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Electron Mass in an Atom Is Less than Rest Mass

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Abstract

Einstein's energy-momentum relationship is a formula that typifies the special theory of relativity (STR). According to the STR, when the velocity of a moving body increases, so does the mass of the body. The STR asserts that the mass of a body depends of the velocity at which the body moves. However, when energy is imparted to a body, this relation holds because kinetic energy increases. When the motion of an electron in an atom is discussed at the level of classical quantum theory, the kinetic energy of the electron is increased due to the emission of energy. At this time, the relativistic energy of the electron decreases, and the mass of the electron also decreases. The STR is not applicable to an electron in an atom. This paper derives an energy-momentum relationship applicable to an electron in an atom. The formula which determines the mass of an electron in an atom is also derived by using that relationship.

Keywords

Einstein's Energy-Momentum Relationship, Relativistic Energy, Electron Mass, Bohr's Quantum Condition, Potential Energy

1. Introduction

According to the special theory of relativity (STR), the following relation holds between the energy and momentum of a body moving in free space [1].

$$\left(mc^2\right)^2=\left(m_0c^2\right)^2+c^2p^2. \tag{1}$$

Here, m_0c^2 is the rest mass energy of the body. And mc^2 is the relativistic energy.

In the STR, there is the following relationship between m_0c^2 and mc^2.

$$mc^2=\frac{m_0c^2}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}. \tag{2}$$

The STR predicts that, when the velocity of a moving body increases, the mass will also increase. However, it is more accurate to say that when the velocity of a body increases, the mass also increases because the relativistic energy increases.

Incidentally, Einstein and Sommerfeld defined the relativistic kinetic energy K_{re} as follows [2].

$$K_{\mathrm{re}} = mc^2 - m_0 c^2. \tag{3}$$

The "re" subscript of K_{re} stands for "relativistic".

The relativistic kinetic energy of a body is given by the difference between the relativistic energy and rest mass energy of the body.

Today, almost all physicists believe that the STR is applicable even in the atom. They also believe that, when the kinetic energy of an electron in an atom increases, the mass of the electron also increases. However, this is totally wrong.

In Section 2 of this paper, it is confirmed that, even when the kinetic energy of an electron in an atom increases, the relativistic energy decreases. Then, in Section 4, a formula applicable to an electron in an atom is derived to take the place of Formula (2) for STR mass.

2. Energies of an Electron in a Hydrogen Atom

Consider the situation where an electron is at rest at a point in free space. According to the uncertainty principle, it is impossible for an electron to be absolutely stationary, but this paper does not touch on that problem.

Now, consider a situation where this electron is attracted electrostatically by a proton (nucleus of a hydrogen atom), and a hydrogen atom is formed.

An electron that has entered the region of a hydrogen atom increases its kinetic energy in a discrete manner. Also, it emits a photon with the same energy as the increased amount of kinetic energy.

The following relationship holds if the energy of a photon emitted from an electron is taken to be $h\nu$.

$$\Delta K = h\nu. \tag{4}$$

Here, ΔK indicates the increase in kinetic energy.

Now, what is the source which supplies the kinetic energy acquired by the electron, and the energy of the emitted photon? In classical quantum theory, the total mechanical energy of the hydrogen atom is indicated as follows.

$$E_n = K_n + V(r_n), \quad E_n < 0. \tag{5}$$

Here, n is the principal quantum number.

Here, the energy becomes negative because energy is taken to be zero when the electron is at rest at a position infinitely distant from the proton. However, the electron in this state actually has a rest mass energy of $m_{\mathrm{e}}c^2$.

For the law of conservation of energy to hold, the following relation must hold between these energies.

$$V(r) + h\nu + K = 0. \tag{6}$$

Formula (6) shows that the source of the electron's kinetic energy, and the energy of the emitted photon, is the potential energy of the electron. However, potential energy has a name, but no real substance.

The author presented the following formula as a formula indicating the relationship between the rest mass energy and potential energy of the electron in a hydrogen atom [3] [4].

$$V(r) = -\Delta m_e c^2. \tag{7}$$

Here, the reduction in rest mass energy of the electron was expressed as $-\Delta m_e c^2$.

Also, if the law of energy conservation is taken into account, then the following relationship holds.

$$-\Delta m_e c^2 + h\nu + K = 0. \tag{8}$$

When describing the motion of a bound electron in a hydrogen atom, a term must be included in that equation for the potential energy. From this $E_{\text{ab},n}$ and $m_n c^2$ can be defined as follows.

$$E_{\text{ab},n} = m_n c^2 = m_e c^2 + V(r_n) + K_{\text{re},n}, \quad n = 1, 2, \cdots. \tag{9}$$

Here, m_n is the relativistic mass of the electron. $E_{\text{ab},n}$ gives the relativistic energy of the electron, but this is also the absolute energy of the electron. The "ab" subscript of $E_{\text{ab},n}$ stands for "absolute". Also, this paper defines $K_{\text{re},n}$ as the relativistic kinetic energy of an electron.

The following relation also holds when Formulas (8) and (9) are taken into account.

$$m_n c^2 = m_e c^2 - K_{\text{re},n} = m_e c^2 + \frac{V(r_n)}{2} = m_e c^2 - \frac{\Delta m_e c^2}{2}. \tag{10}$$

The relativistic energy of an electron in a hydrogen atom $m_n c^2$ becomes smaller than the rest mass energy $m_e c^2$. That is,

$$m_n c^2 < m_e c^2. \tag{11}$$

The behavior of an electron inside an atom, where there is potential energy, cannot be described with the relationship of Einstein (1). Caution is necessary because it is completely overlooked in Formula (11).

Now, referring to Formula (3), it is natural to define the relativistic kinetic energy of an electron in a hydrogen atom as follows [5].

$$K_{\text{re},n} = -E_{\text{re},n} = m_e c^2 - m_n c^2. \tag{12}$$

This paper defines $E_{\text{re},n}$ as the relativistic energy levels of the hydrogen atom.

The relationships of these energies of an electron in a hydrogen atom are as indicated in the following **Figure 1**.

3. Kinetic Energy of a Body Derived from Einstein's Energy-Momentum Relationship

If the relativistic kinetic energy of a body is defined with Formula (3), then it is possible to derive yet another formula for K_{re}.

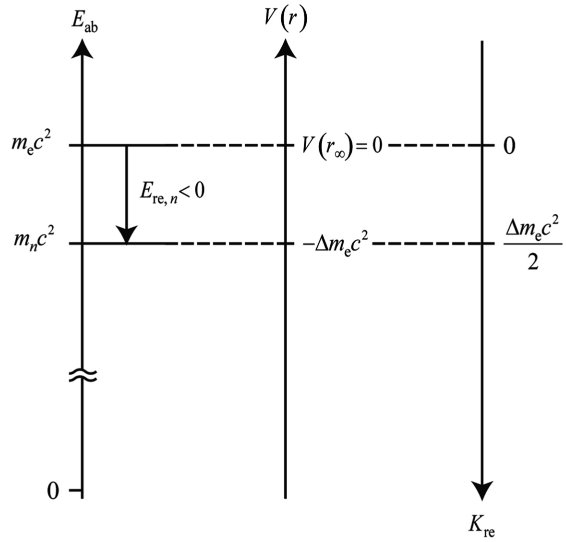

Figure 1. $E_{\text{re},n}$ describes how much the relativistic energy of the electron has decreased from the rest mass energy ($E_{\text{re},n}$ corresponds to 1/2 the decrease in rest mass energy of the electron). In contrast, $m_n c^2$ corresponds to the remainder of the relativistic energy of the electron.

Now, Formula (1) is rewritten as follows.

$$\left(mc^2\right)^2 = m_0^2 c^4 + \left(m^2 c^4 - m_0^2 c^4\right). \tag{13}$$

Comparing Formulas (1) and (13), the relativistic momentum p_{re} can be defined as follows.

$$p_{\text{re}}^2 = m^2 c^2 - m_0^2 c^2. \tag{14}$$

Hence,

$$p_{\text{re}}^2 = \left(m + m_0\right)\left(mc^2 - m_0 c^2\right). \tag{15}$$

The following relation holds due to Formulas (3) and (15).

$$K_{\text{re}} = \frac{p_{\text{re}}^2}{m + m_0}. \tag{16}$$

Based on the above discussion, it was found that the relativistic kinetic energy of a body moving in isolated systems in free space can be described with Formulas (3) and (16).

4. An Energy-Momentum Relationship Applicable to an Electron in a Hydrogen Atom

Einstein's energy-momentum relationship (1) holds when the energy absorbed by a body changes into kinetic energy of that body. However, an electron in a hydrogen atom increases its kinetic energy by emitting energy. The STR is not applicable in this kind of situation.

Thus, referring to Formula (16), we assume the relativistic kinetic energy of an electron in a hydrogen atom to be as follows.

$$K_{\mathrm{re},n}=\frac{p_{\mathrm{re},n}^{2}}{m_{\mathrm{e}}+m_{n}}. \tag{17}$$

Then the right sides of Formula (12) and (17) are joined, yielding the following.

$$m_{\mathrm{e}}c^{2}-m_{n}c^{2}=\frac{p_{\mathrm{re},n}^{2}}{m_{\mathrm{e}}+m_{n}}. \tag{18}$$

Rearranging, the following relation is derived.

$$\left(m_{n}c^{2}\right)^{2}+c^{2}p_{\mathrm{re},n}^{2}=\left(m_{\mathrm{e}}c^{2}\right)^{2}. \tag{19}$$

This energy-momentum relationship is applicable to an electron in a hydrogen atom. Please see the other paper for the mathematical derivation of Formula (19) [6] [7].

Next, the relation of m_{e} and m_{n} is found from Formula (19).

Using the formula for momentum $p_{\mathrm{re},n}=m_{n}v_{n}$, Formula (19) is as follows.

$$c^{2}\left(m_{n}v_{n}\right)^{2}+\left(m_{n}c^{2}\right)^{2}=\left(m_{\mathrm{e}}c^{2}\right)^{2}. \tag{20}$$

If Formula (20) is solved for $m_{n}c^{2}$, the following relation is obtained.

$$m_{n}c^{2}=\frac{m_{\mathrm{e}}c^{2}}{\left(1+\frac{v_{n}^{2}}{c^{2}}\right)^{1/2}}. \tag{21}$$

The mass of the electron in a hydrogen atom decreases when its velocity increases.

Incidentally, the discreteness of energy must be incorporated into Formula (21) to make it into a formula of quantum theory.

Previously, the author has derived the fact that the following relation holds (Appendix)

$$\frac{v_{n}}{c}=\frac{\alpha}{n}. \tag{22}$$

By using the relation in Formula (22), Formula (21) can be written as follows.

$$m_{n}c^{2}=\frac{m_{\mathrm{e}}c^{2}}{\left(1+\frac{\alpha^{2}}{n^{2}}\right)^{1/2}}. \tag{23}$$

5. Discussion

The STR tells us that, when the velocity of a moving body increases, so does the mass of that body. In the STR, the mass of a body depends on the velocity of that body. That is true is in certain sense, but, strictly speaking, not correct.

If the velocity of the body increases in Formula (2), the relativistic energy of the body increases, and thus the mass of the body also increases.

In the STR, the relativistic energy increases together with the velocity of a body, and thus there was no problem, even with the previous explanation. However, increase or decrease in mass depends not on the body's velocity, but on its relativistic energy.

In the case of an electron in a hydrogen atom, kinetic energy increases but relativistic energy decreases when velocity increases.

Here, we confirm that what determines the mass of a body is not the velocity of the body but rather its relativistic energy.

6. Conclusions

The relativistic energy of an electron in a hydrogen atom is less than the rest mass energy. That is,

$$m_n c^2 < m_{\mathrm{e}} c^2. \tag{24}$$

Einstein's relationship (1) is not applicable to an electron in an atom.

The following formula is an energy-momentum relationship applicable in a hydrogen atom.

$$\left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{25}$$

The following formula can be derived by solving Formula (25) for m_n.

$$m_n c^2 = \frac{m_{\mathrm{e}} c^2}{\left(1 + \frac{v_n^2}{c^2}\right)^{1/2}}. \tag{26}$$

However, the discreteness of energy that is characteristic of quantum theory is not incorporated into Formula (26).

Thus the following formula was arrived at by using the relation in Formula (22).

$$m_n = \frac{m_{\mathrm{e}}}{\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2}}. \tag{27}$$

Generally speaking, it is thought that the mass of a certain body depends on the velocity at which that body moves. However, what actually determines the mass of a body is not velocity but the relativistic energy of that body.

Even if an electron in a hydrogen atom increases in velocity and kinetic energy, it's relativistic energy decreases, and thus the mass of the electron becomes less than the rest mass.

However, the only quantum number included in Formula (27) is the principal quantum number n. In the future, Formula (27) will have to be improved to a formula including quantum numbers other than n.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

Formula (22) has already been derived in a past paper [8] [9]. Therefore, an explanation is provided in the Appendix of this paper.

Bohr's orbital radius $r_{\mathrm{BO},n}$ is normally described with the following formula.

$$r_{\mathrm{BO},n} = 4\pi\varepsilon_0 \frac{\hbar^2}{m_\mathrm{e} e^2} n^2. \tag{A1}$$

Bohr thought the following quantum condition was necessary to find the energy levels of the hydrogen atom.

$$m_\mathrm{e} v_n \cdot 2\pi r_{\mathrm{BO},n} = 2\pi n\hbar. \tag{A2}$$

In Bohr's theory, the energy levels of the hydrogen atom is treated non-relativistically, and thus here the momentum of the electron is taken to be $m_\mathrm{e} v$. Also, the Planck constant $\hbar$ can be written as follows [10].

$$\hbar = \frac{h}{2\pi} = \frac{m_\mathrm{e} c \lambda_\mathrm{C}}{2\pi}. \tag{A3}$$

λ_C is the Compton wavelength of the electron.

When Formula (A3) is used, the fine-structure constant α can be expressed as follows.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{e^2}{2\varepsilon_0 m_\mathrm{e} c^2 \lambda_\mathrm{C}}. \tag{A4}$$

Also, the classical electron radius r_e is defined as follows.

$$r_\mathrm{e} = \frac{e^2}{4\pi\varepsilon_0 m_\mathrm{e} c^2}. \tag{A5}$$

If r_e/α is calculated here,

$$\frac{r_\mathrm{e}}{\alpha} = \frac{\lambda_\mathrm{C}}{2\pi}. \tag{A6}$$

If Formula (A1) is written using r_e and α, the result is as follows.

$$r_{\mathrm{BO},n} = 4\pi\varepsilon_0 \frac{\hbar^2}{m_\mathrm{e} e^2} n^2 = \frac{e^2}{4\pi\varepsilon_0 m_\mathrm{e} c^2} \left(\frac{4\pi\varepsilon_0 \hbar c}{e^2} \right)^2 n^2 = \frac{r_\mathrm{e}}{\alpha^2} n^2. \tag{A7}$$

Formula (A7) containing r_e is superior to Formula (A1) from a physical standpoint.

Next, if $\hbar$ in Formula (A3) and $r_{\mathrm{BO},n}$ in Formula (A7) are substituted into Formula (A2),

$$m_\mathrm{e} v_n \cdot 2\pi \frac{r_\mathrm{e}}{\alpha^2} n^2 = 2\pi n \frac{m_\mathrm{e} c \lambda_\mathrm{C}}{2\pi}. \tag{A8}$$

If Formula (A6) is also used, then Formula (A8) can be written as follows.

$$m_\mathrm{e} v_n \cdot 2\pi \frac{r_\mathrm{e}}{\alpha^2} n^2 = 2\pi n \frac{m_\mathrm{e} c r_\mathrm{e}}{\alpha}. \tag{A9}$$

From this, the following relationship can be derived.

$$\frac{v_n}{c} = \frac{\alpha}{n}. \tag{A10}$$

Due to Formula (A10), it is possible to identify discontinuous states that are permissible in terms of quantum mechanics in the continuous motions of classical theory.