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Existence and Stability of Solutions for a Class of Fractional Impulsive Differential Equations with Atangana-Baleanu-Caputo Derivative

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Abstract

In this paper, we are concerned with the existence of solutions to a class of Atangana-Baleanu-Caputo impulsive fractional differential equation. The existence and uniqueness of the solution of the fractional differential equation are obtained by Banach and Krasnoselakii fixed point theorems, and sufficient conditions for the existence and uniqueness of the solution are also developed. In addition, the Hyers-Ulam stability of the solution is considered. At last, an example is given to illustrate the main results.

Keywords

Fractional Differential Equation, Fixed Point Theorem, Existence of Solutions

1. Introduction

Fractional differential equations are generalizations of integer differential equations and more accurate than integer models in reflecting certain properties of things [1]-[6]. Fractional differential equations have a wide range of applications in the fields of fluid dynamics, stochastic equations, and control systems [7] [8] [9] [10] [11].

Atangana-Baleanu-Caputo (ABC) is a fractional derivative centered on the Mittag-Leffler function proposed by Atangana and Baleanu [12]. The nondeterministic nature of ABC derivatives with Mittag-Leffler kernel can effectively take into account the nondeterministic dynamics and more appropriately capture various features of real systems [13] [14] [15] [16]. For instance, Salari and Ghanbari [17] studied the existence and multiplicity of the following ABC fractional derivative boundary value problems

$$\begin{cases} {}^{ABC}\mathcal{D}_1^{\gamma}\left({}^{ABC}\mathcal{D}_{\tau}^{\gamma}y(\tau)\right)-vh(\tau,y(\tau))=0,\tau\in[0,1],\\ y(0)=y(1)=0, \end{cases}$$

where ${}^{ABC}\mathcal{D}_1^{\gamma}$ is ABC fractional derivative, $\frac{1}{2}<\gamma<1$, $v>0$, $h:[0,1]\times\mathbb{R}\to\mathbb{R}$ is L^1-Caratheodory function. The existence and numerical estimation of solutions to the equations are investigated by means of variational methods.

In addition, impulse differential equations are suitable models for describing real processes that rapidly deviate from the state at a specific moment in time, which cannot be expressed by classical differential equations. Impulse differential equations have a wide range of applications in many fields such as physics, management science, population dynamics and so on [18] [19] [20]. Some physical problems have sudden changes and discontinuous jumps, and in order to model and study these problems in depth, many researchers and scientists have taken a keen interest in the study of fractional differential equations with impulses. For example, Yang and Zhao [21] studied existence and optimal controls of the following non-autonomous impulsive integro-differential evolution equation

$$\begin{cases} x'(t)=A(t)x(t)+f(t,x(t),Gx(t))+B(t)u(t),t\in J:=[0,b],t\neq t_k,\\ \Delta x(t_k)=x(t_k^+)-x(t_k^-)=I_k(x(t_k)),k=1,2,\cdots,m,\\ x(0)+g(x)=x_0, \end{cases}$$

where $A(t):D(A)\subseteq X\to X$ is a family of densely defined and closed linear opertor which generates an evolution system $\{U(t,s):0\le s\le t\le b\}$ on *X*, $\{B(t):t\ge 0\}$ is a family of linear operators from *Y* to *X*. The existence of weak solutions to the equation was proved through the Krasnoselakii's fixed point theorem. Subsequently, a minimization sequence was constructed to prove the existence of optimal control pairs for differential evolution control systems.

Nowadays, although some achievements have been made in the study of fractional differential equations with impulse conditions, few studies have been made on the existence of solutions to fractional differential equations with Atangana-Baleanu-Caputo derivative. Therefore, inspired from the above contributions, the existence of solutions to a class of fractional differential equations of type ABC with impulse conditions is studied on the following problem

$$\begin{cases} {}^{ABC}_{\ \ 0}D_t^{\alpha}u(t)=f\left(t,u(t),{}^{ABC}_{\ \ 0}D_t^{\alpha}u(t)\right),t\in J=[0,1],0<\alpha\le 1,\\ \Delta u(t_k)=I_k(u(t_k)),k=1,2,\cdots,m,\\ au(0)+bu(1)=0, \end{cases} \tag{1}$$

where ${}^{ABC}_{\ \ 0}D_t^{\alpha}$ represents the Atangana-Baleanu-Caputo fractional derivative, and $f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given continuous functions, $\Delta u(t_k)=u(t_k^+)-u(t_k^-)$, and $u(t_k^+),u(t_k^-)$ are the left and right limits of $u(t)$ respectively. Assuming that $u(t)$ is left-continuous at $t=t_k$, that is $u(t_k)=u(t_k^-)$. Let $I_k\in C(J,\mathbb{R})$,

$0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, and $J_0=[0,t_1], J_1=(t_1,t_2],\cdots,J_m=(t_m,1]$. We obtain that there is at least one solution by the Banach and Krasnoselakii fixed point theorems. In addition, an example is given to illustrate the validity of the main results.

The rest of this paper is organized as follows. In Section 2, some basic definitions, lemmas, and theorems are presented. Then in Section 3, the existence of a solution for the nonlinear fractional differential equation is obtained by Banach and Krasnoselakii fixed point theorems. Section 4 discusses the Hyers-Ulam stability of solutions. At last, an example is given in the last part to support our study.

2. Preliminaries

In this section, we recollect some basic concepts of fractional calculus and auxiliary lemmas, which will be used throughout this paper. Suppose that $X=C(J,\mathbb{R})$ is a Banach space and equipped with the norm $\|u\|=\max_{t\in J}|u(t)|$.

Definition 2.1 [12] *Let* $u\in H'(c,d)$, $c<d$, *and* $\alpha\in(0,1)$. *The ABC fractional derivative for function u of order* α *is defined as*

$$ {}_{0}^{ABC}D_t^{\alpha}u(t)=\frac{B(\alpha)}{1-\alpha}\int_0^t u'(\xi)E_\alpha\left(\frac{-\alpha(t-\xi)^\alpha}{1-\alpha}\right)\mathrm{d}\xi, $$

where $B(\alpha)$ is a normalized function and $B(0)=B(1)=1$, E_α denotes Mittag-Leffler function.

Definition 2.2 [22] *Let u be a function, then the AB fractional integral of order* $\alpha\in(0,1)$ *is defined by*

$$ {}_{0}^{AB}I_t^{\alpha}u(t)=\frac{1-\alpha}{B(\alpha)}u(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t u(\xi)(t-\xi)^{\alpha-1}\mathrm{d}\xi. $$

Lemma 2.3 [23] *For* $\alpha\in(n,n+1], n=0,1,2,\cdots$, *the following result hold*

$$ {}_{0}^{AB}I_t^{\alpha}\,{}_{0}^{ABC}D_t^{\alpha}u(t)=u(t)+c_0+c_1t+\cdots+c_nt^n, $$

where c_i is unknown arbitrary constants, $i=0,1,\cdots,n$.

Theorem 2.4 [24] (*Banach fixed point theorem*) *Let X be a Banach space,* $G_R\subset X$ *be closed, and* $T:G_R\to G_R$ *is a contraction operator, then* T *has a unique fixed point.*

Theorem 2.5 [25] (*Krasnoselakii fixed point theorem*) *Let W be a nonempty, convex, and closed subset of X. Consider two operators* W_1,W_2 *satisfy*

1) $W_1(v_1)+W_2(v_2)\in W$ for all $v_1,v_2\in W$,

2) W_2 is a contraction operator,

3) W_1 is continuous and compact,

then there exists at least one solution $w\in X$, such that $W_1(w)+W_2(w)=w$.

Lemma 2.6 *Let* $h\in X$, *then the unique solution for the following problem*

$$ \begin{cases} {}_{0}^{ABC}D_t^{\alpha}u(t)=h(t), t\in J=[0,1], 0<\alpha\le 1,\\ \Delta u(t_k)=I_k(u(t_k)), k=1,2,\cdots,m,\\ au(0)+bu(1)=0, \end{cases} \tag{2} $$

is

$$\begin{aligned}u(t)=&\frac{1-\alpha}{B(\alpha)}h(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi\\&+\frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}h(\xi)\mathrm{d}\xi\\&+\frac{b}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi\\&+\frac{b}{a+b}\sum_{i=1}^{m}I_i(u(t_i))\\&+\frac{1-\alpha}{(a+b)B(\alpha)}(ah(0)+bh(1)).\end{aligned}\tag{3}$$

Proof Applying ${}_{0}^{AB}I_t^{\alpha_i}$ to both side of ${}_{0}^{ABC}D_t^{\alpha}u(t)=h(t)$, then the equation (2) can be written in the following equivalent form

$$u(t)={}_{0}^{AB}I_t^{\alpha}h(t)-c_0,$$

where c_0 is a unknown constant.

When $t\in J_0$, from Lemma 2.6 it follows that

$$u(t)=\frac{1-\alpha}{B(\alpha)}h(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t h(\xi)(t-\xi)^{\alpha-1}\mathrm{d}\xi-c_0.$$

When $t\in J_1$,

$$u(t)=\frac{1-\alpha}{B(\alpha)}h(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^t h(\xi)(t-\xi)^{\alpha-1}\mathrm{d}\xi-d_0,$$

$$u(t_1^-)=\frac{1-\alpha}{B(\alpha)}h(t_1^-)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t h(\xi)(t-\xi)^{\alpha-1}\mathrm{d}\xi-c_0,$$

$$u(t_1^+)=\frac{1-\alpha}{B(\alpha)}h(t_1^+)-d_0,$$

from $\Delta u(t_1)=u(t_1^+)-u(t_1^-)=I_1(u(t_1))$, we get

$$\begin{aligned}&\frac{1-\alpha}{B(\alpha)}h(t_1^+)-d_0\\&=\frac{1-\alpha}{B(\alpha)}h(t_1^-)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{t_1}(t_1-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi-c_0+I_1(u(t_1)),\end{aligned}$$

from the continuity of h it follows that

$$-d_0=\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{t_1}(t_1-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi+I_1(u(t_1))+c_0.$$

Thus, when $t\in J_1$, we have

$$\begin{aligned}u(t)=&\frac{1-\alpha}{B(\alpha)}h(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^t h(\xi)(t-\xi)^{\alpha-1}\mathrm{d}\xi\\&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^{t_1}(t_1-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi+I_1(u(t_1))+c_0,\end{aligned}$$

Using the same method, when $t\in J_k$, we can get

$$u(t)=\frac{1-\alpha}{B(\alpha)}h(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi+\sum_{i=1}^{k}I_i(u(t_i))-c_0. \quad (4)$$

By $au(0)+bu(1)=0$, we have

$$c_0=\frac{a(1-\alpha)}{(a+b)B(\alpha)}h(0)+\frac{b}{a+b}\left\{\frac{1-\alpha}{B(\alpha)h(1)}+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi\right.$$
$$\left.+\frac{1}{B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}h(\xi)\mathrm{d}\xi+\sum_{i=1}^{k}I_i(u(t_i))\right\}.$$

Hence, inserting the values of c_0, we get the solution (3). This completes the proof. □

3. Main Results

In this section, we transform the problem (1) into the fixed point of the operator *H*, and then prove the existence of solutions of the problem (1) by using Banach and Krasnoselakii fixed point theorems.

In order to obtain the main results of this paper, the following hypotheses must be satisfied.

(H1) There exist constants K_1, K_2 such that for any $u(t), v(t)\in C(J,\mathbb{R})$, we have

$$\left|f(t,u(t),v(t))-f(t,\bar{u}(t),\bar{v}(t))\right|\le K_1(|u-\bar{u}|+|v-\bar{v}|),$$
$$\left|I_k(u(t_k))-I_k(v(t_k))\right|\le K_2|u-v|;$$

(H2) There exist constants A_g, B_g, C_g, such that

$$|f(t,u,v)|\le A_g+B_g|u|+C_g|v|;$$

(H3) Since $I_k(u(t_k))\in C(J,R)$, then there exists a constant M_1, such that

$$\left|I_k(u(t_k))\right|\le M_1.$$

Let $H:X\to X$ be defined by

$$Hu(t)=\frac{1-\alpha}{B(\alpha)}f\left(t,u(t),{}^{ABC}_{0}D^{\alpha}_{t}u(t)\right)$$
$$+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}f\left(\xi,u(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)\right)\mathrm{d}\xi$$
$$+\frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}f\left(\xi,u(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)\right)\mathrm{d}\xi+\frac{b}{a+b}\sum_{i=1}^{m}I_i(u(t_i))$$
$$+\frac{b}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}f\left(\xi,u(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)\right)\mathrm{d}\xi$$
$$+\frac{1-\alpha}{(a+b)B(\alpha)}\left(af\left(0,u(0),{}^{ABC}_{0}D^{\alpha}_{0}u(0)\right)+bf\left(1,u(1),{}^{ABC}_{0}D^{\alpha}_{1}u(1)\right)\right).$$

Theorem 3.1 *Assume that* (H1) hold, then Equation (1) has the unique solution if

$$\frac{2K_1(a+b)(1-\alpha)\Gamma(\alpha+1)+mK_2(1-K_1B(\alpha))\Gamma(\alpha+1)+K_1[(a+b)\alpha+b+m]}{(a+b)(1-K_1)B(\alpha)\Gamma(\alpha+1)}<1.$$

Proof In order to prove the uniqueness and existence of the solution to problem (1), we first show that H is a compression operator, assuming that $u,\bar{u}\in X$,

$$\begin{aligned}
&\|Hu-H\bar{u}\| \\
&=\max_{t\in J}|Hu-H\bar{u}| \\
&=\max_{t\in J}\Bigg|\frac{1-\alpha}{B(\alpha)}\Big[f\big(t,u(t),{}^{ABC}_{0}D^{\alpha}_{t}u(t)\big)-f\big(t,\bar{u}(t),{}^{ABC}_{0}D^{\alpha}_{t}\bar{u}(t)\big)\Big] \\
&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}\Big[f\big(\xi,u(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)\big)-f\big(\xi,\bar{u}(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}\bar{u}(\xi)\big)\Big]\mathrm{d}\xi \\
&+\frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}\Big[f\big(\xi,u(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)\big)-f\big(\xi,\bar{u}(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}\bar{u}(\xi)\big)\Big]\mathrm{d}\xi \\
&+\frac{1}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}\Big[f\big(\xi,u(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)\big) \\
&-f\big(\xi,\bar{u}(\xi),{}^{ABC}_{0}D^{\alpha}_{\xi}\bar{u}(\xi)\big)\Big]\mathrm{d}\xi+\frac{1}{a+b}\sum_{i=1}^{m}\Big[I_i\big(u(t_i)\big)-I_i\big(\bar{u}(t_i)\big)\Big] \\
&+\frac{b(1-\alpha)}{(a+b)B(\alpha)}\Big[f\big(0,u(0),{}^{ABC}_{0}D^{\alpha}_{0}u(0)\big)-f\big(0,\bar{u}(0),{}^{ABC}_{0}D^{\alpha}_{0}\bar{u}(0)\big)\Big] \\
&+\frac{b(1-\alpha)}{(a+b)B(\alpha)}\Big[f\big(1,u(1),{}^{ABC}_{0}D^{\alpha}_{1}u(1)\big)-f\big(1,\bar{u}(1),{}^{ABC}_{0}D^{\alpha}_{1}\bar{u}(1)\big)\Big]\Bigg| \\
&\le\frac{K_1(1-\alpha)}{B(\alpha)}\Big[|u(t)-\bar{u}(t)|+\big|{}^{ABC}_{0}D^{\alpha}_{t}u(t)-{}^{ABC}_{0}D^{\alpha}_{t}\bar{u}(t)\big|\Big] \\
&+\frac{K_1\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}\Big[|u(\xi)-\bar{u}(\xi)|+\big|{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)-{}^{ABC}_{0}D^{\alpha}_{\xi}\bar{u}(\xi)\big|\Big]\mathrm{d}\xi \\
&+\frac{bK_1}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}\Big[|u(\xi)-\bar{u}(\xi)|+\big|{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)-{}^{ABC}_{0}D^{\alpha}_{\xi}\bar{u}(\xi)\big|\Big]\mathrm{d}\xi \\
&+\frac{K_1}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}\Big[|u(\xi)-\bar{u}(\xi)|+\big|{}^{ABC}_{0}D^{\alpha}_{\xi}u(\xi)-{}^{ABC}_{0}D^{\alpha}_{\xi}\bar{u}(\xi)\big|\Big]\mathrm{d}\xi \\
&+\frac{mK_2}{a+b}|u(t)-\bar{u}(t)|+\frac{aK_1(1-\alpha)}{(a+b)B(\alpha)}\Big[|u(0)-\bar{u}(0)|+\big|{}^{ABC}_{0}D^{\alpha}_{0}u(0)-{}^{ABC}_{0}D^{\alpha}_{0}\bar{u}(0)\big|\Big] \\
&+\frac{K_1b(1-\alpha)}{(a+b)B(\alpha)}\Big[|u(1)-\bar{u}(1)|+\big|{}^{ABC}_{0}D^{\alpha}_{1}u(1)-{}^{ABC}_{0}D^{\alpha}_{1}\bar{u}(1)\big|\Big].
\end{aligned}$$

Now, we have

$$\begin{aligned}
\big|{}^{ABC}_{0}D^{\alpha}_{t}u(t)-{}^{ABC}_{0}D^{\alpha}_{t}\bar{u}(t)\big|&=\big|f\big(t,u(t),{}^{ABC}_{0}D^{\alpha}_{t}u(t)\big)-f\big(t,\bar{u}(t),{}^{ABC}_{0}D^{\alpha}_{t}\bar{u}(t)\big)\big| \\
&\le K_1\Big(|u-\bar{u}|+\big|{}^{ABC}_{0}D^{\alpha}_{t}u(t)-{}^{ABC}_{0}D^{\alpha}_{t}\bar{u}(t)\big|\Big),
\end{aligned}$$

which implies

$$\big|{}^{ABC}_{0}D^{\alpha}_{t}u(t)-{}^{ABC}_{0}D^{\alpha}_{t}\bar{u}(t)\big|\le\frac{K_1}{1-K_1}|u-\bar{u}|. \tag{5}$$

From Equation (5), we get

$$
\begin{aligned}
&\left\|Hu-H\bar{u}\right\| \\
&\le \left|\frac{K_1(1-\alpha)}{B(\alpha)}\left[|u-\bar{u}|+\frac{K_1}{1-K_1}|u-\bar{u}|\right]\right. \\
&+\frac{K_1\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}\left[|u-\bar{u}|+\frac{K_1}{1-K_1}|u-\bar{u}|\right]\mathrm{d}\xi \\
&+\frac{bK_1}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}\left[|u-\bar{u}|+\frac{K_1}{1-K_1}|u-\bar{u}|\right]\mathrm{d}\xi \\
&+\frac{K_1}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\left[|u-\bar{u}|+\frac{K_1}{1-K_1}|u-\bar{u}|\right]\mathrm{d}\xi \\
&+\frac{mK_2}{a+b}|u-\bar{u}|+\frac{aK_1(1-\alpha)}{(a+b)B(\alpha)}\left[|u-\bar{u}|+\frac{K_1}{1-K_1}|u-\bar{u}|\right] \\
&\left.+\frac{bK_1(1-\alpha)}{(a+b)B(\alpha)}\left[|u-\bar{u}|+\frac{K_1}{1-K_1}|u-\bar{u}|\right]\right\| \\
&\le\frac{2K_1(a+b)(1-\alpha)\Gamma(\alpha+1)+mK_2(1-K_1B(\alpha))\Gamma(\alpha+1)}{(a+b)(1-K_1)B(\alpha)\Gamma(\alpha+1)}|u-\bar{u}| \\
&+\frac{K_1\left[(a+b)\alpha+b+m\right]}{(a+b)(1-K_1)B(\alpha)\Gamma(\alpha+1)}|u-\bar{u}|.
\end{aligned}
$$

Thus, the operator H is a contraction map, by Theorem 2.4, the problem (1) has a unique solution. □

By theorem 3.1, H is defined under the consideration of Krasnoselskii's fixed point theorem as follows:

$$Hu(t)=W_1u(t)+W_2u(t),$$

where

$$
\begin{aligned}
W_1u(t)=&\frac{1-\alpha}{B(\alpha)}f\left(t,u(t),{}^{ABC}_{0}D_t^{\alpha}u(t)\right)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}f\left(\xi,u(\xi),{}^{ABC}_{0}D_\xi^{\alpha}u(\xi)\right)\mathrm{d}\xi \\
&+\frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}f\left(\xi,u(\xi),{}^{ABC}_{0}D_\xi^{\alpha}u(\xi)\right)\mathrm{d}\xi \\
&+\frac{1-\alpha}{(a+b)B(\alpha)}\left(af\left(0,u(0),{}^{ABC}_{0}D_0^{\alpha}u(0)\right)+bf\left(1,u(1),{}^{ABC}_{0}D_1^{\alpha}u(1)\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
W_2u(t)=&\frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}f\left(\xi,u(\xi),{}^{ABC}_{0}D_\xi^{\alpha}u(\xi)\right)\mathrm{d}\xi \\
&+\frac{b}{a+b}\sum_{i=1}^{m}I_i\left(u(t_i)\right).
\end{aligned}
$$

Theorem 3.2 *Assume that* (H1)-(H2) hold, then system (1) has at least one solution with the condition

$$\frac{bmK_1}{(a+b)B(\alpha)\Gamma(\alpha+1)(1-K_1)}+\frac{bmK_2}{a+b}<1.$$

Proof The proof is divided into two steps.

Step 1: W_2 is a compression map.

Let $W_2 = \{u \in X : \|u\| \le r\}$ be a closed bounded set and $u, \bar{u} \in W_2$. Now

$$\begin{aligned}
&\|W_2 u - W_2 \bar{u}\| \\
&= \max_{t\in J} |W_2 u(t) - W_2 \bar{u}(t)| \\
&= \frac{b}{(a+b)B(\alpha)\Gamma(\alpha)} \sum_{t_i-1}^{t_i} \int_{t_i}^{1} (t_i - \xi)^{\alpha-1} \Big| f\left(\xi, u(\xi), {}^{ABC}_{\ 0}D^{\alpha}_{\xi} u(\xi)\right) \\
&\quad - f\left(\xi, \bar{u}(\xi), {}^{ABC}_{\ 0}D^{\alpha}_{\xi} \bar{u}(\xi)\right)\Big| \mathrm{d}\xi + \frac{b}{a+b}\sum_{i=1}^{m} \left| I_i(u(t_i)) - I_i(\bar{u}(t_i)) \right| \\
&\le \frac{bm}{(a+b)B(\alpha)\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{\alpha-1} \left[K_1 \left(|u - \bar{u}| + \left| {}^{ABC}_{\ 0}D^{\alpha}_{t} u - {}^{ABC}_{\ 0}D^{\alpha}_{t} \bar{u} \right| \right) \right] \mathrm{d}\xi \\
&\quad + \frac{bmK_2}{a+b} |u - \bar{u}| \\
&\le \frac{bm}{(a+b)B(\alpha)\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} (t_i - \xi)^{\alpha-1} \left[K_1 \left(|u - \bar{u}| + \frac{K_1}{1-K_1} \right) \right] \mathrm{d}\xi + \frac{bmK_2}{a+b} |u - \bar{u}| \\
&\le \left[\frac{bmK_1}{(a+b)B(\alpha)\Gamma(\alpha+1)(1-K_1)} + \frac{bmK_2}{a+b} \right] |u - \bar{u}|.
\end{aligned}$$

Hence, W_2 is a contraction map.

Step 2: W_1 is continuous and compact.

Firstly, we prove that W_1 is bounded.

$$\begin{aligned}
\|W_1 u\| &= \max_{t\in J} |W_1 u(t)| \\
&= \max_{t\in J} \Bigg| \frac{1-\alpha}{B(\alpha)} f\left(t, u(t), {}^{ABC}_{\ 0}D^{\alpha}_{t} u(t)\right) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{t_1}^{t} f\left(\xi, u(\xi), {}^{ABC}_{\ 0}D^{\alpha}_{\xi} u(\xi)\right) \mathrm{d}\xi \\
&\quad + \frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)} \int_{t_i}^{1} (1-\xi)^{\alpha-1} f\left(\xi, u(\xi), {}^{ABC}_{\ 0}D^{\alpha}_{\xi} u(\xi)\right) \mathrm{d}\xi \\
&\quad + \frac{1-\alpha}{(a+b)B(\alpha)} \left(af\left(0, u(0), {}^{ABC}_{\ 0}D^{\alpha}_{0} u(0)\right) + bf\left(1, u(1), {}^{ABC}_{\ 0}D^{\alpha}_{1} u(1)\right) \right) \Bigg| \\
&\le \frac{1-\alpha}{B(\alpha)} \left[A_g + B_g |u(t)| + C_g \left| {}^{ABC}_{\ 0}D^{\alpha}_{t} u(t) \right| \right] \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{t_1}^{t} (t-\xi)^{\alpha-1} \left[A_g + B_g |u(t)| + C_g \left| {}^{ABC}_{\ 0}D^{\alpha}_{t} u(t) \right| \right] \mathrm{d}\xi \\
&\quad + \frac{1-\alpha}{(a+b)B(\alpha)} \Big(a\left[A_g + B_g |u(0)| + C_g \left| {}^{ABC}_{\ 0}D^{\alpha}_{0} u(0) \right| \right] \\
&\quad + b\left[A_g + B_g |u(1)| + C_g \left| {}^{ABC}_{\ 0}D^{\alpha}_{1} u(1) \right| \right] \Big).
\end{aligned} \tag{6}$$

In addition

$$\begin{aligned}
\left| {}^{ABC}_{\ 0}D^{\alpha}_{t} u(t) \right| &= \left| f\left(t, u(t), {}^{ABC}_{\ 0}D^{\alpha}_{t} u(t)\right) \right| \\
&\le A_g + B_g |u(t)| + C_g \left| {}^{ABC}_{\ 0}D^{\alpha}_{t} u(t) \right|,
\end{aligned}$$

this implies

$$\left| {}^{ABC}_{\ 0}D^{\alpha}_{t} u(t) \right| \le \frac{A_g}{1-C_g} + \frac{B_g}{1-C_g} |u(t)|. \tag{7}$$

Substituting Equation (7) into Equation (6), we get

$$\|W_1u\| \le \left[\frac{2(1-\alpha)(a+b)\Gamma(\alpha)+a+2b}{(a+b)B(\alpha)\Gamma(\alpha)}\right]\left[A_g + B_g r + \frac{C_g A_g}{1-C_g} + \frac{B_g C_g r}{1-C_g}\right].$$

Thus, W_1 is bounded. Now to prove equicontinuity for W_1, let $\delta,\eta \in J$, then

$$\begin{aligned}
&\left|W_1u(\delta)-W_1u(\eta)\right| \\
&\le \frac{1-\alpha}{B(\alpha)}\left| f\left(\delta,u(\delta),{}^{ABC}_{\ 0}D^{\alpha}_{\delta}u(\delta)\right) - f\left(\eta,u(\eta),{}^{ABC}_{\ 0}D^{\alpha}_{\eta}u(\eta)\right)\right| \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{\eta}\left[(\delta-\xi)^{\alpha-1}-(\eta-\xi)^{\alpha-1}\right] f\left(\xi,u(\xi),{}^{ABC}_{\ 0}D^{\alpha}_{\xi}u(\xi)\right)\mathrm{d}\xi \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\eta}^{\delta}(\delta-\eta)^{\alpha-1} f\left(\xi,u(\xi),{}^{ABC}_{\ 0}D^{\alpha}_{\xi}u(\xi)\right)\mathrm{d}\xi \\
&\le \frac{1-\alpha}{B(\alpha)}\left[K_1\left(\left|u(\delta)-u(\eta)\right| + \left|{}^{ABC}_{\ 0}D^{\alpha}_{\delta}u(\delta) - {}^{ABC}_{\ 0}D^{\alpha}_{\eta}u(\eta)\right|\right)\right] \\
&\quad - \frac{(\delta-\eta)^{\alpha}}{B(\alpha)\Gamma(\alpha)} + \frac{(\delta-t_1)^{\alpha}}{B(\alpha)\Gamma(\alpha)} - \frac{(\eta-t_1)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \\
&\quad + \frac{(\delta-\eta)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\left[A_g + B_g\left|u(\eta)\right| + C_g\,{}^{ABC}_{\ 0}D^{\alpha}_{\eta}u(\eta)\right] \\
&\le \frac{1-\alpha}{B(\alpha)}\left[K_1\left(\left|u(\delta)-u(\eta)\right| + \left|{}^{ABC}_{\ 0}D^{\alpha}_{\delta}u(\delta) - {}^{ABC}_{\ 0}D^{\alpha}_{\eta}u(\eta)\right|\right)\right] \\
&\quad + \frac{1}{B(\alpha)\Gamma(\alpha)}\left[-(\delta-\eta)^{\alpha} + (\delta-t_1)^{\alpha} - (\eta-t_1)^{\alpha}\right. \\
&\quad \left. + (\delta-\eta)^{\alpha}\left(A_g + B_g + \frac{A_g C_g}{1-C_g} + \frac{B_g C_g r}{1-C_g}\right)\right].
\end{aligned}$$

This shows that $W_1u(\delta)-W_1u(\eta)\to 0$, as $\delta\to\eta$.

Therefore, by Arzela-Ascoli theorem, W_1 is completely continuous. According to the above steps together with Krasnoselakii fixed point theorem, the operator H has at least one fixed point, which implies that the problem (1) has at least one solution on $[0,1]$. □

4. Hyers-Ulam Stability

In this section, we study the Hyers-Ulam stability of (1). Let $\varepsilon > 0$, and consider the following inequalities

$$\begin{cases} \left\|{}^{ABC}_{\ 0}D^{\alpha}_{t}\tilde{u}(t) - f\left(t,\tilde{u}(t),{}^{ABC}_{\ 0}D^{\alpha}_{t}\tilde{u}(t)\right)\right\| \le \varepsilon,\ t\in J=[0,1],\ 0<\alpha\le 1, \\ \left\|\Delta\tilde{u}(t_k) - I_k\left(\tilde{u}(t_k)\right)\right\| \le \varepsilon, k=1,2,\cdots,m. \end{cases} \tag{8}$$

Remark 4.1 *A function* $\tilde{u}\in X$ *is the solution of inequality* (8) *if and only if there exist function* $g_1(t), g_2(t)\in X$ *such that for* $t\in[0,1]$, $g_1(t)\le\varepsilon_1, g_2(t)\le\varepsilon_1$ *and*

$$\begin{cases} {}^{ABC}_{\ \ 0}D_t^{\alpha}\tilde{u}(t) = f\left(t,\tilde{u}(t), {}^{ABC}_{\ \ 0}D_t^{\alpha}\tilde{u}(t)\right) + g_1(t),\ t \in J = [0,1], 0 < \alpha \le 1, \\ \Delta\tilde{u}(t_k) = I_k\left(\tilde{u}(t_k)\right) + g_2(t_k), k = 1,2,\cdots,m. \end{cases} \quad (9)$$

Definition 4.2 [22] *Equation* (1) *is Hyers-Ulam stable if there exists* $\nu > 0$ *such that for each solution* $\tilde{u} \in X$ *of the inequlity* (8), *there exists a solution* $u \in X$ *of Equation* (1) *such that*

$$\left\|\tilde{u}(t) - u(t)\right\| \le \nu(K_1, K_2)\varepsilon.$$

Theorem 4.3 *If assumptions* $(H1)$-$(H3)$ *are fulfilled, the Equation* (1) *is Hyers-Ulam stable with respect to* ε.

Proof Based on Remark 4.1, we can say that the solution of the Equation (9) is

$$\begin{aligned}\tilde{u}(t) &= \frac{1-\alpha}{B(\alpha)}\left[f\left(t,\tilde{u}(t), {}^{ABC}_{\ \ 0}D_t^{\alpha}\tilde{u}(t)\right) + g_1(t)\right] \\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}\left[f\left(\xi,\tilde{u}(\xi), {}^{ABC}_{\ \ 0}D_\xi^{\alpha}\tilde{u}(\xi)\right) + g_1(\xi)\right]\mathrm{d}\xi \\ &+ \frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}\left[f\left(\xi,\tilde{u}(\xi), {}^{ABC}_{\ \ 0}D_\xi^{\alpha}\tilde{u}(\xi)\right) + g_1(\xi)\right]\mathrm{d}\xi \\ &+ \frac{b}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}\left[f\left(\xi,\tilde{u}(\xi), {}^{ABC}_{\ \ 0}D_\xi^{\alpha}\tilde{u}(\xi)\right) + g_1(\xi)\right]\mathrm{d}\xi \\ &+ \frac{b}{a+b}\sum_{i=1}^{m}\left[I_i\left(\tilde{u}(t_i)\right) + g_2(t)\right] + \frac{1-\alpha}{(a+b)B(\alpha)}\Big[\left(af\left(0,\tilde{u}(0), {}^{ABC}_{\ \ 0}D_0^{\alpha}\tilde{u}(0)\right) + g_1(1)\right) \\ &+ b\left(f\left(1,\tilde{u}(1), {}^{ABC}_{\ \ 0}D_1^{\alpha}\tilde{u}(1)\right) + g_1(1)\right)\Big].\end{aligned}$$

Let $\tilde{u}$ be the solution of inequality (8), then for every $t \in I$, we obtain

$$\begin{aligned}&\Bigg\|\tilde{u}(t) - \frac{1-\alpha}{B(\alpha)}f\left(t,\tilde{u}(t), {}^{ABC}_{\ \ 0}D_t^{\alpha}\tilde{u}(t)\right) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}f\left(\xi,\tilde{u}(\xi), {}^{ABC}_{\ \ 0}D_\xi^{\alpha}\tilde{u}(\xi)\right)\mathrm{d}\xi \\ &+ \frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}f\left(\xi,\tilde{u}(\xi), {}^{ABC}_{\ \ 0}D_\xi^{\alpha}\tilde{u}(\xi)\right)\mathrm{d}\xi + \frac{b}{a+b}\sum_{i=1}^{m}I_i\left(\tilde{u}(t_i)\right) \\ &+ \frac{b}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}f\left(\xi,\tilde{u}(\xi), {}^{ABC}_{\ \ 0}D_\xi^{\alpha}\tilde{u}(\xi)\right)\mathrm{d}\xi \\ &+ \frac{1-\alpha}{(a+b)B(\alpha)}\left(af\left(0,\tilde{u}(0), {}^{ABC}_{\ \ 0}D_0^{\alpha}\tilde{u}(0)\right) + bf\left(1,\tilde{u}(1), {}^{ABC}_{\ \ 0}D_1^{\alpha}\tilde{u}(1)\right)\right)\Bigg\| \\ &\le \frac{b}{B(\alpha)}g_1(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}g_1(\xi)\mathrm{d}\xi \\ &\quad + \frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}g_1(\xi)\mathrm{d}\xi + \frac{b}{a+b}\sum_{i=1}^{m}g_2(t_k) \\ &\quad + \frac{b}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}g_1(\xi)\mathrm{d}\xi + \frac{1-\alpha}{(a+b)B(\alpha)}\left[ag_1(0) + bg_1(0)\right] \\ &\le \left[\frac{2\alpha(1-\alpha)(a+b)\Gamma(\alpha) + \alpha(a+b) + b\alpha + mb\alpha B(\alpha)\Gamma(\alpha) + bm}{\alpha(a+b)B(\alpha)\Gamma(\alpha)}\right]\varepsilon_1.\end{aligned}$$

Therefore, for each $t \in I$, we get

$$\begin{aligned}
&\left\|\tilde{u}(t)-u(t)\right\| \\
&=\Bigg\|\tilde{u}(t)-\frac{1-\alpha}{B(\alpha)}f\left(t,u(t),{}_{0}^{ABC}D_{t}^{\alpha}u(t)\right)-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}f\left(\xi,u(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}u(\xi)\right)\mathrm{d}\xi \\
&-\frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}f\left(\xi,u(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}u(\xi)\right)\mathrm{d}\xi+\frac{b}{a+b}\sum_{i=1}^{m}I_i\left(u(t_i)\right) \\
&-\frac{b}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}f\left(\xi,u(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}u(\xi)\right)\mathrm{d}\xi \\
&-\frac{1-\alpha}{(a+b)B(\alpha)}\left(af\left(0,u(0),{}_{0}^{ABC}D_{0}^{\alpha}u(0)\right)+bf\left(1,u(1),{}_{0}^{ABC}D_{1}^{\alpha}u(1)\right)\right)\Bigg\| \\
&\le\left[\frac{2\alpha(1-\alpha)(a+b)\Gamma(\alpha)+\alpha(a+b)+b\alpha+mb\alpha B(\alpha)\Gamma(\alpha)+bm}{\alpha(a+b)B(\alpha)\Gamma(\alpha)}\right]\varepsilon_1 \\
&+\frac{1-\alpha}{B(\alpha)}\left\|f\left(t,\tilde{u}(t),{}_{0}^{ABC}D_{t}^{\alpha}\tilde{u}(t)\right)-f\left(t,u(t),{}_{0}^{ABC}D_{t}^{\alpha}u(t)\right)\right\| \\
&+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_1}^{t}(t-\xi)^{\alpha-1}\left\|f\left(\xi,\tilde{u}(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}\tilde{u}(\xi)\right)-f\left(\xi,u(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}u(\xi)\right)\right\|\mathrm{d}\xi \\
&+\frac{b\alpha}{(a+b)B(\alpha)\Gamma(\alpha)}\int_{t_i}^{1}(1-\xi)^{\alpha-1}\Big\|f\left(\xi,u(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}\tilde{u}(\xi)\right) \\
&-f\left(\xi,\tilde{u}(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}u(\xi)\right)\Big\|\mathrm{d}\xi+\frac{b}{a+b}\sum_{i=1}^{m}\left\|I_i\left(\tilde{u}(t_i)\right)-I_i\left(u(t_i)\right)\right\| \\
&+\frac{b}{(a+b)B(\alpha)\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-\xi)^{\alpha-1}\Big\|f\left(\xi,\tilde{u}(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}\tilde{u}(\xi)\right) \\
&-f\left(\xi,u(\xi),{}_{0}^{ABC}D_{\xi}^{\alpha}u(\xi)\right)\Big\|\mathrm{d}\xi \\
&+\frac{1-\alpha}{(a+b)B(\alpha)}\left[a\left\|f\left(0,\tilde{u}(0),{}_{1}^{ABC}D_{0}^{\alpha}\tilde{u}(0)\right)-f\left(0,u(0),{}_{1}^{ABC}D_{0}^{\alpha}u(0)\right)\right\|\right] \\
&+\frac{1-\alpha}{(a+b)B(\alpha)}\left[b\left\|f\left(1,\tilde{u}(1),{}_{1}^{ABC}D_{1}^{\alpha}\tilde{u}(1)\right)-f\left(1,u(1),{}_{1}^{ABC}D_{1}^{\alpha}u(1)\right)\right\|\right].
\end{aligned}$$

Assume $\varepsilon=\max\{1,\varepsilon_1\}$, according to theorem 3.1, it can be concluded that

$$\begin{aligned}
&\left\|\tilde{u}(t)-u(t)\right\| \\
&\le\left[\frac{2\alpha(1-\alpha)(a+b)\Gamma(\alpha)+\alpha(a+b)+b\alpha+mb\alpha B(\alpha)\Gamma(\alpha)+bm}{\alpha(a+b)B(\alpha)\Gamma(\alpha)}\right]\varepsilon_1 \\
&+\frac{2K_1(a+b)(1-\alpha)\Gamma(\alpha+1)+mK_2\left(1-K_1B(\alpha)\right)\Gamma(\alpha+1)+K_1\left[(a+b)\alpha+b+m\right]}{(a+b)(1-K_1)B(\alpha)\Gamma(\alpha+1)} \\
&=\Bigg[\frac{2\alpha(1-\alpha)(a+b)\Gamma(\alpha)+\alpha(a+b)+b\alpha+mb\alpha B(\alpha)\Gamma(\alpha)+bm}{\alpha(a+b)B(\alpha)\Gamma(\alpha)} \\
&+\frac{2K_1(a+b)(1-\alpha)\Gamma(\alpha+1)+mK_2\left(1-K_1B(\alpha)\right)\Gamma(\alpha+1)+K_1\left[(a+b)\alpha+b+m\right]}{(a+b)(1-K_1)B(\alpha)\Gamma(\alpha+1)}\Bigg]\varepsilon \\
&:=\nu(K_1,K_2)\varepsilon.
\end{aligned}$$

Hence, by Definition 4.2, we can obtain Equation (1) is Hyers-Ulam stable. □

5. Example

In this section, an example is provided to verify the validity of the investigated results.

Example 5.1 *Consider the following fractional differential equation*

$$\begin{cases} {}^{ABC}_{0}D_t^{\frac{2}{3}}u(t) = \dfrac{\cos t}{(t+7)^2}\dfrac{u^2(t)}{1+u^2(t)} + \dfrac{\sin t}{(t+7)^2}\dfrac{{}^{ABC}_{0}D_t^{\frac{2}{3}}u(t)}{1+u^2(t)}, t \in J = [0,1], \\ \Delta u(t_k) = \dfrac{u(t_k)}{|40+u(t_k)|}, k = 1,2,\cdots,m, \\ u(0) + \dfrac{1}{2}u(1) = 0, \end{cases} \tag{10}$$

where $\alpha = \frac{2}{3}$, $a = 1$, $b = \frac{1}{2}$, $B(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$,

$$f\left(t, u(t), {}^{ABC}_{0}D_t^{\alpha}u(t)\right) = \frac{\cos t}{(t+7)^2}\frac{u^2(t)}{1+u^2(t)} + \frac{\sin t}{(t+7)^2}\frac{{}^{ABC}_{0}D_t^{\frac{2}{3}}u(t)}{1+u^2(t)},$$

$$I_k u(t_k) = \frac{u(t_k)}{40+u(t_k)}.$$

Then, for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$, and $t \in J$, we have

$$\left|f(t,u(t),v(t)) - f(t,\bar{u}(t),\bar{v}(t))\right| \le \frac{1}{25}\left(|u-\bar{u}| + |v-\bar{v}|\right),$$

$$\left|I_k(u(t_k)) - I_k(v(t_k))\right| \le \frac{1}{40}|u-v|.$$

From this, we can obtain $K_1 = \frac{1}{25}$, $K_2 = \frac{1}{40}$, when $m = 5$, we have

$$\frac{2K_1(a+b)(1-\alpha)\Gamma(\alpha+1) + mK_2(1-K_1B(\alpha))\Gamma(\alpha+1) + K_1[(a+b)\alpha+b+m]}{(a+b)(1-K_1)B(\alpha)\Gamma(\alpha+1)}$$
$$= 0.34850344608358 < 1,$$

$$\frac{bmK_1}{(a+b)B(\alpha)\Gamma(\alpha+1)(1-K_1)} + \frac{bmK_2}{a+b} = 0.134835722031918 < 1.$$

Therefore, from Theorem 3.1, Theorem 3.2 and Theorem 4.3, it can be concluded that example 5.1 has at least one solution and it is Hyers-Ulam stable.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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