Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2023, 11, 3366-3376
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Minimum Hosoya Index of a Kind of Tetracyclic Graph

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How to cite this paper: Jiu, X.J. (2023) The Minimum Hosoya Index of a Kind of Tetracyclic Graph. *Journal of Applied Mathematics and Physics*, **11**, 3366-3376.
https://doi.org/10.4236/jamp.2023.1111215

Received: October 14, 2023
Accepted: November 5, 2023
Published: November 8, 2023

Abstract

Let $\mathscr{T}_n^m$ be a graph with n vertices and m edges. The sum of absolute value of all coefficients of matching polynomial is called Hosoya index. In this paper, we determine 2^{nd} to 4^{th} minimum Hosoya index of a kind of tetracyclic graph, with $m = n + 3$.

Keywords

Matching Polynomial, Hosoya Index, Tetracyclic Graph, Extremal Graph

1. Introduction

The total number of matchings of agraph is a graphic invariant which is important in structural chemistry. In the chemistry literature this graphic invariant is called the *Hosoya index* of a molecular graph. It was applied to correlations with boiling points, entropies, calculated bond orders, as well as for coding of chemical structures [1] [2] [3]. Therefore, the ordering of molecular graphs in terms of their Hosoya indices is of interest in chemical thermodynamics. Let $G = (V(G), E(G))$ be a graph with vertex $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \cdots, e_n\}$.

The matching polynomial of G is defined as

$$\mu(G, x) = \sum_{k \ge 0} (-1)^k m(G, k) x^{n-2k},$$

where $m(G, k)$ the number of its k-matchings. It is convenient to denote $m(G, 0) = 1$ and $m(G, k) = 0$ for $k > [n/2]$. Its theory is well elaborated [4] [5]. The Hosoya index of G, denoted by $Z(G)$, is defined as the sum of all the numbers of its matchings, namely

$$Z(G) = \sum_{k \ge 0} m(G, k).$$

Let $\mathscr{G}_{n,m}$ be the collection of connected simple graphs of order *n* and size *m*. Checking the structure of G in $\mathscr{G}_{n,m}$, it is easy to see that if $m = n-1, n, n+1, n+2, n+3, \cdots, n^2 - n$, then G contains at least $m-n+1$ cycles. And these graphs is called *unicyclic graphs*, *bicyclic graphs*, *tricyclic graphs*, *tetracyclic graphs*, $\cdots$, respectively. Liu *et al.* [6] determined the *tetracyclic graphs* has at least 4 cycles andat most 15 cycles but has no 9 cycles.

The first chemical application of $Z(G)$ was proposed in 1971 by a chemist Hosoya, which was used to describe the thermodynamic properties of saturated hydrocarbons. Wanger and Gutman [7] gave a summary of the Hosoya index of graphs. Hosoya index conducted important research on the progress of its research. And Wanger [1] proved among all *n*-vertex, the path P_n has the maximum Hosoya index and the star S_n has the minimum Hosoya index. Ou [8] and [9] studied the unicyclic has the maximum and minimum Hosoya index. Deng [10] and [11] studied the bicyclic has the maximum and minimum Hosoya index. Huang *et al.* [12] give sharp bounds on the Hosoya index for connected graphs of fixed size. Liu *et al.* [13] determined the maximum Hosoya index of unicyclic graphs with *n* vertices and diameter 3 or 4. Their results somewhat answer a question proposed by Wagner and Gutman. In 2010 for unicyclic graphs with small diameter. Liu *et al.* [14] determined the maximum Hosoya index of tricyclic graphs and the correspond-ing extremal graphs. Li *et al.* [15] determined the minimum Hosoya index of tricyclic graphs and the corresponding extremal graphs.

In this paper, we are organized as follows. In Section 1, we present some preliminaries and list of some previously known results about Hosoya indices of graphs. In Section 1, we determine the second fourth Hosoya indices of a kind of tetracyclic graph. In final section, we give a brief summary of this paper.

2. Preliminaries

In this section, we introduced some notations and definitions form traditional graph theory, not described here, we refer to [16]. We present some definitions and lemmas to prove the main results later.

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \cdots, e_n\}$. Let $\mathscr{G}_{n,m}$ be the collection of connected simple graphs of order *n* and size *m*. Checking the structure of G in $\mathscr{G}_{n,m}$, it is easy to see that if $m = n-1, n, n+1, n+2, n+3, \cdots, n^2 - n$, then G contains at least $m-n+1$ cycles. And these graphs is called *unicyclic graphs*, *bicyclic graphs*, *tricyclic graphs*, *tetracyclic graphs*, $\cdots$. If $W \subset V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them. Similarly, if $E \subset E(G)$, we denote by $G - E$ the subgraph of G obtained by deleting the edges of *E*. If $W = \{v\}$ and $E = \{xy\}$, we write $G - v$ and $G - xy$ instead of $G - \{v\}$ and $G - \{xy\}$, respectively. Denote the neighborhood of $v \in V(G)$ by $N(v) = N_G(v)$; and let $N[v] = N(v) \cup \{v\}$. $d_G(v) = |N_G(v)|$ is vertex v of degree in *G*. Through out

the paper we denote by P_n, C_n, F_n, T_n, S_n the n-vertex graph equals to the path, cycle, forest, tree, star, let S_n^+ be the graph obtained by two vertex attaching to two pedant vertex in S_n, respectively. For two connected graphs G_1, G_2 with $V(G_1) \cap V(G_2) = \{v\}$, let $G = G_1 v G_2$ be a graph defined by
$V(G) = V(G_1) \cup V(G_2)$, $V(G_1) \cap V(G_2) = \{v\}$, $E(G) = E(G_1) \cup E(G_2)$.

In the following we introduce some graph transformation which does not increase the Hosoya index of a graph.

Lemma 2.1. ([14]) The Hosoya index of a graph satisfies the following identities:

(i) If $v \in V(G)$. Then

$$Z(G) = Z(G-v) + \sum_{u \in N_G} (v) Z(G-u-v).$$

(ii) If $uv \in E(G)$. Then

$$Z(G) = Z(G-uv) + Z(G-u-v).$$

(iii) If $G_1, G_2, \cdots, G_k$ are the connected components of a graph G. Then

$$Z(G) = \prod_{i=1}^{k} Z(G_i).$$

Definition 2.1. Suppose that $uv \in E(G)$, $N_G(u) = \{v, w_1, w_2, \cdots, w_s\}$, where $d(w_i) = 1 (1 \le i \le s)$. Let $G^* = G - \{uw_1, uw_2, \cdots, uw_s\} + \{vw_1, vw_2, \cdots, vw_s\}$ as shown in **Figure 1**. We designate the transformation from G to $G_i^* (i = 1, 2)$ in **Figure 1** as of type *I*.

Lemma 2.2. [10] Let G and G^* be two graphs with n vertices defined in Definition 2.1. Then $Z(G) > Z(G^*)$.

Definition 2.2. Let H, X and Y be three connected graphs. Suppose that u, v are two vertices of H, v' is a vertex of X, u' is a vertex of Y. Let G be the graph obtained from H, X, Y by identifying v with v' and u with u', respectively. Let G_1^* be the graph obtained from H, X and Y by identifying vertices v, v' and u', and let G_2^* be the graph obtained from H, X and Y by identifying vertices u, v' and u', see **Figure 2**. We designate the transformation from G to $G_i^* (i = 1, 2)$ in **Figure 2** as of type *II*.

Lemma 2.3. [18] Let G_1^* and G_2^* be three graphs with n vertices defined in Definition 2.2. Then $Z(G) > Z(G_1^*)$ or $Z(G) > Z(G_2^*)$.

Definition 2.3. Let G_0 be a non-trivial connected graph and $u_0 \in V(G_0)$. Assume that $H \cong C_3$ and $u, v \in V(H)$. Suppose that $G = (G_0 \triangleright u_0 = u \triangleleft H)$.

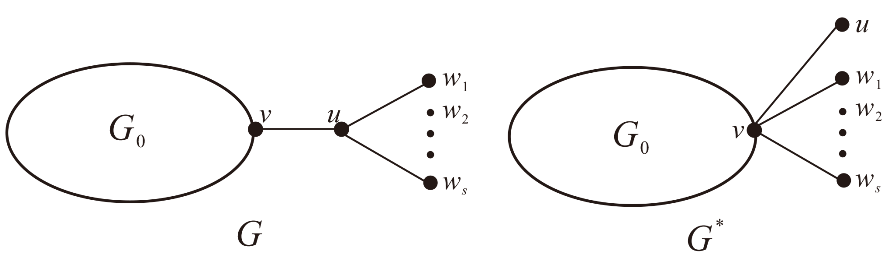

Figure 1. Graphs G, G^*.

Suppose that T is a star tree of order n, whose center vertex is w. If $G_1 = (G \triangleright u = w \triangleleft T)$, $G_2 = G \triangleright v = w \triangleleft T$, we designate the transformation from G_1 to G_2 in **Figure 3** as of type *III*.

Lemma 2.4. [17] Let G_1 and G_2 be three graphs with n vertices defined in Definition 2.3. Then $Z(G_1) < Z(G_2)$.

Definition 2.4. Let G be a graph with k vertices, and let $P_k = x_1, x_2, \cdots, x_k$ $(k \geq 3)$ be a path in $d_{G(x_i)} = 2(i = 1, 2, \cdots, k-1)$. Let G^* be a graph of order n is obtained from G by deleting x_2x_3 and adding x_1x_3, see **Figure 4**. We designate the transformation from G to G^* in **Figure 4** as of type *IV*.

Lemma 2.5. [18] Let G and G^* be two graphs with n vertices defined in Definition 2.4. Then $Z(G) > Z(G^*)$.

Lemma 2.6. [19] Let G be a graph, and let $u, v \in V(G)$. Suppose that $G_{s,t}$ be a graph obtained from G by attaching s, t pendant vertices to v and u, respectively. Then

$$Z(G_{s+i,t-i}) < Z(G_{s,t}), 1 \leq i \leq t; Z(G_{s-i,t+i}) < Z(G_{s,t}), 1 \leq i \leq s.$$

3. The Minimum Hosoya Index of a Kind of Tetracyclic Graph

Pan [20] determined the minimum Hosoya index among all graphs of n vertices

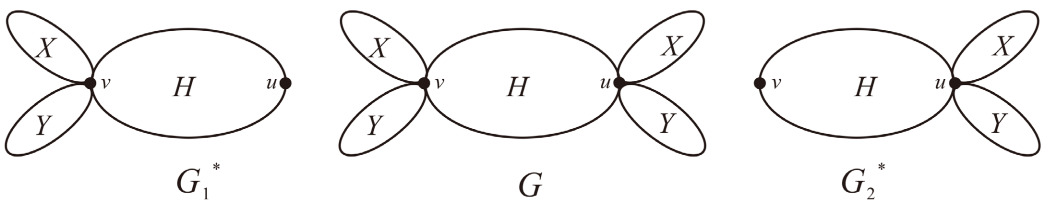

Figure 2. Graphs G_1^*, G, G_2^*.

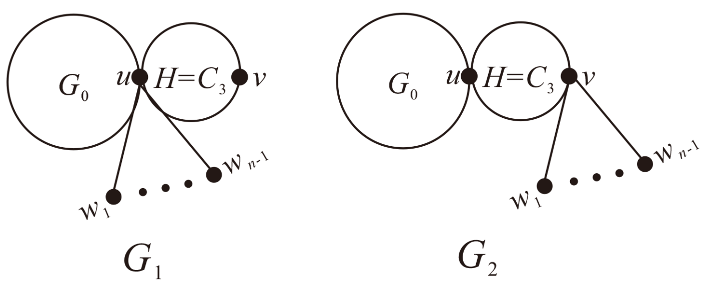

Figure 3. Graphs G_1, G_2.

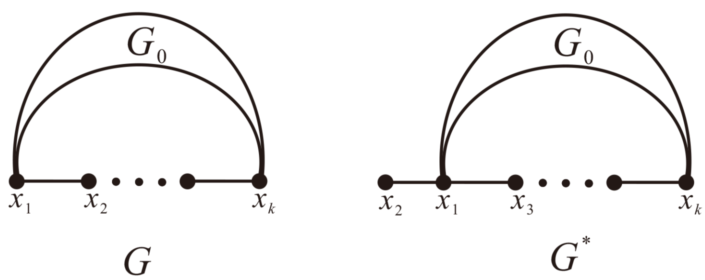

Figure 4. Graphs G, G^*.

and m edges, where $n+3 \le m \le 2n-3$. In the paper, we characterize the 2nd to 4th minimum Hosoya index of a kind of tetracyclic Graph, with $m = n+3$.

The following a kind of tetracyclic graphs and extremal graph defined as follows:

- $p,q,m,l \ge 1, r \ge 0$, let $F_n^{n+3}(p,q,r,m,l)$ be the graph consisting of two given vertices joined by five disjoint paths whose order are p,q,r,m,l, respectively, where $p,q,r,m,l \ge 0$ and most one of them is 0. The resulting graph can be seen in Figure 5.

- $n \ge 7$, Let $F_{n+3}^{*}(1,1,0,1,1,n-6)$ be the graph obtained by adding $n-6$ pendent vertices to one of two vertices of degree 4 in Figure 5.

Theorem 3.1. [20] Let $G \in \mathscr{T}_n^{n+3}$ be a tetracyclic graph with $n(n \ge 7)$ vertices. $Z(G) \ge Z\left(F_{n+3}^{*}(1,1,0,1,1,n-6)\right) = 5n-8$.

We Determine 2nd to 4th Minimum Hosoya Index of a Kind of Tetracyclic Graph

By Theorem 3.1 and Lemmas 2.2, 2.3, 2.4 and 2.5, we can obtain a fact as follows. Let $G \in \left\{\mathscr{T}_n^{n+3} - F_{n+3}^{*}(1,1,0,1,1,n-6)\right\}$ with n vertices. By repeated applications of transformations *I*, *II*, *III* and *IV* presented in Definitions 2.1, 2.2, 2.3 and 2.4, respectively. We can transform G into $F_{n+3}^{*}(1,1,0,1,1,n-6)$. That is, there exist graphs $G^{(i)}$ for $0 \le i \le l$ such that

$$G = G^0 \hookrightarrow G^1 \hookrightarrow G^2 \hookrightarrow \cdots \hookrightarrow G^{l-1} \hookrightarrow G^l = F_{n+3}^{*}(1,1,0,1,1,n-6), \quad (1)$$

where $G^{(l-1)} \ne F_{n+3}^{*}(1,1,0,1,1,n-6)$. This implies that $G^{(l-1)}$ has six possible structures, see Figure 6.

Lemma 3.1. Let $A_1(s,t)$ be a graph with $n = s+t+6$ vertices. If $n \ge 8, s \ge 1, t \ge 1$, in $A_1(s,t)$. Then $Z(A_1(s,t)) \ge 6n-15$, where the equality holds if and only if $A_1(s,t) \cong A_1(1,n-7)$.

Proof. By lemma 2.1 and 2.6, we have,

$Z(A_1(s,t)) \ge Z(A_1(1,n-7)) = Z(A_1(n-7,1)) = 6n-15$,

$Z(A_1(s,t)) \ge Z(A_1(2,n-8)) = Z(A_1(n-8,2)) = 7n-24$. So

$Z(A_1(s,t)) \ge Z(A_1(1,n-7)) = Z(A_1(n-7,1)) = 6n-15$. □

Lemma 3.2. Let $A_2(s,t)$ be a graph with $n = s+t+6$ vertices. If $n \ge 8, s \ge 1, t \ge 1$, in $A_2(s,t)$. Then $Z(A_2(s,t)) \ge 9n-27$, where the equality

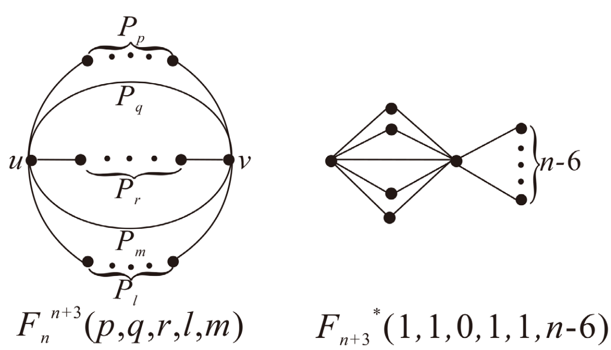

Figure 5. Graphs $F_n^{n+3}(p,q,r,m,l), F_{n+3}^{*}(1,1,0,1,1,n-6)$.

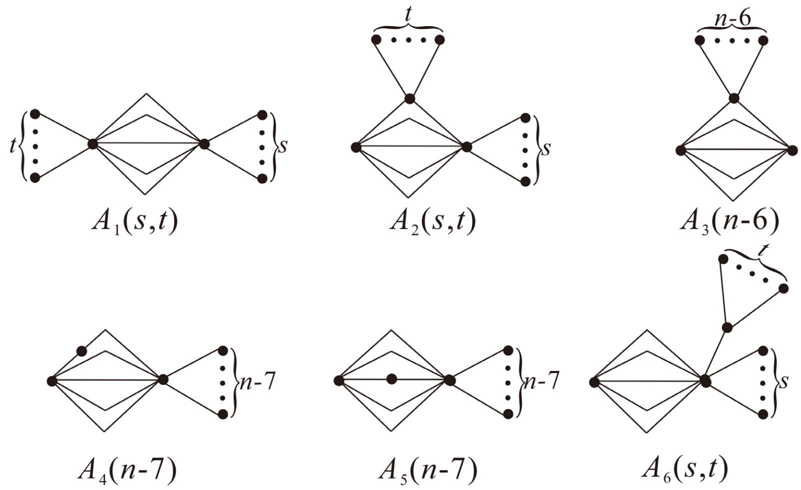

Figure 6. Graphs $A_1(s,t), A_2(s,t), A_3(n-6), A_4(n-7), A_5(n-7), A_6(s,t)$.

holds if and only if $A_2(s,t) \cong A_2(n-7,1)$.

Proof. By lemma 2.1 and 2.6, we have, $Z(A_2(s,t)) \ge Z(A_2(n-7,1)) = 9n-27$ or $Z(A_2(s,t)) \ge Z(A_2(1,n-7)) = 17n-92$. So $Z(A_2(s,t)) \ge Z(A_1(1,n-7)) > Z(A_2(n-7,1))$.□

Lemma 3.3. Let $A_6(s,t)$ be a graph with $n = s+t+7$ vertices. If $n \ge 9, s \ge 0, t \ge 1$, in $A_6(s,t)$. Then $Z(A_6(s,t)) \ge 17n-33$, where the equality holds if and only if $A_6(s,t) \cong A_6(n-8,1)$.

Proof. By lemma 2.1 and 2.6, we have,
$Z(A_6(s,t)) \ge Z(A_6(n-8,1)) = 17n-33$ or
$Z(A_6(s,t)) \ge Z(A_6(0,n-7)) = 18n-99$. So
$Z(A_6(s,t)) \ge Z(A_6(0,n-7)) > Z(A_6(n-8,1))$.□

Theorem 3.2. Let $G \in \mathscr{T}_n^{n+3}$ with $n \ge 7$ vertices. Then $Z(G) \ge 6n-10 > 6n-11 > 6n-15$, where the equality holds if and only if $G \cong A_1(1,n-7)$.

Proof. By lemma 2.1 and 2.6, we have, $Z(A_3(n-6)) = 12n-61$, $Z(A_4(n-7)) = 6n-11$ and $Z(A_5(n-7)) = 6n-10$. Combing Theorem 3.1, Lemmas 3.1, 3.2 and 3.3, (1) and arguments as above, we get that $Z(G) \ge 6n-10 > 6n-11 > 6n-15$.□

Now we characterize the extremal graphs with the third minimal Hosoya index in $\mathscr{T}_n^{n+3}$ By (1), we know that the extremal graphs with the third minimal Hosoya index will be yielded in $G^{(l-1)}$ or $G^{(l-2)}$. By the reverse operations of *I*, *II*, *III* and *IV*, we determine the structures of graphs $A_1(s,t), A_4(n-7)$ and $A_5(n-7)$ in $G^{(l-2)}$. And we also determine the lower bounds of Hosoya indices of these graphs.

By the reverse operations of *I*, *II*, *III* and *IV*, we can obtain that the structures of graphs $A_1(s,t)$ in $G^{(l-2)}$ is isomorphic to one of graphs $A_1^1(s,t,u)$, $A_1^2(s,t)$ $A_1^3(s,t)$, $A_1^4(s,t)$ and $A_1^5(s,t,u)$, see Figure 7.

Lemma 3.4. Let $A_1^1(s,t,u)$ be a graph with $n = s+t+u+6$ vertices. If $n \ge 9, s \ge 1, t \ge 1$ and $u \ge 1$. Then $Z(A_1^1(s,t,u)) \ge 11n-40$, where the equality

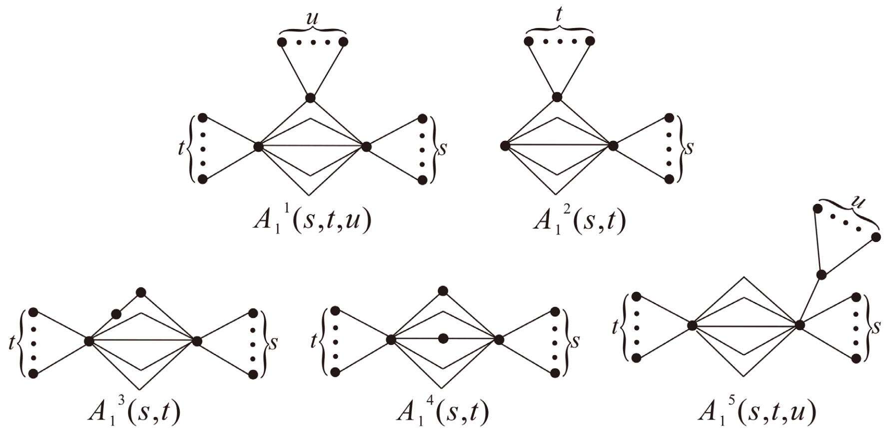

Figure 7. Graphs $A_1^1(s,t,u)$, $A_1^2(s,t)$ $A_1^3(s,t)$, $A_1^4(s,t)$, $A_1^5(s,t,u)$.

holds if and only if $A_1^1(s,t,u) \cong A_1^1(1,n-8,1)$ or $A_1^1(n-8,1,1)$.

Proof. Assume that two of *s,t* and *u* are equal to 1 in $A_1^1(s,t,u)$. By Lemma 2.6, $Z(A_1^1(1,n-8,1)) = 11n-40$, $Z(A_1^1(n-8,1,1)) = 11n-40$, $Z(A_1^1(1,1,n-8)) = 22n-143$.

Assume that at most one of *s,t* and *u* are equal to 1 in $A_1^1(s,t,u)$. By Lemma 2.6, we get that $\begin{aligned} Z(A_1^1(s,t,u)) &= 5n+10u+st+4tu+3su+stu-30 \\ &= s(3u-6)+(4u-6)t+(st+4)u+st-30 \end{aligned}$. Suppose that $\begin{aligned} f(s,t,u) &= 5n+10u+st+4tu+3su+stu-30-11n+40 \\ &= s(3u-6)+(4u-6)t+(st+4)u+st+28 \end{aligned}$. If $s=1, u\ge 2$ and $t\ge 2$, then $f(s,t,u) = t(5u-5)+7u+28>0$. If $u=1, t\ge 2$ and $s\ge 2$, then $f(s,t,u) = s(2t-3)+2t+32>0$. If $t=1, s\ge 2$ and $u\ge 2$, then $f(s,t,u) = s(4u-5)+8u+22>0$. By arguments as above, we have $Z(A_1^1(s,t,u)) \ge 11n-40$. where the equality holds if and only if $A_1^1(s,t,u) \cong A_1^1(1,n-8,1)$ or $A_1^1(n-8,1,1)$.□

Lemma 3.5. Let $A_1^3(s,t)$ be a graph with $n=s+t+7$ vertices. If $n\ge 8, s\ge 1, t\ge 1$, in $A_1^3(s,t)$. Then $Z(A_1^3(s,t)) \ge 10n-40$, where the equality holds if and only if $A_1^3(s,t) \cong A_1^3(1,n-8)$ or $A_1^3(n-8,1)$.

Proof. By lemma 2.1 and 2.6, we have, $Z(A_1^3(s,t)) \ge Z(A_1^3(n-8,1)) = 10n-40$ or $Z(A_1^3(s,t)) \ge Z(A_1^3(1,n-8)) = 10n-40$. So $Z(A_1^3(s,t)) \ge 10n-40$, where the equality holds if and only if $A_1^3(s,t) \cong A_1^3(1,n-8)$ or $A_1^3(n-8,1)$.□

Lemma 3.6. Let $A_1^4(s,t)$ be a graph with $n=s+t+7$ vertices. If $n\ge 8, s\ge 1, t\ge 1$, in $A_1^4(s,t)$. Then $Z(A_1^4(s,t)) \ge 7n-19$, where the equality holds if and only if $A_1^4(s,t) \cong A_1^4(1,n-8)$ or $A_1^4(n-8,1)$.

Proof. By lemma 2.1 and 2.6, we have, $Z(A_1^4(s,t)) \ge Z(A_1^4(n-8,1)) = 7n-19$ or $Z(A_1^4(s,t)) \ge Z(A_1^4(1,n-8)) = 7n-19$. So $Z(A_1^4(s,t)) \ge 7n-19$, where the equality holds if and only if $A_1^4(s,t) \cong A_1^4(1,n-8)$ or $A_1^4(n-8,1)$.□

Lemma 3.7. Let $A_1^5(s,t,u)$ be a graph with $n=s+t+u+7$ vertices. If $n\ge 10, s\ge 0, t\ge 0$ and $u\ge 0$. then $Z(A_1^5(s,t,u)) \ge 9n-27$, where the equality holds if and only if $A_1^5(s,t,u) \cong A_1^5(1,n-7,0)$.

Proof. By lemma 2.1 and 2.6, firstly, $s \ge 0, t \ge 0, u \ge 0$, we need to discuss the following two cases: assume that at most one of *s*,*t* and *u* are equal to 1 in $A_1^5(s,t,u)$, has three subcases:

(*I*) If $s = 0, t = 1$ or $u = 1$,

(*i*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(0,1,n-8)) = 22n-179$,

(*ii*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(0,n-8,1)) = 19n-117$.

(*II*) If $u = 0, t = 1$ or $s = 1$, has two subcases:

(*i*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(0,1,n-7)) = 6n-15$,

(*ii*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(n-8,0,1)) = 6n-15$.

(*III*) If $t = 0, s = 1$ or $u = 1$, has two subcases:

(*i*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(1,0,n-8)) = 18n-56$,

(*ii*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(n-8,0,1)) = 17n-33$.

Assume that two of *s*,*t* and *u* are equal to 1 in $A_1^5(s,t,u)$, has three subcases:

(*i*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(1,1,n-9)) = 26n-186$,

(*ii*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(1,n-9,1)) = 27n-159$.

(*iii*) $Z(A_1^5(s,t,u)) \ge Z(A_1^5(n-9,1,1)) = 21n-156$.□

Theorem 3.3. Let $G \in \mathscr{T}_n^{n+3}$ with $n \ge 7$ vertices. Then $Z(G) \ge 7n-19 > 6n-10 > 6n-11 > 6n-15$.

Proof. By lemma 2.1 and 2.6, we have, $Z(A_1^2(s,t)) = Z(A_2(s,t)) \ge Z(A_2(1,n-7)) = 9n-27$, Combing Theorem 3.1, 3.2, Lemmas 3.4, 3.5, 3.6, 3.7, (1) and arguments as above, we get that $Z(G) \ge 7n-19 > 6n-10 > 6n-11 > 6n-15$.□

Similary, by repeated applications of transformations *I*, *II*, *III* and *IV*, we are considering $A_3(n-7)$. This implies that $G^{(l-2)}$ has six possible structures, see Figure 8.

Lemma 3.8. Let $A_3^2(s,t)$ be a graph with $n = s+t+7$ vertices. If $n \ge 9, s \ge 1, t \ge 1$, in $A_3^2(s,t)$. Then $Z(A_3^2(s,t)) \ge 11n-45$, where the equality

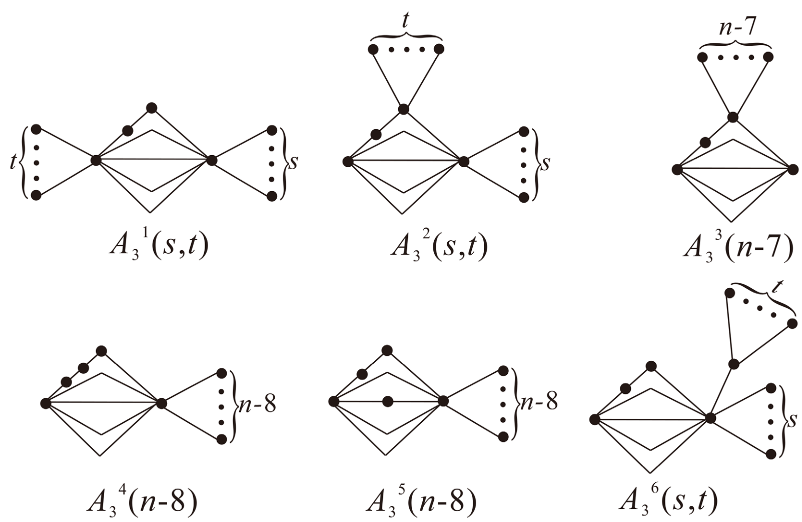

Figure 8. Graphs $A_3^1(s,t)$, $A_3^2(s,t)$ $A_3^3(n-7)$, $A_3^4(n-8)$, $A_3^5(n-8)$, $A_3^6(s,t)$.

holds if and only if $A_3^2(s,t) \cong A_3^2(n-8,1)$.

Proof. By lemma 2.1 and 2.6, we have,

$Z\left(A_3^2(s,t)\right) \ge Z\left(A_3^2(n-8,1)\right) = 11n-45$ or

$Z\left(A_3^2(s,t)\right) \ge Z\left(A_3^2(1,n-8)\right) = 14n-75$. So

$Z\left(A_3^2(s,t)\right) \ge Z\left(A_3^2(1,n-8)\right) > Z\left(A_3^2(n-8,1)\right)$.□

Lemma 3.9. Let $A_3^6(s,t)$ be a graph with $n = s+t+8$ vertices. If $n \ge 10, s \ge 0, t \ge 1$, in $A_3^6(s,t)$. Then $A_3^6(s,t) \ge 9n-30$, where the equality holds if and only if $A_3^6(s,t) \cong A_3^6(n-9,1)$.

Proof. By lemma 2.1 and 2.6, we have, $Z\left(A_3^6(s,t)\right) \ge Z\left(A_3^6(n-9,1)\right) = 9n-30$ or $Z\left(A_3^6(s,t)\right) \ge Z\left(A_3^6(0,n-8)\right) = 31n-236$. So

$Z\left(A_3^6(s,t)\right) \ge Z\left(A_3^6(0,n-8)\right) > Z\left(A_3^6(n-9,1)\right)$.□

Theorem 3.4. Let $G \in \mathscr{T}_n^{n+3}$ with $n \ge 11$ vertices. Then

$Z(G) \ge 9n-30 > 7n-19 > 6n-10 > 6n-11 > 6n-15$.

Proof. By lemma 2.1 and 2.6, we have, $Z\left(A_3^1(s,t)\right) = Z\left(A_1^3(s,t)\right) \ge 10n-40$, $Z\left(A_3^3(n-7)\right) = 12n-45$, $Z\left(A_3^4(n-8)\right) = 14n-53$, $Z\left(A_3^5(n-8)\right) = 11n-35$, Combing Theorem 3.1, 3.2, 3.3, Lemmas 3.8, 3.9, (1) and arguments as above, we get that $Z(G) \ge 9n-30 > 7n-19 > 6n-10 > 6n-11 > 6n-15$.□

Similary, by repeated applications of transformations *I*, *II*, *III* and *IV*, we are considering $A_4(n-7)$. This implies that $G^{(l-2)}$ has six possible structures, see Figure 9.

Lemma 3.10. Let $A_4^2(s,t)$ be a graph with $n = s+t+7$ vertices. If $n \ge 9, s \ge 1, t \ge 1$, in $A_4^2(s,t)$. Then $Z\left(A_4^2(s,t)\right) \ge 8n-14$, where the equality holds if and only if $A_4^2(s,t) \cong A_4^2(n-8,1)$.

Proof. By lemma 2.1 and 2.6, we have, $Z\left(A_4^2(s,t)\right) \ge Z\left(A_4^2(n-8,1)\right) = 8n-14$ or $Z\left(A_4^2(s,t)\right) \ge Z\left(A_4^2(1,n-8)\right) = 11n-52$. So

$Z\left(A_4^2(s,t)\right) \ge Z\left(A_4^2(1,n-8)\right) > Z\left(A_4^2(n-8,1)\right)$.□

Lemma 3.11. Let $A_4^6(s,t)$ be a graph with $n = s+t+8$ vertices. If

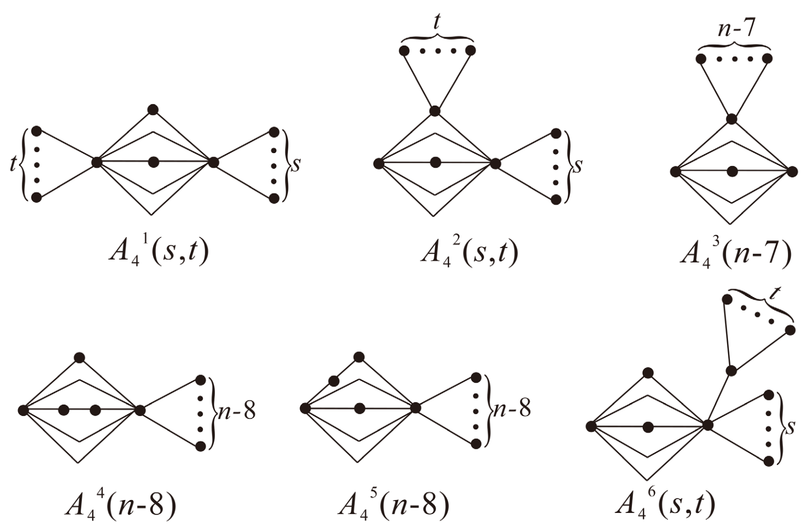

Figure 9. Graphs $A_4^1(s,t)$, $A_4^2(s,t)$ $A_4^3(n-7)$, $A_4^4(n-8)$, $A_4^5(n-8)$, $A_4^6(s,t)$.

$n \ge 10, s \ge 0, t \ge 1$, in $A_4^6(s,t)$. Then $A_4^6(s,t) \ge 11n - 37$, where the equality holds if and only if $A_4^6(s,t) \cong A_4^6(n-9,1)$.

Proof. By lemma 2.1 and 2.6, we have,

$Z(A_4^6(s,t)) \ge Z(A_4^6(n-9,1)) = 11n - 37$ or

$Z(A_4^6(s,t)) \ge Z(A_4^6(0,n-8)) = 24n - 154$. So

$Z(A_4^6(s,t)) \ge Z(A_4^6(0,n-8)) > Z(A_4^6(n-9,1))$.□

Theorem 3.5. Let $G \in \mathscr{T}_n^{n+3}$ with $n \ge 7$ vertices. Then

$Z(G) \ge 9n - 30 > 7n - 19 > 6n - 10 > 6n - 11 > 6n - 15$.

Proof. By lemma 2.1 and 2.6, we have, $Z(A_4^1(s,t)) = Z(A_1^4(s,t)) \ge 7n - 19$, $Z(A_4^3(n-7)) = 16n - 73$, $Z(A_4^4(n-8)) = 10n - 31$, $Z(A_4^6(n-8)) = 11n - 35$, Combing Theorem 3.1, 3.2, 3.3, 3.4, Lemmas 3.10, 3.11, (1) and arguments as above, we get that

$Z(G) \ge 9n - 30 > 8n - 14 > 7n - 19 > 6n - 10 > 6n - 11 > 6n - 15$.□

Theorem 3.6. Let $G \in \mathscr{T}_n^{n+3}$ with $n \ge 7$ vertices. Then

$$Z(G) \ge 6n - 10 = Z(A_4(n-7)) > 6n - 11 = Z(A_5(n-7)) > 6n - 15 = Z(A_1(1,n-7)) > 5n - 8 = Z(F_{n+3}^*(1,1,0,1,1,n-6)).$$

4. Conclusions

Combing Theorem 3.1, 3.2, 3.3, 3.4, 3.5 we obtain.

Let $G \in \mathscr{T}_n^{n+3}$ with $n \ge 7$ vertices. Then

$$Z(G) \ge 6n - 10 = Z(A_4(n-7)) > 6n - 11 = Z(A_5(n-7)) > 6n - 15 = Z(A_1(1,n-7)) > 5n - 8 = Z(F_{n+3}^*(1,1,0,1,1,n-6))$$

. In this paper, we determine the second to fourth minimal Hosoya indices in a kind of tetracyclica graph. This method has not been cited yet, and it is innovative in terms of method. Using this method can solve other graphics and knowledge in the field of graph theory, so promoting the development of graph theory research, respectively. Some new topological indicators in graph theory are closely related to hosoya indicators, laying the foundation for these studies.

Fund Projects

Project No 07M2022003.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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