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Single Mode Periodic Wave Trains in Self-Gravitating Dusty Plasma

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Abstract

In this paper, we consider the dynamics of modulated waves in an unmagnetized, non-isothermal self-gravitating dusty plasma model. The varying charge on the moving dust, as it moves in and out of regions of differing electron and ion densities (due to changes in the electrostatic potential), will be out of phase with the equilibrium charge. The effect of the dust is to increase the phase velocity of the ion-acoustic (IA) waves *i.e.* decrease the Landau damping. In the low-amplitude limit and weak damping, we apply the reductive perturbation method on the model that resulted to the complex cubic Ginzburg-Landau (CCGL) equation. From these results, it is observed that, the plasma parameters strongly influence the properties of the solitary wave solution namely, the amplitude and the width. The effects of non-isothermal electrons, gravity, dust charge fluctuations and drifting motion on the ion-acoustic solitary waves are discussed with application in astrophysical contexts. It is also observed that the number of charges residing on the dust grains increases the modulational stability of the plane waves in the plasma, thus, enhancing the generation of modulated waves.

Keywords

Ginzburg-Landau Equation, Weierstrass Function, Modulated Waves, Self-Gravitating Dusty Plasma

1. Introduction

Dust is quite common throughout the universe and represents much of the solid matter in it. On the other hand, the gaseous component of matter is often ionized (at least partially), and thus the dust co-exists with plasma and forms a “dusty plasma” [1] [2] [3] [4]. The motion of a dust particle can be influenced by a large number of different effects, such as gravity, radiation pressure, plasma or gas drag and electromagnetic forces. The motion is further complicated by the fact that the size of a dust particle may change with time and that the dust charge is changing if its velocity varies or if the ambient conditions, such as plasma temperature and radiation field, change. The effect of the dust is to increase the phase velocity of the ion-acoustic (IA) waves. This effect has been shown to damp the coherent oscillations of electro-statically supported dust rings [5] and to damp the oscillations of levitated dust particles in plasma sheaths at surfaces of solid bodies [6]. The dust charge variation may cause damping of the modes in the dusty plasma system [7]. It is well-known that most of the astrophysical objects and proto-stars contain a significant fraction of dust particulates, which can be charged due to a variety of processes. Specifically, we note that charged dust grains in interstellar spaces, circumstellar disks, supernova remnants, dark molecular clouds, nebulae, etc., are held under the combined influence of electromagnetic and gravitational forces [3] [8] [9] [10]. Accordingly, the knowledge of collective processes in gravitating cosmic dusty plasma is rather essential for understanding the dynamics and transport of charged dust grains.

For the purposes of this work we define a dusty plasma as an ensemble of dust particles immersed in a (perhaps partially ionized) plasma consisting of electrons, ions, and neutrals. Dust particles can be charged by various means, for example, by photo-ionization or absorption of charged particles. If such charged dust particles exist in a plasma, *i.e.* in a conducting fluid, the interaction between the particles and externally applied electric and magnetic fields (e.g. a planetary magnetic field) is modified by the presence of the plasma [1]. For instance, a negatively charged dust, or for that matter any negatively charged particle, will be surrounded by a plasma which is not charge neutral everywhere but has a positive charge density near the grain because it tends to attract the positive ions and repel the negative electrons. This positive charge density partially screens out the negative dust charge and reduces the strength of the interaction. This "debye screening" effect is a unique consequence of the fact that the dust is immersed in a plasma [1] [2].

Highly charged massive dust grains present in a plasma may exhibit charge fluctuation in response to certain types of oscillations incorporated to the plasma. Under this situation, the grain charge becomes a time dependent and self-consistent variable [11] [12]. The consequent modification in the collective properties of a dusty plasma in response to the variation of charge is studied for non complicated plasma systems [13] [14]. It may be noted that the existence of IA wave on a very slow time scale of dust dynamics was investigated for the first time by Rao *et al.* [15]. They also showed the formation of rarefaction type IA soliton solution in a simple dusty plasma system. Similarly, Ma and Liu [16] discussed the existence of rarefaction IA soliton solution in a plasma in presence of dust charge fluctuation. Using a reductive perturbation theory, Xie *et al.* [17] derived small amplitude IA soliton and double layers in dusty plasma with varying dust charges and they had shown that only rarefaction waves exist when the

Mach number lies within an appropriate regime depending on the system parameters.

On the other hand, Schekinov [18] studied analytically the non-linear properties of IA waves in a dusty plasma consisting of cold dust grains of constant charge and non-isothermal ions. Mamun [19] studied non-linear small amplitude IA waves considering non-isothermal ions. The effect of non-linear dust grain charging on large amplitude electrostatic waves in a dusty plasma with trapped ions has been studied by Nejoh [20]. Kakati and Goswami [13] studied non-linear shock-like IA waves considering non-isothermal ions and adiabatic dust charge variations using the reductive perturbation technique. El-Labany *et al.* [21] revisited the same problem and studied the critical density solitary waves and small amplitude IA waves in hot dusty plasma with non-isothermal ions. Also, the effect of non-adiabatic dust charge variations on non-linear IA waves with non-isothermal ions has been investigated by Ghosh *et al.* [22]. The amplitude and width of such solitary waves are shown to be significantly changed by the gravitational effects on the dusty plasma.

The presence of charged massive dust grains can significantly modify the linear and non-linear wave propagation through plasma. When the size of the dust grains becomes considerable, the gravitational effects of dust grains become important though the effect is certainly negligible for electrons and ions. In fact, a number of authors have considered non-linear wave propagation in self-gravitating dusty plasma where there is a competition between gravitational attraction and electrostatic repulsion between the charged grains, apart from other electromagnetic effects. It has been found that the gravitational effect can also significantly influence the non-linear wave propagation through dusty plasma [23].

The objective of the present paper is to study the existence and characteristics of ion-acoustic waves in un-magnetized, self-gravitating dusty plasmas consisting of warm positive ions, non-isothermal electrons and dust particles with charge variations. Thus we apply for the first time the cylindrical coordinate system to a self-gravitating dusty plasma model and derive the complex cubic Ginzburg-Landau (CCGL) equation. The main motivation of the present work being to investigate the effects of dust charge fluctuations and non-isothermal electrons on the non-linear waves associated with the IA waves in dusty plasmas. By using the reductive perturbation technique, the small amplitude IA waves are described by the complex cubic Ginzburg-Landau (CCGL) equation. It is shown that the amplitude of the waves increases with the population of fast or non-isothermal electrons. We have also examined the modification of electrostatic dusty plasma wave spectra due to the presence of a gravitational force, which basically acts on the dust fluid. It is found that both the electrostatic fields and equilibrium density in homogeneities contribute to the propagation and stability of IA waves in self-gravitating dusty plasma systems. Propagation and damping of ion-acoustic waves have been investigated both theoretically and experimentally [24] [25] [26] [27]. It is well known that the charged dust grains in interstellar medium, dark molecular clouds, supernova remnants, nebulae are held under the combined influence of electromagnetic and gravitational forces [1] [28]. Hence the consideration of charged dust particles in self-gravitating plasma can lead to new information to understand the various mechanisms in interstellar spaces [29] [30] [31]. Also, the gravitational attractions on the dusty plasma particles cause damping of sub-luminal (slower) waves propagating in the plasma systems.

The rest of the paper is organized in the following manner. In Section II, we present in general the hydrodynamic equations for the ion and dust fluids, density distribution for the non-isothermal electrons and the model equation of the dust fluid has been obtained from the dust continuity and dust momentum equations along with Poisson's equation which relates the self-gravitational potential and the dust mass density. Section III is devoted to the derivation of the complex cubic Ginzburg-Landau (CCGL) equation using reductive perturbation method and the single mode periodic solutions of the CCGL equation. We shall revisit elliptic equation methods and apply them to obtain periodic solutions of the CCGL equation in terms of Jacobi elliptic function and Weierstrass elliptic function. In Section IV, we present the stability analyses of the plane wave and periodic wave solutions. In Section V, we present a brief conclusion and the possible applications of our investigation.

2. Basic Equations and Mathematical Model

We consider fully ionized and un-magnetized dusty plasma consisting of positive ion fluid, non-isothermal electrons and charge fluctuating dust particles. The non-linear dynamics or behaviour of IA waves, whose phase velocity is much smaller (larger) than the electron (ion) speed, propagating in such a dusty plasma system as governed by the gravitational force acting on the electrons and the ions is neglected, because m_e, $m_i \ll m_d$. The dynamics of the dust fluid is governed by the continuity equation and the equation of motion for the dust described by:

$$\frac{\partial n_d}{\partial t}+\vec{\nabla}\cdot\left(n_d\vec{v}_d\right)=0. \tag{1}$$

$$\frac{\partial \vec{v}_d}{\partial x}+\left(\vec{v}_d\cdot\vec{\nabla}\right)\vec{v}_d+\frac{\sigma K_B T_d}{\mu_d n_d}\vec{\nabla}n_d=\frac{Z_d}{\mu_d}\vec{\nabla}\phi-\vec{\nabla}\psi. \tag{2}$$

where n_d and v_d are the perturbations in the dust number density and the dust fluid velocity, respectively, and ψ and ϕ are the perturbed gravitational potential and electrostatic potential. Z_d is the number of electrons residing onto the dust grain surface normalized by its equilibrium value Z_{d0}, $\sigma = T_i/T_e$ is the non-isothermal temperature ratio and $\mu_d = m_d/m_i$. The system of equations (1) and (2) of the non-linear dust dynamics is closed by the two Poisson's equations, one for electrostatic potential, ϕ and the other for gravitational potential, ψ.

$$\vec{\nabla}^2\phi = n_e - n_i + Z_d n_d. \tag{3}$$

$$\vec{\nabla}^2\psi = 4\pi G\left(m_e n_e + m_i n_i + m_d n_d\right). \tag{4}$$

The number density perturbations of the electron and ion fluids, in the local approximation are given by the Maxwell Boltzmann's distributions [32]

$$n_e = n_{e0}\exp\left(\frac{e\phi}{K_B T_e}\right) = n_{e0}\exp\left(\alpha\phi\right). \tag{5}$$

$$n_i = n_{i0}\exp\left(-\frac{e\phi}{K_B T_i}\right) = n_{i0}\exp\left(-\beta\phi\right). \tag{6}$$

Here, n_{e0} and n_{i0} are the densities for the electrons and the ions at $\phi = 0$. T_e and T_i are the plasma temperatures for the electrons and ions, respectively.

We note that the dust grain is charged by the plasma currents at the grain surface. This means that Z_d is not constant but varies with space and time. We also consider a simple situation in which the charging current originates from the collections of electrons and ions hitting the dust grain surface.

The charge neutrality condition at equilibrium requires that:

$$n_{i0} = n_{e0} + Z_d n_{d0}. \tag{7}$$

where n_{e0}, n_{i0} and n_{d0} are the unperturbed electron, ion, and dust number densities, respectively, and Z_{d0} is the unperturbed number of charges residing on the dust grains measured in unit of the electron charge [33].

3. The CCGL Equation and Single Mode Periodic Wave Trains

To study electrostatic solitary structures in the un-magnetized gravitating dusty plasma mode under consideration, we construct a weakly non-linear theory of IA waves with small but finite amplitude. The scaling of the independent variables leads to the derivation of the evolution equation of the system which in this case is the complex cubic Ginzburg-Landau (CCGL) equation.

3.1. The CCGL Equation of the System

Solitons and periodic wave models describe mechanical processes that occur in non-linear systems e.g. dusty plasma. The model under consideration is based on the assumption that the dust grains are seen as cylindrical-shaped particles. In cylindrical geometry, the operators take the form below:

$$\begin{cases} \vec{\nabla} = \dfrac{\partial}{\partial r}\cdot\hat{e}_r + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}\cdot\hat{e}_\theta + \dfrac{\partial}{\partial z}\cdot\hat{e}_z, \\ \vec{v}_d = v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z, \\ \hat{e}_r\cdot\hat{e}_r = \hat{e}_\theta\cdot\hat{e}_\theta = 1, \\ \hat{e}_r\cdot\hat{e}_\theta = \hat{e}_\theta\cdot\hat{e}_r = 0 \end{cases} \tag{8}$$

This leads to the normalized form of Equations (1)-(4) expressed in cylindrical geometry.

The evolution equation can be obtained by the multiple-scale expansion method. This method was first applied in [34] and has been used in many branches of science. In order to study the dynamics of small amplitude IA waves in the presence of dust charge variation, we derive an evolution equation from the system of Equations (1)-(4), employing the reductive perturbation technique [35] by introducing the stretched coordinates [36] $\xi = \varepsilon^{1/2}(kr - Mt)$, $\tau = \varepsilon^{3/2}t$ and $\zeta = \varepsilon^{-1/2}\theta$, where ε is a small parameter and M is the Mach number (which represents the ratio of the speed of the object (dust grains) relative to the fluid (plasma) to the speed of acoustic waves relative to the fluid). The variables v_r, v_θ, n_d, n_e, n_i, ψ, and ϕ are expanded in a power series as follows:

$$\begin{cases} v_r = \varepsilon v_{r1} + \varepsilon^2 v_{r2} + \varepsilon^3 v_{r3} + \varepsilon^4 v_{r4} + \cdots, \\ v_\theta = \varepsilon^{3/2} v_{\theta 1} + \varepsilon^{5/2} v_{\theta 2} + \varepsilon^{7/2} v_{\theta 3} + \varepsilon^{9/2} v_{\theta 4} + \cdots, \\ n_d = n_{d0} + \varepsilon n_{d1} + \varepsilon^2 n_{d2} + \varepsilon^3 n_{d3} + \varepsilon^4 n_{d4} + \cdots, \\ n_e = n_{e0} + \varepsilon n_{e1} + \varepsilon^2 n_{e2} + \varepsilon^3 n_{e3} + \varepsilon^4 n_{e4} + \cdots, \\ n_i = n_{i0} + \varepsilon n_{i1} + \varepsilon^2 n_{i2} + \varepsilon^3 n_{i3} + \varepsilon^4 n_{i4} + \cdots, \\ \psi = \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \varepsilon^3 \psi_3 + \varepsilon^4 \psi_4 + \cdots, \\ \phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \varepsilon^4 \phi_4 + \cdots \end{cases} \tag{9}$$

We substitute these expansions into Equations (1)-(7) expressed in cylindrical geometry and collect terms at different powers of the parameter, ε. At the lowest order, for the continuity and the momentum equations, we obtain:

$$v_{r1} = \frac{M}{n_{d0}} n_{d1}, \quad \psi = M v_{r1} - \frac{\sigma K_B T_d}{\mu_d n_{d0}} n_{d1} + \frac{Z_d}{\mu_d}\phi. \tag{10}$$

while the two Poisson's equations yield:

$$n_{e1} - n_{i1} + Z_d n_{d1} = 0. \tag{11}$$

$$m_{e1} n_{e1} + m_i n_{i1} + m_d n_{d1} = 0. \tag{12}$$

The Boltzmann's distribution of ion and electron number densities can be expressed in the following form:

$$\begin{cases} n_{e1} = \alpha \phi_1 n_{e0}, \\ n_{i1} = -\beta \phi_1 n_{i0}, \\ n_{e2} = \alpha \phi_2 + \dfrac{\alpha^2}{2}\phi_1^2, \\ n_{i2} = \beta \phi_2 + \dfrac{\beta^2}{2}\phi_1^2 \end{cases} \tag{13}$$

For the next higher order terms of both the continuity and the momentum equations, we obtain:

$$\frac{\partial n_{d1}}{\partial \tau} - M\frac{\partial n_{d2}}{\partial \xi} + \frac{2Mn_{d1}}{n_{d0}}\frac{\partial n_{d1}}{\partial \xi} + n_{d0}\frac{\partial v_{r2}}{\partial \xi} + \frac{1}{\tau}n_{d1} + \frac{n_{d0}}{M}\frac{1}{\tau}\frac{\partial v_{\theta 1}}{\partial \zeta} = 0. \tag{14}$$

$$\frac{M}{n_{d0}}\frac{\partial n_{d1}}{\partial \tau} - M\frac{\partial v_{r2}}{\partial \xi} + \frac{M^2}{n_{d0}^2}n_{d1}\frac{\partial n_{d1}}{\partial \xi} + \frac{\sigma K_B T_d}{\mu_d n_{d0}}\frac{\partial n_{d2}}{\partial \xi} - \frac{Z_d}{\mu_d}\frac{\partial \phi_2}{\partial \xi} + \frac{\partial \psi_2}{\partial \xi} = 0. \tag{15}$$

Taking second derivatives on Equations (14) and (15) with respect to the stretched coordinates result to an anonymous fourth order partial differential equation which can be expressed in terms of the electrostatic potential, ϕ_1 only.

As we are interested in weakly non-linear propagating wave solutions of the self-gravitating dusty plasma, we apply the reductive perturbation technique. To achieve this goal, we introduce a small parameter ε and proceed with the substitutions $a_2 \to \varepsilon^2 a_2$, $a_3 \to \varepsilon^2 a_3$, $a_4 \to \varepsilon a_4$, $a_5 \to \varepsilon^3 a_5$, $a_6 \to a_6$, $a_7 \to a_7$, $a_8 \to a_8$, $a_9 \to \varepsilon^2 a_9$, into the anonymous equation (**Appendix**).

Next, we consider the following variable transformations:

$$\chi = \varepsilon(\xi - \tau), \quad s = \varepsilon^3 \tau. \tag{16}$$

where ε and ε^3 are chosen in such a way as to balance the effects of non-linearity and damping. Using these transformations, terms of order ε^4 give the modified Burgers Korteweg-de Vries (MBKdV) equation.

$$\frac{\partial \phi_1}{\partial s} = \frac{N}{2}\phi_1^2 - \frac{F}{2}\phi_1 \frac{\partial \phi_1}{\partial \chi} + \frac{S}{2}\phi_1^2 \frac{\partial \phi_1}{\partial \chi} + \frac{R}{2}\frac{\partial^3 \phi_1}{\partial \chi^3} - \frac{H}{2}\frac{\partial^2 \phi_1}{\partial \chi^2} + \frac{W}{2}\frac{\partial^2}{\partial \chi^2}\left(\phi_1 \frac{\partial \phi_1}{\partial \chi}\right). \tag{17}$$

If we assume a low-amplitude oscillation of the electrostatic potential of the dusty plasma, *i.e.* $\phi_1 = \delta\phi_1 (\delta \ll 1)$ and a weak damping such that $N \to \delta N$ and $H \to \delta^2 H$, Equation (17) can be written as:

$$\begin{aligned}\frac{\partial \phi_1}{\partial s} = \frac{N}{2}\delta^2\phi_1^2 - \frac{F}{2}\delta\phi_1 \frac{\partial \phi_1}{\partial \chi} + \frac{S}{2}\delta^2\phi_1^2 \frac{\partial \phi_1}{\partial \chi} + \frac{R}{2}\frac{\partial^3 \phi_1}{\partial \chi^3} \\ - \frac{H}{2}\delta^2\frac{\partial^2 \phi_1}{\partial \chi^2} + \frac{W}{2}\delta\frac{\partial^2}{\partial \chi^2}\left(\phi_1 \frac{\partial \phi_1}{\partial \chi}\right).\end{aligned} \tag{18}$$

The multiple-scale expansion is a perturbation technique in which both the carrier waves and the amplitude are treated in the continuum limit [37] [38]. It is thus incumbent on us to use this technique to obtain the evolution equation of the MBKdV equation when the non-linearity and damping are balanced. The method involves introducing two time and spatial scales *i.e.* the fast time and spatial scales for the oscillations and the slow time and spatial scales for the envelope amplitude. This method has been used to derive the non-linear Schrödinger equation from the KdV equation in Ref. (38). We introduce a new time scale $T_i = \delta^i s$ and a space scale $X_i = \delta^i \chi$, with each value of T_i and X_i being treated as an independent variable. This leads to a perturbation series of operators from all the independent variables:

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial T_0} + \delta\frac{\partial}{\partial T_1} + \delta^2\frac{\partial}{\partial T_2}, \tag{19}$$

$$\frac{\partial}{\partial \chi} = \frac{\partial}{\partial X_0} + \delta\frac{\partial}{\partial X_1}.$$

According to the multiple-scale expansion method, the ansatz for the solution ϕ_1 must be consistent with the series expansion of the differential operators in powers of the small parameter, δ [37] [38] adopted in Equation (19). Let us therefore write ϕ_1 as a perturbation series and consider only terms to the first order in, δ:

$$\phi_1 = A\mathrm{e}^{i\Phi} + A^*\mathrm{e}^{-i\Phi} + \delta\left(C + B\mathrm{e}^{2i\Phi} + B^*\mathrm{e}^{-2i\Phi}\right). \tag{20}$$

where the amplitudes A, C and B as well as their corresponding complex conjugates A^*, C^* and B^* are functions of (T_1, T_2, and X_1) and $\Phi = kX_0 - \omega T_0$, with k the normal mode wave number and ω the angular velocity of the wave. We then substitute Equation (19), Equation (20) into Equation (18) and look for relations between terms of same order in δ proportional to $e^{\pm i\Phi}$, $e^{\pm 2i\Phi}$.

To the order δ^0, the annihilation of terms in $e^{\pm i\Phi}$ gives the group velocity dispersion relation of linear waves:

$$\omega = \frac{Rk^3}{2}. \tag{21}$$

To the order δ^1, the cancellation of terms in $e^{\pm i\Phi}$ gives

$$\frac{\partial A}{\partial T_1} + \frac{3Rk^2}{2}\frac{\partial A}{\partial X_1} = 0. \tag{22}$$

To the order δ^1, the annihilation of terms in $e^{\pm 2i\Phi}$ gives

$$B = \frac{Fk + 2Wk^3 - iN}{6k^3} A^2. \tag{23}$$

To the second order in the perturbation δ^2, terms with zero exponential dependence yield:

$$\frac{\partial C}{\partial T_1} - F\frac{\partial |A|^2}{\partial X_1} = 0, \quad C = \frac{2F}{3Rk^2}|A|^2. \tag{24}$$

To the second-order approximation δ^2, annihilation of terms in $e^{\pm i\Phi}$ gives the complex cubic Ginzburg-Landau (CCGL) equation:

$$i\frac{\partial A}{\partial T_2} - P\frac{\partial^2 A}{\partial X_1^2} + Q|A|^2 A + i\rho A = 0. \tag{25}$$

Equation (25) is the CCGL equation and generally speaking, it represents one of the most studied non-linear equations in the physics world today. This equation describes the evolution of the envelope amplitude of the electrostatic field potential, ϕ_1 in the one-dimensional cylindrical dusty plasma.

The imaginary term in the CCGL equation causes damping of the amplitude. The non-linearity (self-trapping), the damping and the dispersion coefficients $Q = Q_r + iQ_i$, ρ and P, respectively, are expressed in terms of dusty plasma parameters:

$$Q_r = \frac{5F^2}{12Rk} + \frac{11FWk}{12R} - \frac{kS}{2} - \frac{N^2}{12Rk^2}, \tag{26}$$

$$Q_i = -\frac{FN}{2Rk^2} - \frac{7WN}{12R},$$

$$\rho = \frac{Hk^2}{2}, \quad P = \frac{3Rk}{2}.$$

The variations of constants P, Q_r, Q_i, ρ and of the product PQ_r, PQ_i with respect to the wave vector, k are represented in **Figure 1** and **Figure 2**. Since the dispersion coefficient is real, the modulational instability depends on the sign of the product PQ_r. According to Benjamin-Feir instability criterion, plane waves

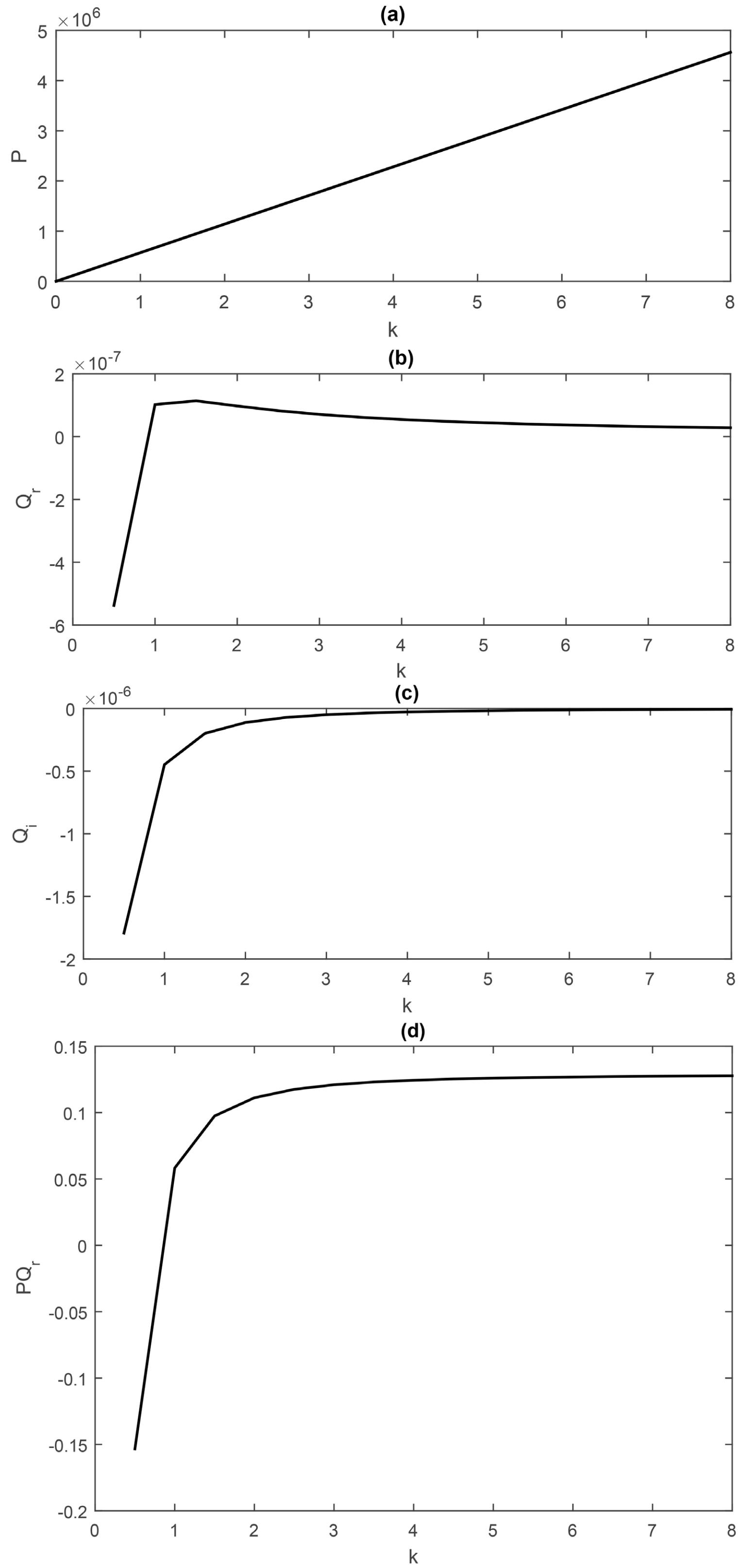


Figure 1. Variation of coefficients (a) P, (b) Q_r, (c) Q_i, (d) PQ_r, in terms of the wave vector, k of the carrier wave for the parameters $\delta = 0.05$, $\sigma = 0.0125$, $\varepsilon = 0.8$, $Z_d = 70000000$.

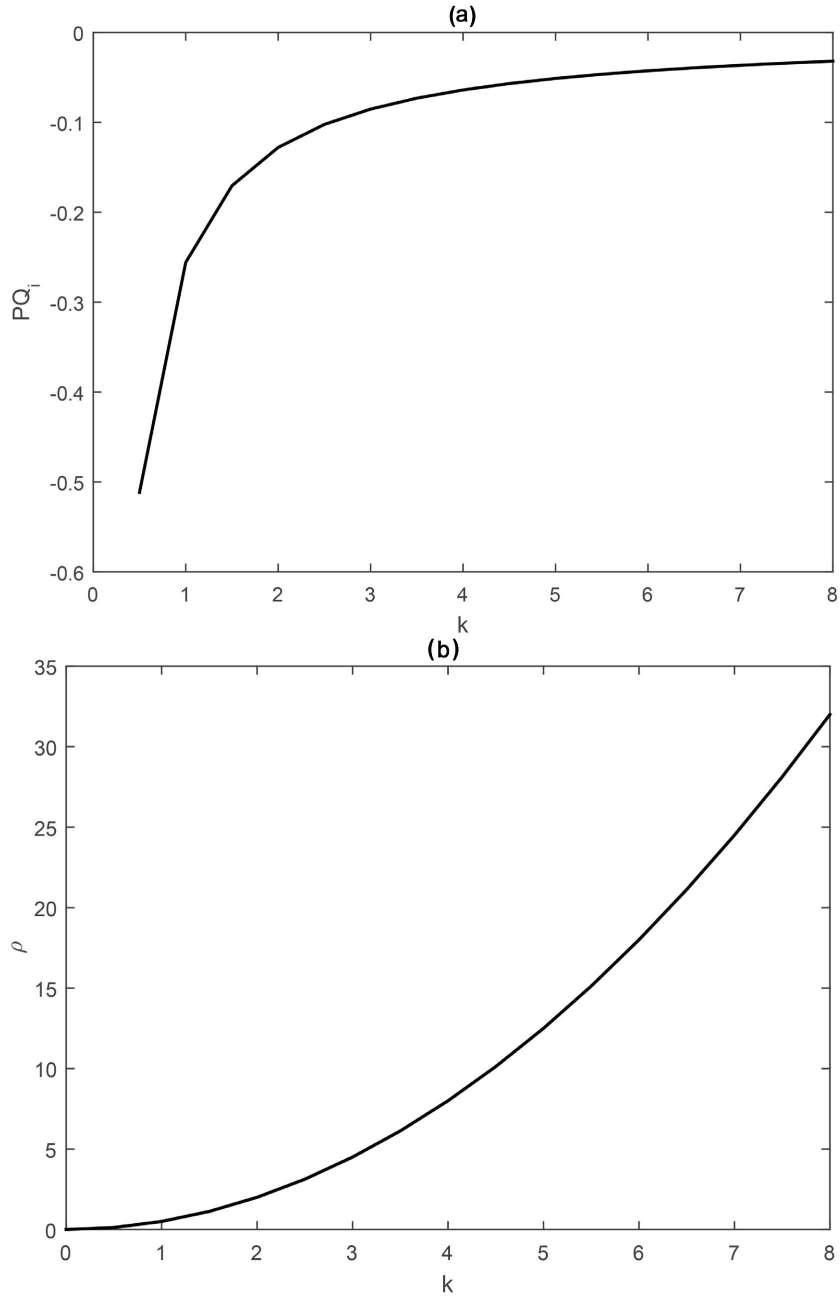


Figure 2. Variation of coefficients (a) PQ_i, (b) ρ, in terms of the wave vector, k of the carrier wave for the parameters $\delta = 0.05$, $\sigma = 0.0125$, $\varepsilon = 0.8$, $Z_d = 70000000$.

are unstable for positive values of PQ_r, while they are stable for negative values. In the present case, the complex cubic Ginzburg-Landau (CCGL) equation obtained from the un-magnetized, non-isothermal self-gravitating dusty plasma model clearly indicates that information encoding and transmission in the form of an electromechanical wave travelling along the system can also emerge as modulated structures.

More importantly, the sign of PQ_r determines the nature of the solutions of Equation (25). If PQ_r is positive, then Equation (25) is said to admit a stable envelope soliton solution that has a vanishing amplitude as $|X_1| \to \infty$ and corresponds to a small-amplitude pulse. However, if the product PQ_r is less than zero, then a dark (envelope hole) soliton will propagate with finite amplitude as $|X_1| \to \infty$ [38] [39]. Thus, one expects to find in the dusty plasma spatially localized pulses for any wave carrier whose wave number is in the positive range of PQ_r.

3.2. Periodic Wave Solutions of the CCGL Equation

The above complex cubic Ginzburg-Landau (CCGL) Equation (25) is a universal equation that is of major importance in continuum mechanics, plasma physics and optics. From a fundamental point of view, the CCGL equation permits us to investigate a travelling-wave profile characterized by an abrupt rise to the excited state and a drop back down to the refractory state with periodic pulse generation observed in the soliton theory [40].

The perturbation techniques developed for soliton solutions include the adiabatic perturbation method, the perturbed inverse scattering method, homogeneous balance method, the Lie-transform method and the variational method [40]. Since we are dealing with the complex cubic Ginzburg-Landau (CCGL) equation, it is incumbent on us to look for a propagating wave solution of the form: [41] [42]

$$A(X_1, T_2) = Y(\varsigma)\mathrm{e}^{i\theta(\varsigma, T_2)}. \tag{27}$$

where $\varsigma = X_1 - v_g T_2$. Substituting Equation (27) into Equation (25), we obtain:

$$v_g Y\theta_\varsigma - PY_{\varsigma\varsigma} + P\theta_\varsigma^2 Y + Q_r Y^3 - \theta_{T_2} Y = 0, \tag{28}$$

$$-v_g Y_\varsigma - PY\theta_{\varsigma\varsigma} - 2PY_\varsigma\theta_\varsigma + Q_i Y^3 + \rho Y = 0. \tag{29}$$

A further simplification of Equations (28) and (29) may be achieved if we assume that $z = \theta_\varsigma$, $\varphi = Y^2$, $\Psi = Y^2 z$ and $\theta_{T_2} = B = \text{constant}$. Then (28) and (29) become [43]:

$$2\varphi\varphi_{\varsigma\varsigma} - \varphi_\varsigma^2 - 4\Psi^2 - \frac{4v_g}{P}\varphi\Psi - \frac{4Q_r}{P}\varphi^3 + \frac{4B}{P}\varphi^2 = 0, \tag{30}$$

$$\Psi_\varsigma + \frac{v_g}{2P}\varphi_\varsigma - \frac{\rho}{P}\varphi - \frac{Q_i}{P}\varphi^2 = 0. \tag{31}$$

In the nonlinear Schrödinger equation (NLSE) limit, $P_i = Q_i = \rho = 0$, Equation (31) is integrated, giving $\Psi = \frac{v_g}{2P}\Psi + C$, C = constant.

When $v_g = 0$, Equation (30) becomes:

$$2\varphi\varphi_{\varsigma\varsigma} - \varphi_\varsigma^2 - 4C^2 - \frac{4Q_r}{P}\varphi^3 + \frac{4B}{P}\varphi^2 = 0. \tag{32}$$

And when $v_g \neq 0$, $C = 0$ Equation (30) becomes:

$$2\varphi\varphi_{\varsigma\varsigma} - \varphi_\varsigma^2 + \left(\frac{v_g^2}{P^2} + \frac{4B}{P}\right)\varphi^2 - \frac{4Q_r}{P}\varphi^3 = 0. \tag{33}$$

Let us now introduce the Miura transformation to further simplify Equations (30) and (31). We assume that $\Psi = \varphi_{\varsigma}$. Substituting in Equation (31) gives an Abel's type equation. When $v_g = 0$, Equation (30) reduces to the form:

$$\varphi_{\varsigma}^2 - \left(\frac{2\rho}{5P} + \frac{4B}{P}\right)\varphi^2 + \left(\frac{4Q_r}{5P} - \frac{2Q_i}{5P}\right)\varphi^3 = 0. \tag{34}$$

And when $v_g \neq 0$, Equation (30) becomes:

$$\varphi_{\varsigma}^2 + \frac{v_g}{P}\varphi\varphi_{\varsigma} - \left(\frac{2\rho}{5P} + \frac{4B}{P}\right)\varphi^2 + \left(\frac{4Q_r}{5P} - \frac{2Q_i}{5P}\right)\varphi^3 = 0. \tag{35}$$

For the periodic profiles, we reduce Equations (32)-(35) into a Weierstrass's type function [44] [45] (**Appendix**). We consider the periodic solutions of Equations (32)-(35) to be the explicit solutions of Equation (25) which is the complex cubic Ginzburg-Landau (CCGL) equation. For Equation (32), we obtain the solution as

$$A(X_1, T_2) = \sqrt{\frac{2B}{3Q_r} + \frac{2P}{Q_r}\eta(\varsigma, g_2, g_3)}\,\mathrm{e}^{i\theta_{\varsigma}}. \tag{36}$$

For Equation (33), the solution becomes:

$$A(X_1, T_2) = \sqrt{\left(\frac{v_g^2}{6PQ_r} + \frac{2B}{3Q_r}\right) + \frac{2P}{Q_r}\eta(\varsigma, g_2, g_3)}\,\mathrm{e}^{i\theta_{\varsigma}}. \tag{37}$$

For Equation (34), we have:

$$A(X_1, T_2) = \sqrt{\gamma}\,\mathrm{e}^{i\theta_{\varsigma}}, \tag{38}$$

$$\gamma = -\frac{\rho + 2B}{3Q_i - 6Q_r} + \frac{10P}{Q_i - 2Q_r}\eta(\varsigma, g_2, g_3).$$

For Equation (35), we obtain: (**Appendix**)

$$A(X_1, T_2) = \sqrt{\gamma}\,\mathrm{e}^{i\theta_{\varsigma}}, \tag{39}$$

$$\gamma = -\left(\frac{b_1}{3b_2} + \frac{g_2}{6b_0}\right) + \frac{3}{b_2}\eta(\varsigma, g_2, g_3) - \frac{3}{2b_0b_2}\eta_{\varsigma}(\varsigma, g_2, g_3).$$

With g_2 and g_3 the invariants and

$$\eta(\varsigma, g_2, g_3) = e_2 - (e_2 - e_3)\,\mathrm{cn}^2\left(\sqrt{e_1 - e_3}\,\varsigma, \kappa\right). \tag{40}$$

Where e_1, e_2 and e_3 are the roots of the cubic equations.

It is important to note that the primary reason for employing the multi-scale analysis is to obtain low-amplitude modulated waves of the self-gravitating dusty plasma model Equation (18) and this approximate periodic solution is given by Equation (20) which admits elliptic solutions of the form Equation (40).

Wave modulations in one-dimensional an-harmonic systems are studied by the use of a perturbation method established by Taniuti and Yajima [46]. We consider a system of non-linear wave equations which admits, in a linear approximation, a plane wave solution with high-frequency oscillation (e.g. carrier waves). Then, for the wave of small but finite amplitude (e.g. IA waves), we investigate how slowly varying parts of the wave such as the amplitude are modulated by non-linear self-interactions. A system of equations to determine the evolution of the slowly varying parts in the lowest order of an asymptotic expansion is obtained. A stretching transformation shows that, in the lowest order of an asymptotic expansion, the original system of equations can be reduced to a tractable, single, non-linear equation to determine the amplitude modulation. One interesting result is that the non-linearly modulated wave must be accompanied by the slowly varying wave which tends to stabilize the modulated wave.

The single solutions of the CCGLE in both the NLSE limit and the CCGLE with damping term are plotted in **Figure 3**. It should be noted that for the modulus, $m=1$, the periodic solutions describe a single dip in the NLS equation limit (Equations (32) and (33)) and a single pulse or kink in the CCGL equation limit (Equations (34) and (35)). In **Figure 4**, we observe the periodic profiles for the squares of the moduli which illustrate the evolution of the periodic wave trains described by Equations (32)-(35). **Figure 5** shows the periodic envelope profiles for the model Equation (20).

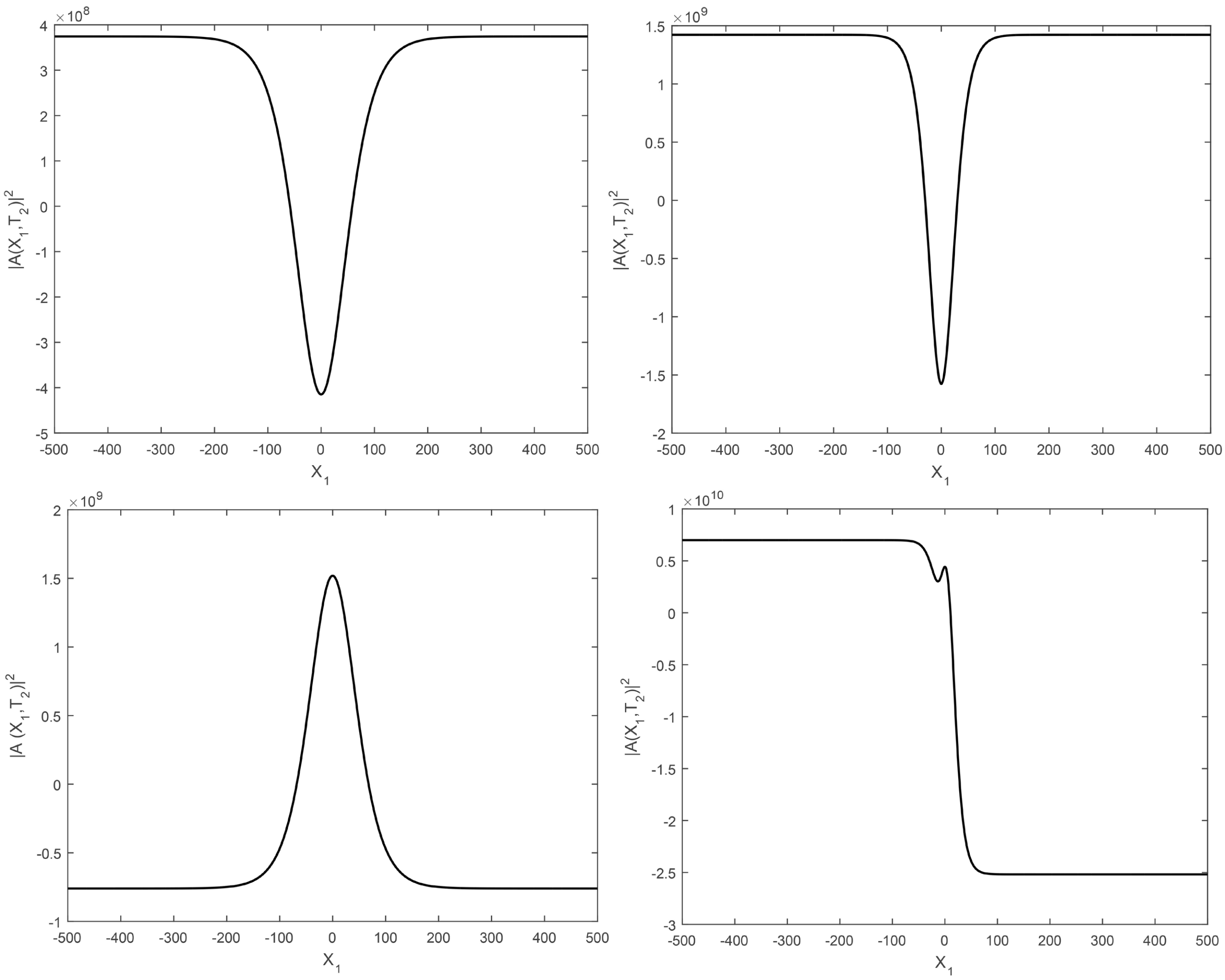


Figure 3. The single wave profiles (solutions), $A(X_1,T_2)$ of Equations (32)-(35) for the parameters, $k=0.08$, integration constant $K=1.0$, $\delta=0.05$, $\sigma=0.0125$, $B=100\text{ rad/s}$, modulus, $m=1$, $\varepsilon=0.8$, damping $\rho=0.0032$, $T_2=0.0$ and $Z_d=100000$.

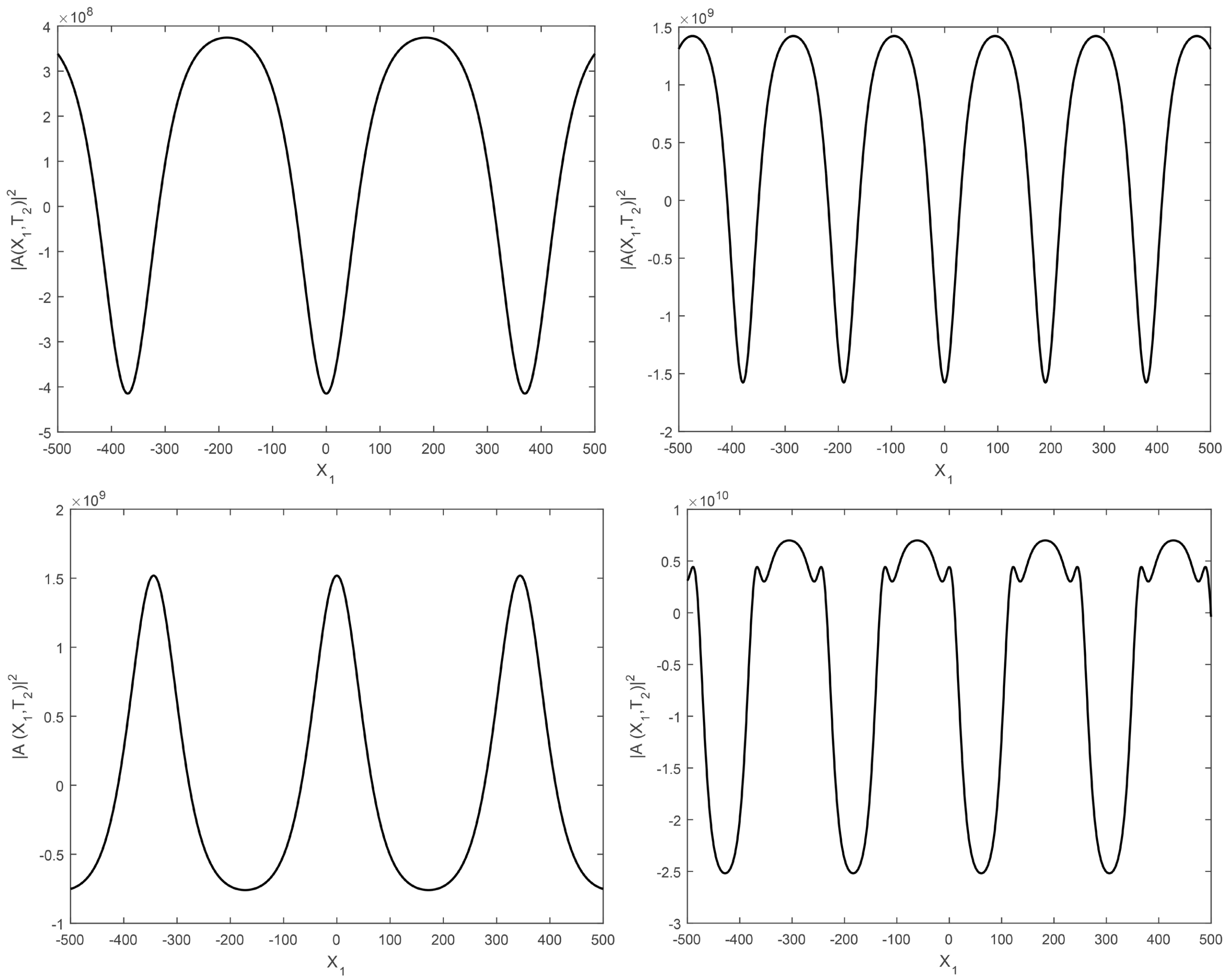
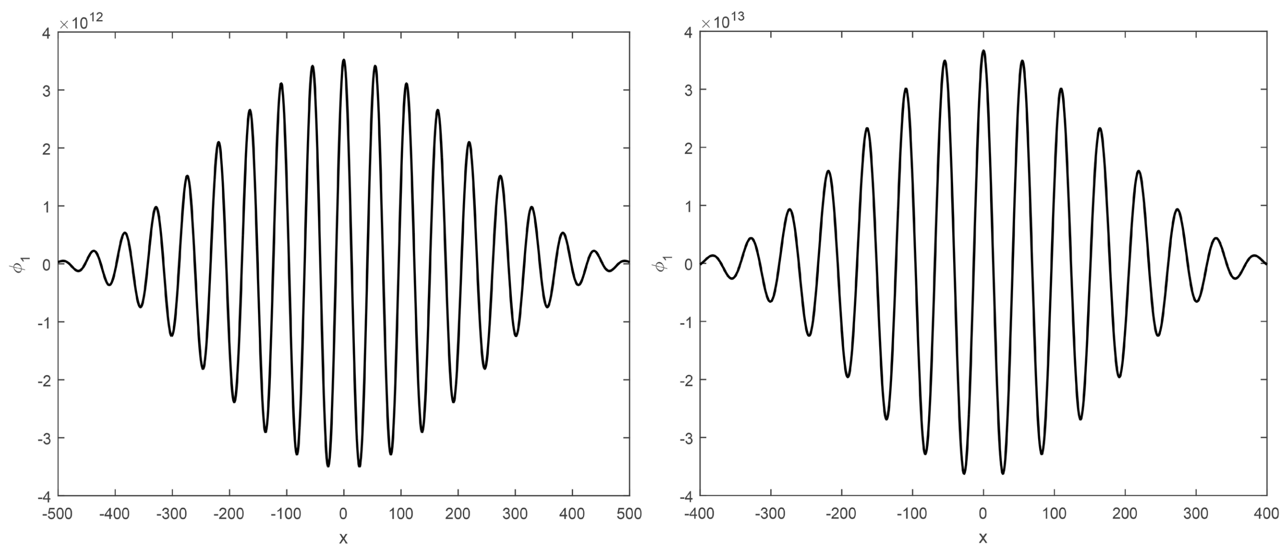


Figure 4. The square of the modulus of the periodic wave profiles, $|A(X_1,T_2)|^2$ for the parameters $k=0.08$, integration constant $K=1.0$, $\delta=0.05$, $\sigma=0.0125$, $B=100\ \text{rad/s}$, modulus, $m=0.95$, $\varepsilon=0.8$, damping $\rho=0.0032$ and $Z_d=100000$ with $T_2=0.0$ and neglecting the mass of the electrons (*i.e.* $m_e=0\ \text{kg}$).



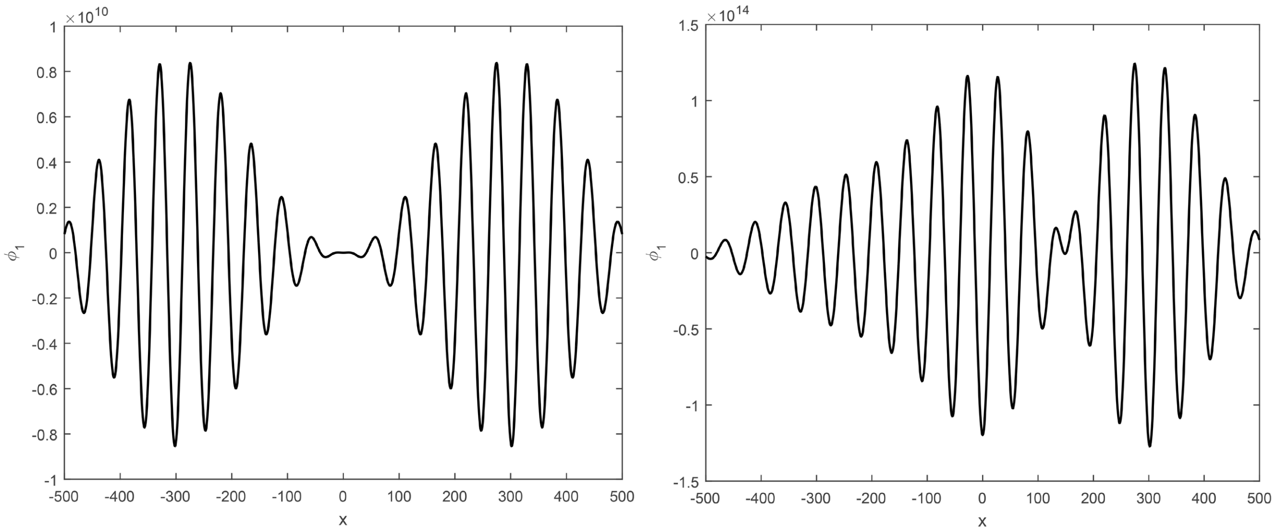


Figure 5. The wave profiles, ϕ_1 of the model Equation (20) for the parameters $k = 0.08$, $\delta = 0.05$, $\sigma = 0.0125$, $B = 100\ \text{rad/s}$, modulus, $m = 0.95$, $\varepsilon = 0.8$, damping, $\rho = 0.0032$, $T_2 = 0.0$ and $Z_d = 100000$ neglecting the mass of the electrons (*i.e.* $m_e = 0\ \text{kg}$).

4. Stability Analysis of the Wave Solutions

In the previous section, we obtained periodic wave solutions of the CCGL equation in the un-magnetized self-gravitating dusty plasma model. To discuss the stability of these solutions, we superimpose a small perturbation on the solutions and analyse the evolution of the perturbation. Stability analysis is an important technique in the study of non-linear dynamical systems because it provides an effective way of testing the robustness of the wave trains against small perturbation in the amplitude [46]. As shown by Benjamin and Feir [47] in the context of fluid dynamics, such stability/instability can be a precursor of localized periodic wave trains in systems exhibiting weak non-linearity or weak damping.

4.1. Stability Analysis of the Single Wave Solutions

The instability of the plane waves will generate amplitude modulated waves in the system. Thus, to discuss the stability of these solutions, we search for plane wave solutions in the form:

$$A(X_1, T_2) = A_0 e^{i(\nu X_1 - \Lambda T_2)}. \tag{41}$$

Where ν is the wave *number* and Λ the wave *angular frequency*.

We substitute Equation (41) in to Equation (25) to obtain the non-linear dispersion relation of the plane waves.

$$\Lambda = -P\nu^2 - Q_r A_0^2. \tag{42}$$

To examine the linear stability of the plane waves, we consider small perturbations $a(X_1, T_2)$ and $\theta(X_1, T_2)$ on the amplitude and phase, respectively, of the plane waves such that the solution Equation (41) can be rewritten as:

$$A(X_1, T_2) = \left[A_0 + a(X_1, T_2)\right] e^{i\left[\nu X_1 - \Lambda T_2 + \theta(X_1, T_2)\right]}. \tag{43}$$

Where the perturbation amplitude $a(X_1,T_2)$ is small compared to the plane wave amplitude.

Inserting Equation (43) in the CCGL Equation (25), neglecting the non-linear terms and retaining only linear terms in the perturbations and separating in to the real and imaginary parts lead to the following two coupled linear differential equations:

$$-A_0\frac{\partial\theta}{\partial T_2}-P\frac{\partial^2 a}{\partial X_1^2}+2P\nu A_0\frac{\partial\theta}{\partial X_1}+2QrA_0^2a=0,$$

$$\frac{\partial a}{\partial T_2}-PA_0\frac{\partial^2\theta}{\partial X_1^2}-2P\nu\frac{\partial a}{\partial X_1}-2\rho a=0. \tag{44}$$

The system Equation (44) admits solutions of the form:

$$a(X_1,T_2)=a_0\mathrm{e}^{i(KX_1-\Omega T_2)}+c.c, \tag{45}$$

$$\theta(X_1,T_2)=\theta_0\mathrm{e}^{i(KX_1-\Omega T_2)}+c.c. \tag{46}$$

Where a_0 and θ_0 are constants and K and Ω are the modulation wave vector and frequency, respectively. Substituting Equations (45) and (46) into Equation (44) yields the following linear homogeneous system for a_0 and θ_0 which can be represented as a 2 × 2 matrix.

$$\begin{pmatrix} 2Q_rA_0^2+PK^2 & iA_0(\Omega+2P\nu K) \\ -2\rho-i(\Omega+2P\nu K) & PA_0K^2 \end{pmatrix}\begin{pmatrix} a_0 \\ \theta_0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system will allow non-trivial solutions if the determinant of the matrix is equal to zero. The eigenvalues from the characteristic equation are two distinct dispersion relations for the perturbations.

$$\Omega(K)=-2P\nu K+i\rho\pm\sqrt{-\rho^2+P^2K^4+2PQ_rA_0^2K^2}. \tag{47}$$

For the plane waves to be modulationally unstable, the modulation frequency $\Omega(K)$ must be a complex function. It follows that the plane waves will be unstable if $\mathrm{Im}[\Omega(K)]<0$.

$$i\rho\pm i\sqrt{-\rho^2+P^2K^4+2PQ_rA_0^2K^2}<0. \tag{48}$$

For the ideal dusty plasma model, where there is no damping, (*i.e.* $\rho=0$) with very large values of the amplitude, A_0 the condition Equation (48) reduces to:

$$\frac{Q_r}{P}>\frac{K^2}{2A_0^2},\quad PQ_r>0. \tag{49}$$

Which is the Benjamin-Feir instability criterion for the non-linear Schrödinger equation or the ideal dusty plasma model [48]. When this condition is fulfilled, the amplitude of perturbation grows exponentially with time making the plane waves to become *unstable* in the region. We define the local growth rate of the modulational instability or modulation gain as $G=\mathrm{Im}[\Omega(K)]$ [49]. In **Figure 6**, we plot the modulation gain as a function of the modulation wave number, K for different values of the charge, Z_d. The growth rate initially increases with K

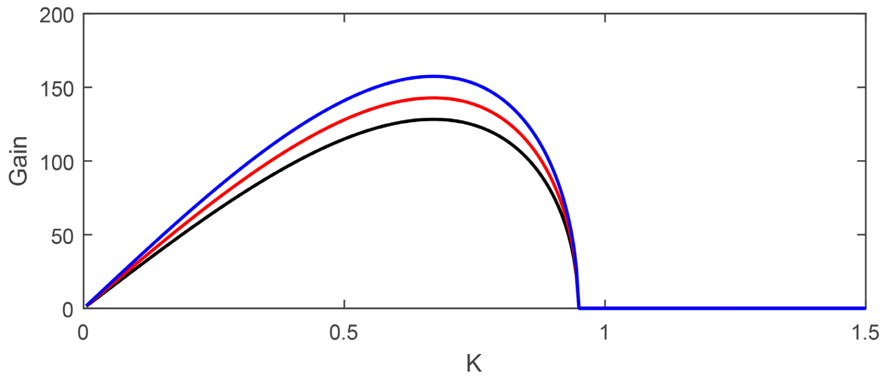


Figure 6. (Colour On-line) Gain spectrum for different values of charge residing on the dust grains for $Z_d = 100$ (black), $Z_d = 50000$ (red), $Z_d = 100000$ (blue) and the parameters $k = 0.08$, $\delta = 0.05$, $\sigma = 0.0125$, $\varepsilon = 0.8$, $K_c = 0.9$.

and becomes maximum, then reduces to zero. We observe that changes in Z_d affect the modulation gain in the system. In fact, the gain is seen to grow with an increase in Z_d meaning that the number of charge residing on the dust grains increases the modulational instability in the plasma. As a consequence the plane wave solutions of the damped CCGL equation become unstable and will eventually break up.

4.2. Stability of the Periodic Wave Solutions

To discuss the stability of these solutions, one must superimpose a small perturbation on this solution and analyse the evolution of the perturbation. In this case, according to the linear stability analysis, the solution is considered to be of the form:

$$A = \left(A_0\left(X_1\right) + \varepsilon A_1\left(X_1\right)\right)\mathrm{e}^{iv_g T_2}. \tag{50}$$

where ε is a small parameter that separates the solution trains and the perturbation $A_1\left(X_1\right)$. After introducing this perturbed solution Equation (50) into the CCGL Equation (25), it is found that the solution and the perturbation obey the following equations at various orders of ε: At order ε^0,

$$P\frac{\partial^2 A_0}{\partial X_1^2} - \left(Q_r + iQ_i\right)A_0^3 + v_g A_0 - i\rho A_0 = 0. \tag{51}$$

At order ε^1,

$$P\frac{\partial^2 A_1}{\partial X_1^2} - 3\left(Q_r + iQ_i\right)A_0^2 A_1 + v_g A_1 - i\rho A_1 = 0. \tag{52}$$

From Equation (51), the solution at zeroth-order (ε^0) can be obtained by using the Weierstrass elliptic function method [44] as described previously, and it is given by:

$$A_0\left(X_1\right) = \sqrt{-\frac{C_2}{3} + \frac{4}{\alpha_2}\eta\left(\varsigma, g_2, g_3\right)}. \tag{53}$$

Which is also a solution of Equation (25).

In Equation (52), the divergence of the solution A_0 can be prevented by finding the exact solution of Equation (32) in A_1. To obtain this solvability condition, we use the transformation $Y = \lambda(X_1 - X_0)$ in Equation (53) and substitute in Equation (52). The solvability condition of Equation (52) can be expressed in term of Lamé's type equation of the second kind as:

$$\frac{\partial^2 A_1}{\partial Y^2} + \left(h(g_2) - \frac{12Q_r}{P\alpha_2} \mathrm{sn}^2(Y) \right) A_1 = 0. \tag{54}$$

Where $h(g_2) = \dfrac{8Q_r\lambda^2 + C_2Q_r\alpha_2 + v_g\alpha_2}{P\lambda^2\alpha_2}$ with $\lambda = \sqrt{\dfrac{3}{2}\sqrt{\dfrac{g_2}{3}}}$.

Equation (54) has solution A_1 that can propagate in the dusty plasma system without influencing its dynamics. Thus, the solution will be stable if we can find a function $A_1(Y)$ such that Equation (54) is satisfied. If such a function does not exist, then the solution $A = A_0 + \varepsilon A_1$ will no longer be a solution of the CCGL Equation (25) and thus, the solution A_0 will be unstable.

5. Conclusions

In this study a completely integrable, modulated, non-linear evolution equation has been derived for the ion-acoustic wave propagating in un-magnetized, self-gravitating dusty plasma system. Dusty plasma contains charged grains whose motion is influenced not only by gravity and radiation pressure but also by plasma drag and electromagnetic forces. Even though the electromagnetic forces are usually small, they cause several interesting effects such as resonant orbit perturbations, modification of density wave dispersion characteristics in planetary rings and angular momentum transport in planetary rings. The charge to mass ratio is a complicated function of plasma parameters and the number density of dust particles. Only a relatively small number of cases have been investigated in detail.

To describe the non-linear propagation of ion-acoustic waves through un-magnetized, self-gravitating dusty plasma, we have derived an uncoupled complex cubic Ginzburg-Landau (CCGL) Equation (25) from the modified Burgers Korteweg-de Vries (MBKdV) equation which includes the effects of gravity, dust charge fluctuation as well as that of non-isothermal electrons. The spatial and time directions are scaled symmetrically allowing weak perturbation. Using the reductive perturbation scheme in the semi-discrete limit or approximation and the Weierstrass elliptic function, the exact algebraic periodic wave solutions are calculated.

Our results show that in such a dusty plasma, solitary and periodic waves can exist and the effects of the non-isothermal electrons and dust charge fluctuations modify the structure of the IA waves. Such plasmas may exist in both space environments and laboratory. We can draw the following conclusions from the investigation: 1) The dust charging effect is of crucial importance in the sense that the dust charge number drastically changes due to the parameters such as the floating potential of dust particles, electrostatic potential of the plasma, ion to electron density ratio dust to ion mass ratio, trapped electron temperature, ion temperature, and Mach number. The region for existence of non-linear waves varies due to the ion to electron density ratio and floating potential of dust grain particles. An increase of the Mach number increases the dust charge number. Also, as the electrons evolve toward their thermodynamic equilibrium, the amplitude of the wave increases. 2) The dependence of the charge variation of dust particles on the trapped electron temperature, ion density and temperature is observed in the dusty plasmas. The effect of the ion temperature decreases the dust charge number. Since the effect of the ion temperature affects the characteristics of the collective motion of the plasma, this effect is important in understanding non-linear waves propagating in dusty plasmas. 3) The phase velocity and frequency of unstable modes increase with relative to the negatively charged dust grains.

The stability or instability of the periodic solution is found to depend on the sign of the coefficients P and Q_r in the complex cubic Ginzburg-Landau (CCGL) equation. The non-isothermal parameter and gravitational effects all enter into the factors P and Q_r in a complicated way. Thus the non-isothermal parameter and the gravitational effects involved in P and Q_r will significantly influence the properties of the periodic solution including amplitude, speed, width as well as its stability. It is to be noted that the non-isothermality of the electrons has no effect on the linear properties of the wave but has a significant effect on the non-linear properties of the wave. The model considered in this paper can be easily applied to study the effects of different types of electron distributions on the non-linear propagation of ion-acoustic waves in self-gravitating dusty plasma.

To crown it all, the most important achievement in this paper is that instead of coupled non-linear equations as obtained by earlier authors an uncoupled non-linear equation has been obtained which is advantageous. Also, the model considered here can be easily extended to study the effects of different types of electron distributions on the non-linear propagation of ion-acoustic waves in both magnetized and un-magnetized dusty plasmas. In this situation, our results are important in understanding the charging mechanism of the streaming of dust grain particles and confirming the existence of arbitrary amplitude electrostatic ion acoustic waves in dusty plasmas.

Finally, it should be pointed out that, being a very complex system, a dusty plasma includes numerous effects that cannot be investigated in general in every analysis. Rather, for the sake of clarity and simplicity, these effects are studied separately, yet one should always keep in mind that the real physical picture is usually not so clear. To mention just a few, like the size distribution of dust particles, dust temperature variation as a result of permanent bombing by electrons and ions, shadow force effects again resulting from the attachment of electrons and ions, coagulation of dust particles and the redistribution of mass and size of grains image charge effects, etc. However, discussions on these fine and tiny, yet important effects are out of the scope of the present paper, and should be the subject of some independent studies.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

We neglect the mass of the electrons, $m_e = 0$ to have:

$$\begin{cases} F = a_2 - 2a_8, \\ S = \dfrac{a_3}{2} - 2a_9, \\ R = a_6, \\ H = a_4, \\ W = a_7, \\ N = a_5 \end{cases} \tag{55}$$

With:

$$\begin{cases} \sigma_0 = \dfrac{\omega_1 - \omega_0 \delta_0}{Z_d \omega_1 - 4\pi G m_d \delta_0}, \\ \sigma_1 = \dfrac{\omega_1 \delta_1 - \omega_2 \delta_0}{Z_d \omega_1 - 4\pi G m_d \delta_0}, \\ \sigma_2 = \dfrac{4\pi G m_d \delta_1 - Z_d \omega_2}{4\pi G m_d - Z_d \omega_0} \end{cases} \tag{56}$$

And:

$$\begin{cases} a_2 = \dfrac{2Ma_1}{n_{d0}}, \\ a_3 = \dfrac{2M^2 a_1^2}{n_{d0}^2}, \\ a_4 = 1, \\ a_5 = \sigma_2, \\ a_6 = \dfrac{M\sigma_0}{a_1}, \\ a_7 = 2q\sigma_0, \\ a_8 = \dfrac{M\sigma_1}{a_1}, \\ a_9 = 2q\sigma_1 \end{cases} \tag{57}$$

For:

$$\begin{cases} \delta_0 = \alpha n_{e0} - \beta n_{i0}, \\ p = \dfrac{Z_d}{\mu_d}, \\ \delta_1 = \dfrac{1}{2}\left(\alpha n_{e0} - \beta n_{i0}\right), \\ a_1 = \dfrac{\beta n_{i0}}{\mu_d}, \\ q = \dfrac{\sigma K_B T_d}{\mu_d n_{d0}}, \\ \omega_0 = \dfrac{M^2 a_1}{n_{d0}} - q a_1 + p, \\ \omega_1 = 4\pi G m_i n_i \beta, \\ \omega_2 = 2\pi G m_i n_i \beta^2 \end{cases} \tag{58}$$

We proceed to obtain the periodic wave solutions of Equations (32)-(35).

For Equation (32), we let $\varphi = S^2$ to obtain a differential equation which can be reduced to a Weierstrass's form equation by setting $S = \frac{4}{\alpha_2}\eta(\varsigma, g_2, g_3) - \frac{C_2}{3}$ with invariant $g_2 = \frac{4B^2}{P^2} - \frac{4KQ_r}{P}$, $\alpha_2 = \frac{2Q_r}{P}$, $C_2 = -\frac{2B}{Q_r}$ Equation (51).

For Equation (33), we let $\varphi = S^2$ and by setting $S = \frac{4}{\alpha_2}\eta(\varsigma, g_2, g_3) - \frac{C_2}{3}$ with invariant $g_2 = \left(\frac{v_g^2}{2P^2} + \frac{2B}{P}\right)^2 - \frac{4KQ_r}{P}$, $\alpha_2 = \frac{2Q_r}{P}$, $C_2 = -\left(\frac{2B}{Q_r} + \frac{v_g^2}{2PQ_r}\right)$ [50].

For Equation (34), we set $\varphi = \frac{4}{\alpha_2}\eta(\varsigma, g_2, g_3) - \frac{C_2}{3}$ to reduced the equation into a Weierstrass's form equation with invariant $g_2 = \frac{1}{12}\left(\frac{2\rho + 4B}{5P}\right)^2$, $\alpha_2 = \frac{2Q_i - 4Q_r}{5P}$, $C_2 = \frac{\rho + 2B}{Q_i - 2Q_r}$.

For Equation (35), we apply the homogeneous balance method (HBM). We suppose that Equation (35) takes the following ansatz solution [45]

$$\varphi = A_0 + A_1\eta(\varsigma, g_2, g_3) + A_2\eta_\varsigma(\varsigma, g_2, g_3). \quad (59)$$

Where

$$\begin{cases} A_0 = -\frac{b_1}{3b_2} - \frac{g_2}{6b_0}, \\ A_1 = \frac{3}{b_2}, \\ A_2 = -\frac{3}{2b_0 b_2}, \\ b_0 = \frac{v_g}{P}, \\ b_1 = \frac{2\rho}{5P}, \\ b_2 = \frac{2Q_i - 4Q_r}{5P}, \\ g_2 = \frac{8}{3}b_0^4 - \frac{32}{9}b_0^4 b_2 \end{cases} \quad (60)$$