

$$\begin{aligned}E'(t)=&\sum_{x\in\Omega^0}\left(1-\frac{S^*}{S}\right)\left(d\Delta_\omega^p S+\left(\lambda-\mu S-\beta SI\right)\right)\\&+\sum_{x\in\Omega^0}\left(1-\frac{I^*}{I}\right)\left(d\Delta_\omega^p I+\left(\beta SI-(\mu+\alpha)I\right)\right).\end{aligned}\tag{23}$$

In view of Lemma 2.1, it is easy to see that $\sum_{x\in\Omega^0}\Delta_\omega^p S=0$, thus we have

$$\begin{aligned}&\sum_{x\in\Omega^0}\left(1-\frac{S^*}{S}\right)\Delta_\omega^p S=-\sum_{x\in\Omega^0}\frac{S^*}{S}\Delta_\omega^p S\\&=\frac{S^*}{2}\sum_{x\in\Omega^0}\left(\frac{1}{S(y)}-\frac{1}{S(x)}\right)\left|S(y)-S(x)\right|^{p-2}\left(S(y)-S(x)\right)\omega(x,y)\\&=-\frac{S^*}{2}\sum_{x\in\Omega^0}\frac{\left|S(y)-S(x)\right|^p}{S(x)S(y)}\omega(x,y)\leq 0.\end{aligned}\tag{24}$$

Similarly, we have $\sum_{x\in\Omega^0}\left(1-\frac{I^*}{I}\right)\Delta_\omega^p I\leq 0$. Plugging this and (24) into (23), we have

$$\begin{aligned}E'(t)&\leq\sum_{x\in\Omega^0}\left(1-\frac{S^*}{S}\right)(\lambda-\mu S-\beta SI)+\sum_{x\in\Omega^0}\left(1-\frac{I^*}{I}\right)\left(\beta SI-(\mu+\alpha)I\right)\\&\leq\sum_{x\in\Omega^0}\left[-\frac{\mu\left(S-S^*\right)^2}{S}-\beta\left(S-S^*\right)\left(I-I^*\right)-\frac{\beta I^*\left(S-S^*\right)^2}{S}+\beta\left(S-S^*\right)\left(I-I^*\right)\right]\\&\leq-\sum_{x\in\Omega^0}\frac{\mu+\beta I^*}{S}\left(S-S^*\right)^2\leq 0\end{aligned}\tag{25}$$

for all $S,I\geq 0$. By applying the Lyapunov-LaSalle invariance principle [8], we have $\lim_{t\to+\infty}(S,I)=\left(S^*,I^*\right)$ uniformly for $x\in\Omega^0$, which complete the proof.

4. Conclusion

Define the basic reproduction number $r_0:=\frac{\lambda\beta}{\mu(\mu+\alpha)}$, and the endemic equilibrium $\left(S^*,I^*\right)$ with $S^*=\frac{\mu+\alpha}{\beta}$, $I^*=\frac{\lambda}{\mu+\alpha}-\frac{\mu}{\beta}$. We prove that the disease-free equilibrium $\left(\frac{\lambda}{\mu},0\right)$ is globally asymptotically stable if $r_0<1$, while the endemic equilibrium $\left(S^*,I^*\right)$ is globally asymptotically stable if $r_0>1$. Our results extend the stability results of SIR models with graph Laplacian ($p=2$) studied in [2] to general graph *p*-Laplacian with $p>2$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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