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The Extended Non-Elementary Amplitude Functions as Solutions to the Damped Pendulum Equation, the Van der Pol Equation, the Damped Duffing Equation, the Lienard Equation and the Lorenz Equations

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Abstract

In this paper, we define some non-elementary amplitude functions that are giving solutions to some well-known second-order nonlinear ODEs and the Lorenz equations, but not the chaos case. We are giving the solutions a name, a symbol and putting them into a group of functions and into the context of other functions. These solutions are equal to the amplitude, or upper limit of integration in a non-elementary integral that can be arbitrary. In order to define solutions to some short second-order nonlinear ODEs, we will make an extension to the general amplitude function. The only disadvantage is that the first derivative to these solutions contains an integral that disappear at the second derivation. We will also do a second extension: the two-integral amplitude function. With this extension we have the solution to a system of ODEs having a very strange behavior. Using the extended amplitude functions, we can define solutions to many short second-order nonlinear ODEs.

Keywords

Non-Elementary Functions, Second-Order Nonlinear Autonomous ODE, Damped Pendulum Equation, Van der Pol Equation, Damped Duffing Equation, Lienard Equation, Lorenz System

1. Introduction

On page 1 in the book [1], we find the sentences: very few ordinary differential equations have explicit solutions expressible in finite terms. This is not because ingenuity fails, but because the repertory of standard functions (polynomials, exp, sin and so on) in terms of which solutions may be expressed is too limited to accommodate the variety of differential equations encountered in practice.

This is the main reason for this work. It should be possible to do something with this problem. If we don't have enough tools in our mathematical toolbox, we must make the tools first. For this problem, we will attempt to define some new functions.

Wolfram Math World describes three nonlinear second-order ODEs that have the Jacobi elliptic functions *sn*, *cn* and *dn* as solutions. Define a solution $x(t) = cn(t)$ and differentiate twice, and you will obtain the ODE:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \left(2k^2 - 1\right)x - 2k^2 x^3,\ \ 0 \le k < 1 \tag{1}$$

And if we use the Jacobi amplitude function $am(t,k)$ as a solution $x(t)$ and differentiate twice, we will obtain the ODE:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k^2 \sin(x)\cos(x) \tag{2}$$

This causes us to think that other second-order nonlinear ODEs have functions made by the same methods than Jacobi elliptic functions, as their solutions. It should be possible to make more non-elementary functions by changing the non-elementary integral. The solutions defined in this paper are more "simple" than other solutions I have seen to this kind of problems.

We will use the methods described by Armitage and Eberlein [2] in their book Elliptic Functions, especially Section 1.6 and 1.7. They apply what they call the Abel's methods. "Eberlein sought to relate the ideas of Abel to the later work of Jacobi."

The Jacobi amplitude function, $am(u,k)$, is giving solution to the undamped pendulum equation [3]. Salas is writing: an analytical approximated solution to the differential equation describing the oscillations of the damped nonlinear pendulum at large angels is presented. The solution is expressed in terms of the Jacobi elliptic function *sd*. [4]

The Jacobi elliptic function, $cn(u,k)$, is giving solution to the undamped Duffing equation [3]. Equation (1) is a special case of this equation. Theotokoglou is writing: whereas weakly nonlinear oscillators with weak damping can be solved approximately using techniques such as, averaging or multiple-scaling, exact solutions of damped oscillators with strong nonlinearities include. It is proved that there is no exact analytic solution for the damped Duffing oscillator without linear stiffness terms (NLD equation). This unsolvability is due to the fact that the problem leads to an Abel equation of the normal form, which does not admit exact analytic solutions in terms of known functions for *f(x)* arbitrary. Only special cases of equations of this type can be analytically solved [5].

Salas is writing about "Approximate Solutions to a Generalized Van der Pol Equation in Plasma Oscillations" [6].

Kudryashov has published "Analytical Solutions of the Lorenz System". From the Abstract we can read: the main result of the work is the classification of the elliptic solutions expressed via the Wieierstrass function. It is shown that most of the elliptic solutions are degenerated and expressed via trigonometric functions. However, two solutions of the Lorenz system can be expressed via the elliptic functions [7].

Markakis has found exact solutions for certain nonlinear autonomous ODEs of the second order, by reducing them to Abel equations of the first kind [8].

The damped case of the pendulum equation and the damped Duffing equation is a problem, and so also with some other short second-order nonlinear ODEs containing the variable *x* and its first derivative. What we have read above causes us to think that also the damped cases have solutions as elliptic functions, or close related functions.

In [9] we have defined the complex expo-elliptic function, or the complex μ-function:

$$M = M(u) = \mathrm{e}^{\lambda\varphi}, \quad \lambda = a + ib, \quad a, b \in R, \quad i = \sqrt{-1}, \quad \varphi = \varphi(u) \tag{3}$$

$$\mathrm{e}^{\lambda\varphi} = \mathrm{e}^{a\varphi}\left(\cos(b\varphi) + i\sin(b\varphi)\right) \tag{4}$$

In this equation we find the expo-elliptic function $\mathrm{e}^{a\varphi}$, the solutions in *The Jef-Family*: $\cos\varphi$ and $\sin\varphi$, and the amplitude function: $amp(t) = \varphi = \varphi(t)$. Three groups of non-elementary functions that are very useful as solutions to second-order nonlinear ODEs.

These non-elementary functions have elementary functions as special cases: If $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1$, is $\mathrm{e}^{a\varphi} = \mathrm{e}^{at}$, $\cos\varphi = \cos t$, $\sin\varphi = \sin t$, $\varphi = t + C$.

Let us try the third group, the amplitude functions. This is the shortest solution, where $x(t)$ is equal to the amplitude φ.

2. Consideration of the Problem

Consider three well-known second-order nonlinear ODEs:

1) The damped pendulum equation, as we find it in some textbooks:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2\sin x = 0 \tag{5}$$

2) The damped Duffing equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + ax + bx^3 = 0 \tag{6}$$

3) The Van der Pol equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon\frac{\mathrm{d}x}{\mathrm{d}t}\left(1 - x^2\right) + x = 0 \tag{7}$$

What is common for these three second-order ODEs? They are short. They contain a function of the variable *x*, and its first and second derivative in the same equation:

$$f(x),\ \frac{\mathrm{d}x}{\mathrm{d}t},\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

Is it possible to define a solution $x(t)$, a non-elementary function, that meet these conditions? Using our mathematical tools, it should be possible.

The shortest solutions of the three functions under (4), are the amplitude functions, where $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}$

Let us try to define an amplitude function that is giving solution to the damped pendulum equation.

$$x(t)=amp(t)=\varphi=\varphi(t) \tag{8}$$

$$\text{Choose } \frac{\mathrm{d}\varphi}{\mathrm{d}t}=\varphi+\sin\varphi \tag{9}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=x+\sin x \tag{10}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{\mathrm{d}x}{\mathrm{d}t}+\cos x\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}t}+\cos x(x+\sin x) \tag{11}$$

This second-order ODE is not short enough. What are we doing now?

If one of the terms in the second derivative not shall contain $\frac{\mathrm{d}x}{\mathrm{d}t}$, it must be replaced with an expression of the first derivative, as done above. If the second-order ODE shall be a very short equation with $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $f(x)$, must $\frac{\mathrm{d}x}{\mathrm{d}t}$ contain an integral of $f(x)$.

$$\text{F. ex.: } \frac{\mathrm{d}x}{\mathrm{d}t}=x+\int f(x)\mathrm{d}t \tag{12}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{\mathrm{d}x}{\mathrm{d}t}+f(x) \tag{13}$$

And the purpose is achieved.

If it is not possible to find a solution to a distinct second order nonlinear ODE, working in the traditional ways, why not just define a solution? How must this solution look like, and where in the landscape of functions can we place this solution? If we are thinking that every reasonable problem has a solution, let us try to define solutions to some of the well-known ODEs, like the damped pendulum equation. We can choose to define a group of functions expressed in a such way that the first derivative to these functions contains an integral. This is the only disadvantage. If we accept such functions, we can define solutions to many well-known second order ODEs.

3. The Amplitude Functions

This group of functions I have named *amplitude functions*, after the Jacobi amplitude function $am(u,k)$, that is an example of this group of functions. Common for the amplitude functions is the solution $x(t)=\varphi=\varphi(t)$, that is the amplitude or upper limit of integration in a non-elementary integral that can be arbitrary.

The function $am(u,k)$ is equal to the Jacobi amplitude φ, and is defined as an integral:

$$am(u,k) = \int_0^u dn(u',k)\,\mathrm{d}u' \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} am(u,k) = dn(u,k) \tag{15}$$

The second-order nonlinear differential equation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k^2 \sin x \cos x,\ \ 0 \le k < 1 \tag{16}$$

has $am(t,k)$ as solution: $x(t) = am(t,k) = \varphi = \varphi(t) = \int_0^t dn(u,k)\,\mathrm{d}u$ (17)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = dn(t) = \sqrt{1-k^2\sin^2\varphi} \tag{18}$$

One more differentiation gives Equation (16).

3.1. Definition of the General Amplitude Function

Define a function $amp(u) = \varphi = \varphi(u) = \int_0^u adn(u')\,\mathrm{d}u', -\infty < u, \varphi < \infty$ (19)

where φ is the amplitude, or upper limit of integration in a non-elementary integral.

$$\frac{\mathrm{d}}{\mathrm{d}u} amp(u) = \frac{\mathrm{d}\varphi}{\mathrm{d}u} = adn(u) = f(\varphi) \tag{20}$$

The function $adn(u)$ is the same function as in the general *Jef-Family* [9].

The function f can be arbitrary. It may also be a square root or a fraction. We have named the derivative to $amp(u)$ for $adn(u)$ in order to show the relationship to $dn(u,k)$.

When $f(\varphi) = \sqrt{1-k^2\sin^2\varphi}$, is $adn(u) = dn(u,k)$ and $amp(u) = am(u,k)$ (21)

Definition 3.1: The amplitude functions.

If a solution $x(t)$ is equal to the amplitude or upper limit of integration in a non-elementary integral, then this solution is an amplitude function, and denoted $x(t) = amp(t)$.

3.2. The *amp*-Function as Solution to Second-Order Nonlinear ODE

Consider the solution $x(t) = amp(t) = \varphi = \varphi(t) = \int_0^t adn(u)\,\mathrm{d}u$ (22)

Such as the Jacobi amplitude function is defined as an integral, every amplitude function is defined as a non-elementary integral.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = adn(t) \tag{23}$$

$$\text{When } \frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1\text{, is } \varphi = \int 1\mathrm{d}t = t + C \tag{24}$$

We will give an example:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \sin x - x = 0 \tag{25}$$

We take it step by step:

$$\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\mathrm{d}t = \int (x - \sin x)\mathrm{d}t \tag{26}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + c_1 = \sqrt{x^2 + 2\cos x + C} \tag{27}$$

Choose $c_1 = 0$ and define a solution $x(t) = amp(t) = \varphi = \varphi(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{\varphi^2 + 2\cos\varphi + C} \tag{28}$$

$$\text{Inverting: } \frac{\mathrm{d}t}{\mathrm{d}\varphi} = \frac{1}{\sqrt{\varphi^2 + 2\cos\varphi + C}} \tag{29}$$

$$t = t(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{\theta^2 + 2\cos\theta + C}} \tag{30}$$

Here we see that the solution $x(t) = \varphi = \varphi(t)$ is the amplitude, or upper limit of integration in a non-elementary integral. The solution $x(t)$ to Equation (25) is an amplitude function.

We will make an extension to Definition 3.1:

Definition 3.2: The one-integral amplitude function.

If a solution $x(t)$ is equal to the amplitude, or upper limit of integration in a non-elementary integral, and the first derivative of this solution contains an integral, then this solution is an example of the one-integral amplitude functions.

Using Definition 3.2 we will try to find solutions to some well-known second order nonlinear ODEs.

Realizing the quantity of the amplitude functions, I have chosen to denote each amplitude function with a letter and a number, f. ex. amp_{D1}.

3.2.1. The Damped Pendulum Equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \lambda\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 \sin x = 0 \tag{31}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\lambda\frac{\mathrm{d}x}{\mathrm{d}t} - \omega^2 \sin x \tag{32}$$

$$\int \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\mathrm{d}t = -\lambda\int\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t - \omega^2\int \sin x\mathrm{d}t \tag{33}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + c_1 = -\lambda(x + c_2) - \omega^2\int \sin x\mathrm{d}t \tag{34}$$

Put $C = -c_1 - \lambda c_2$

Then we define a solution $x(t) = amp(t) = \varphi = \varphi(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=C-\lambda\varphi-\omega^2\int_0^{\varphi}\sin\theta\mathrm{d}\theta \tag{35}$$

Inverting:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi}=\frac{1}{C-\lambda\varphi-\omega^2\int_0^{\varphi}\sin\theta\mathrm{d}\theta} \tag{36}$$

$$t=t(\varphi)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{C-\lambda\theta-\omega^2\int_0^{\varphi}\sin\theta\mathrm{d}\theta} \tag{37}$$

And here we can see that the solution to the damped pendulum equation is an amplitude function, a member in the group of extended amplitude functions, where the first derivative of the function contains an integral. We can give this function the symbol amp_P, after the pendulum.

$$x(t)=amp_P(t)=\varphi=\varphi(t)=\int_0^{t}adn_P(u)\mathrm{d}u \tag{38}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}amp_P(t)=adn_P(t)=C-\lambda\varphi-\omega^2\int_0^{\varphi}\sin\theta\mathrm{d}\theta \tag{39}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}amp_P(t)=C-\lambda x-\omega^2\int_0^{x}\sin\theta\mathrm{d}\theta \tag{40}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\lambda\frac{\mathrm{d}x}{\mathrm{d}t}-\omega^2\sin x \tag{41}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\lambda x+\omega^2\sin x=0$$

$x(t)=amp_P(t)$ is a solution to the damped pendulum equation.

3.2.2. Van der Pol Equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\varepsilon\left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}t}+x=0 \tag{42}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\varepsilon\left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}t}-x \tag{43}$$

$$\int\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathrm{d}t=-\varepsilon\int\left(x^2-1\right)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t-\int x\mathrm{d}t \tag{44}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}+c_1=-\varepsilon\left(\frac{x^3}{3}-x+c_2\right)-\int x\mathrm{d}t \tag{45}$$

Put $C=-c_1-\varepsilon c_2$

Define a solution $x(t)=amp(t)=\varphi=\varphi(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=C-\varepsilon\left(\frac{\varphi^3}{3}-\varphi\right)-\int_0^{\varphi}\theta\mathrm{d}\theta \tag{46}$$

Inverting:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi}=\frac{1}{C-\varepsilon\left(\frac{\varphi^3}{3}-\varphi\right)-\int_0^{\varphi}\theta\mathrm{d}\theta} \tag{47}$$

$$t = t(\varphi) = \int_0^\varphi \frac{\mathrm{d}\theta}{C - \varepsilon\left(\frac{\theta^3}{3} - \theta\right) - \int_0^\varphi \theta \mathrm{d}\theta} \tag{48}$$

And again, the solution to Van der Pol equation can be placed in the group of the extended amplitude functions, and we can denote it amp_V, after Van der Pol.

$$x(t) = amp_V(t) = \varphi = \varphi(t) = \int_0^t adn_V(u)\mathrm{d}u \tag{49}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} amp_V(t) = adn_V(t) = C - \varepsilon\left(\frac{\varphi^3}{3} - \varphi\right) - \int_0^\varphi \theta \mathrm{d}\theta \tag{50}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} amp_V(t) = C - \varepsilon\left(\frac{x^3}{3} - x\right) - \int_0^x \theta \mathrm{d}\theta \tag{51}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\varepsilon\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} - x \tag{52}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \varepsilon\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

$x(t) = amp_V(t)$ is a solution to Van der Pol equation.

3.2.3. The Damped Duffing Equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + ax + bx^3 = 0 \tag{53}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k\frac{\mathrm{d}x}{\mathrm{d}t} - ax - bx^3 \tag{54}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + c_1 = -kx - kc_2 - \int\left(ax + bx^3\right)\mathrm{d}t \tag{55}$$

$$\text{Put } -c_1 - kc_2 = C$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = C - kx - \int\left(ax + bx^3\right)\mathrm{d}t \tag{56}$$

Define a solution $x(t) = amp(t) = \varphi = \varphi(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = C - k\varphi - \int_0^\varphi\left(a\theta + b\theta^3\right)\mathrm{d}\theta \tag{57}$$

Inverting:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi} = \frac{1}{C - k\varphi - \int_0^\varphi\left(a\theta + b\theta^3\right)\mathrm{d}\theta} \tag{58}$$

$$t = t(\varphi) = \int_0^\varphi \frac{\mathrm{d}\theta}{C - k\theta - \int_0^\varphi\left(a\theta + b\theta^3\right)\mathrm{d}\theta} \tag{59}$$

The solution is an amplitude function, denoted amp_D, after Duffing.

$$x(t) = amp_D(t) = \varphi = \varphi(t) = \int_0^t adn_D(u)\mathrm{d}u \tag{60}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}amp_D\left(t\right)=adn_D\left(t\right)=C-k\varphi-\int_0^{\varphi}\left(a\theta+b\theta^3\right)\mathrm{d}\theta \quad (61)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}amp_D\left(t\right)=C-kx-\int_0^{x}\left(a\theta+b\theta^3\right)\mathrm{d}\theta \quad (62)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-k\frac{\mathrm{d}x}{\mathrm{d}t}-ax-bx^3 \quad (63)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+k\frac{\mathrm{d}x}{\mathrm{d}t}+ax+bx^3=0$$

$x\left(t\right)=amp_D\left(t\right)$ is a solution to the damped Duffing equation.

3.2.4. Lorenz Equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\sigma\left(y-x\right) \quad (64)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\rho x-y-xz \quad (65)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=-\beta z+xy \quad (66)$$

As we find it in some textbooks.

In order to give chaos-behavior, the values of the parameters are:

$\sigma=10,\rho=28,\beta=\frac{8}{3}$.

Let us try to use the solution that we have found to the damped Duffing equation: $x\left(t\right)=amp_D\left(t\right)$.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=D-kx-\int_0^{x}\left(a\theta+b\theta^3\right)\mathrm{d}\theta \quad (67)$$

I have used D for the integration-constant.

Add and subtract cx:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=D+cx-kx-cx-\int_0^{x}\left(a\theta+b\theta^3\right)\mathrm{d}\theta \quad (68)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=cx+hy \quad \text{if:} \quad (69)$$

$$y\left(t\right)=\frac{D}{h}-\frac{\left(k+c\right)x}{h}-\frac{1}{h}\int_0^{x}\left(a\theta+b\theta^3\right)\mathrm{d}\theta \quad (70)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}\left(k+c\right)\left(cx+hy\right)-\frac{1}{h}ax-\frac{b}{h}x^3 \quad (71)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{h}\left(c\left(k+c\right)+a\right)x-\left(k+c\right)y-xz\text{, if: } z\left(t\right)=\frac{b}{h}x^2 \quad (72)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{2b}{h}x\left(cx+hy\right)=2cz+2bxy \quad (73)$$

Then we have the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=cx+hy \quad (74)$$

$$\frac{dy}{dt} = -\frac{1}{h}\left(kc + c^2 + a\right)x - \left(k + c\right)y - xz \tag{75}$$

$$\frac{dz}{dt} = 2cz + 2bxy \tag{76}$$

This system has the solutions:

$$x\left(t\right) = amp_D\left(t\right) = \varphi = \varphi\left(t\right) \tag{77}$$

$$y\left(t\right) = \frac{D}{h} - \frac{\left(k + c\right)\varphi}{h} - \frac{1}{h}\int_0^{\varphi}\left(a\theta + b\theta^3\right)d\theta \tag{78}$$

$$z\left(t\right) = \frac{b}{h}\varphi^2 \tag{79}$$

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}amp_D\left(t\right) = \frac{d\varphi}{dt}\\ &= D - k\varphi - \int_0^{\varphi}\left(a\theta + b\theta^3\right)d\theta + cx\left(t\right) - cx\left(t\right)\\ &= cx\left(t\right) + hy\left(t\right)\end{aligned} \tag{80}$$

$$\begin{aligned}\frac{dy}{dt} &= -\frac{1}{h}\left(k + c\right)\frac{d\varphi}{dt} - \frac{1}{h}\left(a\varphi + b\varphi^3\right)\\ &= -\frac{1}{h}\left(k + c\right)\left(cx\left(t\right) + hy\left(t\right)\right) - \frac{a}{h}x\left(t\right) - \frac{b}{h}x\left(t\right)^3\\ &= -\frac{1}{h}\left(kc + c^2 + a\right)x\left(t\right) - \left(k + c\right)y\left(t\right) - x\left(t\right)z\left(t\right)\end{aligned} \tag{81}$$

$$\frac{dz}{dt} = 2\frac{b}{h}\varphi\frac{d\varphi}{dt} = 2\frac{b}{h}\varphi\left(cx\left(t\right) + hy\left(t\right)\right) = 2cz\left(t\right) + 2bx\left(t\right)y\left(t\right) \tag{82}$$

The solution $x\left(t\right) = amp_D\left(t\right)$ is a solution to the Lorenz equations.

Choose these values for the parameters:

$c = -10, h = 10, k = 11, a = -270, b = 0.5$.

This gives the system:

$$\frac{dx}{dt} = -10x + 10y \tag{83}$$

$$\frac{dy}{dt} = 28x - y - xz \tag{84}$$

$$\frac{dz}{dt} = -20z + xy \tag{85}$$

This system has three equilibrium points, two spiral sink and one saddle-point, no chaos. Only the parameter to the z-term in the third equation don't have the value that is giving chaos behavior. The amplitude function amp_D is giving solution to the Lorenz equations, but not to the chaos cases.

3.2.5. The Lienard Equation

$$\frac{d^2x}{dt^2} + f\left(x\right)\frac{dx}{dt} + g\left(x\right) = 0 \tag{86}$$

$$\int\frac{d^2x}{dt^2}dt = -\int f\left(x\right)\frac{dx}{dt}dt - \int g\left(x\right)dt \tag{87}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}+c_1=-F(x)-c_2-\int g(x)\mathrm{d}t \tag{88}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)+c_2\right)=f(x)$$

Choose $-c_2-c_1=C$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=C-F(x)-\int g(x)\mathrm{d}t \tag{89}$$

Define an amplitude function as solution:

$$x(t)=amp(t)=\varphi=\varphi(t) \tag{90}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=C-F(\varphi)-\int_0^{\varphi}g(\theta)\mathrm{d}\theta \tag{91}$$

Inverting:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi}=\frac{1}{C-F(\varphi)-\int_0^{\varphi}g(\theta)\mathrm{d}\theta} \tag{92}$$

$$t=t(\varphi)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{C-F(\theta)-\int_0^{\varphi}g(\theta)\mathrm{d}\theta} \tag{93}$$

The solution to Lienard equation is an amplitude function, that we may name amp_L after Lienard.

$$x(t)=amp_L(t)=\varphi=\varphi(t)=\int_0^t adn_L(u)\mathrm{d}u \tag{94}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}amp_L(t)=adn_L(t)=C-F(\varphi)-\int_0^{\varphi}g(\theta)\mathrm{d}\theta \tag{95}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}amp_L(t)=C-F(x)-\int_0^{x}g(\theta)\mathrm{d}\theta \tag{96}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-f(x)\frac{\mathrm{d}x}{\mathrm{d}t}-g(x) \tag{97}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+f(x)\frac{\mathrm{d}x}{\mathrm{d}t}+g(x)=0$$

The solution $x(t)=amp_L(t)$ is a solution to the Lienard equation.

3.3. The Two-Integrals Amplitude Function

We will make one more extension to the amplitude functions.

Definition 3.3: The two-integrals amplitude function:

If a solution $x(t)$ is equal to the amplitude, or upper limit of integration in a non-elementary integral, and the first derivative of this solution contain two integrals, where one integral disappear at the second derivation and the second integral disappear at the third derivation, then this solution is an example of the two-integrals amplitude functions.

We will give an example. Consider the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y \tag{98}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y\left(4x^2 - 1\right) - x - xz \tag{99}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -x^2\left(1 - x^2\right) \tag{100}$$

This system has a behavior that reminds of the Lorenz attractor.

Define an integral function u:

$$u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{C + k\left(\frac{a}{3}\theta^3 + \frac{b}{2}\theta^2 + c\theta\right) + \int_0^{\varphi}\left(r\theta + s^2\theta\int_0^{\varphi}\theta^2\left(p^2 - \theta^2\right)\mathrm{d}\theta\right)\mathrm{d}\theta} \tag{101}$$

C is an arbitrary constant. $-\infty < a, b, c, k, p, r, s < \infty$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{1}{C + k\left(\frac{a}{3}\varphi^3 + \frac{b}{2}\varphi^2 + c\varphi\right) + \int_0^{\varphi}\left(r\theta + s^2\theta\int_0^{\varphi}\theta^2\left(p^2 - \theta^2\right)\mathrm{d}\theta\right)\mathrm{d}\theta} \tag{102}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = C + k\left(\frac{a}{3}\varphi^3 + \frac{b}{2}\varphi^2 + c\varphi\right) + \int_0^{\varphi}\left(r\theta + s^2\theta\int_0^{\varphi}\theta^2\left(p^2 - \theta^2\right)\mathrm{d}\theta\right)\mathrm{d}\theta \tag{103}$$

Define an amplitude function as solution:

$$x(t) = amp(t) = \varphi = \varphi(t) \tag{104}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = C + k\left(\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right) + \int_0^{x}\left(r\theta + s^2\theta\int_0^{x}\theta^2\left(p^2 - \theta^2\right)\mathrm{d}\theta\right)\mathrm{d}\theta = y \tag{105}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k\left(ax^2 + bx + c\right)\frac{\mathrm{d}x}{\mathrm{d}t} + rx + s^2x\int_0^{x}\theta^2\left(p^2 - \theta^2\right)\mathrm{d}\theta \tag{106}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(ax^2 + bx + c\right) + rx + sxz \tag{107}$$

$$\text{If} \quad z(t) = s\int_0^{x}\theta^2\left(p^2 - \theta^2\right)\mathrm{d}\theta \tag{108}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = sx^2\left(p^2 - x^2\right) \tag{109}$$

This system has the equilibrium points: $(0,0,0), \left(\pm p, 0, -\frac{r}{s}\right)$

Choose $k = -1, a = 4, b = 0, c = -1, r = -1, s = -1$ and we become a system that I have given the number (3429):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \tag{110}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y\left(4x^2 - 1\right) - x - xz \tag{111}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -x^2\left(p^2 - x^2\right) \tag{112}$$

The equilibrium points to this system are: $(0,0,0)$ and $(\pm p, 0, -1)$.

See **Figure 1**, where $p = 1$, and the initial values are (−0.5, 0, −0.4). This system has a behavior that reminds of the behavior of the Lorenz system.

The trajectories are arriving the left spiral disc from the right at different places on the disc, spiraling away from the equilibrium point, leaving the spiral disc on the left, turning to the spiral disc on the right, making some rotations and returning to the first spiral, every time choosing a new path beside the other paths, and so on. The equilibrium points are unstable.

See **Figure 2** where $p = 2$. Four trajectories are arriving the spiral disc with equilibrium point (2, 0, −1).

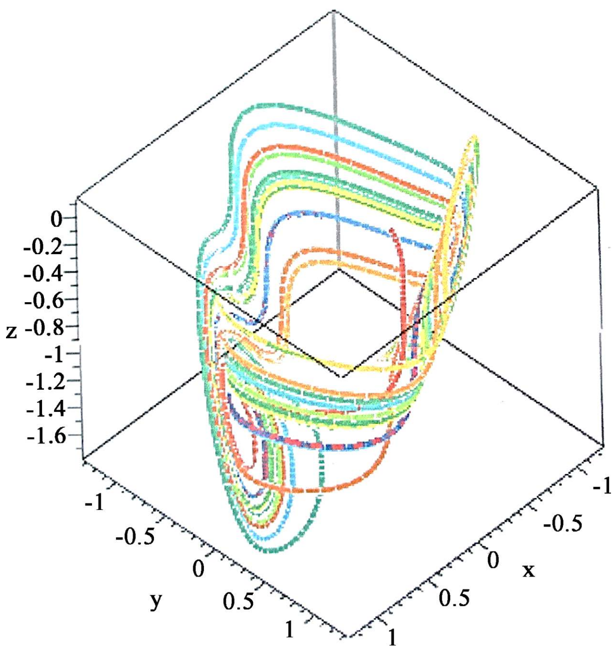

Figure 1. Phase space of a strange system, $p = 1$.

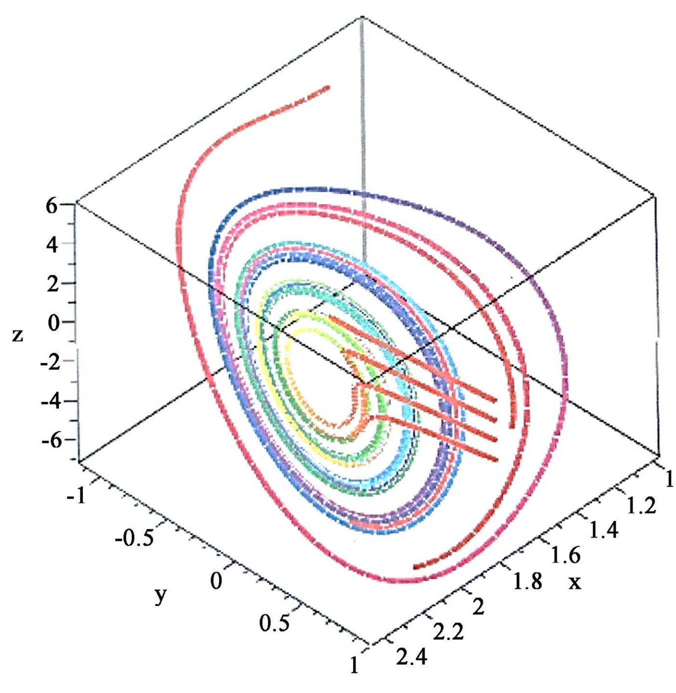

Figure 2. Four trajectories are arriving a spiral disc at different places on the disc.

There should be an attractor somewhere. When $p=1$, it can be difficult to find it. See **Figure 3**, where $p=1$ and $t=0\cdots590$. This attractor is weak.

Choosing greater values of the parameter *p*, it is easier to find the attractor. In **Figure 4** we can see a limit cycle surrounding the two spirals with equilibrium points $\left(\sqrt{5},0,-1\right)$, $\left(-\sqrt{5},0,-1\right)$.

When the parameter $p=3$, we can see two 3D limit cycles intersecting as a link. Each of these attractors exist independent of the other one. Bifurcation with

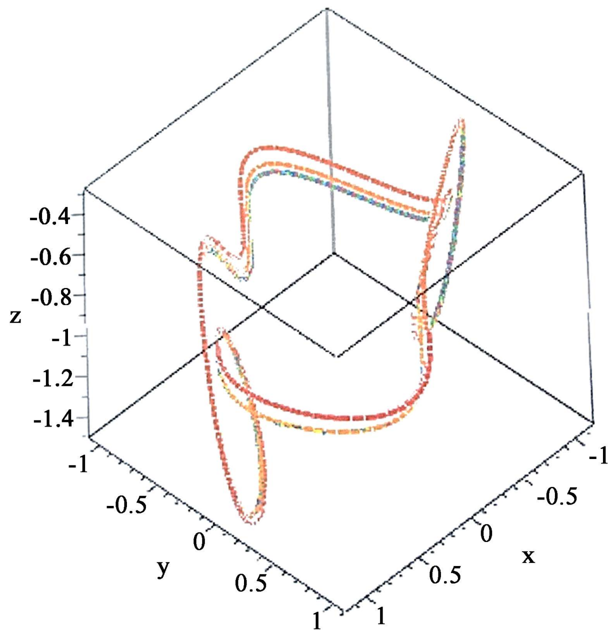

Figure 3. The attractor when $p=1$.

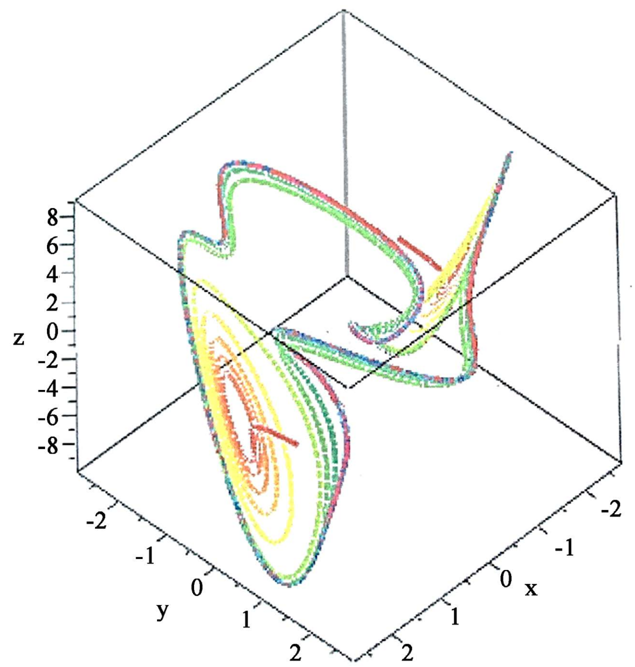

Figure 4. The attractor when $p^2=5$.

the parameter *p*. One limit cycle has been divided into two limit cycles. See **Figure 5** showing solution curves that are spiraling around two equilibrium points in increasing spirals, approaching two limit cycles that are intersecting two times "above" and "under" each other.

In **Figure 6**, the parameter $p = 5$. The two limit cycles are intersecting several times in different heights. **Figure 7** is showing the same attractors and the spiral discs.

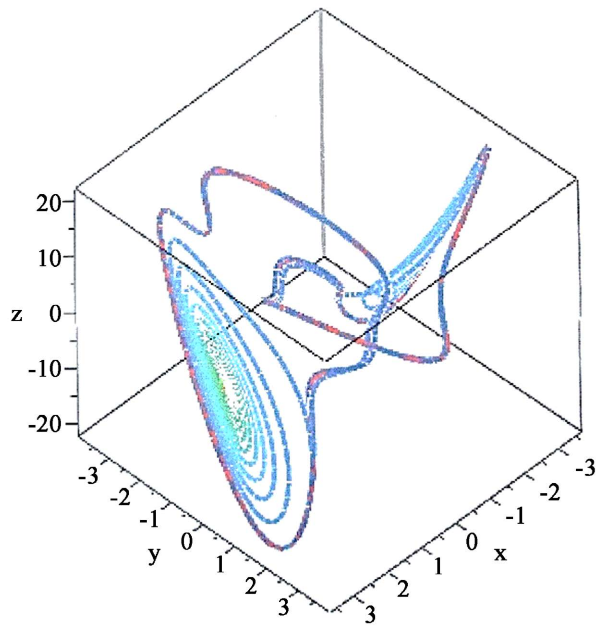

Figure 5. Two limit cycles are intersecting above each other.

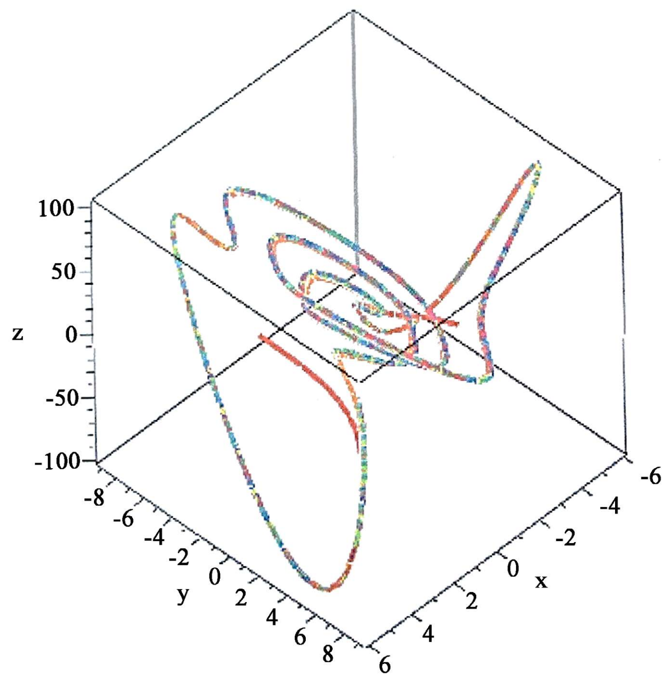

Figure 6. Two really strange and strong attractors, $p = 5$.

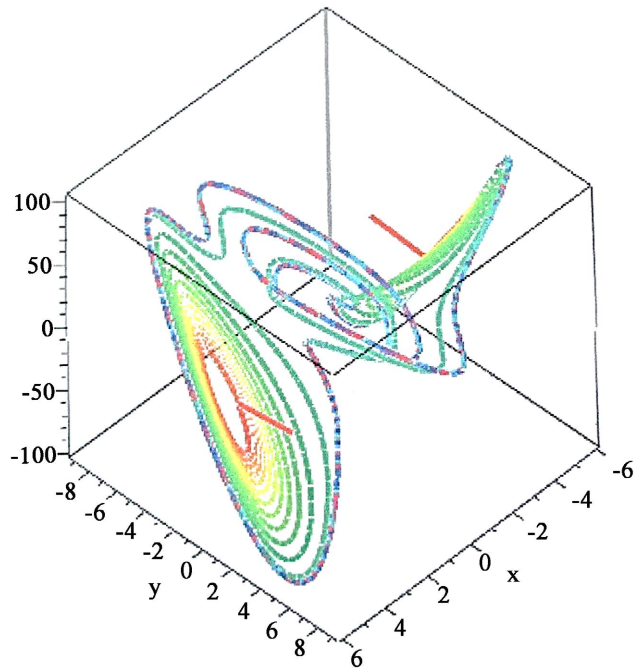

Figure 7. Two spirals and two 3D limit cycles intersecting several times.

Let us now study the eigenvalues to the linearized system (3429) near the equilibrium points. The qualitative behavior is determined by the eigenvalues [10]. Define three functions *f*, *g* and *h*, so that $f(x,y,z)=y$, $g(x,y,z)=-y(4x^2-1)-xz-x$ and $h(x,y,z)=-x^2(p^2-x^2)$.

$$\text{Jacobian matrix: } J=\begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0,z_0) & \frac{\partial f}{\partial y}(x_0,y_0,z_0) & \frac{\partial f}{\partial z}(x_0,y_0,z_0) \\ \frac{\partial g}{\partial x}(x_0,y_0,z_0) & \frac{\partial g}{\partial y}(x_0,y_0,z_0) & \frac{\partial g}{\partial z}(x_0,y_0,z_0) \\ \frac{\partial h}{\partial x}(x_0,y_0,z_0) & \frac{\partial h}{\partial y}(x_0,y_0,z_0) & \frac{\partial h}{\partial z}(x_0,y_0,z_0) \end{bmatrix} \tag{113}$$

$$\text{When } \ p=1, \ \det\left(J(1,0,-1)-\lambda I\right)=\det\begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & -3-\lambda & -1 \\ 2 & 0 & 0-\lambda \end{bmatrix} \tag{114}$$

The roots of the characteristic polynomial $\lambda^3+3\lambda^2+2=0$ are $\lambda_1 \approx -3.2$, $\lambda_{2,3} \approx 0.1 \pm 0.8i$.

According to the first eigenvalue, the equilibrium point $(1,0,-1)$ is a sink. The two other eigenvalues are complex with a small positive real part. This implies spiral source, as the figures are showing. A such equilibrium point is classified as a spiral saddle, that has a line of solutions that tend toward the equilibrium point as *t* increases and a plane of solutions that spiral toward the equilibrium point as *t* decreases, just as in the case with the Lorenz system. Solutions approach the equilibrium point quickly along the direction of the eigenvector of the negative eigenvalue, then spiral slowly away along the plane corresponding to the complex eigenvalues [10].

$$\det\left(J(0,0,0)-\lambda I\right)=\det\begin{bmatrix}0-\lambda & 1 & 0\\ -1 & 1-\lambda & 0\\ 0 & 0 & 0-\lambda\end{bmatrix} \tag{115}$$

The roots of the characteristic polynomial $\lambda\left(\lambda^2-\lambda+1\right)=0$ are $\lambda_1=0$, and $\lambda_{2,3}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$. Along the z-axis all derivatives are zero. The z-axis is an unstable equilibrium line. The solutions spiral away from (0, 0, 0) when *t* increase, on a plane made up of all combinations of the eigenvectors corresponding to the complex eigenvalues [10].

The amplitude function that is giving solution to system (3429), can we denote amp_B.

$$x(t)=amp_B(t)=\varphi=\varphi(t)=\int_0^t adn_B(u)\,\mathrm{d}u \tag{116}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}amp_B(t)=adn_B(t)\\ &=C+k\left(\frac{a}{3}\varphi^3+\frac{b}{2}\varphi^2+c\varphi\right)+\int_0^\varphi\left(r\theta+s^2\theta\int_0^\varphi\theta^2\left(p^2-\theta^2\right)\mathrm{d}\theta\right)\mathrm{d}\theta\end{aligned} \tag{117}$$

$$y(t)=C+k\left(\frac{a}{3}\varphi^3+\frac{b}{2}\varphi^2+c\varphi\right)+\int_0^\varphi\left(r\theta+s^2\theta\int_0^\varphi\theta^2\left(p^2-\theta^2\right)\mathrm{d}\theta\right)\mathrm{d}\theta \tag{118}$$

$$z(t)=s\int_0^\varphi\theta^2\left(p^2-\theta^2\right)\mathrm{d}\theta \tag{119}$$

The solution to system (3429) is a two-integrals amplitude function.

4. Conclusions

In this paper we have defined an extension to the amplitude functions, where the first derivative contains an integral. Using the extended amplitude functions, or the one-integral amplitude functions, we can define solutions to many short second-order nonlinear ODEs containing the first derivative of the variable, like

The damped pendulum equation;

Van der Pol equation;

The damped Duffing equation;

The Lienard equation;

The Lorenz system, unfortunately not the chaos case.

We have placed the solutions in the group of functions named *the extended amplitude functions*, and into a context of the elliptic functions.

The method is to do the integration of the second-order equation and define an amplitude function as solution. If $t=t(\varphi)$ is an integral where φ is the amplitude or upper limit of integration, then the solution is an amplitude function, and $x(t)=amp(t)=\varphi=\varphi(t)$.

There are much more to investigate in system (3429), and I hope somebody will find it interesting. This system has more strange behavior that is not told in this paper, f. ex. when the parameter $p=\sqrt{8}$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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