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Multi-Barycenter Mechanics, *N*-Body Problem and Rotation of Galaxies and Stars

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Abstract

In the present paper, the establishment of a systematic multi-barycenter mechanics is based on the multi-particle mechanics. The new theory perfects the basic theoretical system of classical mechanics, which finds the law of mutual interaction between particle groups, reveals the limitations of Newton's third law, discovers the principle of the intrinsic relationship between gravity and tidal force, reasonably interprets the origin and change laws for the rotation angular momentum of galaxies and stars and so on. By applying new theory, the multi-body problem can be transformed into a special two-body problem and for which an approximate solution method is proposed, the motion law of each particle can be roughly obtained.

Keywords

Multi-Barycenter Mechanics, Translation Principle for a Vector System, Variation Principle for a Vector, Origin and Change Laws for the Rotation Angular Momentum of Galaxies and Stars, Multi-Body Problem

1. Introduction

In the basic theory of mechanics, there are some branches, such as the particle mechanics, the multi-particle mechanics, mechanics of rigid bodies, analytical mechanics and so on. It seems that there are no basic principles and laws that need to be supplemented. But classical mechanics has only studied the interactions between particles, and has made almost no theoretical study of the interactions between groups of particles. As a result, classical mechanics is ambiguous, one-sided and even contradictory when analyzing the phenomenon of interaction between groups of particles, which restricts the development of astrophysics, solar physics, geophysics and other disciplines.

For instance, the origin of the angular momentum of galactic rotation is thought to be caused by the tidal force of the galaxies and surrounding celestial bodies [1] [2], whereas the origin of the angular momentum of the rotation of the stars in the solar system is a sensitive issue that has been evaded by various related professional books and literature [3] [4] [5] [6] [7], since tidal forces are thought to be responsible for slowing down the Earth's rotation [8] [9] [10]. The force of interaction between celestial bodies is gravitational and obeys Newton's third law; gravitation is always considered to pass through the center of mass of an object, so how can a force that passes through the center of mass of an object cause a change in the object's rotation? Is it complete to analyze the interactions between celestial bodies by tidal forces, which are merely a component force of gravitation?

Newtonian mechanics effectively solves the two-body problem, but is helpless in the multi-body problem, can an effective approximate solution method be proposed? Why are galaxies universally disk-shaped? Why are the orbits of a large number of stars within galaxies generally conical? Why do the stars within a galaxy all rotate in the same direction? Why are the directions of rotational angular momentum and angular velocities of the stars within a galaxy so different? These important questions cannot be effectively addressed within the framework of existing classical mechanics.

In order to satisfactorily solve all the issues raised above, this paper will comprehensively and in-depth study the laws of mechanics for n particle groups in a system ($2 \le n < \infty$), that is to establish a new branching system: **multi-barycenter mechanics**.

In this paper we use bolded italicized letters to denote vectors, e.g., $\boldsymbol{r}$ for a position vector, $\boldsymbol{M}$ for a force moment or a force moment of couple, and $\boldsymbol{J}$ for an angular momentum or a momentum moment of couple. If not otherwise stated, all mechanics systems we study are in an inertial frame.

2. Multi-Barycenter Mechanics (1)

Before studying the multi-barycenter mechanics, we need to research the translation principle for a vector system.

If position vectors of vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ about a fixed point O are $\boldsymbol{r}_P$ and $\boldsymbol{r}_Q$ respectively, and

$$\boldsymbol{r}_{PQ} = \overrightarrow{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_P = -\overrightarrow{QP} = -\boldsymbol{r}_{QP},$$

then the vector moments of $\boldsymbol{P}$ and $\boldsymbol{Q}$ about O are $\boldsymbol{r}_P \times \boldsymbol{P}$ and $\boldsymbol{r}_Q \times \boldsymbol{Q}$ severally. If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are equal in magnitude and opposite in direction, then they are a vector moment of couple $\boldsymbol{r}_{QP} \times \boldsymbol{P}$ which has nothing to do with the reference point.

Vector is a quantity which has magnitude and direction. By its definition, a vector does not change its size and direction when it is translated arbitrarily in a coordinate system, that is, the vector does not change. Due to the existence of the vector moment, the translational displacements of the vector will make a change to its vector moment about a reference point. So, in general, a vector can not be arbitrarily translated in a coordinate system. While a vector moment of couple has nothing to do with a reference point, it can be arbitrarily translated in a coordinate system.

In the study of mechanics, sometimes we need to move vectors parallelly. Such as in Figure 1, at a certain moment, we shift $\boldsymbol{P}$ from the point B to the point A parallelly, relative to a fixed point O, $\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{AB}$, so

$$\boldsymbol{r}_B \times \boldsymbol{P} = \boldsymbol{r}_A \times \boldsymbol{P} + \boldsymbol{r}_{AB} \times \boldsymbol{P} \tag{1}$$

The meaning of Equation (1) is that at any moment, the vector $\boldsymbol{P}$ at the point B is translated to the point A parallelly, its vector moment about O is changed from $\boldsymbol{r}_B \times \boldsymbol{P}$ to $\boldsymbol{r}_A \times \boldsymbol{P}$, that is to say, a vector moment of couple $\boldsymbol{r}_{AB} \times \boldsymbol{P}$ no relation with O is reduced. In order not to change the translation effect of $\boldsymbol{P}$, it is needed to add $\boldsymbol{r}_{AB} \times \boldsymbol{P}$.

Some vectors are related to the movement of the reference point, such as the linear momentum; if a vector $\boldsymbol{P}$ has nothing to do with the reference point movement, the point O can be selected arbitrarily.

If at any time vectors $\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots, \boldsymbol{P}_n$ at points $B_1, B_2, \cdots, B_n$ were translated parallelly to points $A_1, A_2, \cdots, A_n$ respectively, in order not to change the translation effect, according to Equation (1), the vector moment of $\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots, \boldsymbol{P}_n$ about any fixed point O needs to be added a vector moment of couple $\sum_{i=1}^{n} \boldsymbol{r}_{AB_i} \times \boldsymbol{P}_i$. Since we often shift a vector system parallelly to a point such as the center of mass, then propose Theorem 1:

Theorem 1. *If vectors $\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots, \boldsymbol{P}_n$ at points $B_1, B_2, \cdots, B_n$ are translated parallelly to a point A at any time, in order not to change the translation effect, the total vector moment of $\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots, \boldsymbol{P}_n$ relative to any fixed point O needs to add a vector moment of couple $\sum_{i=1}^{n} \boldsymbol{r}_{AB_i} \times \boldsymbol{P}_i$.*

Similar to the previous analysis, the proof of Theorem 1 is not difficult, readers could try it on their own.

Theorem 1 is universally applicable to every vector, for example, both a force and a linear momentum are vectors, and their translational principles can be acquired directly.

Translation principle for a force system. *If forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_n$ at points*

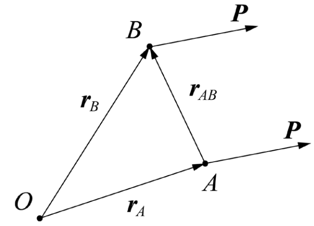

Figure 1. Translate a vector.

$B_1, B_2, \cdots, B_n$ are translated parallelly to a point A at any time, in order not to change the mechanical effect, the total force moment of $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_n$ about any fixed point O needs to add a force moment of couple $\boldsymbol{M}' = \sum_{i=1}^{n} \boldsymbol{r}_{AB_i} \times \boldsymbol{F}_i$.

Translation principle for a linear momentum system. *If linear momentums $\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots, \boldsymbol{P}_n$ at points $B_1, B_2, \cdots, B_n$ are translated parallelly to a point A at any time, in order not to change the mechanical effect, the total angular momentum of $\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots, \boldsymbol{P}_n$ about any fixed point O needs to add a momentum moment of couple $\boldsymbol{J}' = \sum_{i=1}^{n} \boldsymbol{r}_{AB_i} \times \boldsymbol{P}_i$.*

Translation principle for a force system can be implemented to the multi-particle mechanics. If there is a system of particles $P_1, P_2, \cdots, P_n$ with masses $m_1, m_2, \cdots, m_n$, linear momentums are $\boldsymbol{P}_1(t), \boldsymbol{P}_2(t), \cdots, \boldsymbol{P}_n(t)$, position vectors are $\boldsymbol{r}_1(t), \boldsymbol{r}_2(t), \cdots, \boldsymbol{r}_n(t)$ about a fixed point *O* and $\boldsymbol{r}_{C_1}(t), \boldsymbol{r}_{C_2}(t), \cdots, \boldsymbol{r}_{C_n}(t)$ about the center of mass severally. The position vector of the barycenter *C* about *O* is $\boldsymbol{r}_C(t)$, so $\boldsymbol{r}_i = \boldsymbol{r}_C + \boldsymbol{r}_{C_i}$, ($i = 1, 2, \cdots, n$). Set the resultant external force acting on the *i*-th particle is $\boldsymbol{F}_i$, at some moment $\boldsymbol{F}_i$ are translated parallelly to the center of mass *C*, according to the translation principle for a force system the total torque of $\boldsymbol{F}_i$ about *O* need to increase a moment of couple $\boldsymbol{M}' = \sum_{i=1}^{n} \boldsymbol{r}_{C_i} \times \boldsymbol{F}_i$. Let the resultant force of $\boldsymbol{F}_i$ translated to the center of mass is $\boldsymbol{F}_C$, so the motion equation of the barycenter is:

$$m\ddot{\boldsymbol{r}}_C = \sum_{i=1}^{n} \boldsymbol{F}_i = \boldsymbol{F}_C \tag{2}$$

where $m = m_1 + m_2 + \cdots + m_n$.

The reason why we propose the translation principle of a linear momentum system is that the movement law of a barycenter is the variation law of the barycenter's linear momentum, which is each particle linear momentum translated parallelly to the center of mass. The angular momentum of the system in the zero momentum frame is $\boldsymbol{J}' = \sum_{i=1}^{n} \boldsymbol{r}_{C_i} \times \boldsymbol{P}_i$, relative to any fixed point *O* in an inertial frame, it is $\boldsymbol{J}'$ plus the angular momentum of the barycenter about *O*.

If there are n_S interacting particles in a mechanics system, it can be studied as a particle group which has a center of mass or *n* particle groups which have *n* barycenters, so the multi-barycenter mechanics and the multi-particle mechanics are not only independent but internal unity. Set numbers of particles in each particle group are $n_1, n_2, \cdots, n_n$ respectively, namely:

$$n_S = n_1 + n_2 + \cdots + n_n \tag{3}$$

Let the total mass of the *i*-th particle group is m_i, ($i = 1, 2, \cdots, n$), the *j*-th particle in the *i*-th particle group has mass m_{i_j}, its position vector about a fixed point *O* is $\boldsymbol{r}_{i_j}$, then the position vector $\boldsymbol{r}_{C_i}$ of the barycenter C_i of the *i*-th particle group satisfy

$$\boldsymbol{r}_{C_i} = \frac{\sum_{j=1}^{n_i} m_{i_j} \boldsymbol{r}_{i_j}}{\sum_{j=1}^{n_i} m_{i_j}} = \frac{\sum_{j=1}^{n_i} m_{i_j} \boldsymbol{r}_{i_j}}{m_i}$$

Namely

$$m_i \boldsymbol{r}_{C_i} = \sum_{j=1}^{n_i} m_{i_j} \boldsymbol{r}_{i_j} \tag{4}$$

Because there are n barycenters in the system, we call such a mechanics system **a barycenter group**. The total mass of n_S particles in the mechanical system is m, the position vector of the total barycenter C about O is $\boldsymbol{r}_C$, then

$$\boldsymbol{r}_C = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n_i} m_{i_j} \boldsymbol{r}_{i_j}}{\sum_{i=1}^{n} \sum_{j=1}^{n_i} m_{i_j}} = \frac{\sum_{i=1}^{n} m_i \boldsymbol{r}_{C_i}}{m}$$

So

$$m \boldsymbol{r}_C = \sum_{i=1}^{n} m_i \boldsymbol{r}_{C_i} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} m_{i_j} \boldsymbol{r}_{i_j} \tag{5}$$

First we study the simplest kind of barycenter group, in which there are only two particle groups and a particle group consisting of only one particle. The mutual interactions between the particles satisfy Newton's third law and Theorem 2 is proposed as follows.

Theorem 2. *There are n particles in a barycenter group* ($n \geq 2$). *The interaction rule for an arbitrary particle A with the particle group B consisting of the rest particles is: the resultant forces of the mutual interaction between A and the centre of mass of B are equal in magnitude, opposite in direction and effecting a force moment of couple* $\boldsymbol{M}_Z$. $\boldsymbol{M}_Z$ *and* $\boldsymbol{M}_{BA}$, *which A acts upon B about its barycenter* C_B, *are equal in magnitude and opposite in direction.*

Proof. The mutual interactions between particles of B are the internal forces of B, the mutual interactions between A and any particle of B are the external forces of B. Set the force that the i-th particle of B acts upon A is $\boldsymbol{f}_{Ai}$, the force that A acts upon the i-th particle of B is $\boldsymbol{f}_{iA}$. $\boldsymbol{f}_{Ai}$ and $\boldsymbol{f}_{iA}$ are equal in magnitude, opposite in direction and along the straight line joining the two particles, ($i = 1, 2, \cdots, n-1$), namely

$$\boldsymbol{f}_{Ai} = -\boldsymbol{f}_{iA} \tag{6}$$

So the resultant external force which particle group B acts upon A and the resultant external force which A acts upon the particle group B satisfy

$$\sum_{i=1}^{n-1} \boldsymbol{f}_{Ai} = -\sum_{i=1}^{n-1} \boldsymbol{f}_{iA} \tag{7}$$

As A is a mass point, so the resultant external force $\boldsymbol{F}_A$ which B acts upon it is

$$\boldsymbol{F}_A = \sum_{i=1}^{n-1} \boldsymbol{f}_{Ai}$$

According to the translation principle for a force system, translating $\boldsymbol{f}_{iA}$ parallelly to the barycenter C_B of B, the translation effect not only produces a resultant force $\boldsymbol{F}_B$, and produces a moment of couple $\boldsymbol{M}_{BA}$:

$$\boldsymbol{M}_{BA} = \sum_{i=1}^{n} \boldsymbol{r}_{C_B i} \times \boldsymbol{f}_{iA}, \tag{8}$$

where $\boldsymbol{r}_{C_B i}$ is the position vector of the i-th particle about C_B, distinctly $\boldsymbol{M}_{BA}$ is equal to the torque which A acts upon B about C_B, and

$$\boldsymbol{F}_A = \sum_{i=1}^{n-1} \boldsymbol{f}_{Ai} = -\sum_{i=1}^{n-1} \boldsymbol{f}_{iA} = -\boldsymbol{F}_B$$

So

$$\boldsymbol{F}_A = -\boldsymbol{F}_B \tag{9}$$

Namely $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ are equal in magnitude and opposite in direction. Usually $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ are not in the same straight line and produce a moment of couple $\boldsymbol{M}_Z = \boldsymbol{r}_{AC_B} \times \boldsymbol{F}_B$, so the total torque acting upon the system is $\boldsymbol{M}_Z + \boldsymbol{M}_{BA}$. The barycenter group composed of A and B can be regard as a particle group D consisting of n particles, since the mutual interactions between n particles belong to the internal forces of D, so about any fixed point O, the total torque generated by internal forces is zero, then regardless of whether the system is isolated, we have

$$\boldsymbol{M}_Z + \boldsymbol{M}_{BA} \equiv 0 \tag{10}$$

So Theorem 2 is proved. $\boldsymbol{M}_Z$ and $\boldsymbol{M}_{BA}$ are generally changing over time, but Equation (10) hold eternally. □

According to Theorem 2, in general, a mass point has a force and torque effect on a mass point group. To describe the mechanical effect more clearly and succinctly, it can be reduced to a force acting on the center of mass and a force moment of couple acting on the mass point group. The force will change the motion state of the center of mass and the force moment of couple will change the rotation state of the mass point group.

Here is a typical case to illustrate Theorem 2, in **Figure 2** the mass of three particles P, Q and S are all m, the mutual interaction is their gravitation. Set P and Q form a particle group B, G is the gravitational constant, the resultant force $\boldsymbol{F}_S = \frac{2Gm^2}{5\sqrt{5}}\boldsymbol{e}_x + Gm^2\left(1+\frac{1}{5\sqrt{5}}\right)\boldsymbol{e}_y$ acting upon S is not along the straight line joining S and the centre of mass of B obviously, namely $\boldsymbol{M}_Z \neq 0$, we get $\boldsymbol{M}_{BS} \neq 0$ by Equation (10). So the interaction which S exerts on B are force and torque at the same time.

Below we further analyze $\boldsymbol{M}_{BA}$, set

$$\boldsymbol{f}'_{iA} = m_i\left(\boldsymbol{a}_{iA} - \boldsymbol{a}_{C_B}\right) = \boldsymbol{f}_{iA} - m_i\ddot{\boldsymbol{r}}_{C_B}$$

where $\boldsymbol{a}_{C_B}$ is the accelerated speed of C_B, $\boldsymbol{a}_{iA}$ is the accelerated speed of the i-th particle in B which subjected to $\boldsymbol{f}_{iA}$, so

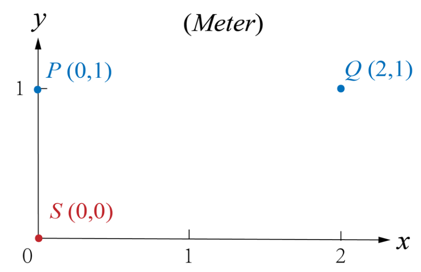

Figure 2. Three particles form a barycenter group.

$$\boldsymbol{f}_{iA} = \boldsymbol{f}'_{iA} + m_i \ddot{\boldsymbol{r}}_{C_B} \tag{11}$$

For $\sum_{i=1}^{n} m_i \boldsymbol{r}_{C_B i} = 0$, according to Equation (8)

$$\boldsymbol{M}_{BA} = \sum_{i=1}^{n} \boldsymbol{r}_{C_B i} \times \boldsymbol{f}_{iA} = \sum_{i=1}^{n} \boldsymbol{r}_{C_B i} \times \boldsymbol{f}'_{iA} - \ddot{\boldsymbol{r}}_{C_B} \times \sum_{i=1}^{n} m_i \boldsymbol{r}_{C_B i}$$

We get

$$\boldsymbol{M}_{BA} = \sum_{i=1}^{n} \boldsymbol{r}_{C_B i} \times \boldsymbol{f}_{iA} = \sum_{i=1}^{n} \boldsymbol{r}_{C_B i} \times \boldsymbol{f}'_{iA} \tag{12}$$

If the mutual interaction between the particles is gravitation, $\boldsymbol{f}'_{iA}$ is the tidal force which A acts upon the i-th particle in the particle group B, Equation (12) explains the force moment of couple which A acts upon B is equal to the total torque which A acts upon each particle in B about the center of mass C_B and equals the total torque of tidal force which A acts upon each particle in B about C_B.

According to Theorem 2, we can propose Theorem 3:

Theorem 3. *There are n particle groups in a barycenter group, the interaction rule between any two particle groups A and B is: the resultant forces* $\boldsymbol{F}_A$ *and* $\boldsymbol{F}_B$ *of the mutual interaction between their barycenters are equal in magnitude, opposite in direction, and resulting in a force moment of couple* $\boldsymbol{M}_Z$*; the force moment of couple* $\boldsymbol{M}_{AB}$ *which B acts upon A about the barycenter* C_A *and the force moment of couple* $\boldsymbol{M}_{BA}$ *which A acts upon B about the barycenter* C_B *satisfy* $\boldsymbol{M}_Z + \boldsymbol{M}_{AB} + \boldsymbol{M}_{BA} \equiv 0$.

Proof. Assuming n particle groups in a barycenter group which has n_S particles, the mutual interactions between any two particle groups A and B belong to each other's external forces, set A has n_A particles and B has n_B particles, the force $\boldsymbol{f}_{ij}^{(A)}$ is the j-th particle in B acting upon the i-th particle in A, the force $\boldsymbol{f}_{ji}^{(B)}$ is the i-th particle in A acting upon the j-th particle in B, $\boldsymbol{f}_{ij}^{(A)}$ and $\boldsymbol{f}_{ji}^{(B)}$ are equal in magnitude, opposite in direction and along the straight line joining the two particles, namely

$$\boldsymbol{f}_{ij}^{(A)} = -\boldsymbol{f}_{ji}^{(B)} \tag{13}$$

According to the translation principle for a force system, the resultant external forces $\boldsymbol{F}_A$ which the particles of B acting upon the barycenter of A and the resultant external forces $\boldsymbol{F}_B$ which the particles of A acting upon the barycenter of B satisfy

$$\boldsymbol{F}_A = \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} \boldsymbol{f}_{ij}^{(A)} = \sum_{i=1}^{n_A} \boldsymbol{F}_{iA} = -\sum_{j=1}^{n_B}\sum_{i=1}^{n_A} \boldsymbol{f}_{ji}^{(B)} = -\sum_{j=1}^{n_B} \boldsymbol{F}_{jB} = -\boldsymbol{F}_B, \tag{14}$$

where the force $\boldsymbol{F}_{iA}$ is all the particles in B acting upon the i-th particle in A and the force $\boldsymbol{F}_{jB}$ is all the particles in A acting upon the j-th particle in B. According to Equation (14), $\boldsymbol{F}_A$ and $\boldsymbol{F}_B$ are equal in magnitude and opposite in direction, set their moment of couple is $\boldsymbol{M}_Z$. By the translation principle for a force system, the external forces $\boldsymbol{F}_{iA}$ which B acts upon A are translated parallelly to the barycenter C_A of A will produce a moment of couple $\boldsymbol{M}_{AB}$

$$\boldsymbol{M}_{AB} = \sum_{i=1}^{n_A} \boldsymbol{r}_{C_A i} \times \boldsymbol{F}_{iA}, \tag{15}$$

where $\boldsymbol{r}_{C_A i}$ is the position vector of the i-th particle about C_A, $\boldsymbol{M}_{AB}$ is equal to the torque which B acts upon A about C_A. The external forces $\boldsymbol{F}_{jB}$ which A acts upon B are translated parallelly to the barycenter C_B of B will produce a moment of couple $\boldsymbol{M}_{BA}$

$$\boldsymbol{M}_{BA} = \sum_{j=1}^{n_B} \boldsymbol{r}_{C_B j} \times \boldsymbol{F}_{jB}, \tag{16}$$

where $\boldsymbol{r}_{C_B j}$ is the position vector of the j-th particle about C_B, $\boldsymbol{M}_{BA}$ is equal to the torque which A acts upon B about C_B. A and B can be as a system D, the total torque acting upon D is $\boldsymbol{M}_Z + \boldsymbol{M}_{AB} + \boldsymbol{M}_{BA}$, D is essentially a particle group too, the mutual interactions between particles within it belong to the internal forces of D, the total torque about any fixed reference point generated by internal forces is zero, so regardless of whether the system is isolated,

$$\boldsymbol{M}_Z + \boldsymbol{M}_{AB} + \boldsymbol{M}_{BA} \equiv 0 \tag{17}$$

Theorem 3 then proved. □

According to Theorem 3, in general, the mutual interaction between any two particle groups A and B can be reduced to the forces between their barycenters and the force moment of couple between them, and the interaction resultant forces are not on the same straight line. Equations (15), (16) show that $\boldsymbol{M}_{AB}$ and $\boldsymbol{M}_{BA}$ are usually not equal in magnitude and opposite in direction. Since any object in reality is a group of particles, therefore, Newton's third law is sometimes established strictly, and sometimes it is approximately established.

Let's take a typical case to explain Theorem 3. In **Figure 3**, the mass of four particles P, Q, R and S are all m, the interaction is their mutual gravitation, set P, Q form a particle group B, and R, S form another particle group A, the resultant force $\boldsymbol{F}_A = \dfrac{2Gm^2}{5\sqrt{5}}\boldsymbol{e}_x + Gm^2\left(1 + \dfrac{1}{5\sqrt{5}} + \dfrac{1}{\sqrt{2}}\right)\boldsymbol{e}_y$ acting upon A is not along the straight line joining barycenters of A and B obviously, namely $\boldsymbol{M}_Z \neq 0$, we can obtain $\boldsymbol{M}_{AB} + \boldsymbol{M}_{BA} \neq 0$ by Equation (17). Then the mutual interactions between A and B are force and moment of couple at the same time.

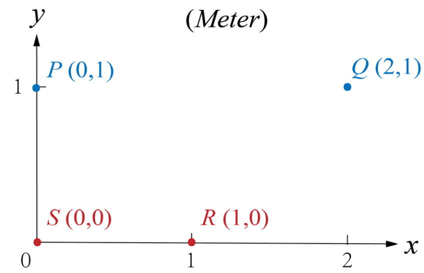

Figure 3. Four particles form a barycenter group.

Now we, set

$$\boldsymbol{F}_{iA}' = m_i\left(\boldsymbol{a}_{iA} - \boldsymbol{a}_{C_A}\right) = \boldsymbol{F}_{iA} - m_i\ddot{\boldsymbol{r}}_{C_A},$$

where $\boldsymbol{a}_{C_A}$ is the accelerated speed of C_A, $\boldsymbol{a}_{iA}$ is the accelerated speed of the i-th particle in A which subjected to $\boldsymbol{F}_{iA}$, so

$$\boldsymbol{F}_{iA} = \boldsymbol{F}_{iA}' + m_i\ddot{\boldsymbol{r}}_{C_A} \tag{18}$$

For $\sum_{i=1}^{n_A} m_i \boldsymbol{r}_{C_A i} = \boldsymbol{0}$, according to Equation (15)

$$\boldsymbol{M}_{AB} = \sum_{i=1}^{n_A} \boldsymbol{r}_{C_A i} \times \boldsymbol{F}_{iA} = \sum_{i=1}^{n_A} \boldsymbol{r}_{C_A i} \times \boldsymbol{F}_{iA}' - \ddot{\boldsymbol{r}}_{C_A} \times \sum_{i=1}^{n_A} m_i \boldsymbol{r}_{C_A i}$$

We get

$$\boldsymbol{M}_{AB} = \sum_{i=1}^{n_A} \boldsymbol{r}_{C_A i} \times \boldsymbol{F}_{iA} = \sum_{i=1}^{n_A} \boldsymbol{r}_{C_A i} \times \boldsymbol{F}_{iA}' \tag{19}$$

Set

$$\boldsymbol{F}_{jB}' = m_j\left(\boldsymbol{a}_{jB} - \boldsymbol{a}_{C_B}\right) = \boldsymbol{F}_{jB} - m_j\ddot{\boldsymbol{r}}_{C_B} \Rightarrow \boldsymbol{F}_{jB} = \boldsymbol{F}_{jB}' + m_j\ddot{\boldsymbol{r}}_{C_B}, \tag{20}$$

where $\boldsymbol{a}_{C_B}$ is the accelerated speed of C_B, $\boldsymbol{a}_{jB}$ is the accelerated speed of the j-th particle in B which subjected to $\boldsymbol{F}_{jB}$, similarly can be obtained

$$\boldsymbol{M}_{BA} = \sum_{j=1}^{n_B} \boldsymbol{r}_{C_B j} \times \boldsymbol{F}_{jB} = \sum_{j=1}^{n_B} \boldsymbol{r}_{C_B j} \times \boldsymbol{F}_{jB}' \tag{21}$$

If the mutual interaction between the particles is gravitation, $\boldsymbol{F}_{iA}'$ is the tidal force which B acts upon the i-th particle in the particle group A. Equation (19) explains the force moment of couple which B acts upon A is equal to the total torque which B acts upon each particle in A about the center of mass C_A and equals the total torque of tidal force which B acts upon each particle in A about C_A.

If A is a particle, B is a spherical symmetry rigid body, it is not difficult to prove $\boldsymbol{M}_Z = \boldsymbol{M}_{BA} \equiv 0$; *if A is a particle group, B is a spherical symmetry rigid body, it is not difficult to prove* $\boldsymbol{M}_{BA} = \boldsymbol{M}_Z + \boldsymbol{M}_{AB} \equiv 0$; *if A and B are both spherical symmetry rigid bodies, it is not difficult to prove* $\boldsymbol{M}_Z = \boldsymbol{M}_{BA} = \boldsymbol{M}_{AB} \equiv 0$. Readers can try to prove the three laws themselves.

If there are n particle groups in a barycenter group, we analyze the law of energy change caused by the interaction forces between any two particle groups

A and B. For the sake of simplicity, we assume that all the work done by the interaction forces between them translates into the kinetic energy of each other. Set $\mathrm{d}T_{AB}$ is the differential of the kinetic energy of the particle group A caused by the particle group B and $\mathrm{d}T_{BA}$ is the differential of the kinetic energy of B caused by A. So

$$\mathrm{d}T_{AB} = \sum_{i=1}^{n_A} \boldsymbol{F}_{iA} \cdot \mathrm{d}\boldsymbol{r}_{iA}, \mathrm{d}T_{BA} = \sum_{j=1}^{n_B} \boldsymbol{F}_{jB} \cdot \mathrm{d}\boldsymbol{r}_{jB} \tag{22}$$

Set $\mathrm{d}T'_{AB}$ be the differential of the kinetic energy of A in the zero momentum frame with the origin C_A, which caused by B, namely

$$\mathrm{d}T'_{AB} = \sum_{i=1}^{n_A} \boldsymbol{F}_{iA} \cdot \mathrm{d}\boldsymbol{r}_{C_A i} \tag{23}$$

According to Equation (18)

$$\mathrm{d}T'_{AB} = \sum_{i=1}^{n_A} \boldsymbol{F}_{iA} \cdot \mathrm{d}\boldsymbol{r}_{C_A i} = \sum_{i=1}^{n_A} \boldsymbol{F}'_{iA} \cdot \mathrm{d}\boldsymbol{r}_{C_A i} + \ddot{\boldsymbol{r}}_{C_A} \cdot \mathrm{d}\left(\sum_{i=1}^{n_A} m_i \boldsymbol{r}_{C_A i}\right)$$

Then

$$\mathrm{d}T'_{AB} = \sum_{i=1}^{n_A} \boldsymbol{F}_{iA} \cdot \mathrm{d}\boldsymbol{r}_{C_A i} = \sum_{i=1}^{n_A} \boldsymbol{F}'_{iA} \cdot \mathrm{d}\boldsymbol{r}_{C_A i} \tag{24}$$

Similarly can be obtained

$$\mathrm{d}T'_{BA} = \sum_{j=1}^{n_B} \boldsymbol{F}_{jB} \cdot \mathrm{d}\boldsymbol{r}_{C_B j} = \sum_{j=1}^{n_B} \boldsymbol{F}'_{jB} \cdot \mathrm{d}\boldsymbol{r}_{C_B j} \tag{25}$$

In general, it is obviously $\mathrm{d}T_{AB} \neq \mathrm{d}T_{BA}$, $\mathrm{d}T'_{AB} \neq \mathrm{d}T'_{BA}$. If the mutual interaction between the particles is gravitation, $\boldsymbol{F}'_{iA}$ is the tidal force which B acts upon the i-th particle in A. Equation (24) explains that the differential of the kinetic energy of A in the zero momentum frame with the origin C_A, which caused by B, is equal to the sum of the elementary work which the total external forces of B acting upon each particle in A about the center-of-mass frame of A do, and equals the sum of the elementary work which the tidal forces of B acting upon each particle in A about the center-of-mass frame of A do.

Because a star is non-rigid body, some of the kinetic energy caused by the work, which the forces of other stars acting upon it about its center-of-mass frame do, is converted into the star's heat energy due to friction and collision. Therefore, Equations (24), (25) reveal an important source of the thermal energy inside a star.

According to Equations (19), (24), we can propose Theorem 4:

Theorem 4. *There are n particle groups in a barycenter group, A and B are arbitrary two particle groups in the system. If the mutual interaction between particles is gravitation, then:*

The force moment of couple which B acts upon A is equal to the total torque which B acts upon each particle in A about the center of mass C_A *and equals the total torque of tidal force which B acts upon each particle in A about* C_A*.*

The differential of the kinetic energy of A in the zero momentum frame with *the origin C_A, which caused by B, is equal to the sum of the elementary work which the total external forces of B acting upon each particle in A about the center-of-mass frame of A do, and equals the sum of the elementary work which the tidal forces of B acting upon each particle in A about the center-of-mass frame of A do.*

3. The Origin and Variation Laws for the Rotation Angular Momentum of Galaxies and Stars

In order to reveal the origin and variation laws for the rotation angular momentum of galaxies and stars, we first propose the variation principle for a vector.

Theorem 5. *If two vectors* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *satisfy*

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}=\boldsymbol{B} \tag{26}$$

Then the direction of $\boldsymbol{A}$ *will change towards* $\boldsymbol{B}$ *with time.*

Proof. According to Equation (26) we can get

$$\mathrm{d}\boldsymbol{A}=\boldsymbol{B}\mathrm{d}t \Rightarrow \boldsymbol{A}(t+\mathrm{d}t)-\boldsymbol{A}(t)=\boldsymbol{B}\mathrm{d}t$$

The directions of $\boldsymbol{B}\mathrm{d}t$ and $\boldsymbol{B}$ are same, if the directions of $\boldsymbol{A}(t)$ and $\boldsymbol{B}$ are same or opposite, according to the superposition principle for vectors we can get $\boldsymbol{A}(t+\mathrm{d}t)$ will both change towards $\boldsymbol{B}$.

Set the angle between $\boldsymbol{A}(t)$ and $\boldsymbol{B}$ is φ, ($0<\varphi<\pi$), the angle between $\boldsymbol{A}(t+\mathrm{d}t)$ and $\boldsymbol{B}$ is ψ, and the angle between $\boldsymbol{A}(t)$ and $\boldsymbol{A}(t+\mathrm{d}t)$ is θ. Obviously, θ is quite small but greater than zero. The relations between vectors $\boldsymbol{A}(t)$, $\boldsymbol{A}(t+\mathrm{d}t)$ and $\boldsymbol{B}\mathrm{d}t$ are shown in **Figure 4**. We translate $\boldsymbol{B}\mathrm{d}t$ parallelly to the intersection point of $\boldsymbol{A}(t)$ and $\boldsymbol{A}(t+\mathrm{d}t)$, could find $\varphi-\psi=\theta$ as shown in **Figure 5**, namely the direction of $\boldsymbol{A}$ constantly approaching $\boldsymbol{B}$ with time. So Theorem 5 proved. □

From the above analysis, it is not difficult to further conclude: *If the angle* φ *between* $\boldsymbol{A}(t)$ *and* $\boldsymbol{B}$ *is less than* $\pi/2$, *then* $|\boldsymbol{A}(t)|$ *will become larger with*

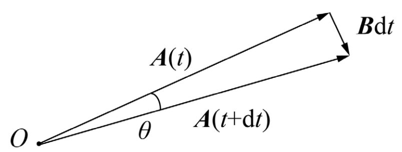

Figure 4. The relationship between $\boldsymbol{A}(t)$, $\boldsymbol{A}(t+\mathrm{d}t)$ and $\boldsymbol{B}\mathrm{d}t$ is shown graphically.

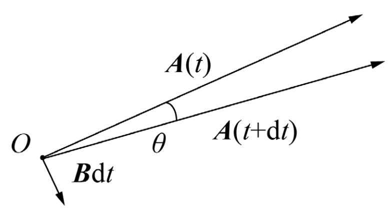

Figure 5. The direction of the vector **A** is shown graphically as it changes with time.

time; if $\varphi = \pi/2$, $|\boldsymbol{A}(t)|$ will not change; if $\varphi > \pi/2$, $|\boldsymbol{A}(t)|$ will be smaller over time.

When we study the motion state of any particle group *A*, we can use all other substances in the entire universe as the second particle group *B*. According to Theorem 3, under normal circumstances, *A* is subject to the force $\boldsymbol{F}_A$ upon its barycenter and the moment of couple $\boldsymbol{M}_{AB}$ from *B*. If *A* is in equilibrium, $\boldsymbol{F}_A = 0$, $\boldsymbol{M}_{AB} = 0$, that is, linear momentum and angular momentum of *A* are conserved. $\boldsymbol{F}_A$ is the only reason for the movement change of the barycenter of *A*, $\boldsymbol{M}_{AB}$ is the only reason for the change of *A*'s angular momentum, combined with Theorem 5, we can analyze the origin and variation of the angular momentum of galaxies and stars.

It exists that various types of galaxies rotate about their center of mass. Galactic rotation curves measured the earliest were normal spiral galaxies [11] [12] [13] [14] [15]. Later, the rotation curves of other types galaxies were also measured and discussed successively, such as SB galaxy [16] [17], E galaxies, S0 galaxy, Irr galaxy, etc [18] [19] [20] [21].

It is generally believed that the angular momentum of galactic rotation is obtained through the mutual interaction of the tides of the surrounding celestial bodies [1] [2]. This is a qualitative analysis with hypothetical components, which does not explain why the tidal force can generate torque, what the relationship between the torque generated by gravity and the torque generated by tidal force is, what the relationship that the torque of the mutual interaction between galaxies and surrounding celestial bodies satisfy is and so on.

According to multi-barycenter mechanics, the entire universe can be considered as a barycenter group and each galaxy a particle group. On the basis of Theorem 3 and 4, the mutual interaction between particle groups generally produces a force moment of couple, which explains why universally galaxies rotate around their barycenter. The force moment of couple is not only equal to the total torque of the external force acting upon each particle in the galaxy about its barycenter, but also equal to the total torque of the tidal force acting upon each particle in the galaxy about its barycenter. The direction of the total moment of couple $\boldsymbol{M}$ acting upon a galaxy is always uniquely determined. According to Theorem 5, the direction of the angular momentum $\boldsymbol{J}$ of the galaxy will constantly change towards the direction of $\boldsymbol{M}$ over time. Although the magnitude and orientation of $\boldsymbol{M}$ vary with time, it always makes star's motion orbit in the galaxy constantly converge to a same plane, and the revolution directions of every star's motion orbit tend to be consistent. So, after a long enough time, under the action of the total moment of couple $\boldsymbol{M}$, galaxies are generally flat and the revolution directions of the internal stars are generally the same.

As for the irregular shape of some galaxies, the reasons may be: 1, for some reasons the total moment of couple acting upon a galaxy is too small, for example, it is very far away from other galaxies. 2, two galaxies are colliding with each other, or the time after the collision between two galaxies is not long enough. 3, galaxies are newly formed.

Until 1939, the nonuniform rotation of the Earth was finally confirmed [8]. In almost all geophysics books and related literatures [3] [4] [22] [23], the origin of the angular momentum of the Earth's rotation is avoided. When analyzing the influence of the tidal force on rotation, it is generally believed that the tidal force will always slow the angular speed of the earth rotation [8] [9] [10]. In the literatures about rotation of the Sun and the solar system's planets and satellites, there is almost no mention about the origin of the angular momentum of their rotation [5] [6] [24] [25]. Let's analyze the problem below.

We can also think of the entire universe as a barycenter group and treat each star as a particle group. According to Theorem 2 and 3, the force moment of couple produced by the mutual interaction between the particle groups reveals the reason for the star rotation around its barycenter. There is no doubt that the collisions with other stars are also the cause. If there is a large amount of liquid water on the surface of the planet, tidal-induced changes in the distribution of matter at the same time have an important negative effect on the angular velocity of rotation. Due to the universal existence of the force moment of couple, a star is not a rigid body and the continuous change of its material distribution will cause the successive variation of moment of inertia, the principal axis of inertia and the interaction with other planets. So, the changes in the rotation of a star are generally more complicated.

The observation of changes in the rotation of the Earth and the Sun is an important content of geophysics and solar physics and has important practical significance. For example, the earthquake is related to changes in the angular velocity of the Earth's rotation [26]. Many observations have proved the complexity of the changes in the state of rotation of the Earth and the Sun [24] [27]. According to the previous analysis, it can be concluded that within classical mechanics, the angular momentum of a star's rotation has two origins: 1, it is affected by the force moment of couple of other stars and matter in the universe. 2, it collides with other stars or objects. There are three reasons for the change of the star's rotation state: 1, the effect of the force moment of couple of other stars and matter in the universe. 2, changes in the material distribution of the star. 3, the collision with other stars or objects.

4. Simulation Solution for *N*-Body Problem

Multi-body problem has always been the focus of mechanical research [28] [29], numerical simulation is a major research method because of the inability to obtain exact solutions in general [30] [31]. Suppose an isolated system with mass m, it has n interacting particles. If a particle A has mass m_A, velocity $\boldsymbol{v}_A$ and position vector $\boldsymbol{r}_A$ about a fixed point O; the rest of the particles $P_1, P_2, \cdots, P_{n-1}$ with masses $m_1, m_2, \cdots, m_{n-1}$, velocities $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{n-1}$ and position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_{n-1}$ about O respectively. Set $P_1, P_2, \cdots, P_{n-1}$ form a particle group B with the center of mass C_B, mass m_B and $m_B = m - m_A$. The speed $\boldsymbol{v}_B$ of C_B

satisfy

$$\boldsymbol{v}_B = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 + \cdots + m_{n-1}\boldsymbol{v}_{n-1}}{m_B}$$

According to Theorem 2, the interaction forces of A and C_B are equal in magnitude , opposite in direction, and generally not in a same straight line, namely

$$\boldsymbol{F}_A = \sum_{i=1}^{n-1}\boldsymbol{F}_{Ai} = \sum_{i=1}^{n-1} G\frac{m_A m_{P_i}}{r_{AP_i}^2}\frac{\boldsymbol{r}_{AP_i}}{r_{AP_i}} = -\boldsymbol{F}_{C_B}, \tag{27}$$

where $\boldsymbol{F}_{Ai}$ is the force which P_i acts upon A, $\boldsymbol{F}_A$ is the resultant force which the particle group B acts upon A, $\boldsymbol{F}_{C_B}$ is the force which A acts upon the barycenter C_B, so the N-body problem is transformed into a special two-body problem of A and C_B. If the laws of motion of A and C_B can be solved, we can use the similar method to solve the movement laws of all other particles in the system.

The approximate simulation is calculated as follows:

On the line of $\boldsymbol{F}_A$, we suppose there is a virtual D with mass of m_D and initial velocity $\boldsymbol{v}_{D0}$, set

$$\boldsymbol{v}_{D0} = \boldsymbol{v}_{B0}, m_D = m_B = \sum_{i=1}^{n-1} m_i \tag{28}$$

Set the interaction force between A and D to meet the inverse-square law, namely

$$\boldsymbol{F}_A = \sum_{i=1}^{n-1} G\frac{m_A m_{P_i}}{r_{AP_i}^2}\frac{\boldsymbol{r}_{AP_i}}{r_{AP_i}} = \frac{k_A}{r_{AD}^2}\frac{\boldsymbol{r}_{AD}}{r_{AD}} = -\boldsymbol{F}_D = -\boldsymbol{F}_{C_B} \tag{29}$$

The numerical value of k_A could be chosen appropriately by experimental datum, according to Equation (29) we have

$$\boldsymbol{r}_{AD} = \sqrt{\frac{k_A}{\left|\sum_{i=1}^{n-1} G\frac{m_A m_{P_i}}{r_{AP_i}^2}\frac{\boldsymbol{r}_{AP_i}}{r_{AP_i}}\right|}}\frac{\boldsymbol{r}_{AD}}{r_{AD}} \tag{30}$$

Using Equation (30) we can solve the initial position vector $\boldsymbol{r}_{D0}$ of D. As $\boldsymbol{F}_A$ and $\boldsymbol{F}_D$ in the same straight line, the centre of mass of A and D is in the active line of $\boldsymbol{F}_A$ distinctly, and its mass is m, its initial position vector can be solved by $\boldsymbol{r}_{A0}$ and $\boldsymbol{r}_{D0}$, we call the barycenter of A and D the **corresponding barycenter** for A, written as C_A, set $\overrightarrow{C_A A} = \boldsymbol{r}_1$, $\overrightarrow{C_A D} = \boldsymbol{r}_2$, so the kinetic equation of A about C_A is:

$$m_A\ddot{\boldsymbol{r}}_1 = \frac{k_A}{(r_1 + r_2)^2}\frac{\boldsymbol{r}_1}{r_1} \tag{31}$$

In the zero momentum frame with the origin C_A

$$m_A\boldsymbol{r}_1 + m_D\boldsymbol{r}_2 = 0 \Rightarrow m_A r_1 = m_D r_2 \Rightarrow r_1 + r_2 = \frac{m_A + m_D}{m_D}r_1 = \frac{m_A + m_D}{m_A}r_2$$

Then

$$m_A\ddot{\boldsymbol{r}}_1 = \frac{k_A m_D^2}{\left(m_A + m_D\right)^2 r_1^2}\frac{\boldsymbol{r}_1}{r_1} \tag{32}$$

Similarly the kinetic equation of D about C_A could be get

$$m_D\ddot{\boldsymbol{r}}_2 = \frac{k_A m_A^2}{\left(m_A + m_D\right)^2 r_2^2}\frac{\boldsymbol{r}_2}{r_2} \tag{33}$$

Equations (32), (33) are two typical central force problems, according to Binet equation, A and D around C_A for conic curve movement, their orbits are determined by k_A and the initial value.

Since the initial states of n particles in the isolated system are known, and A is a randomly selected particle, the law of motion of other particles can be obtained similarly.

Based on the above analysis, we can conclude the following laws:

1. *An isolated system composed of n particles with the gravitational interaction, an arbitrary particle A is approximately around its corresponding barycenter* C_A *for conic curve movement.*

2. *For the system composed of 3 or more particles, the barycenter C of the system and the corresponding barycenter* C_i *of each particle have the same mass, but generally different the position vectors.*

The above rules are obtained under the premise of rough simulation and need to be further amended according to experiments. Since gravitational forces between an object and another object are generally not on a same straight line, the solution for the two-body problem is actually an approximation too.

5. Multi-Barycenter Mechanics (2)

In addition to the above-mentioned unique laws, multi-barycenter mechanics has some similar laws to multi-particle mechanics yet.

If there are n particle groups in a mechanics system, numbers of particles in each particle group are $n_1, n_2, \cdots, n_n$ respectively, ($n_S = n_1 + n_2 + \cdots + n_n$). Set the total mass of the i-th particle group is m_i, ($i = 1, 2, \cdots, n$), the position vector of the barycenter C_i about a fixed point O is $\boldsymbol{r}_{C_i}$. The j-th particle in the i-th particle group has mass m_{i_j}, its position vector is $\boldsymbol{r}_{i_j}$. The total mass of n_S particles is m, the position vector of the total barycenter C about O is $\boldsymbol{r}_C$, and

$$m\boldsymbol{r}_C = \sum_{i=1}^{n} m_i \boldsymbol{r}_{C_i} = \sum_{i=1}^{n}\sum_{j=1}^{n_i} m_{i_j}\boldsymbol{r}_{i_j} \tag{5}$$

Differentiating Equation (5) with respect to t, we get

$$m\dot{\boldsymbol{r}}_C = \boldsymbol{p}_C = \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_{C_i} = \sum_{i=1}^{n}\boldsymbol{p}_{C_i} = \sum_{i=1}^{n}\sum_{j=1}^{n_i} m_{i_j}\dot{\boldsymbol{r}}_{i_j} = \sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{p}_{i_j}$$

Then

$$\boldsymbol{p}_C = \sum_{i=1}^{n} \boldsymbol{p}_{C_i} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \boldsymbol{p}_{i_j} \tag{34}$$

Equation (34) explains that the linear momentum $\boldsymbol{p}_C$ of the total center of mass of the barycenter group is equal to the sum of the barycenters linear momentum of each particle group, and equal to the sum of each particle linear momentum in the system. Differentiating Equation (5) twice with respect to *t*, we have

$$m\ddot{\boldsymbol{r}}_C = \sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_{C_i} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} m_{i_j} \ddot{\boldsymbol{r}}_{i_j} \tag{35}$$

For the barycenter group, the mutual interactions between each particle group are the internal forces of the system, but for every particle group, the interaction forces with other particle groups are the external forces. Set the resultant external force acting upon the total barycenter C of the system is $\boldsymbol{F}_C$, the resultant external force acting upon the barycenter C_i of the i-th particle group is $\boldsymbol{F}_{C_i}$, and the resultant external force acting upon the j-th particle in the i-th particle group is $\boldsymbol{F}_{i_j}^{(\mathrm{e})}$, according to motion equations of a particle and a particle group, we can obtain

$$m\ddot{\boldsymbol{r}}_C = \boldsymbol{F}_C, m_i \ddot{\boldsymbol{r}}_{C_i} = \boldsymbol{F}_{C_i}, m_{i_j} \ddot{\boldsymbol{r}}_{i_j} = \boldsymbol{F}_{i_j}^{(\mathrm{e})} \tag{36}$$

Combining Equation (35) we have

$$\boldsymbol{F}_C = \sum_{i=1}^{n} \boldsymbol{F}_{C_i} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \boldsymbol{F}_{i_j}^{(\mathrm{e})} \tag{37}$$

Namely the resultant external force acting upon C is equal to the sum of external forces acting upon C_i, and is equal to the total external force acting upon each particle.

Using Equations (35), (37), we can get the motion principle for a barycenter group:

Motion principle for a barycenter group: *There are n particle groups in a mechanical system, if the total external force which they are subject to is* $\sum_{i=1}^{n} \sum_{j=1}^{n_i} \boldsymbol{F}_{i_j}^{(\mathrm{e})} = \boldsymbol{F}_C$, *then*

$$\sum_{i=1}^{n} m_i \frac{\mathrm{d}^2 \boldsymbol{r}_{C_i}}{\mathrm{d}t^2} = m \frac{\mathrm{d}^2 \boldsymbol{r}_C}{\mathrm{d}t^2} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} m_{i_j} \frac{\mathrm{d}^2 \boldsymbol{r}_{i_j}}{\mathrm{d}t^2} = \boldsymbol{F}_C \tag{38}$$

Proof. The mechanical system can be regarded as a particle group, according to the translation principle for a force system and Equation (2), we get:

$$m \frac{\mathrm{d}^2 \boldsymbol{r}_C}{\mathrm{d}t^2} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \boldsymbol{F}_{i_j}^{(\mathrm{e})} = \boldsymbol{F}_C \tag{39}$$

According to Equation (5), we have:

$$m\frac{\mathrm{d}^2\boldsymbol{r}_C}{\mathrm{d}t^2}=m\frac{\mathrm{d}^2\left(\frac{\sum_{i=1}^{n}m_i\boldsymbol{r}_{C_i}}{m}\right)}{\mathrm{d}t^2}=\sum_{i=1}^{n}m_i\frac{\mathrm{d}^2\boldsymbol{r}_{C_i}}{\mathrm{d}t^2}$$
$$=m\frac{\mathrm{d}^2\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{n_i}m_{i_j}\boldsymbol{r}_{i_j}}{m}\right)}{\mathrm{d}t^2}=\sum_{i=1}^{n}\sum_{j=1}^{n_i}m_{i_j}\frac{\mathrm{d}^2\boldsymbol{r}_{i_j}}{\mathrm{d}t^2}$$

So the principle proved. □

According to the motion principle for a barycenter group, we can get the linear momentum principle for a barycenter group and the conservation law of the linear momentum for a barycenter group:

Linear momentum principle for a barycenter group: *In any motion of a barycenter group, the rate of increase of the total linear momentum of all barycenters is equal to the total external forces acting upon each particle, namely*

$$\frac{\mathrm{d}\sum_{i=1}^{n}\boldsymbol{p}_{C_i}}{\mathrm{d}t}=\boldsymbol{F}_C \tag{40}$$

Proof. According to the motion principle for a barycenter group:

$$\sum_{i=1}^{n}m_i\frac{\mathrm{d}^2\boldsymbol{r}_{C_i}}{\mathrm{d}t^2}=\boldsymbol{F}_C=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}m_i\frac{\mathrm{d}\boldsymbol{r}_{C_i}}{\mathrm{d}t}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}m_i\boldsymbol{v}_{C_i}\right)=\frac{\mathrm{d}\sum_{i=1}^{n}\boldsymbol{p}_{C_i}}{\mathrm{d}t},$$

where $\boldsymbol{p}_{C_i}$ is the linear momentum of the barycenter of the *i*-th barycenter group, so the principle proved. □

Conservation principle of linear momentum for a barycenter group: *In any motion of an isolated barycenter group, the total linear momentum of all barycenters is conserved, that is*

$$\sum_{i=1}^{n}\boldsymbol{p}_{C_i}=K \tag{41}$$

Proof. According to the linear momentum principle for a barycenter group, when $\boldsymbol{F}_C=0$,

$$\frac{\mathrm{d}\sum_{i=1}^{n}\boldsymbol{p}_{C_i}}{\mathrm{d}t}=0\Rightarrow\sum_{i=1}^{n}\boldsymbol{p}_{C_i}=K$$

where K is a constant quantity. So the principle proved. □

Let us analyze the total kinetic energy formula for a barycenter group. Set the position vector of the barycenter C_i of the *i*-th particle group about a fixed point O is $\boldsymbol{r}_{C_i}$; the *j*-th particle in the *i*-th particle group has mass m_{i_j}, its position vector about O is $\boldsymbol{r}_{i_j}$ and about C_i is $\boldsymbol{r}'_{i_j}$, the kinetic energy of the particle is $\frac{1}{2}m_{i_j}\dot{\boldsymbol{r}}_{i_j}^2=\frac{1}{2}m_{i_j}\left(\dot{\boldsymbol{r}}_{C_i}+\dot{\boldsymbol{r}}'_{i_j}\right)^2$, and $\sum_{j=1}^{n_i}m_{i_j}\boldsymbol{r}'_{i_j}=\sum_{j=1}^{n_i}m_{i_j}\dot{\boldsymbol{r}}'_{i_j}=0$, then

$$
\begin{aligned}
T &= \sum_{i=1}^{n}\sum_{j=1}^{n_i}\frac{1}{2}m_{i_j}\dot{\boldsymbol{r}}_{i_j}^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}m_{i_j}\left(\dot{\boldsymbol{r}}_{C_i} + \dot{\boldsymbol{r}}'_{i_j}\right)^2 \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n_i}m_{i_j}\left(\dot{\boldsymbol{r}}_{C_i}^2 + \left(\dot{\boldsymbol{r}}'_{i_j}\right)^2 + 2\dot{\boldsymbol{r}}_{C_i}\dot{\boldsymbol{r}}'_{i_j}\right) \\
&= \left(\frac{1}{2}m_1\dot{\boldsymbol{r}}_{C_1}^2 + \frac{1}{2}\sum_{j=1}^{n_1}m_{1_j}\left(\dot{\boldsymbol{r}}'_{1_j}\right)^2 + \dot{\boldsymbol{r}}_{C_1}\sum_{j=1}^{n_1}m_{1_j}\dot{\boldsymbol{r}}'_{1_j}\right) \\
&\quad + \left(\frac{1}{2}m_2\dot{\boldsymbol{r}}_{C_2}^2 + \frac{1}{2}\sum_{j=1}^{n_2}m_{2_j}\left(\dot{\boldsymbol{r}}'_{2_j}\right)^2 + \dot{\boldsymbol{r}}_{C_2}\sum_{j=1}^{n_2}m_{2_j}\dot{\boldsymbol{r}}'_{2_j}\right) \\
&\quad + \cdots + \left(\frac{1}{2}m_n\dot{\boldsymbol{r}}_{C_n}^2 + \frac{1}{2}\sum_{j=1}^{n_n}m_{n_j}\left(\dot{\boldsymbol{r}}'_{n_j}\right)^2 + \dot{\boldsymbol{r}}_{C_n}\sum_{j=1}^{n_n}m_{n_j}\dot{\boldsymbol{r}}'_{n_j}\right) \\
&= \left(\frac{1}{2}m_1\dot{\boldsymbol{r}}_{C_1}^2 + \frac{1}{2}\sum_{j=1}^{n_1}m_{1_j}\left(\dot{\boldsymbol{r}}'_{1_j}\right)^2\right) + \left(\frac{1}{2}m_2\dot{\boldsymbol{r}}_{C_2}^2 + \frac{1}{2}\sum_{j=1}^{n_2}m_{2_j}\left(\dot{\boldsymbol{r}}'_{2_j}\right)^2\right) + \cdots \\
&\quad + \left(\frac{1}{2}m_n\dot{\boldsymbol{r}}_{C_n}^2 + \frac{1}{2}\sum_{j=1}^{n_n}m_{n_j}\left(\dot{\boldsymbol{r}}'_{n_j}\right)^2\right)
\end{aligned}
$$

So

$$T = \sum_{i=1}^{n}\frac{1}{2}m_i\dot{\boldsymbol{r}}_{C_i}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\frac{1}{2}m_{i_j}\left(\dot{\boldsymbol{r}}'_{i_j}\right)^2 \tag{42}$$

That is, the total kinetic energy of a barycenter group is equal to the sum of kinetic energy of each barycenter and the kinetic energy of each particle group about its barycenter.

Since a barycenter group can be regarded as a particle group, the forms of their kinetic energy principle are the same. The kinetic energy principle of the i-th particle group relative to its barycenter can also be obtained in a similar way as follows

$$\mathrm{d}\sum_{j=1}^{n_i}\left(\frac{1}{2}m_{i_j}\left(\dot{\boldsymbol{r}}'_{i_j}\right)^2\right) = \sum_{j=1}^{n_i}\boldsymbol{F}_{i_j}^{(\mathrm{e})}\cdot\mathrm{d}\boldsymbol{r}'_{i_j} + \sum_{j=1}^{n_i}\boldsymbol{F}_{i_j}^{(\mathrm{i})}\cdot\mathrm{d}\boldsymbol{r}'_{i_j} \tag{43}$$

where the total external and internal forces acting upon the j-th particle in the i-th particle group are $\boldsymbol{F}_{i_j}^{(\mathrm{e})}$ and $\boldsymbol{F}_{i_j}^{(\mathrm{i})}$ respectively, by Equation (43), we obtain

$$\mathrm{d}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\left(\frac{1}{2}m_{i_j}\left(\dot{\boldsymbol{r}}'_{i_j}\right)^2\right) = \sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{F}_{i_j}^{(\mathrm{e})}\cdot\mathrm{d}\boldsymbol{r}'_{i_j} + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{F}_{i_j}^{(\mathrm{i})}\cdot\mathrm{d}\boldsymbol{r}'_{i_j} \tag{44}$$

For

$$\boldsymbol{r}_{i_j} = \boldsymbol{r}_{C_i} + \boldsymbol{r}'_{i_j},\ \dot{\boldsymbol{r}}_{i_j} = \dot{\boldsymbol{r}}_{C_i} + \dot{\boldsymbol{r}}'_{i_j},\ \ddot{\boldsymbol{r}}_{i_j} = \ddot{\boldsymbol{r}}_{C_i} + \ddot{\boldsymbol{r}}'_{i_j} \tag{45}$$

The angular momentum $\boldsymbol{J}$ of a barycenter group about a fixed point O is

$$
\begin{aligned}
\boldsymbol{J} &= \sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{r}_{i_j}\times m_{i_j}\dot{\boldsymbol{r}}_{i_j} = \sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{r}_{C_i}\times\boldsymbol{P}_{i_j} + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{r}'_{i_j}\times m_{i_j}\left(\dot{\boldsymbol{r}}_{C_i} + \dot{\boldsymbol{r}}'_{i_j}\right) \\
&= \sum_{i=1}^{n}\boldsymbol{r}_{C_i}\times\boldsymbol{P}_{C_i} - \sum_{i=1}^{n}\dot{\boldsymbol{r}}_{C_i}\times\sum_{j=1}^{n_i}m_{i_j}\boldsymbol{r}'_{i_j} + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{r}'_{i_j}\times\boldsymbol{P}'_{i_j} \\
&= \sum_{i=1}^{n}\boldsymbol{r}_{C_i}\times\boldsymbol{P}_{C_i} + \sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{r}'_{i_j}\times\boldsymbol{P}'_{i_j} = \sum_{i=1}^{n}\left(\boldsymbol{J}_{C_i} + \boldsymbol{J}'_i\right)
\end{aligned}
$$

So

$$\boldsymbol{J}=\sum_{i=1}^{n}\left(\boldsymbol{J}_{C_i}+\boldsymbol{J}'_i\right),\boldsymbol{J}_{C_i}=\boldsymbol{r}_{C_i}\times\boldsymbol{P}_{C_i},\boldsymbol{J}'_i=\sum_{j=1}^{n_i}\boldsymbol{r}'_{i_j}\times\boldsymbol{P}'_{i_j} \tag{46}$$

where $\boldsymbol{J}_{C_i}$ is the angular momentum of C_i about O, $\boldsymbol{P}'_{i_j}$ is the linear momentum of the j-th particle about C_i in the i-th particle group, $\boldsymbol{J}'_i$ is the total angular momentum of the i-th particle group about C_i. Equation (46) reveals the total angular momentum of a barycenter group about any fixed point O is equal to that the total angular momentum of each barycenter C_i about O plus the total angular momentum of each particle group about its barycenter.

The total torque $\boldsymbol{M}$ acting upon the barycenter group is

$$\begin{aligned}\boldsymbol{M}&=\sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{r}_{i_j}\times\boldsymbol{F}_{i_j}\\&=\sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{r}_{C_i}\times\boldsymbol{F}_{i_j}+\sum_{i=1}^{n}\sum_{j=1}^{n_i}\boldsymbol{r}'_{i_j}\times\boldsymbol{F}_{i_j}\\&=\sum_{i=1}^{n}\boldsymbol{r}_{C_i}\times\boldsymbol{F}_{C_i}+\sum_{i=1}^{n}\boldsymbol{M}'_i\\&=\sum_{i=1}^{n}\left(\boldsymbol{M}_{C_i}+\boldsymbol{M}'_i\right)\end{aligned}$$

So

$$\boldsymbol{M}=\sum_{i=1}^{n}\left(\boldsymbol{M}_{C_i}+\boldsymbol{M}'_i\right),\boldsymbol{M}_{C_i}=\boldsymbol{r}_{C_i}\times\boldsymbol{F}_{C_i},\boldsymbol{M}'_i=\sum_{j=1}^{n_i}\boldsymbol{r}'_{i_j}\times\boldsymbol{F}_{i_j} \tag{47}$$

The physical meaning of Equation (47) is a barycenter group is subjected to external forces $\boldsymbol{F}_{i_j}$, the total torque of $\boldsymbol{F}_{i_j}$ about any fixed point O is equal to that the total torque of external forces $\boldsymbol{F}_{C_i}$ acting upon n barycenters about O plus the total torque of $\boldsymbol{F}_{i_j}$ about the barycenter C_i.

A barycenter group can be regarded as a particle group, so $\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}=\boldsymbol{M}$ established, combining with Equation (47), we have

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}=\boldsymbol{M}=\sum_{i=1}^{n}\boldsymbol{M}_{C_i}+\sum_{i=1}^{n}\boldsymbol{M}'_i \tag{48}$$

Since

$$\frac{\mathrm{d}\boldsymbol{J}_{C_i}}{\mathrm{d}t}=\frac{\mathrm{d}\left(\boldsymbol{r}_{C_i}\times m_i\frac{\mathrm{d}\boldsymbol{r}_{C_i}}{\mathrm{d}t}\right)}{\mathrm{d}t}=\frac{\mathrm{d}\boldsymbol{r}_{C_i}}{\mathrm{d}t}\times m_i\frac{\mathrm{d}\boldsymbol{r}_{C_i}}{\mathrm{d}t}+\boldsymbol{r}_{C_i}\times m\frac{\mathrm{d}^2\boldsymbol{r}_{C_i}}{\mathrm{d}t^2}=\boldsymbol{r}_{C_i}\times\boldsymbol{F}_{C_i}=\boldsymbol{M}_{C_i}$$

Then

$$\frac{\mathrm{d}\boldsymbol{J}_{C_i}}{\mathrm{d}t}=\boldsymbol{M}_{C_i} \tag{49}$$

That is, about O the rate of increase of the angular momentum of the i-th barycenter is equal to the torque of the total external force acting upon C_i.

According to Equation (45), we can get

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{J}_i'}{\mathrm{d}t} &= \frac{\mathrm{d}\left(\sum_{j=1}^{n_i}\boldsymbol{r}_{i_j}'\times\boldsymbol{P}_{i_j}'\right)}{\mathrm{d}t} = \frac{\mathrm{d}\left(\sum_{j=1}^{n_i}\boldsymbol{r}_{i_j}'\times m_{i_j}\dfrac{\mathrm{d}\boldsymbol{r}_{i_j}'}{\mathrm{d}t}\right)}{\mathrm{d}t} \\
&= \sum_{j=1}^{n_i}\frac{\mathrm{d}\boldsymbol{r}_{i_j}'}{\mathrm{d}t}\times m_{i_j}\frac{\mathrm{d}\boldsymbol{r}_{i_j}'}{\mathrm{d}t} + \sum_{j=1}^{n_i}\boldsymbol{r}_{i_j}'\times m_{i_j}\frac{\mathrm{d}^2\boldsymbol{r}_{i_j}'}{\mathrm{d}t^2} \\
&= \sum_{j=1}^{n_i}\boldsymbol{r}_{i_j}'\times m_{i_j}\left(\ddot{\boldsymbol{r}}_{i_j}-\ddot{\boldsymbol{r}}_{C_i}\right) = \sum_{j=1}^{n_i}\boldsymbol{r}_{i_j}'\times\boldsymbol{F}_{i_j} + \ddot{\boldsymbol{r}}_{C_i}\times\sum_{j=1}^{n_i}m_{i_j}\boldsymbol{r}_{i_j}' \\
&= \sum_{j=1}^{n_i}\boldsymbol{r}_{i_j}'\times\boldsymbol{F}_{i_j} = \boldsymbol{M}_i'
\end{aligned}
$$

Namely

$$\frac{\mathrm{d}\boldsymbol{J}_i'}{\mathrm{d}t} = \boldsymbol{M}_i' \tag{50}$$

That is, the rate of increase of the total angular momentum of the i-th particle group about its barycenter C_i is equal to the total torque about C_i of the external forces acting upon each particle.

According to the above rules, angular momentum laws for a barycenter group can be summarized as follows:

1. About any fixed point O in an inertial frame, the rate of increase of the angular momentum $\boldsymbol{J}$ *of a barycenter group is equal to the torque* $\boldsymbol{M}$ *of the total external force acting upon each particle; the rate of increase of the angular momentum* $\boldsymbol{J}_{C_i}$ *of the i-th barycenter is equal to the torque* $\boldsymbol{M}_{C_i}$ *of the total external force acting upon* C_i*.*

2. The rate of increase of the total angular momentum $\boldsymbol{J}_i'$ *of the i-th particle group about its barycenter* C_i *is equal to the total torque* $\boldsymbol{M}_i'$ *about* C_i *of the external forces* $\boldsymbol{F}_{i_j}$ *acting upon each particle.*

3. About any fixed point O in an inertial frame, the total angular momentum of a barycenter group is equal to that the total angular momentum of each barycenter C_i *about O plus the total angular momentum of each particle group about its barycenter.*

4. A barycenter group is subjected to external forces $\boldsymbol{F}_{i_j}$*, the total torque of* $\boldsymbol{F}_{i_j}$ *about any fixed point O is equal to that the total torque of external forces* $\boldsymbol{F}_{C_i}$ *acting upon n barycenters about O plus the total torque of* $\boldsymbol{F}_{i_j}$ *about barycenter* C_i*.*

6. Discussions and Conclusions

The study and understanding of the laws of physics by humans is always deepening [32] [33]. Mutual interactions between any actual objects are the interactions between particle groups. Thus, it is necessary to study the mechanics laws for the system composed of multiple particle groups. This paper fills the theoretical gap in this area of classical mechanics.

On the basis of multi-particle mechanics, we have established the multi-barycenter mechanics and proposed the motion principle, the linear momentum principle, conservation principle of linear momentum etc. for a barycenter group. Any object in reality is a group of particles, such as the universe, galaxies, stars, molecules, atoms and so on. Even basic particles without the internal structure can be regarded as a particle group consisting of one particle. In nature, mutual interactions between any actual objects are the interactions between particle groups, thus any mechanical system in the real world is a multi-barycenter mechanical system. According to the laws of multi-barycenter mechanics, we will have an overall clear understanding of any complex mechanical system.

Theorem 1 presented in this thesis reveals the translation principles for a vector system. Using Theorem 2, the conventional multi-body problem can be transformed into a special two-body problem, by the method of simulation calculation, the motion law of each particle can be roughly obtained. According to Theorem 3, in general, the mutual interactions between any two particle groups are force and moment of couple at the same time, and the interaction resultant forces are not on a same straight line. Therefore, Newton's third law is sometimes established strictly, and sometimes it is approximately established. Theorem 4 finds the principle of the internal relationship between gravity and tidal force. Combined with Theorem 5 about the variation principle for a vector, the origin and change laws for the rotation angular momentum of galaxies and stars are clear.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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