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Deterministic and Stochastic Analysis of a New Rumor Propagation Model with Nonlinear Propagation Rate in Social Network

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Abstract

This paper presents a study on a new rumor propagation model with nonlinear propagation rate and secondary propagation rate. We divide the total population into three groups, the ignorant, the spreader and the aware. The nonlinear incidence rate describes the psychological impact of certain serious rumors on social groups when the number of individuals spreading rumors becomes larger. The main contributions of this work are the development of a new rumor propagation model and some results of deterministic and stochastic analysis of the rumor propagation model. The results show the influence of nonlinear propagation rate and stochastic fluctuation on the dynamic behavior of the rumor propagation model by using Lyapunov function method and stochastic related knowledge. Numerical examples and simulation results are given to illustrate the results obtained.

Keywords

Rumor Model, Nonlinear Incidence Rate, Secondary Propagation Rate, Stochastic Fluctuation

1. Introduction

Nowadays social networks like Wechat, Twitter, Facebook are used by extremely large number of people. With the development of communication technology and the wide use of social platforms, the speed of information transmission has been significantly improved, and the amount of information is expanding day by day. However, in so much information, rumors are mixed in and highly confusing. A rumor is basically a "circulating story of questionable authenticity that is superficially plausible but difficult to verify" [1] [2]. The spread of rumors has brought great inconvenience to society and individuals [3]. Nowadays with the development of social science, mathematics and computer tools, the application of nonlinear dynamics theory has been used to explain the spread of rumors in social networks [4]. Kendall and Daly [5] proposed a sociological mathematical model that divides the total population into three subgroups: ignorant people who have not heard of the rumor, spreaders who spread the rumor, and stiflers who have lost interest in the rumor and stopped spreading them. Jain *et al.* [6] used epidemiological modeling techniques to study the mathematical model of news communication and proposed rumor detection and verification criteria for this model. Moumita Ghosh *et al.* [7] studied dynamics and control of delayed rumor propagation through social networks and discussed the Hopf bifurcation with respect to delay and transmission rate.

However, in the known rumor propagation model, the nonlinear logical growth rate and the secondary of rumor propagation have not been considered. In this paper, we consider the logical growth of communication groups. As we all know, the rumor spread quickly at first. But after a period of time, the spread speed will slow down. Therefore, in our rumor propagation model, we consider the nonlinear rumor propagation growth rate. By calculating the influence threshold of rumor, the conditions of rumor prevalence and final disappearance are given. In this paper, we also give the secondary propagation rate of rumors. A controversial topic, even if one has a certain understanding of the facts, is likely to be influenced by the surrounding social environment and become a rumor spreader again. And there are some social networks to guide trolls. The secondary propagation rate is really important to effectively reflect the reality.

This paper is organized as follows. In Section 2, we give the formulation of the model. In Section 3, we show the existence, uniqueness and condition of stability of equilibria. And we discuss the global stability and the exponential stability of the rumor model. In Section 4, we show the asymptotic behavior around the rumor-free equilibria in the stochastic rumor model. In Section 5 we have discussed the asymptotic behavior around the rumor-existing equilibrium and studied the ergodicity. Numerical simulations are presented in Section 6. Finally some conclusions are obtained in Section 7.

2. Model Formulation

Combined with the existing rumor propagation model [8]-[14] and epidemic model [15] [16] [17] [18] [19], in this paper we established the following model by introducing the influence of the logical growth of the spreader group, nonlinear incidence rate and random fluctuation on rumor propagation. Logistic growth happens when the growth rate slows down as the population tends to reach a maximum sustainable value C which is called the carrying capacity. Moreover the total population is bounded, which is proved later. Therefore after a period of time, rumor propagation does not increase and it becomes asymptotic to the constant C.

We consider the rumor propagation rate is $\frac{ky(t)}{1+\alpha y^h(t)}$, where $ky(t)$ represents the intensity of the rumor, $\frac{1}{1+\alpha y^h(t)}$ show that the social rumor spread rate changes when the number of spreaders becomes huge. People who aware the absurdity of rumors, may become the spreaders of rumors due to the influence of the social environment around them. Therefore, we introduce the secondary rumor propagation rate.

We consider some assumptions:

- The spreader and the aware will never come back to ignorant class.
- The aware who realizes that the information is controversial may be disturbed by the surrounding environment and become a spreader of rumors.

Taking into account the above assumptions, we can formulate the model as follows:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = b - dx(t) - \frac{kx(t)y(t)}{1+\alpha y^h(t)}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{kx(t)y(t)}{1+\alpha y^h(t)} - (d+\mu)y(t) + rz(t), \\ \frac{\mathrm{d}z}{\mathrm{d}t} = \mu y(t) - dz(t) - rz(t). \end{cases} \tag{2.1}$$

We use $T(t) = x(t) + y(t) + z(t)$ to represent the total number of accounts on the social network at time t. All the parameters in Equation (2.1) are considered as positive constants and their definitions are given in **Table 1**.

Stochastic fluctuations in the social environment have many effects on people's psychology and affecting the spread of rumors. In this paper, we assume that both the nonlinear propagation rate and the secondary propagation rate are affected by environmental fluctuations. We propose the corresponding stochastic model as follows:

$$\begin{cases} \mathrm{d}x = \left[b - dx(t) - \frac{kx(t)y(t)}{1+\alpha y^h(t)}\right]\mathrm{d}t - \frac{\sigma_1 x(t)y(t)}{1+\alpha y^h(t)}\mathrm{d}B_1(t), \\ \mathrm{d}y = \left[\frac{kx(t)y(t)}{1+\alpha y^h(t)} - (d+\mu)y(t) + rz(t)\right]\mathrm{d}t \\ \qquad + \frac{\sigma_1 x(t)y(t)}{1+\alpha y^h(t)}\mathrm{d}B_1(t) + \sigma_2 z(t)y(t)\mathrm{d}B_2(t), \\ \mathrm{d}z = \left[\mu y(t) - dz(t) - rz(t)\right]\mathrm{d}t - \sigma_2 z(t)y(t)\mathrm{d}B_2(t). \end{cases} \tag{2.2}$$

In Equation (2.2), $B_i(t)$, $i = 1,2$ are independent Brownian movement defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. R_+^3 is the set of 3-dimensional real column vector with non-negative elements.

3. Some Results of the Extinction and Persistence

In this section, we will discuss the persistence and extinction of the rumor model.

Table 1. Parameter description.

Parameter	Parameter definition
b	The total number of social network users at the beginning.
d	The rate of the social network users becomes inactive to the rumor.
k	Proportionality constant.
μ	The rate of the spreader become the aware.
r	The rate of the aware become the spreader due to the social environment around them.
α	The saturated rates.

Considering the characteristic of Equation (2.1) and Equation (2.2), we have the following conclusions.

Lemma 3.1. *For the solution* $(x(t), y(t), z(t))$ *of Equation (2.1) and Equation (2.2) with initial value* $(x(t_0), y(t_0), z(t_0)) \in \mathcal{R}_+^3$, *we can obtain that*

$$\max\left\{\limsup_{t\to\infty} x(t), \limsup_{t\to\infty} y(t), \limsup_{t\to\infty} z(t)\right\} \le \frac{b}{d}.$$

Proof. Summing up the three equations in Equation (2.1) and Equation (2.2), we have

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = b - dN(t).$$

Obviously, it can be obtained by calculation

$$T(t) = \frac{b}{d}\left(1 - \mathrm{e}^{d(t_0 - t)}\right) + T(t_0)\,\mathrm{e}^{d(t_0 - t)}, \tag{3.1}$$

and take the limit of both sides of Equation (3.1), we achieve

$$\lim_{t\to\infty} T(t) = \frac{b}{d}.$$

The proof is complete.□

The boundness and non-negative of the solution of Equation (2.1) and Equation (2.2) indicate that the models have realistic meaning. We denote

$$I = \left\{(x(t), y(t), z(t)) : x(t) + y(t) + z(t) \equiv \frac{b}{d}, x(t) \ge 0, y(t) \ge 0, z(t) \ge 0\right\}$$

is a invariant region for Equation (2.1) and Equation (2.2).

From the invariant region *I*, we can get the following equivalent model of Equation (2.1)

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{ky(t)}{1 + \alpha y^h(t)}\left[\dfrac{b}{d} - y(t) - z(t)\right] - (d + \mu)y(t) + rz(t) \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = \mu y(t) - (d + r)z(t). \end{cases} \tag{3.2}$$

We would mainly study the existence, uniqueness, and the stability of equilibria of Equation (3.2) in order to obtain the dynamical properties of Equation

(2.1).

Lemma 3.2. $I=\left\{\left(x(t),y(t),z(t)\right):x(t)+y(t)+z(t)\equiv\frac{b}{d},x(t)\ge 0,y(t)\ge 0,z(t)\ge 0\right\}$ *is an absorbing set of model 2.1 in the first quadrant.*

Proof. From model 3.2, we can see that on the line $y(t)=0$, $\frac{\mathrm{d}y}{\mathrm{d}t}>0$, and on the line $R=0$, $\frac{\mathrm{d}z}{\mathrm{d}t}>0$. Hence, no orbit of model 3.2 can exit from the first quadrant, with the boundary $z=0$ and $y=0$. From the proposition of the region *I* and Lemma 3.1, we can get

$$\lim_{t\to\infty}y(t)\le\lim_{t\to\infty}T(t)=\frac{b}{d},\quad \lim_{t\to\infty}z(t)\le\lim_{t\to\infty}T(t)=\frac{b}{d}.$$

Thus

$$\begin{aligned}\left.\frac{\mathrm{d}(y+z)}{\mathrm{d}t}\right|_{y+z=\frac{b}{d}}&=\left.\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{y+z=\frac{b}{d}}\\&=\left.\frac{ky(t)}{1+\alpha y^{h}(t)}\left[\frac{b}{d}-y(t)-z(t)\right]-d(y+z)\right|_{y+z=\frac{b}{d}}\\&=-b<0.\end{aligned}\tag{3.3}$$

Equation (3.3) shows that the orbit of model 3.2 getting at the boundary $y(t)+z(t)=\frac{b}{d}$ must go into the interior of *I*. Thus, the region *I* is an absorbing set of model 3.2, in the first quadrant. The proof is complete.□

Next we show that the threshold value of the rumor model 2. is

$$R_0=\frac{kbr+kbd+\mu rd}{d(\mu+d)(r+d)}.$$

Lemma 3.3. (1) *If* $R_0\le 1$*, the model 3.2 has a unique equilibrium* $E_0=(0,0)$ *in the first quadrant.*

(2) If $R_0>1$, the model 3.2 has two equilibria in the first quadrant, which are E_0 and $E^*=\left(y^*,z^*\right)$, where $y^*,z^*>0$.

Proof. Apparently, model 3.2 has an equilibrium $E_0=(0,0)$. And it has a positive equilibrium $E^*=\left(y^*,z^*\right)$, if and only if $E^*=\left(y^*,z^*\right)$ satisfies the following equation

$$\begin{cases}\frac{ky}{1+\alpha y^{h}}\left(\frac{b}{d}-y-z\right)-(d+\mu)y+rz=0,\\ \mu y-(d+r)z=0.\end{cases}\tag{3.4}$$

By calculation, the above Equation (3.4) is equivalent to

$$\left(\frac{r\mu}{r+d}-(d+\mu)\right)\alpha y^{h}-\left(\frac{k\mu}{r+d}+k\right)y=(d+\mu)(1-R_0).\tag{3.5}$$

Obviously, we can get that $\frac{r\mu}{r+d}-(d+\mu)<0$. Then if $R_0\le 1$, Equation (3.5) has no positive solution. So the conclusion in case (1) holds. Case (2) is demonstrated as follows. Set $H(y)=\left(\frac{r\mu}{r+d}-(d+\mu)\right)\alpha y^h-\left(\frac{k\mu}{r+d}+k\right)y$. We have that

$$\frac{\mathrm{d}H}{\mathrm{d}y}=\left(\frac{r\mu}{r+d}-(d+\mu)\right)\alpha h y^{h-1}-\left(\frac{k\mu}{r+d}+k\right).$$

We could easily get that $\frac{\mathrm{d}H}{\mathrm{d}y}<0$, and $H(y)$ is strictly monotone decreasing for $y\ge 0$.

Thus, we can obtain that

$$H\left(\frac{b}{d}\right)<(d+\mu)(1-R_0)<H(0)=0.$$

Therefore, case (2) can be obtained.□

Theorem 3.1. *There exists a unique solution* $(x(t),y(t),z(t))$ *of model 2.2 for* $t\ge 0$ *with any initial value* $(x(0),y(0),z(0))\in R_+^3$ *and the solution will remain in* R_+^3 *with probability one.*

Proof. According to the local Lipschitz continuity of the coefficient of model 2.2, it can be achieved that there exists a unique local solution solution $(x(t),y(t),z(t))$ on $[0,\omega_e)$ with an initial value $(x(t),y(t),z(t))\in R_+^3$, where ω_e represents the explosion time. To prove the globality of the solution, we have to show that $\omega_e=\infty$ a.s.. We suppose that there exist $M_0\ge 1$ is sufficiently large such that $x(0),y(0),z(0)$ all in the interval $\left[\frac{1}{M_0},M_0\right]$. For each integer $k\ge M_0$, we define the stopping time

$$\omega_k=\inf\left\{t\in[0,\omega_e):\min\{x(t),y(t),z(t)\}\le\frac{1}{k}\text{ or }\max\{x(t),y(t),z(t)\}\ge k\right\}.$$

Then ω_k increases as $k\to\infty$. Denote $\omega_\infty=\lim_{k\to\infty}\omega_k$. Obviously, $\omega_\infty\le\omega_e$. Next, we prove that $\omega_\infty=\infty$ a.s. If it is not true, then there is constant $\varepsilon\in(0,1)$, such that $P\{\omega_\infty<\infty\}>\varepsilon$. Thus there are two constants, integer $k_1>M_0$ and $T>0$ satisfying

$$P\{\omega_k\le T\}\ge\varepsilon,$$

for all $k>k_1$. Define a C^2-function $V:R_+^3\to R_+$ by

$$V(x,y,z)=\left(x-a-a\ln\frac{x}{a}\right)+(y-1-\ln y)+(z-1-\ln z), \tag{3.6}$$

where a is a constant that will be given later. The nonnegativity of this function (3.6) can be seen from $y-1-\ln y\ge 0$, for all $y>0$. By Itô's formula, we achieve

$$\begin{aligned}\mathrm{d}V(x,y,z)&=\left(1-\frac{a}{x}\right)\mathrm{d}x+\left(1-\frac{1}{y}\right)\mathrm{d}y+\left(1-\frac{1}{z}\right)\mathrm{d}z\\&\quad+\frac{1}{2}\cdot\frac{a}{x^2}\frac{\sigma_1^2x^2y^2}{(1+\alpha y_h)^2}\mathrm{d}t+\frac{1}{2}\cdot\frac{1}{y^2}\frac{\sigma_1^2x^2y^2}{(1+\alpha y_h)^2}\mathrm{d}t\end{aligned}$$

$$+\frac{1}{2}\cdot\frac{1}{y^2}\sigma_2^2 y^2 z^2 \mathrm{d}t+\frac{1}{2}\cdot\frac{1}{z^2}\sigma_2^2 y^2 z^2 \mathrm{d}t$$
$$=\mathcal{L}V(x,y,z)\mathrm{d}t+\frac{a\sigma_1 y-\sigma_1 x}{1+\alpha y^h}\mathrm{d}B_1(t)+\sigma_2(y-z)\mathrm{d}B_2(t).$$

Here, $\mathcal{L}V: R_+^3 \to R_+$ and choose a sufficiently small, then we achieve

$$\mathcal{L}V=(b+ad+2d+\mu+r)-d(x+y+z)-\left(\frac{ab}{x}+\frac{kx}{1+\alpha y^h}+\frac{rz}{y}+\frac{\mu y}{z}\right)$$
$$+\frac{aky}{1+\alpha y^h}+\frac{a\sigma_1{}^2 y^2}{2\left(1+\alpha y^h\right)^2}+\frac{\sigma_1{}^2 x^2}{2\left(1+\alpha y^h\right)^2}+\frac{\sigma_2^2\left(z^2+y^2\right)}{2}$$
$$\le(b+ad+2d+\mu+r)+\frac{akb}{d}+\frac{a\sigma_1^2 b^2}{2d^2}+\frac{\sigma_1^2 b^2}{2d^2}+\frac{\sigma_2^2 b^2}{d^2}.$$

Since a is sufficiently small, we could get that

$$\mathcal{L}V\le(b+2d+\mu+r)+\frac{2\sigma_1^2 b^2}{d^2}\le\mathcal{K}, \tag{3.7}$$

where $\mathcal{K}$ is a constant in Equation (3.7). Thus,

$$\mathrm{d}\left(V(x,y,z)\right)\le\mathcal{K}\mathrm{d}t+\frac{a\sigma_1 y-\sigma_1 x}{1+\alpha y^h}\mathrm{d}B_1(t)+\sigma_2(y-z)\mathrm{d}B_2(t). \tag{3.8}$$

Taking integral on the above inequality (3.8) from 0 to $\omega_k\wedge T$, we obtain

$$\int_0^{\omega_k\wedge T}\mathcal{K}\mathrm{d}t+\int_0^{\omega_k\wedge T}\frac{a\sigma_1 y+\sigma_1 x}{1+\alpha y^h}\mathrm{d}B_1(t)+\int_0^{\omega_k\wedge T}\sigma_2(y-z)\mathrm{d}B_2(t),$$

where $\omega_k\wedge T=\min\{\omega_k,T\}$. Then, we can have that

$$E\Big(V\big(x(\omega_k\wedge T),y(\omega_k\wedge T),z(\omega_k\wedge T)\big)\Big)\le V\big(x(0),y(0),z(0)\big)+\mathcal{K}T.$$

Let $\Omega_k=\{\omega_k\le T\}$, then we have $P(\Omega_k)\ge\varepsilon$. For each $\omega\in\Omega_k$, $x(\omega_k,\omega)$, $y(\omega_k,\omega)$, or $z(\omega_k,\omega)$ equals either k or $\frac{1}{k}$, and the nonnegativity properties of the terms in function $V\big(x(t),y(t),z(t)\big)$, we can conclude that

$$V\big(x(\omega_k,\omega),y(\omega_k,\omega),z(\omega_k,\omega)\big)\ge\min\left\{k-1-\ln k,\frac{1}{k}-1+\ln k\right\}.$$

Thus

$$\begin{aligned}&V\big(x(0),y(0),z(0)\big)+\mathcal{K}T\\ &\ge E\left[1_{\Omega_k}(\omega)V\big(x(\omega_k,\omega),y(\omega_k,\omega),z(\omega_k,\omega)\big)\right]\\ &\ge\varepsilon\min\left\{k-1-\ln k,\frac{1}{k}-1+\ln k\right\},\end{aligned}$$

where $1_{\Omega_k}(\omega)$ is the indicator function of Ω_k. Letting $k\to\infty$, we can obtain the contraction

$$\infty>V\big(x(0),y(0),z(0)\big)+\mathcal{K}T\ge\infty.$$

The proof is complete.

By constructing appropriate Lyapunov functions, we can study the extinction and persistence conditions of the rumor propagation model. Above all, we will discuss the globally asymptotically stable in probability and exponentially stable a.s. of solution to the equilibrium $\tilde{E}_0=\left(\frac{b}{d},0,0\right)$.

Theorem 3.2. *Suppose* $\left(x(t),y(t),z(t)\right)$ *be the solution of model 2.2 with the initial value* $\left(x(0),y(0),z(0)\right)\in R_+^3$. *If* $\frac{\left((\sigma_1)^2+(\sigma_2)^2\right)b^2}{d^2}>2(d+\mu)$ *hold, then the trivial solution of model 2.2 is globally asymptotically stable with probability one.*

Proof. Define

$$V_1(x,y,z)=m\left(\frac{b}{d}-x\right)^2+ny^2+z^2. \tag{3.9}$$

Here, $m,n>0$ will be chosen later. Then applying Itô's formula, we get

$$\begin{aligned}
&\mathrm{d}V_1(x,y,z)\\
&=-2m\left(\frac{b}{d}-x\right)\mathrm{d}x+2ny\mathrm{d}y+2z\mathrm{d}z\\
&\quad+\left[\left(2mx-\frac{2mb}{d}\right)\left(b-dx-\frac{kxy}{\left(1+\alpha y^h\right)^2}\right)+2ny\left(\frac{kxy}{1+\alpha y^h}-dy-\mu y+rz\right)\right]\mathrm{d}t\\
&\quad+\left[2z\left(\mu y-dz-rz\right)+\frac{m\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2}+\frac{n\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2}+n\sigma_2^2z^2y^2+\sigma_2^2z^2y^2\right]\mathrm{d}t\\
&\quad-2m\left(x-\frac{b}{d}\right)\frac{\sigma_1xy}{1+\alpha y^h}\mathrm{d}B_1(t)+\frac{2n\sigma_1xy^2}{1+\alpha y^h}\mathrm{d}B_1(t)\\
&\quad+2n\sigma_2zy^2\mathrm{d}B_2(t)-2\sigma_2yz^2\mathrm{d}B_2(t).
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathcal{L}V_1(x,y,z)\\
&=-2mdx^2-\frac{2mkx^2y}{1+\alpha y^h}-\frac{2mb^2}{d^2}-2n(d+\mu)y^2-2dz^2-2rz^2-2dz^2-2rz^2\\
&\quad+4mbx+2nryz+2\mu yz+n\sigma_2^2z^2y^2+\sigma_2^2z^2y^2\\
&\quad+\frac{2mbkxy}{d\left(1+\alpha y^h\right)}+\frac{2nkxy^2}{1+\alpha y^h}+\frac{m\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2}+\frac{n\sigma_1^2x^2y^2}{\left(1+\alpha y^2\right)^2}\\
&\le-2mdx^2-\frac{2mkx^2y}{1+\alpha y^h}-\frac{2mb^2}{d}-2dz^2-2rz^2-2n(d+\mu)y^2\\
&\quad+2n^2+\left[\frac{\sigma_1^2b^4}{d^4}+\frac{\sigma_2^2b^4}{d^4}-2(d+\mu)y^2\right]n+\left(\frac{k^2b^6}{d^6}+\frac{r^2b^4}{d^4}+\frac{2\mu b^2}{d^2}+\frac{\sigma_2^2b^4}{d^4}\right).
\end{aligned}$$

Let $N=\left(\frac{k^2b^6}{d^6}+\frac{r^2b^4}{d^4}+\frac{2\mu b^2}{d^2}+\frac{\sigma_2^2b^4}{d^4}\right)$, then we can get

$$H(n)=2n^2+\left[\frac{\sigma_1^2b^4}{d^4}+\frac{\sigma_2^2b^4}{d^4}-2(d+\mu)y^2\right]n+N. \tag{3.10}$$

The discriminant of $H(n)$ in (3.10) is

$$\Delta=\left[\frac{\sigma_1^2b^4}{d^4}+\frac{\sigma_2^2b^4}{d^4}-2(d+\mu)y^2\right]^2-8N>0.$$

Thus,

$$\left[\frac{\sigma_1^2b^4}{d^4}+\frac{\sigma_2^2b^4}{d^4}-2(d+\mu)y^2\right]^2>8N.$$

The following conclusion can be obtained from calculation:

$$\frac{(\sigma_1^2+\sigma_2^2)b^2}{2d^2(d+\mu)}>1. \tag{3.11}$$

Therefore, when Equation (3.11) is satisfied, we can get that $H(n)<0$ for every $n\in[n_1,n_2]$, where n_1 and n_2 are distinct positive roots for $H(n)$. Then $\mathcal{L}V_1$ is negative definite function if $\frac{(\sigma_1^2+\sigma_2^2)b^2}{2d^2(d+\mu)}>1$. The proof is complete.

Remark 3.1. *For any* $(x(0),y(0),z(0))\in I$ *, if Equation* (3.11) *is satisfied, then the solution of model 2.2 satisfies:* $\lim_{t\to\infty}x(t)=\frac{b}{d}$ *,* $\lim_{t\to\infty}y(t)=0$ *,* $\lim_{t\to\infty}z(t)=0$ *a.s..*

Theorem 3.3. [20] *(Strong law of large numbers) Suppose* $X=\{X_t\}_{t\ge0}$ *is a local continuous martingale which satisfies* $X(0)=0$ *. Then*

$$\lim_{t\to\infty}\langle X,X\rangle_t=\infty\ a.s.\Rightarrow\lim_{t\to\infty}\frac{X_t}{\langle X,X\rangle_t}=0\ a.s.,$$

and

$$\limsup_{t\to\infty}\frac{\langle X,X\rangle_t}{t}<\infty\ a.s.\Rightarrow\lim_{t\to\infty}\frac{X_t}{t}=0\ a.s..$$

Remark 3.2. *The Brownian motion* $B(t)$ *is a square integrable martingale, and its second variations is* $\langle X,X\rangle_t=t\ (t\ge0)$ *. Combined with Strong law of large numbers, it can be known that*

$$\lim_{t\to\infty}\frac{B_t}{t}=0\ a.s..$$

Theorem 3.4. *Suppose* $(x(t),y(t),z(t))$ *is the solution of model 2.2 with any initial value* $(x(0),y(0),z(0))\in R_+^3$ *. Then the solution of model 2.2 obeys:*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left[\left(\frac{b}{d}-x\right)+y+z\right]\le\left(-d+\frac{2kb}{d}\right).$$

Proof. Define a Lyapunov function

$$V_2\left(x(t),y(t),z(t)\right)=\ln\left[\left(\frac{b}{d}-x\right)+y+z\right].$$

Then we have that

$$\mathrm{d}V_2(x,y,z)=\frac{-1}{\frac{b}{d}-x+y+z}\mathrm{d}x+\frac{1}{\frac{b}{d}-x+y+z}\mathrm{d}y$$
$$+\frac{1}{\frac{b}{d}-x+y+z}\mathrm{d}z-\frac{(\sigma_2 zy)^2}{\left(\frac{b}{d}-x+y+z\right)^2}\mathrm{d}t.$$

Thus, we could get that

$$\mathcal{L}V_2(x,y,z)$$
$$=\frac{1}{\frac{b}{d}-x+y+z}\left[-b+dx-dy-dz+\frac{2kxy}{1+\alpha y^h}\right]-\frac{(\sigma_2 zy)^2}{\left(\frac{b}{d}-x+y+z\right)^2}\mathrm{d}t$$
$$\le -d+\frac{2kxy}{\frac{b}{d}-x+y+z}\le -d+\frac{2kx\left(\frac{b}{d}-x-z\right)}{\frac{b}{d}-x+y+z}\le -d+\frac{2kb}{d}.$$

Therefore,

$$\mathrm{d}V_2(x,y,z)\le\left(-d+\frac{2kb}{d}\right)\mathrm{d}t+\frac{2\sigma_1 xy}{\left(\frac{b}{d}-x+y+z\right)(1+\alpha y_h)}\mathrm{d}B_1(t). \quad (3.12)$$

Taking integral on both sides of above Equation (3.12) and divided by *t*. We have that

$$\frac{\ln\left[\frac{b}{d}-x(t)+y(t)+z(t)\right]}{t}$$
$$\le\frac{\ln\left[\frac{b}{d}-x(0)+y(0)+z(0)\right]}{t}+\left(-d+\frac{2kb}{d}\right)+\frac{2\sigma_1 xy}{\left(\frac{b}{d}-x+y+z\right)(1+\alpha y^h)}\mathrm{d}B_1(t) \quad (3.13)$$
$$\le\frac{\ln\left[\frac{b}{d}-x(0)+y(0)+z(0)\right]}{t}+\left(-d+\frac{2kb}{d}\right)+\frac{2\sigma_1 b}{d}\mathrm{d}B_1(t).$$

$W(t)=\int_0^t\frac{2\sigma_1 b}{d}\mathrm{d}B_1(t)$ is a continuous local martingale, then by the strong law of large number for local martingales and above Equation (3.13). We have

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left[\left(\frac{b}{d}-x\right)+y+z\right]\le -d+\frac{2kb}{d}.$$

The proof is complete.□

Remark 3.3. *The trivial solution for stochastic model 2.2 is exponentially stable a.s. in I, if the following condition hold:*

$$\frac{2kb}{d}-d<0.$$

4. Asymptotic behavior around $\tilde{E}_0$

We know that $\tilde{E}_0=\left(\frac{b}{d},0,0\right)$ is the rumor free equilibrium (RFE) of the deterministic model 2.1. However it may be not an equilibrium of the stochastic model 2.2. In this section, we show the average oscillation around $\tilde{E}_0=\left(\frac{b}{d},0,0\right)$ under environmental random perturbation.

Theorem 4.1. *Let* $(x(t),y(t),z(t))$ *be the solution of model 2.2 with any initial value* $(x(0),y(0),z(0))\in R_+^3$. *If* $R_0\le 1$, *then*

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t m_1\left(x(r)-\frac{b}{d}\right)^2+m_2y^2(r)+m_3z^2(r)\mathrm{d}t\le\rho_1,$$

where

$$m_1=\frac{d}{2}-\sigma_1^2\left(-d+\frac{(2d+\mu)^2}{4}+\frac{r^2}{4}\right),$$

$$m_2=\frac{d}{2}-\sigma_1^2\left(-d-\mu+\frac{(2d+\mu)^2}{4}+\frac{r^2}{4}\right),$$

$$m_3=\frac{d}{2}-\frac{\sigma_1^2}{2},$$

$$\rho_1=\frac{b_4}{d_4}\left(\sigma_1^4+\frac{\sigma_1^2\sigma_2^2}{2}+\sigma_1^2+\sigma_2^2\right).$$

Proof. Let $s_1=x-\frac{b}{d}$, $s_2=y$, $s_3=z$. Then, we can change model 2.2 into

$$\begin{cases}\mathrm{d}s_1=\left[b-d\left(s_1+\frac{b}{d}\right)-\frac{k\left(s_1+\frac{b}{d}\right)s_2}{1+\alpha s_2^h}\right]\mathrm{d}t-\frac{\sigma_1\left(s_1+\frac{b}{d}\right)s_2}{1+\alpha s_2^h}\mathrm{d}B_1(t),\\ \mathrm{d}s_2=\left[\frac{k\left(s_1+\frac{b}{d}\right)s_2}{1+\alpha s_2^h}-(d+\mu)s_2+rs_3\right]\mathrm{d}t+\frac{\sigma_1\left(s_1+\frac{b}{d}\right)s_2}{1+\alpha s_2^h}\mathrm{d}B_1(t)+\sigma_2s_2s_3\mathrm{d}B_2(t),\\ \mathrm{d}s_3=[\mu s_2-ds_3-rs_3]-\sigma_2s_2s_3\mathrm{d}B_2(t).\end{cases}$$

Define a C^2-function $W=V_1(s)+\sigma_1^2V_2(s)$. Where $s=(s_1,s_2,s_3)$,

$$V_1(s)=\frac{(s_1+s_2+s_3)^2}{2}$$

$$\text{and } V_2(s)=\frac{(s_1+s_2)^2}{2}.$$

By Itô's formular, we can get that

$$\mathrm{d}V_1 = (s_1+s_2+s_3)(\mathrm{d}s_1+\mathrm{d}s_2+\mathrm{d}s_3) + \frac{\sigma_1^2\left(s_1+\frac{b}{d}\right)^2 s_2^2}{\left(1+\alpha s_2^h\right)^2}\mathrm{d}t + \sigma_2^2 s_2^2 s_3^2 \mathrm{d}t$$
$$\le \left[-\frac{d}{2}s_1^2 - \frac{d}{2}s_2^2 - \frac{d}{2}s_3^2 + \left(\sigma_1^2+\sigma_2^2\right)\frac{b^4}{d^4}\right]\mathrm{d}t$$

and

$$\mathrm{d}V_2 = (s_1+s_2)(\mathrm{d}s_1+\mathrm{d}s_2) + \frac{\sigma_1^2\left(s_1+\frac{b}{d}\right)^2 s_2^2}{\left(1+\alpha s_2^h\right)^2}\mathrm{d}t$$
$$+\frac{\sigma_2^2 s_2^2 s_3^2}{2}\mathrm{d}t + \sigma_1 s_2 s_3 (s_1+s_2)\mathrm{d}B_2(t)$$
$$\le \left[-d+\frac{(2d+\mu)^2}{4}+\frac{r^2}{4}\right]s_1^2\mathrm{d}t + \left[-(d+\mu)+\frac{(2d+\mu)^2}{4}+\frac{r^2}{4}\right]s_2^2\mathrm{d}t$$
$$+\frac{r^2 s_3^2}{2}\mathrm{d}t + \frac{\left(\sigma_1^2+\frac{\sigma_2^2}{2}\right)b^4}{d^4}\mathrm{d}t + \sigma_1 s_2 s_3 (s_1+s_2)\mathrm{d}B_2(t).$$

Therefore

$$\mathrm{d}W = \mathrm{d}V_1 + \sigma_1^2 \mathrm{d}V_2$$
$$\le \left[-\frac{d}{2}+\sigma_1^2\left(-d+\frac{(2d+\mu)^2}{4}+\frac{r^2}{4}\right)\right]s_1^2\mathrm{d}t$$
$$+\left[-\frac{d}{2}+\sigma_1^2\left(-d-\mu+\frac{(2d+\mu)^2}{4}+\frac{r^2}{4}\right)\right]s_2^2\mathrm{d}t \tag{4.1}$$
$$+\left[-\frac{d}{2}+\frac{\sigma_1^2 r^2}{2}\right]s_3^2\mathrm{d}t + \left[\frac{b^4}{d^4}\left(\sigma_1^4+\frac{\sigma_1^2\sigma_2^2}{2}+\sigma_1^2+\sigma_2^2\right)\right]\mathrm{d}t$$
$$+\frac{\sigma_1 b^2 (s_1+s_2)}{d^2}\mathrm{d}B_2(t).$$

Integrating both sides of (4.1) from 0 to t and taking the expectation, we can have that

$$EW(s(t)) - W(s(0)) \le -E\int_0^t m_1 x_1(r)^2 + m_2 x_2(r)^2 + m_3 x_3(r)^2 - \rho_1 \mathrm{d}t. \tag{4.2}$$

From the boundedness of the solution of model 2.2 giveen in Lemma 3.1, we could see that the left side of the above Equation (4.2) is bounded. Thus,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t m_1\left(x(r)-\frac{b}{d}\right)^2 + m_2 y^2(r) + m_3 z^2(r)\mathrm{d}t \le \rho_1.$$

The proof is complete.□

Remark 4.1. *From the above Theorem* 4.1*, when* $R_0 \le 1$ *and the intensity of environmental random disturbance is small enough such that* $m_i > 0, i = 1,2,3$ *, i.e.,*

$$\sigma_1^2\left[-d+\frac{(2d+\mu)^2}{4}+\frac{r^2}{4}\right]<\frac{d}{2},$$
$$\sigma_1^2\left[-(d+\mu)+\frac{(2d+\mu)^2}{4}+\frac{r^2}{4}\right]<\frac{d}{2},$$
$$\sigma_1^2 r^2<d.$$

Then the solution of model (2.2) will oscillate around E_0 *and the oscillation amplitude can be estimated by*

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t m_1\left(x(r)-\frac{b}{d}\right)^2+m_2y^2(r)+m_3z^2(r)\mathrm{d}t\le\rho_1.$$

From a sociological point of view, the rumor will trend to die out when the intensity of stochastic environmental perturbations is small enough.

Particularly, when $\sigma_1=0,\sigma_2=0$, *the model 2.2 has one rumor-free equilibrium* E_0, *which is globally asymptotically, as proved above.*

5. Asymptotic Behavior around E_* and Ergodicity

When $R_0>1$, $E_*=(x^*,y^*,z^*)$ is the globally asymptotically stable rumor existing or prevailing Equilibrium (REE), with $y^*>0$. However it may not be an equilibrium of stochastic model 2.2. In this section, we will study the asymptotic behavior around E_* of the model 2.2 under environmental stochastic perturbation.

Before giving the theorem, let's introduce the lemma which is needed in the proof process.

Lemma 5.1. [21] *Suppose that there exists a bounded domain* $W\subset R^n$, *with regular boundary* Γ *satisfying the following properties.*

(B1) *In the domain W and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix* $D(x)$ *is bounded away from zero.*

(B2) *If* $x\in R^n\setminus W$, *the mean time* τ *at which a path issuing from x reaches the set W is finite and* $\sup_{x\in K}E^x\tau<\infty$ *for every compact subset* $K\in R^n$. *Then the Markov process* $X(t)$ *has a stationary distribution* $\mu(\cdot)$ *with density in* R^n *such that*

$$\lim_{t\to\infty}P(t,x,A)=\mu(A),\ P_x\left\{\lim_{t\to\infty}\int_0^t f(X(t))\mathrm{d}t=\int_{R_n}f(x)\mu(\mathrm{d}x)\right\}=1,$$

for any Borel set $A\subset R_n$ *and* $f(\cdot)$ *is an integrable function with respect to the measure* μ.

Theorem 5.1. *Let* $(x(t),y(t),z(t))$ *be the solution of model 2.2, with any initial value* $(x(0),y(0),z(0))\in R_+^3$. *If* $R_0>1$, *then*

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t n_1(x(r)-x^*)^2+n_2(y(r)-y^*)^2+n_3(z(r)-z^*)^2\mathrm{d}r\le\rho_2,$$

where

$$n_1=\frac{d}{2},\quad n_2=\frac{d}{2},\quad n_3=\frac{d}{2},\quad \rho_2=\frac{\left(\sigma_1^2+\sigma_2^2\right)b^4}{d^4}.$$

Moreover, there exists a stationary distribution $\pi(\cdot)$ *and the solution is ergodic for model 2.2.*

Proof. If $R_0>1$, there is one unique REE, $E_*=\left(x^*,y^*,z^*\right)=\left(x_1,y_1,z_1\right)$, of model 2.2, which satisfies

$$\begin{cases} b-dx_1-\dfrac{kx_1y_1}{1+\alpha y^h}=0, \\ \dfrac{kx_1y_1}{1+\alpha y^h}-\left(d+\mu\right)y_1+rz_1=0, \\ \mu y_1-dz_1-rz_1=0. \end{cases} \tag{5.2}$$

Define a C^2-function $Q(s)=\dfrac{\left(x-x_1+y-y_1+z-z_1\right)^2}{2}$, where $s=(x,y,z)$. By Itô's formula and Eq. (5.1), we compute

$$\begin{aligned} \mathrm{d}Q(s)&=-d\left(x-x_1+y-y_1+z-z_1\right)^2\mathrm{d}t+\frac{\sigma_1^2x^2y^2}{1+\alpha y^h}\mathrm{d}t+\sigma_2^2y^2z^2\mathrm{d}t \\ &\le -d\left[\left(x-x_1\right)^2+\left(y-y_1\right)^2+\left(z-z_1\right)^2\right]\mathrm{d}t \\ &\quad -d\left[2\left(x-x_1\right)\left(y-y_1\right)+2\left(x-x_1\right)\left(z-z_1\right)+2\left(y-y_1\right)\left(z-z_1\right)\right]\mathrm{d}t \\ &\quad +\frac{\sigma_1^2x^2y^2}{1+\alpha y^h}\mathrm{d}t+\sigma_2^2y^2z^2\mathrm{d}t \\ &\le -\frac{d}{2}\left(x-x_1\right)^2-\frac{d}{2}\left(y-y_1\right)^2-\frac{d}{2}\left(z-z_1\right)^2+\frac{\left(\sigma_1^2+\sigma_2^2\right)b^4}{d^4}. \end{aligned} \tag{5.2}$$

Integrating both sides of above Equation (5.2) from 0 to t and taking the expectation, we have that

$$\begin{aligned} &EQ\left(s(t)\right)-Q\left(s(0)\right) \\ &\le E\int_0^t -n_1\left(x(r)-x_1\right)^2-n_2\left(y(r)-y_1\right)^2-n_3\left(z(r)-z_3\right)^2+\rho_2\mathrm{d}r. \end{aligned} \tag{5.3}$$

Note that the boundedness of solution of model 2.2 we can easily obtain that

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t n_1\left(x(r)-x^*\right)^2+n_2\left(y(r)-y^*\right)^2+n_3\left(z(r)-z^*\right)^2\mathrm{d}r\le\rho_2. \tag{5.4}$$

Moreover, if Equation (5.4) hold, we can get that

$$n_1\left(x(r)-x^*\right)^2+n_2\left(y(r)-y^*\right)^2+n_3\left(z(r)-z^*\right)^2\le\rho_2. \tag{5.5}$$

And (5.5) also denotes the ellipsoid domain, which lies entirely in R_+^3. Let U be any open neighborhood of the ellipsoid domain such that its closure $\bar{U}\subset R_+^3$. We can conclude that $\mathcal{L}Q(s)<0$, for any $s\in R_+^3\setminus U$. This implies the second condition in Lemma 5.1. Then we prove that the first condition of Lemma 5.1 is satisfied.

The corresponding diffusion matrix is

$$D=\begin{pmatrix}\frac{\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2} & 0 & 0\\ 0 & \frac{\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2}+\sigma_2^2y^2z^2 & 0\\ 0 & 0 & \sigma_2^2y^2z^2\end{pmatrix}. \tag{5.6}$$

There is a $R=\min\left\{\frac{\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2},\frac{\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2}+\sigma_2^2y^2z^2,\sigma_2^2y^2z^2\right\}$ so that for any $\left(x(0),y(0),z(0)\right)\in\bar{U}$ and $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in R_+^3$, then

$$\begin{aligned}\sum_{i,j=1}^{3}d_{ij}\lambda_i\lambda_j&=\frac{\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2}\lambda_1^2+\left[\frac{\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2}+\sigma_2^2y^2z^2\right]\lambda_2^2+\sigma_2^2y^2z^2\lambda_3^2\\&\geq\min\left\{\frac{\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2},\frac{\sigma_1^2x^2y^2}{\left(1+\alpha y^h\right)^2}+\sigma_2^2y^2z^2,\sigma_2^2y^2z^2\right\}|\lambda|^2.\end{aligned}$$

According to Rayleigh's principle in [22] and Gard' principle in [23], the first condtion in Lemma 5.1 is satisfied. We can draw a conclusion that the model 2.2 has a stationary distribution $\pi(\cdot)$ and the solution is ergodic. The proof is complete.□

6. Example

In this section, we will present some numerical results to validate our theoretical findings.

R_0 is one of the most important parameter expressions, that indicates when a rumor will be completely eliminated or persist for deterministic and random rumor models. In **Figure 1**, we describe the trajectories of stochastic rumor model with different initial values, when $R_0<1$. We can get that when other parameters are the same, only the initial values are different, the rumor-free equilibrium is the same. And the asymptotic behavior around the rumor-free equilibrium. We can achieve the extinction of the rumor.

Picture (a) of **Figure 2** shows that when the conditions in Theorem 5.1 are satisfied and the values of σ_1 and σ_2 are small enough, the asymptotic behavior around E_*. Image (b) indicates that when the values of σ_1 and σ_2 are large, the dynamical behavior around E_*.

From **Figure 1** and **Figure 2**, we can achieve that when $R_0<1$ the rumors eventually died out with the small enough σ_1 and σ_2. As we know the rumors stop at the wise. When $R_0>1$ and the σ_1,σ_2 are small enough, the rumor model has asymptotic behavior around the rumor existing equilibrium.

7. Conclusion

In this paper, we discuss the rumor model with nonlinear propagation rate and

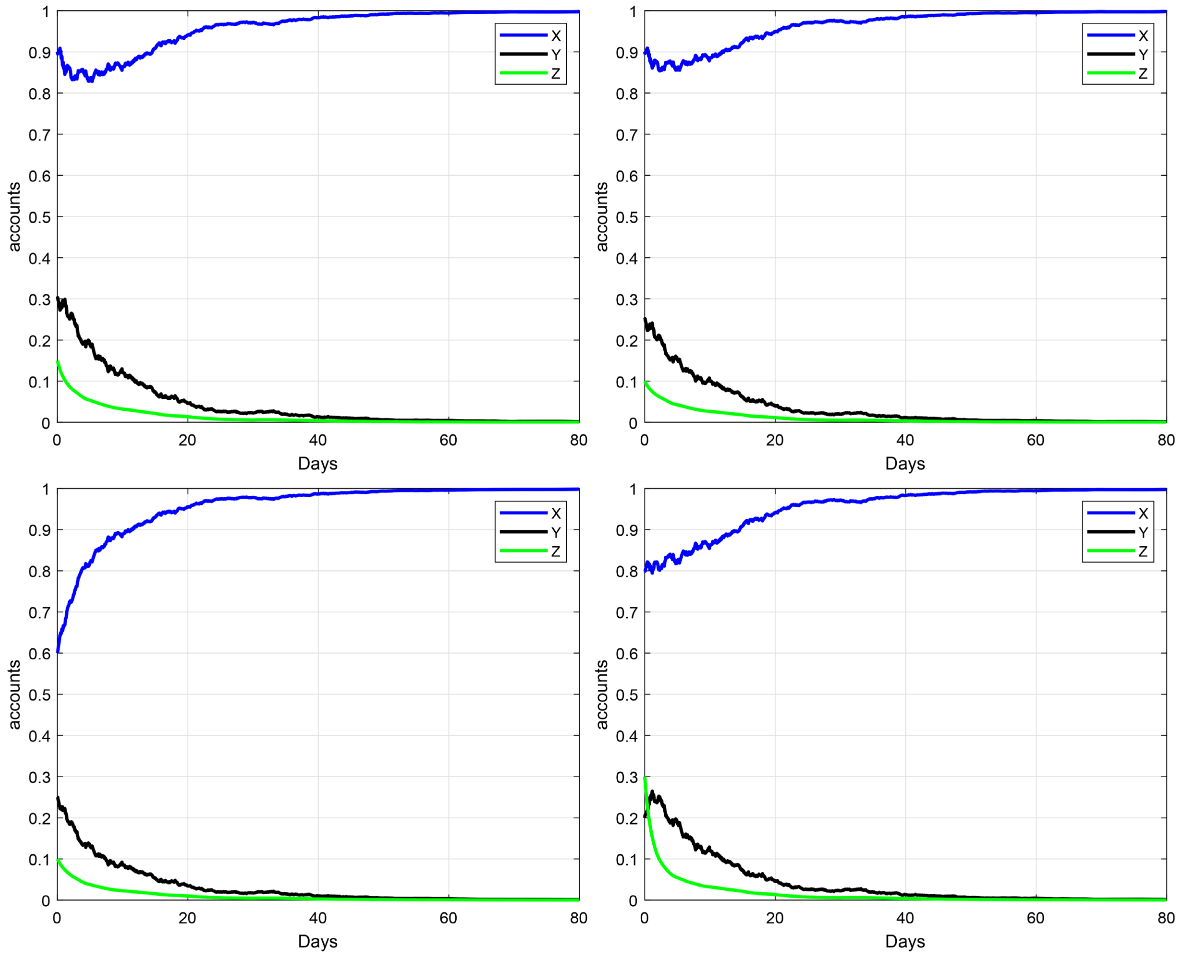
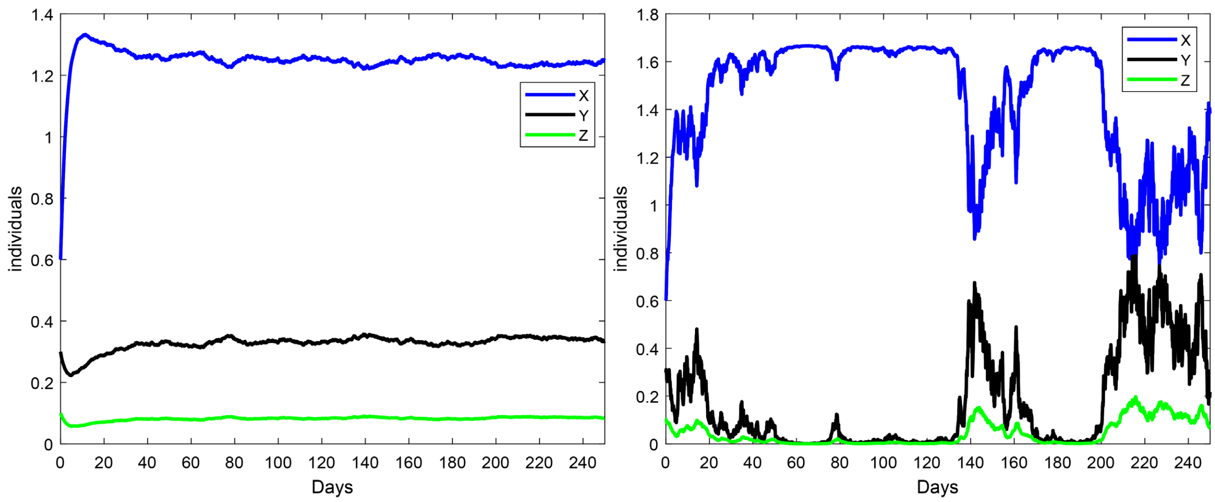


Figure 1. In the above pictures we choose $b = 0.5$, $d = 0.3$, $k = 0.3$, $\mu = 0.2$, $r = 0.5$, $\sigma_1 = \sigma_2 = 0.1$ and $R_0 = 0.8$. Pictures show the trajectories of stochastic rumor model with different initial values.



(a) (b)

Figure 2. In picture (a) and picture (b), we choose $b = 0.5$, $d = 0.3$, $k = 0.3$, $\mu = 0.2$, $r = 0.5$. In picture (a) $\sigma_1 = 0.01$, $\sigma_2 = 0.01$. In picture (b) $\sigma_1 = 0.3$, $\sigma_2 = 0.1$. These pictures show that when $R_0 = 1.33 > 1$, the asymptotic behavior around E_* under different parameter values of σ_1 and σ_2.

secondary propagation rate. The advantage of it is first introducing the nonlinear propagation rate $\frac{ky}{1+\alpha y^h}$ in the rumor propagation model. And the secondary propagation rate indicates that people in a social networks are easily confused by rumors, even though they have a certain understanding of the facts at first. We obtained the existence and boundness of the global solution for model 2.2 and studied the deterministic and stochastic dynamics of the rumor model. The asymptotic behavior around $\tilde{E}_0$ and E_* are derived. Moreover, by constructing Lyapunov functions, we get the existence of an ergodic stationary distribution.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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